A FABER–KRAHN INEQUALITY FOR MIXED LOCAL AND NONLOCAL OPERATORS*

By

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Abstract. We consider the first Dirichlet eigenvalue problem for a mixed local/nonlocal elliptic operator and we establish a quantitative Faber–Krahn inequality. More precisely, we show that balls minimize the first eigenvalue among sets of given volume and we provide a stability result for sets that almost attain the minimum.

1 Introduction

At the end of the XIX century, relying on explicit calculations on suitable domains, John William Strutt, 3rd Baron Rayleigh, conjectured that the ball is the minimizer of the first Dirichlet eigenvalue among the domains of a given volume; see [52]. The confirmation of this conjecture entails a number of interesting physical consequences, such as:

- among all drums of a given surface, the circular drum produces the lowest voice,
- among all the regions of a given volume with the boundary maintained at a constant (say, zero) temperature, the one which dissipates heat at the slowest possible rate is the sphere.

Also, the statement with equal volume constraints gives as a byproduct the one with equal perimeter constraint (thanks to the scaling property of the first eigenvalue and the isoperimetric inequality). In this sense, the first attempt to prove Lord Rayleigh's conjecture dates back to 1918, when Richard Courant established the

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above claim with equal perimeter constraint; see [19]. Then, using rearrangement methods and the variational characterization of eigenvalues, the original conjecture with volume constraint was established independently by Georg Faber and Edgar Krahn; see [33, 47, 48]. See also [42, Chapter 2] and [46]. We refer to [20] for similar results in the context of composite membranes.

Given that balls are actually established to be the unique minimizers for the first eigenvalue under volume constraint (hence if the first eigenvalue is equal to that of the corresponding ball, then the domain must necessarily be a ball), an intense research activity focused on quantitative versions of the Faber–Krahn inequality: roughly speaking, if the eigenvalue is "close to the one of the ball", can one deduce that the domain is also "close to a ball"? Classical results in this direction have been obtained by Wolfhard Hansen and Nikolai Nadirashvili in [41] and Antonios Melas in [49], and sharp bounds in terms of the so-called Fraenkel asymmetry have been obtained recently by Lorenzo Brasco, Guido De Philippis and Bozhidar Velichkov in [12]. See also [2] for some stability results in space forms.

The goal of this paper is to obtain a Faber–Krahn inequality and a quantitative version of it for an elliptic operator of mixed order. More specifically, for the sake of simplicity, we will focus on operators obtained by the superposition of a classical and a fractional Laplacian, namely of operators of the form

$$\mathcal{L} := -\Delta + (-\Delta)^s,$$

with $s \in (0, 1)$ and

$$(-\Delta)^s u(x) := \frac{1}{2}\int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}}\, dy.$$

Operators of this type present interesting mathematical questions, especially due to the lack of scale invariance and in view of the combination of local and nonlocal behaviors; see [1, 3–5, 7, 8, 10, 14–18, 21–24, 29, 32, 35, 43, 44, 53]. Moreover, they possess a concrete interest in applications since they model diffusion patterns with different time scales (loosely speaking, the higher order operator leading the diffusion for small times and the lower order operator becoming predominant for large times) and they arise in bi-modal power-law distribution processes; see [51]. Further applications arise in the theory of optimal searching strategies, biomathematics and animal foraging; see [30, 31] and the references therein. See also [11] for further applications.

In our setting, given a bounded open subset Ω of $\mathbb{R}^n$, we consider the first Dirichlet eigenvalue $\lambda_{\mathcal{L}}(\Omega)$ (see Section 3 for a detailed presentation) and we characterize the optimal set by the following result:

Theorem 1.1 (Faber–Krahn inequality for $\lambda_{\mathcal{L}}(\Omega)$). *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with boundary $\partial\Omega$ of class C^1. Let $m := |\Omega| \in (0, \infty)$, and let $B^{(m)}$ be any Euclidean ball with volume m. Then,*

$$\lambda_{\mathcal{L}}(\Omega) \geq \lambda_{\mathcal{L}}(B^{(m)}). \tag{1.1}$$

Moreover, if the equality holds in (1.1)*, then Ω is a ball.*

A related Faber–Krahn inequality has been recently obtained for radially symmetric, nonnegative and continuous kernels with compact support in [39]. With this respect, the case treated here of singular kernels seems to be new to the best of our knowledge. Additionally, and more importantly, we establish a stability result for inequality (1.1):

Theorem 1.2 (Quantitative Faber–Krahn inequality for $\lambda_{\mathcal{L}}(\Omega)$). *Let $s \in (0, 1/2)$. Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and uniformly convex set with boundary $\partial\Omega$ of class C^3.*

Then, there exists $\varepsilon_0 > 0$ with the following property: if $B \subseteq \mathbb{R}^n$ is a ball with $|B| = |\Omega|$ and

$$\lambda_{\mathcal{L}}(\Omega) \leq (1+\varepsilon)\lambda_{\mathcal{L}}(B), \quad \text{for some } 0 < \varepsilon < \varepsilon_0, \tag{1.2}$$

then there exist two balls $B^{(1)}, B^{(2)} \subset \mathbb{R}^n$ such that $B^{(1)} \subseteq \Omega \subseteq B^{(2)}$ and

$$|B^{(1)}| \geq (1 - C\varepsilon^{\frac{1}{2n}})|\Omega|, \tag{1.3}$$

and

$$|\Omega| \geq (1 - C\varepsilon^{\frac{1}{n(n+1)}})|B^{(2)}|, \tag{1.4}$$

where $C > 0$ is a structural constant.

The proof of Theorem 1.1 is based on the definition of principal eigenvalue of $\mathcal{L}$ and on the Polya–Szegö inequality.

The proof of Theorem 1.2 is more involved and requires some delicate geometric arguments. For this, a careful analysis of the superlevel sets of the principal eigenfunction is needed, combined with the use of a Bonnesen-type inequality, which is a strengthening of the classical isoperimetric inequality.

We stress that, while in Theorem 1.1 we only require the boundary of Ω to be C^1, in Theorem 1.2 we need to ask that the boundary of Ω is of class C^3. This is due to the fact that, in order to prove Theorem 1.2, we have to use the Faber–Krahn inequality in (1.1) for the superlevel sets of the principal eigenfunction and we are able to prove that these sets are convex and with boundary of class C^1 under the condition that Ω has a C^3-boundary; see the forthcoming Lemma 3.7.

The rest of this paper is organized as follows. In Section 2 we introduce the basic notation and the setting in which we work, and we provide some regularity results and an Hopf-type lemma for the operator $\mathcal{L}$. In Section 3 we introduce the notion of principal Dirichlet eigenvalue for $\mathcal{L}$ and we give some regularity results for the associated eigenfunction. Section 3 also contains a result (namely, Lemma 3.7) that proves the convexity of the superlevel sets of the first eigenfunction near the boundary of a convex domain: this result, based on a detailed use of the Inverse Function Theorem, highlights an interesting technical difference with respect to the classical case where one can exploit the convexity of all the level sets of the Dirichlet principal eigenfunction, due to its concavity as a function (which is unknown in the fractional case, see [45]).

Section 4 contains the proofs of Theorems 1.1 and 1.2. The paper finishes with a couple of appendices: in the first one, we discuss the optimality of a geometric lemma needed in the proof of Theorem 1.2, while in the second we prove Theorem 2.8.

2 Basic notions and preliminary results

In this section we properly introduce the relevant definitions and notation which shall be used throughout the rest of the paper. Moreover, we review/establish some regularity results concerning our operator $\mathcal{L} = -\Delta + (-\Delta)^s$.

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with boundary $\partial\Omega$ of class C^1. We then consider the space $\mathbb{X}(\Omega)$ defined as follows:

$$\mathbb{X}(\Omega) := \{u \in H^1(\mathbb{R}^n) : u \equiv 0 \text{ a.e. on } \mathbb{R}^n \setminus \Omega\}.$$

In view of the regularity of $\partial\Omega$, it is well-known that $\mathbb{X}(\Omega)$ can be naturally identified with $H_0^1(\Omega)$; more precisely, we have (see, e.g., [13, Prop. 9.18])

$$(2.1) \qquad u \in \mathbb{X}(\Omega) \Longrightarrow u|_\Omega \in H_0^1(\Omega) \quad \text{and} \quad u \in H_0^1(\Omega) \Longrightarrow u \cdot \mathbf{1}_\Omega \in \mathbb{X}(\Omega),$$

where $\mathbf{1}_\Omega$ denotes the indicator function of Ω. Throughout what follows, we tacitly identify a function $u \in H_0^1(\Omega)$ with its "zero-extension" $\widehat{u} = u \cdot \mathbf{1}_\Omega \in \mathbb{X}(\Omega)$.

We then observe that, as a consequence of (2.1), the set $\mathbb{X}(\Omega)$ is endowed with a structure of real Hilbert space by the scalar product

$$\langle u, v\rangle_{\mathbb{X}(\Omega)} := \int_\Omega \langle \nabla u, \nabla v\rangle \, dx.$$

The norm associated with this scalar product is

$$\|u\|_{\mathbb{X}(\Omega)} := \left(\int_\Omega |\nabla u|^2\,dx\right)^{1/2},$$

and the (linear) map $\mathcal{E}_0 : H_0^1(\Omega) \to \mathbb{X}(\Omega)$ defined by

$$\mathcal{E}_0(u) := u \cdot \mathbf{1}_\Omega$$

turns out to be a bijective isometry between $H_0^1(\Omega)$ and $\mathbb{X}(\Omega)$.

Let $s \in (0, 1)$. On the space $\mathbb{X}(\Omega)$, we consider the bilinear form

$$\mathcal{B}_{\Omega,s}(u, v) := \int_\Omega \langle \nabla u, \nabla v\rangle\,dx + \iint_{\mathbb{R}^{2n}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}}\,dx\,dy;$$

moreover, for every $u \in \mathbb{X}(\Omega)$ we define

$$\mathcal{D}_{\Omega,s}(u) := \mathcal{B}_{\Omega,s}(u, u). \tag{2.2}$$

Remark 2.1. We explicitly notice that $\mathcal{B}_{\Omega,s}$ is well-defined on $\mathbb{X}(\Omega)$ in view of the following facts: given any $u, v \in \mathbb{X}(\Omega)$, by Hölder's inequality we have

$$\iint_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)| \cdot |v(x) - v(y)|}{|x - y|^{n+2s}}\,dx\,dy \le [u]_s \cdot [v]_s, \tag{2.3}$$

where we have used the notation

$$[f]_s := \left(\iint_{\mathbb{R}^{2n}} \frac{|f(x) - f(y)|^2}{|x - y|^{n+2s}}\,dx\,dy\right)^{1/2} \quad \text{for all } f \in H^1(\mathbb{R}^n).$$

Furthermore, for every $f \in H^1(\mathbb{R}^n)$ one has (see, e.g., [26, Proposition 2.2])

$$[f]_s \le c_{n,s}\|f\|_{H^1(\mathbb{R}^n)}. \tag{2.4}$$

Gathering together (2.3) and (2.4), we then get

$$|\mathcal{B}_{\Omega,s}(u, v)| \le c_{n,s}^2 \|u\|_{H^1(\mathbb{R}^n)} \cdot \|v\|_{H^1(\mathbb{R}^n)} < \infty.$$

Using the bilinear form $\mathcal{B}_{\Omega,s}$, we can give the following definition.

Definition 2.2. Let $f \in L^2(\Omega)$. We say that a function $u : \mathbb{R}^n \to \mathbb{R}$ is a weak solution of the $\mathcal{L}$-Dirichlet problem

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega, \\ u \equiv 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{(D)$_f$}$$

if it satisfies the following properties:

(1) $u \in \mathbb{X}(\Omega)$;

(2) for every test function $\varphi \in C_0^\infty(\Omega)$, one has

$$\mathcal{B}_{\Omega,s}(u,\varphi) = \int_\Omega f\varphi\,dx.$$

Remark 2.3. Let $f \in L^2(\Omega)$ and $u \in \mathbb{X}(\Omega)$. Since $C_0^\infty(\Omega)$ is dense in $\mathbb{X}(\Omega)$, we see that u is a weak solution of $(\mathrm{D})_f$ if and only if

$$\mathcal{B}_{\Omega,s}(u,v) = \int_\Omega fv\,dx \quad \text{for every } v \in \mathbb{X}(\Omega).$$

Then, by applying the Lax–Milgram Theorem to the bilinear form $\mathcal{B}_{\Omega,s}$, one can prove the following existence result (see, e.g., [7, Theorem. 1.1]).

Theorem 2.4. *For every $f \in L^2(\Omega)$, there exists a unique weak solution $u_f \in \mathbb{X}(\Omega)$ of $(\mathrm{D})_f$, further satisfying the a-priori estimate*

$$\|u_f\|_{\mathbb{X}(\Omega)} \leq \mathbf{c}_0\|f\|_{L^2(\Omega)}.$$

Here, $\mathbf{c}_0 > 0$ is a constant independent of f.

Since we aim at proving some global regularity results for u_f, it is convenient to fix also the definition of classical solution of problem $(\mathrm{D})_f$. For this, we recall the notation $C_b(\mathbb{R}^n) := C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Definition 2.5. Let $f : \Omega \to \mathbb{R}$. We say that a function $u : \mathbb{R}^n \to \mathbb{R}$ is a classical solution of $(\mathrm{D})_f$ if it satisfies the following properties:

(1) $u \in C_b(\mathbb{R}^n) \cap C^2(\Omega)$;

(2) $u \equiv 0$ pointwise in $\mathbb{R}^n \setminus \Omega$;

(3) $\mathcal{L}u(x) = f(x)$ pointwise for every $x \in \Omega$.

Remark 2.6. (1) We explicitly observe that, if $u \in C_b(\mathbb{R}^n) \cap C^2(\Omega)$, it is possible to compute $\mathcal{L}u(x)$ pointwise for every $x \in \Omega$. Indeed, we have

$$(-\Delta)^s u(x) = -\frac{1}{2}\int_{\mathbb{R}^n} \frac{u(x+z)+u(x-z)-2u(x)}{|z|^{n+2s}}\,dz,$$

and the regularity of u ensures that the "second-order" different quotient

$$z \ni \mathbb{R}^n \mapsto \frac{u(x+z)+u(x-z)-2u(x)}{|z|^{n+2s}}$$

is in $L^1(\mathbb{R}^n)$. To be more precise, for every $x \in \mathbb{R}^n$ we have

$$\begin{aligned}&\frac{|u(x+z)+u(x-z)-2u(x)|}{|z|^{n+2s}}\\ &\quad\leq c_n\Big(\frac{\|u\|_{C^2(B(x,\rho_x))}}{|z|^{n+2s-2}}\cdot \mathbf{1}_{B(0,\rho_x)} + \frac{\|u\|_{L^\infty(\mathbb{R}^n)}}{|z|^{n+2s}}\cdot \mathbf{1}_{\mathbb{R}^n\setminus B(0,\rho_x)}\Big),\end{aligned}$$

where $c_n > 0$ is a suitable constant and $\rho_x > 0$ is such that $B(x,\rho_x) \Subset \Omega$.

(2) Assume that $f \in L^2(\Omega)$, and let $u_f \in \mathbb{X}(\Omega)$ be the (unique) weak solution of $(\mathrm{D})_f$, according to Theorem 2.4. If we further assume that $u_f \in C_b(\mathbb{R}^n) \cap C^2(\Omega)$, we can compute

$$\mathcal{L}u_f(x) = -\Delta u_f(x) + (-\Delta)^s u_f(x) \quad \text{pointwise for every } x \in \Omega.$$

Then, a standard integration-by-parts argument shows that u_f is also a classical solution of $(\mathrm{D})_f$. Conversely, if $u \in C_b(\mathbb{R}^n) \cap C^2(\Omega)$ is a classical solution of $(\mathrm{D})_f$ such that $u \in H^1(\mathbb{R}^n)$, then u is also a weak solution of $(\mathrm{D})_f$.

The first regularity result that we aim to prove is a global $C^{1,\alpha}$-regularity theorem, which relies on the $W^{2,p}$-theory for $\mathcal{L}$ developed by Bensoussan and Lions [6].

Theorem 2.7. *Let $f \in L^\infty(\Omega)$ and let $u_f \in \mathbb{X}(\Omega)$ denote the unique weak solution of $(\mathrm{D})_f$. Moreover, let us assume that $\partial\Omega$ is of class $C^{1,1}$. Then,*

$$u_f \in C^{1,\beta}(\overline{\Omega}) \quad \textit{for some } \beta \in (0,1). \tag{2.5}$$

Proof. Note that $f \in L^p(\Omega)$ for every $p \geq 2$. We utilize [36, Theorem 3.1.22] to obtain $W^{2,p}$-regularity for some $p > n$.[1] By combining this with the Sobolev embedding theorem, we obtain (2.5). □

Furthermore, we recall that the $C^{2,\alpha}$**-regularity of** u_f when $f \in C^\alpha(\overline{\Omega})$ can be deduced by applying [36, Theorem 3.1.12]. We refer to Appendix B for an explicit proof.

Theorem 2.8. *Let $s \in (0, 1/2)$ and $\alpha \in (0,1)$. Suppose that $\partial\Omega$ is of class $C^{2,\alpha}$. If $f \in C^\alpha(\overline{\Omega})$ and if $u_f \in \mathbb{X}(\Omega)$ denotes the unique weak solution of $(\mathrm{D})_f$ (according to Theorem 2.4), then*

$$u_f \in C_b(\mathbb{R}^n) \cap C^{2,\alpha}(\overline{\Omega}).$$

In particular, u_f is a classical solution of $(\mathrm{D})_f$.

We also recall that a regularity result up to the boundary for eigenfunctions of mixed operators has been recently obtained in [25] for radially symmetric, nonnegative and continuous kernels with compact support.

We close this section by stating, for future reference, the following Hopf-type lemma for our operator $\mathcal{L}$. Similar statements for this type of operators have been proved in [36, Theorem 3.1.5], but with the additional assumption that $s \in (0, 1/2)$.

[1] For instance, in the most delicate case $s \in [1/2, 1)$, one can use the setting in [36] with $j(x,\xi) := \xi$, $m(x,\xi) := 1$, $\pi(d\xi) := |\xi|^{-n-2s}\, d\xi$, $\bar{j}(\xi) := \xi$, $\gamma := 2s + \epsilon$ (with $\epsilon > 0$ conveniently small), $\theta := 0$, $\gamma_1 := \gamma$ and $\lambda_1(\xi) := |\xi|^\gamma$.

Alternatively to [36], one can use [6, Theorem 3.2.3].

Theorem 2.9. *Let $c_0 \in \mathbb{R}$ and $\overline{\varepsilon} \in (0,1)$. Let $u \in C_b(\mathbb{R}^n) \cap C^2(\overline{\Omega})$ be such that*

$$\mathcal{L}u \geq 0 \quad \textit{pointwise in } \Omega. \tag{2.6}$$

Let $\xi \in \partial\Omega$. Assume that $u = c_0$ on $B_{\overline{\varepsilon}}(\xi) \cap \partial\Omega$ and that $u \geq c_0$ in $\mathbb{R}^n$. Suppose also that $u \not\equiv c_0$ and that $B_{\overline{\varepsilon}}(\xi) \cap \partial\Omega$ is of class C^2. Finally, let ν be the outer unit normal to $\partial\Omega$ at ξ. Then,

$$\partial_\nu u(\xi) < 0. \tag{2.7}$$

Proof. Without loss of generality, we suppose that $c_0 = 0$ and that ξ coincides with the origin. In particular,

$$u(0) = 0. \tag{2.8}$$

Also, by the regularity of $B_{\overline{\varepsilon}}(\xi) \cap \partial\Omega$, we suppose that there exists $\rho_0 > 0$ such that $B_{\rho_0}(\rho_0 e_n) \subseteq \Omega$ touches the boundary of Ω at the origin. We remark that, by assumption, there exists a point $p \in \mathbb{R}^n$ such that $u(p) > 0$, and therefore, by continuity, there exists a ball $B \subset \mathbb{R}^n$ such that $u > 0$ in B. In light of this and (2.8), we can find $\varepsilon_0 \in (0, \overline{\varepsilon}/4)$ sufficiently small such that

$$B \subset \mathbb{R}^n \setminus B_\varepsilon \quad \text{for all } \varepsilon \in (0, \varepsilon_0].$$

Now, we claim that there exists $\rho \in (0, \rho_0]$ such that

$$(-\Delta)^s u \leq 0 \quad \text{in } B_\rho(\rho e_n). \tag{2.9}$$

To prove this, we suppose by contradiction that for every $k \in \mathbb{N}$ there exists

$$p_k \in B_{1/k}(e_n/k) \subseteq \Omega$$

such that $(-\Delta)^s u(p_k) > 0$. This implies that, for every $\varepsilon \in (0, \varepsilon_0]$,

$$\begin{aligned} 0 &\leq \liminf_{k \to +\infty} \int_{\mathbb{R}^n} \frac{2u(p_k) - u(p_k + y) - u(p_k - y)}{|y|^{n+2s}}\, dy \\ &= \liminf_{k \to +\infty} \left(\int_{B_\varepsilon} \frac{2u(p_k) - u(p_k + y) - u(p_k - y)}{|y|^{n+2s}}\, dy \right. \\ &\qquad \left. + \int_{\mathbb{R}^n \setminus B_\varepsilon} \frac{2u(p_k) - u(p_k + y) - u(p_k - y)}{|y|^{n+2s}}\, dy \right). \end{aligned} \tag{2.10}$$

Notice that $p_k \to 0$ as $k \to +\infty$, and therefore, by the Dominated Convergence

Theorem, one sees that

$$\begin{aligned}\lim_{k\to+\infty}\int_{\mathbb{R}^n\setminus B_\varepsilon}&\frac{2u(p_k)-u(p_k+y)-u(p_k-y)}{|y|^{n+2s}}\,dy\\&=\int_{\mathbb{R}^n\setminus B_\varepsilon}\frac{2u(0)-u(y)-u(-y)}{|y|^{n+2s}}\,dy=-\int_{\mathbb{R}^n\setminus B_\varepsilon}\frac{u(y)+u(-y)}{|y|^{n+2s}}\,dy\\&\le-\int_B\frac{u(y)+u(-y)}{|y|^{n+2s}}\,dy\le-\int_B\frac{u(y)}{|y|^{n+2s}}\,dy\le -c,\end{aligned}$$

for some $c > 0$ independent of ε. Plugging this information into (2.10), we obtain that, for every $\varepsilon \in (0, \varepsilon_0]$,

$$c\le\liminf_{k\to+\infty}\int_{B_\varepsilon}\frac{2u(p_k)-u(p_k+y)-u(p_k-y)}{|y|^{n+2s}}\,dy. \tag{2.11}$$

Now we claim that there exists $\widetilde{u}\in C^{1,1}(B_{2\varepsilon})$ such that

$$\widetilde{u}=u \text{ in } B_{2\varepsilon}\cap\Omega \quad\text{and}\quad \widetilde{u}\le u \text{ in } B_{2\varepsilon}\setminus\Omega. \tag{2.12}$$

To this end, we consider a C^2-diffeomorphism Φ such that

(a) $B_{2\varepsilon}\subseteq\Phi(B_{C_0\varepsilon})$ for a suitable $C_0 > 0$;
(b) $B_{2\varepsilon}\cap\Omega\subseteq\Phi(B_{C_0\varepsilon}\cap\{x_n>0\})\subseteq\Omega$;
(c) $B_{2\varepsilon}\setminus\Omega\subseteq\Phi(B_{C_0\varepsilon}\setminus\{x_n>0\})\subseteq\mathbb{R}^n\setminus\Omega$.

Possibly reducing ε we can also assume that

$$\Phi(B_{C_0\varepsilon})\subseteq B_{\overline{\varepsilon}/2}.$$

For all $x\in B_{C_0\varepsilon}$ we define $U(x):=u(\Phi(x))$ and we notice that U is a $C^{1,1}$ function in $B_{C_0\varepsilon}\cap\{x_n\ge 0\}$ and $U=0$ along $\{x_n=0\}$. Thus, we define

$$\widetilde{U}(x):=\begin{cases}U(x',x_n) & \text{if } x_n>0,\\ -U(x',-x_n) & \text{if } x_n\le 0.\end{cases}$$

We observe that $\widetilde{U}$ is continuous across $\{x_n=0\}$ and that

$$\nabla\widetilde{U}(x)=\begin{cases}\nabla U(x',x_n) & \text{if } x_n>0,\\ \nabla U(x',-x_n) & \text{if } x_n<0.\end{cases}$$

This gives that $\widetilde{U}$ is a C^1 function in $B_{C_0\varepsilon}$. Moreover, we claim that

$$\widetilde{U} \text{ is a } C^{1,1} \text{ function in } B_{C_0\varepsilon}. \tag{2.13}$$

To check this, it suffices to consider $x = (x', x_n)$ and $y = (y', y_n)$ with $x_n > 0 > y_n$ and observe that

$$\begin{aligned}|\nabla\widetilde{U}(x) - \nabla\widetilde{U}(y)| &= |\nabla U(x', x_n) - \nabla U(y', -y_n)| \\ &\le |\nabla U(x', x_n) - \nabla U(x', 0)| + |\nabla U(x', 0) - \nabla U(y', 0)| \\ &\quad + |\nabla U(y', 0) - \nabla U(y', -y_n)| \\ &\le \|U\|_{C^{1,1}(B_{C_0\varepsilon}\cap\{x_n\ge 0\})}(x_n + |x' - y'| - y_n) \\ &\le 2\|U\|_{C^{1,1}(B_{C_0\varepsilon}\cap\{x_n\ge 0\})}|x - y|,\end{aligned}$$

thus establishing (2.13).

Hence, for every $x \in B_{2\varepsilon}$ we define $\widetilde{u}(x) := \widetilde{U}(\Phi^{-1}(x))$ and we infer from (2.13) that $\widetilde{u} \in C^{1,1}(B_{2\varepsilon})$. Furthermore, if $x \in B_{2\varepsilon} \cap \Omega$, then

$$\Phi^{-1}(x) \in B_{C_0\varepsilon} \cap \{x_n > 0\}$$

and, as a result, we have that

$$\widetilde{u}(x) = \widetilde{U}(\Phi^{-1}(x)) = U(\Phi^{-1}(x)) = u(x).$$

If instead $x \in B_{2\varepsilon} \setminus \Omega$, then $\Phi^{-1}(x) \in B_{C_0\varepsilon} \cap \{x_n \le 0\}$. Hence, letting

$$(y', y_n) := \Phi^{-1}(x)$$

and using that $u \ge 0$, we get in this case that

$$\begin{aligned}\widetilde{u}(x) &= \widetilde{U}(\Phi^{-1}(x)) = \widetilde{U}(y', y_n) = -U(y', -y_n) \le 0 \\ &\le U(y', y_n) = U(\Phi^{-1}(x)) = u(x).\end{aligned}$$

These observations complete the proof of (2.12).

By (2.12) it follows that

$$2u(p_k) - u(p_k + y) - u(p_k - y) \le 2\widetilde{u}(p_k) - \widetilde{u}(p_k + y) - \widetilde{u}(p_k - y).$$

Hence, if $y \in B_\varepsilon$,

$$2u(p_k) - u(p_k + y) - u(p_k - y) \le \|\widetilde{u}\|_{C^{1,1}(B_{2\varepsilon})}|y|^2$$

and accordingly

$$\begin{aligned}\int_{B_\varepsilon} \frac{2u(p_k) - u(p_k + y) - u(p_k - y)}{|y|^{n+2s}}\,dy &\le \int_{B_\varepsilon} \|\widetilde{u}\|_{C^{1,1}(B_{2\varepsilon})}|y|^{2-n-2s}\,dy \\ &\le C\|\widetilde{u}\|_{C^{1,1}(B_{2\varepsilon})}\varepsilon^{2-2s}.\end{aligned}$$

This and (2.11) entail that $c \leq C\|\widetilde{u}\|_{C^{1,1}(B_{2\varepsilon})}\varepsilon^{2-2s}$, and we thereby obtain a contradiction by choosing ε conveniently small. This completes the proof of (2.9).

From (2.6) and (2.9) we deduce that there exists $\rho > 0$ such that

$$0 \leq \mathcal{L}u = -\Delta u + (-\Delta)^s u \leq -\Delta u \quad \text{in } B_\rho(\rho e_n).$$

Moreover, we have $u > 0$ in $B_\rho(\rho e_n)$. Indeed, if there exists a point $q \in B_\rho(\rho e_n)$ such that $u(q) = 0$, then by the Maximum Principle (see [7, Theorem 1.4]) we would have that $u \equiv 0$ in $\mathbb{R}^n$, which would contradict our hypothesis. As a consequence, we are in the position of applying the Hopf Lemma for the classical Laplacian (see e.g. [37, Lemma 3.4]) thus obtaining the desired result in (2.7). □

Remark 2.10. We stress that

(2.14) the assumption in Theorem 2.9 that $u(\xi) \leq u(x)$ for all $x \in \mathbb{R}^n$ cannot be weakened by assuming that $u(\xi) \leq u(x)$ for all $x \in \overline{\Omega}$.

As a counterexample, let u be the minimizer of

$$\int_{-1}^{1} |\nabla u(x)|^2 \, dx + \iint_{\mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x - y|^{1+2s}} \, dx \, dy$$

among the functions in $H_0^1(\mathbb{R})$ such that $u = u_0$ outside $(-1, 1)$, for a given smooth function u_0 such that

$$\chi_{(-4,-3)\cup(3,4)} \leq u_0 \leq \chi_{(-5,-2)\cup(2,5)}.$$

By the Sobolev Embedding Theorem, we know that u is continuous in $\mathbb{R}$ and, in view of its minimality property, it satisfies $\mathcal{L}u = 0$ in $(-1, 1)$. Thus, let $\xi \in [-1, 1]$ be such that $u(\xi) = \min_{[-1,1]} u$. We claim that

$$\xi \in (-1, 1). \tag{2.15}$$

Indeed, if not, we would have that $u \geq 0$ in $[-1, 1]$. Since $u \leq 0$ outside $(-1, 1)$ we know from the weak maximum principle (see, e.g., [7, Theorem 1.2]) that

$$u \leq 0 \quad \text{in } [-1, 1],$$

and consequently u vanishes identically in $[-1, 1]$. In particular, the origin would provide an interior maximum for u. Accordingly, by the strong maximum principle (see, e.g., [7, Theorem 1.4]), we gather that u vanishes identically in $\mathbb{R}$. Since this is a contradiction with the values of u in $(-4, -3) \cup (3, 4)$, the proof of (2.15) is completed. Now, from (2.15), we deduce that

$$u'(\xi) = 0. \tag{2.16}$$

We also take $\delta > 0$ such that $\Omega := (\xi, \xi + \delta) \subset (-1, 1)$. By construction, we have that ξ is a minimizing point for u in $\overline{\Omega}$ and that $\mathcal{L}u = 0$ in Ω. Thus, equation (2.16) proves the observation in (2.14).

3 The principal Dirichlet eigenvalue for $\mathcal{L}$

In this section we introduce the notion of principal Dirichlet eigenvalue for $\mathcal{L}$, and we prove some regularity results for the associated eigenfunctions. In what follows, we tacitly inherit all the assumptions and notation introduced so far; in particular, $s \in (0, 1)$ and $\Omega \subseteq \mathbb{R}^n$ is a bounded open set with C^1 boundary.

To begin with, we give the following definition.

Definition 3.1. We define the **principal Dirichlet eigenvalue of $\mathcal{L}$ in Ω** as

$$\lambda_{\mathcal{L}}(\Omega) := \inf\left\{ \frac{\mathcal{D}_{\Omega,s}(u)}{\|u\|^2_{L^2(\Omega)}} : u \in \mathbb{X}(\Omega) \setminus \{0\} \right\} \in [0, \infty), \tag{3.1}$$

where $\mathcal{D}_{\Omega,s}$ is the quadratic form defined in (2.2).

Remark 3.2. Before proceeding we list, for a future reference, some simple properties of $\lambda_{\mathcal{L}}(\cdot)$ which easily follow from its very definition.

(1) For every $x_0 \in \mathbb{R}^n$, one has

$$\lambda_{\mathcal{L}}(\Omega) = \lambda_{\mathcal{L}}(x_0 + \Omega).$$

(2) For every $t > 0$, one has
   (i) $t^{-2s}\lambda_{\mathcal{L}}(\Omega) \leq \lambda_{\mathcal{L}}(t\Omega) \leq t^{-2}\lambda_{\mathcal{L}}(\Omega)$ if $0 < t \leq 1$;
   (ii) $t^{-2}\lambda_{\mathcal{L}}(\Omega) \leq \lambda_{\mathcal{L}}(t\Omega) \leq t^{-2s}\lambda_{\mathcal{L}}(\Omega)$ if $t > 1$.

(3) Let $\lambda_1(\Omega)$ be the principal Dirichlet eigenvalue of $-\Delta$ in Ω, i.e.,

$$\lambda_1(\Omega) := \inf\left\{ \frac{\int_\Omega |\nabla u|^2\,dx}{\|u\|^2_{L^2(\Omega)}} : u \in H^1_0(\Omega) \setminus \{0\} \right\}.$$

   Then, one has the bound

$$\lambda_{\mathcal{L}}(\Omega) \geq \lambda_1(\Omega) > 0.$$

(4) There exists a positive constant $c = c(\Omega) > 0$ such that

$$\lambda_{\mathcal{L}}(\Omega) \leq c\lambda_1(\Omega).$$

Then, by using standard arguments of the Calculus of Variations (see, e.g., the approach in [54, Proposition 9]), it is possible to prove the following result.

Theorem 3.3. *The infimum in* (3.1) *is achieved. More precisely, there exists a unique* $u_0 \in \mathbb{X}(\Omega) \setminus \{0\}$ *such that:*

(1) $\|u_0\|_{L^2(\Omega)} = 1$ *and* $\lambda_{\mathcal{L}}(\Omega) = \mathcal{D}_{\Omega,s}(u_0)$*;*

(2) $u_0 \geq 0$ *a.e. in* Ω*.*

Furthermore, u_0 *is a weak solution of the following problem:*

$$\begin{cases} \mathcal{L}u = \lambda_{\mathcal{L}}(\Omega)u & \text{in } \Omega, \\ u \equiv 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{3.2}$$

namely

$$\begin{aligned} &\int_\Omega \langle \nabla u_0, \nabla v \rangle \, dx + \iint_{\mathbb{R}^{2n}} \frac{(u_0(x) - u_0(y))(v(x) - v(y))}{|x-y|^{n+2s}} \, dx \, dy \\ &\qquad = \lambda_{\mathcal{L}}(\Omega) \int_{\mathbb{R}^n} u_0 v \, dx \quad \text{for every } v \in \mathbb{X}(\Omega). \end{aligned} \tag{3.3}$$

Definition 3.4. We refer to u_0 given by Theorem 3.3 as the **principal eigenfunction of** $\mathcal{L}$ (in Ω).

We devote the rest of this section to prove some regularity properties of u_0. To begin with, we establish the following global boundedness result.

Theorem 3.5. *The principal eigenfunction* u_0 *is globally bounded on* Ω*.*

The proof of Theorem 3.5 relies on the classical method by Stampacchia, and is essentially analogous to that of [7, Theorem 4.7] (see also [28, 55]). However, we present it here with all the details for the sake of completeness.

Proof of Theorem 3.5. To begin with, we arbitrarily fix $\delta > 0$ and we set

$$\widetilde{u}_0 := \sqrt{\delta} u_0.$$

Moreover, for every $k \in \mathbb{N}$, we define $C_k := 1 - 2^{-k}$ and

$$v_k := \widetilde{u}_0 - C_k, \quad w_k := (v_k)_+ := \max\{v_k, 0\}, \quad U_k := \|w_k\|^2_{L^2(\Omega)}.$$

We explicitly point out that, in view of these definitions, one has

(a) $\|\widetilde{u}_0\|^2_{L^2(\Omega)} = \delta \|u_0\|^2_{L^2(\Omega)} = \delta$;

(b) $w_0 = v_0 = \widetilde{u}_0$ (since $C_0 = 0$);

(c) $v_k \geq v_{k+1}$ and $w_k \geq w_{k+1}$ (since $C_k < C_{k+1}$).

We now observe that, since $u_0 \in \mathbb{X}(\Omega) \subseteq H^1(\mathbb{R}^n)$, we have $v_k \in H^1_{\mathrm{loc}}(\mathbb{R}^n)$; furthermore, since $\widetilde{u}_0 = \sqrt{\delta} u_0 \equiv 0$ a.e. in $\mathbb{R}^n \setminus \Omega$, one also has

$$v_k = \widetilde{u}_0 - C_k = -C_k < 0 \quad \text{on } \mathbb{R}^n \setminus \Omega,$$

and thus $w_k = (v_k)_+ \in \mathbb{X}(\Omega)$ (recall that Ω is bounded). We are then entitled to use the function w_k as a test function in (3.3), obtaining

$$\int_\Omega \langle \nabla \widetilde{u}_0, \nabla w_k \rangle \, dx + \iint_{\mathbb{R}^{2n}} \frac{(\widetilde{u}_0(x) - \widetilde{u}_0(y))(w_k(x) - w_k(y))}{|x-y|^{n+2s}} \, dx \, dy = \lambda_{\mathcal{L}}(\Omega) \int_\Omega \widetilde{u}_0 w_k \, dx. \tag{3.4}$$

To proceed further we notice that, for a.e. $x, y \in \mathbb{R}^n$, one has

$$\begin{aligned} |w_k(x) - w_k(y)|^2 &= |(v_k)_+(x) - (v_k)_+(y)|^2 \\ &\le ((v_k)_+(x) - (v_k)_+(y))(v_k(x) - v_k(y)) \\ &= (w_k(x) - w_k(y))(\widetilde{u}_0(x) - \widetilde{u}_0(y)). \end{aligned} \tag{3.5}$$

Moreover, taking into account the definition of w_k, we get

$$\int_\Omega \langle \nabla \widetilde{u}_0, \nabla w_k \rangle \, dx = \int_{\Omega \cap \{\widetilde{u}_0 > C_k\}} \langle \nabla \widetilde{u}_0, \nabla v_k \rangle \, dx = \int_\Omega |\nabla w_k(x)|^2 \, dx. \tag{3.6}$$

Gathering together (3.4), (3.5) and (3.6), we obtain

$$\int_\Omega |\nabla w_k(x)|^2 \, dx \le \lambda_{\mathcal{L}}(\Omega) \int_\Omega \widetilde{u}_0 w_k \, dx. \tag{3.7}$$

We then claim that, for every $k \ge 1$, one has

$$\widetilde{u}_0 < 2^k w_{k-1} \quad \text{on } \{w_k > 0\}. \tag{3.8}$$

Indeed, if $x \in \{w_k > 0\}$, we have $\widetilde{u}_0(x) > C_k > C_{k-1}$; as a consequence,

$$\begin{aligned} 2^k w_{k-1}(x) &\ge (2^k - 1) w_{k-1}(x) = (2^k - 1)(\widetilde{u}_0(x) - C_{k-1}) \\ &= \widetilde{u}_0(x) + (2^k - 2)(\widetilde{u}_0(x) - C_k) > \widetilde{u}_0(x), \end{aligned}$$

which is exactly the claimed (3.8). By combining (3.8) with (3.7), and taking into account that $w_k \le w_{k-1}$ a.e. in $\mathbb{R}^n$, for every $k \ge 1$ we get

$$\begin{aligned} \int_\Omega |\nabla w_k(x)|^2 \, dx &\le \lambda_{\mathcal{L}}(\Omega) \int_{\{w_k > 0\}} \widetilde{u}_0 w_k \, dx \\ &\le 2^k \cdot \lambda_{\mathcal{L}}(\Omega) \int_{\{w_k > 0\}} w_{k-1} w_k \, dx \\ &\le 2^k \cdot \lambda_{\mathcal{L}}(\Omega) \int_\Omega w_{k-1}^2 \, dx = 2^k \cdot \lambda_{\mathcal{L}}(\Omega) \|w_{k-1}\|^2_{L^2(\Omega)} \\ &= 2^k \cdot \lambda_{\mathcal{L}}(\Omega) U_{k-1}. \end{aligned} \tag{3.9}$$

We now turn to estimate from below the left-hand side of (3.9). To this end we first observe that, owing to the very definition of w_k, we have

$$\{w_k > 0\} = \{\widetilde{u}_0 > C_k\} \subseteq \{w_{k-1} > 2^{-k}\} \quad \text{for all } k \geq 1.$$

As a consequence, we obtain

$$\begin{aligned} U_{k-1} &= \int_\Omega w_{k-1}^2\,dx \geq \int_{\{w_{k-1}>2^{-k}\}} w_{k-1}^2\,dx \\ &\geq 2^{-2k}|\{w_{k-1} > 2^{-k}\}| \geq 2^{-2k}|\{w_k > 0\}|. \end{aligned} \tag{3.10}$$

Using the Hölder inequality (with exponents $2^*/2$ and $n/2$, being $2^* := \frac{2n}{n-2}$ the Sobolev exponent), jointly with the Sobolev inequality, from (3.9)–(3.10) we obtain the following estimate for every $k \geq 1$:

$$\begin{aligned} U_k = \|w_k\|^2_{L^2(\Omega)} &\leq \left(\int_\Omega w_k^{2^*}\,dx\right)^{2/2^*} |\{w_k > 0\}|^{2/n} \\ &\leq \mathbf{c}_S \int_\Omega |\nabla w_k|^2\,dx \cdot |\{w_k > 0\}|^{2/n} \\ &\leq \mathbf{c}_S (2^k \lambda_{\mathcal{L}}(\Omega) U_{k-1})(2^{2k} U_{k-1})^{2/n} \\ &= \mathbf{c}'(2^{1+4/n})^{k-1} U_{k-1}^{1+2/n}, \end{aligned} \tag{3.11}$$

where $\mathbf{c}_S$ is the Sobolev constant and $\mathbf{c}' := 2^{1+4/n}\mathbf{c}_S\lambda_{\mathcal{L}}(\Omega)$.

With estimate (3.11) at hand, we are finally ready to complete the proof. Indeed, since $2/n > 0$ and

$$\eta := 2^{1+4/n} > 1,$$

we deduce from [38, Lemma 7.1] that $U_k \to 0$ as $k \to \infty$, provided that

$$U_0 = \|\widetilde{u}_0\|^2_{L^2(\Omega)} = \delta < (\mathbf{c}')^{-n/2}\eta^{-n^2/4}.$$

As a consequence, if $\delta > 0$ is small enough, we obtain

$$0 = \lim_{k\to\infty} U_k = \lim_{k\to\infty} \int_\Omega (\widetilde{u}_0 - C_k)_+^2\,dx = \int_\Omega (\widetilde{u}_0 - 1)_+^2\,dx.$$

Bearing in mind that $\widetilde{u}_0 = \sqrt{\delta}u_0$ (and $u_0 \geq 0$), we then get

$$0 \leq u_0 \leq \frac{1}{\sqrt{\delta}} \quad \text{a.e. in } \Omega,$$

from which we conclude that $u_0 \in L^\infty(\Omega)$. □

Now, if $\partial\Omega$ is sufficiently regular, by combining Theorem 3.5 with Theorems 2.7–2.8 we obtain the next key result.

Theorem 3.6. *Let* $s \in (0, 1)$ *and* $\alpha \in (0, 1)$ *be such that*

$$\alpha + 2s < 1.$$

Let us suppose that $\partial\Omega$ *is of class* $C^{2,\alpha}$. *Denoting by* $u_0 \in \mathbb{X}(\Omega)$ *the principal eigenfunction of* $\mathcal{L}$ *in* Ω *(according to Theorem 3.3), we have*

$$u_0 \in C_b(\mathbb{R}^n) \cap C^{2,\alpha}(\overline{\Omega}).$$

Moreover, u_0 *is a classical solution of* (3.2)*, that is,*

$$\begin{cases} \mathcal{L}u_0 = \lambda_{\mathcal{L}}(\Omega)u_0 & \textit{pointwise in } \Omega, \\ u_0 \equiv 0 & \textit{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{3.12}$$

Furthermore, u_0 *satisfies the following additional properties:*

(1) $u_0 > 0$ *pointwise in* Ω*;*

(2) *denoting by* ν *the external unit normal to* $\partial\Omega$*, we have*

$$\partial_\nu u(\xi) < 0 \quad \textit{for all } \xi \in \partial\Omega. \tag{3.13}$$

Proof. First of all we observe that, setting $f := \lambda_{\mathcal{L}}(\Omega)u_0$, from Theorem 3.5 we infer that $f \in L^\infty(\Omega)$; as a consequence, since u_0 is the (unique) weak solution of

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ u \equiv 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{D$_f$}$$

(and $\partial\Omega$ is of class $C^{2,\alpha}$), we deduce from Theorem 2.7 that

$$u_0 \in C^{1,\beta}(\overline{\Omega}) \quad \text{for every } \beta \in (0, 1).$$

In particular, $f \in C^\alpha(\overline{\Omega})$. In view of this fact, and taking into account our assumptions on s and on $\partial\Omega$, we can apply Theorem 2.8 to u_0: therefore,

$$u_0 \in C_b(\mathbb{R}^n) \cap C^{2,\alpha}(\overline{\Omega}),$$

and u_0 is a classical solution of (D$_f$), that is, u_0 satisfies (3.12). To complete the proof, we now turn to prove the validity of properties (1)–(2).

(1) Since u_0 satisfies (3.12) and $u_0 \geq 0$ in Ω, we have

$$\mathcal{L}u_0 = \lambda_{\mathcal{L}}(\Omega)u_0 \geq 0 \text{ pointwise in } \Omega \text{ and } u_0 \equiv 0 \text{ on } \mathbb{R}^n \setminus \Omega; \tag{3.14}$$

hence, we can invoke the Strong Maximum Principle for $\mathcal{L}$ in [7, Theorem 1.4], ensuring that either $u_0 \equiv 0$ in $\mathbb{R}^n$ or $u_0 > 0$ in Ω. On the other hand, since $u_0 \not\equiv 0$, we conclude that $u_0 > 0$ in Ω, as desired.

(2) Since $u_0 > 0$ in Ω and $u_0 \equiv 0$ on $\partial\Omega$, by (3.14) we derive that
(a) $\mathcal{L}u_0 \geq 0$ pointwise in Ω;
(b) $u_0(\xi) = \min_{\partial\Omega} u_0 = 0$ for every $\xi \in \partial\Omega$.
As a consequence, taking into account that $u_0 \in C^1(\overline{\Omega})$ and $\partial\Omega$ is of class $C^{2,\alpha}$, we are entitled to apply the Hopf-type Theorem 2.9, obtaining that

$$\partial_\nu u_0(\xi) < 0 \quad \text{for all } \xi \in \partial\Omega.$$

This is precisely the desired (3.13), and the proof is complete. □

We want to stress that one can get almost everywhere positivity of u_0 by means of the strong maximum principle for weak solutions proved in [9, 14].

Finally, by exploiting Theorem 3.6, we can prove Lemma 3.7 below. We stress that in the purely local regime, the following lemma has been proved in [57] without any restriction on δ. On the contrary, the same property is not known to hold in the purely fractional case. As a matter of fact, convexity and superharmonicity properties for fractional eigenfunctions can reserve surprisingly severe difficulties; see, e.g., [45].

Lemma 3.7. *Let $s \in (0, 1/2)$ and let us suppose that Ω is uniformly convex, and that $\partial\Omega$ is of class C^3. Then, denoting by u_0 the principal eigenfunction of $\mathcal{L}$ in Ω, there exists $\delta_0 > 0$ such that, for every fixed $\delta \in (0, \delta_0)$, the set*

$$\Omega_\delta := \{x \in \Omega : u_0(x) > \delta\} \quad \textit{is convex.}$$

Moreover, $\partial\Omega_\delta$ is of class C^1.

Proof. We split the proof into three steps.

Step I. In this first step we prove that, if $\delta > 0$ is sufficiently small, the super-level set Ω_δ can be realized as a suitable "deformation" of Ω. To this end we first notice that, owing to the assumptions on Ω, Theorem 3.6 ensures that
(a) $u_0 \in C_b(\mathbb{R}^n) \cap C^{2,\alpha}(\overline{\Omega})$, for every $\alpha \in (0, 1)$ such that $\alpha + 2s < 1$, and u_0 is a classical solution of (3.2);
(b) $u_0 > 0$ pointwise in Ω;
(c) $\partial_\nu u_0(\xi) < 0$ for every $\xi \in \partial\Omega$.
Moreover, since $u_0 \equiv 0$ on $\partial\Omega$, we have

$$\text{either } \nabla u_0 = |\nabla u_0|\nu \text{ or } \nabla u_0 = -|\nabla u_0|\nu \text{ pointwise in } \partial\Omega.$$

Now, since property (c) implies that $\langle \nabla u_0, \nu \rangle < 0$ on $\partial\Omega$, we get

$$\nabla u_0(\xi) = -|\nabla u_0(\xi)|\nu(\xi) \quad \text{for all } \xi \in \partial\Omega; \tag{3.15}$$

as a consequence, since $u_0 \in C^{2,\alpha}(\overline{\Omega})$, again by property (c) we have

$$0 > -a := \max_{\partial\Omega} \partial_\nu u \geq \langle \nabla u_0(\xi), \nu(\xi) \geq -|\nabla u_0(\xi)| \quad \text{for all } \xi \in \partial\Omega. \tag{3.16}$$

Hence, it is possible to find some $d_0 > 0$ such that

$$|\nabla u_0| \geq \frac{a}{2} \quad \text{on } \Omega_0 := \{x \in \Omega : d(x, \partial\Omega) < d_0\}. \tag{3.17}$$

Furthermore, bearing in mind that $\partial\Omega$ is of class C^3, by possibly shrinking d_0 we can also suppose that (see, e.g., [37, Section 14.6])

(i) $d(\cdot) := d(\cdot, \partial\Omega)$ is of class C^3 on Ω_0;

(ii) for every $x \in \Omega_0$ there exists a unique $\xi \in \partial\Omega$ such that

$$x - \xi = -d(x)\nu(\xi).$$

For a fixed $\xi \in \partial\Omega$, we then consider the function

$$F(t) := u_0(\xi - t\nu(\xi)) \quad \text{for } 0 \leq t < d_0.$$

Since u_0 is of class $C^{2,\alpha}$ on $\overline{\Omega}$, by (3.15)–(3.16) we have

$$\begin{aligned} F'(t) &= -\langle \nabla u_0(\xi - t\nu(\xi)), \nu(\xi) \rangle \\ &= -\langle \nabla u_0(\xi), \nu(\xi) \rangle - [\langle \nabla u_0(\xi - t\nu(\xi)) - \nabla u_0(\xi), \nu(\xi) \rangle] \\ &= |\nabla u_0(\xi)| - t\|u\|_{C^2(\overline{\Omega})} \geq a - t\|u\|_{C^2(\overline{\Omega})}; \end{aligned}$$

as a consequence, again by shrinking d_0 if necessary, we obtain

$$F'(t) \geq \frac{a}{2} \quad \text{for all } 0 \leq t < d_0. \tag{3.18}$$

In particular, F is strictly increasing on $[0, d_0)$, and

$$F(t) = F(t) - F(0) \geq \frac{ta}{2} \quad \text{for all } 0 \leq t < d_0. \tag{3.19}$$

To proceed further, we consider the set $\mathcal{O} := \{x \in \Omega : d(x, \partial\Omega) > d_0/2\} \subseteq \Omega$ and we notice that, since $u_0 > 0$ in Ω, we have

$$c_0 := \inf_{\mathcal{O}} u_0 > 0. \tag{3.20}$$

Hence, we define

$$\delta_0 := \frac{1}{2} \min\left\{c_0, \frac{ad_0}{4}\right\} > 0. \tag{3.21}$$

Taking into account (3.19), it is immediate to see that, for every $\delta \in [0, \delta_0)$, there exists a unique point $t = t(\xi, \delta) \in [0, d_0)$ such that

$$F(t(\xi, \delta)) = u_0(\xi - t(\xi, \delta)\nu(\xi)) = \delta;$$

then, we set

$$\mathcal{V}_\delta := \{\xi - t\nu(\xi) : \xi \in \partial\Omega,\ 0 \le t \le t(\xi,\delta)\}, \quad O_\delta := \Omega \setminus \mathcal{V}_\delta;$$

see Figure 1.

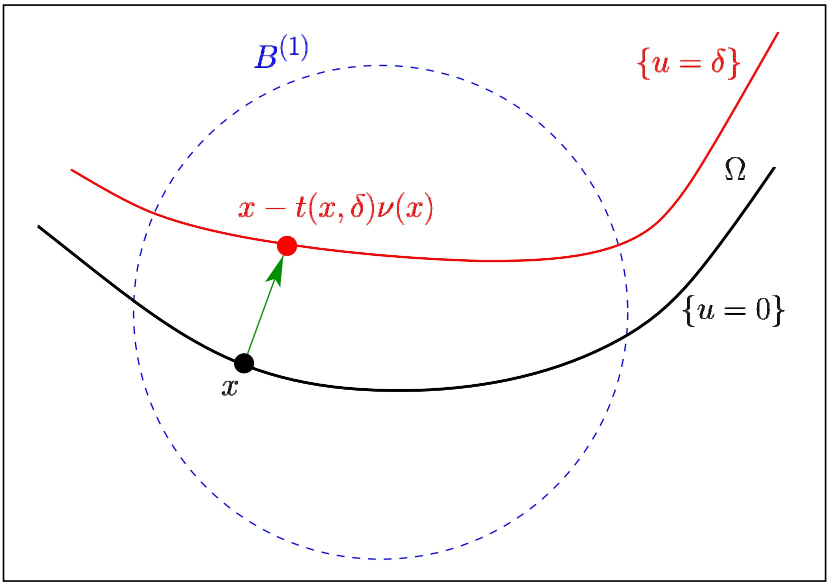

Figure 1. The construction of O_δ.

Before proceeding we highlight, for future reference, the following fact: since $F(0) = 0$ and F is strictly increasing on $[0, d_0)$, we have

$$t(\xi, 0) = 0 \quad \text{for every } \xi \in \partial\Omega. \tag{3.22}$$

We now turn to prove that, if $0 < \delta < \delta_0$, one has

$$O_\delta = \Omega_\delta = \{x \in \Omega : u_0(x) > \delta\}. \tag{3.23}$$

To this end we first observe that, by the very definition of $t(\xi,\delta)$, we have $u_0 \equiv \delta$ on ∂O_δ; moreover, since F is strictly increasing on $(0, d_0)$, one also has

$$u_0(\xi - t\nu(\xi)) = F(t) \le F(t(\xi,\delta)) = \delta \quad \text{for all } 0 \le t \le t(\xi,\delta),$$

and thus $u_0 \le \delta$ out of O_δ. Finally, we show that

$$u(x) > \delta \quad \text{for every } x \in O_\delta.$$

Let then $x \in O_\delta$ be fixed. If $x \in \mathcal{O}_0$, by (3.20)–(3.21) we have

$$u_0(x) \geq c_0 > \delta_0 > \delta.$$

If, instead, $x \notin \mathcal{O}_0$, one has $d(x) = d(x, \partial\Omega) \leq d_0/2$, and thus $x \in \Omega_0$. By property (ii) of Ω_0, we know that there exists a unique $\xi \in \partial\Omega$ such that

$$x = \xi - d(x)\nu(\xi);$$

on the other hand, since $x \in O_\delta = \Omega \setminus \mathcal{V}_\delta$, we necessarily have

$$d_0 > d(x) > t(\xi, \delta).$$

This, together with the strict monotonicity of F on $(0, d_0)$, implies that

$$u_0(x) = u_0(\xi - d(x)\nu(\xi)) = F(d(x)) > F(t(\xi, \delta)) = \delta,$$

and completes the proof of (3.23).

Step II. We now turn to prove that, if $\delta_0 > 0$ is as in Step I and $\delta \in (0, \delta_0)$, the boundary of $\partial\Omega_\delta$ is of class C^1. To this end we observe that, by crucially exploiting identity (3.23) and the very definition of O_δ, one has

$$\partial\Omega_\delta = \partial O_\delta = \{\xi - t(\xi, \delta)\nu(\xi) : \xi \in \partial\Omega\} \subseteq \{x \in \Omega : d(x) < d_0\}. \tag{3.24}$$

This, together with (3.17), shows that

$$|\nabla u_0| \neq 0 \quad \text{on } \partial\Omega_\delta,$$

and thus $\partial\Omega_\delta$ is of class C^1 (actually, it is of class $C^{2,\alpha}$).

Step III. In this last step we prove that, if $\delta \in (0, \delta_0)$ (where δ_0 is as in Step I), the set Ω_δ is convex. As in Step II, we use in a crucial way identity (3.23).

To begin with, we consider a covering of $\partial\Omega$ of finitely many small open balls

$$\{B^{(i)}\}_{i\in\{1,\ldots,N\}}$$

such that for every $i \in \{1, \ldots, N\}$ we can write $2B^{(i)} \cap \partial\Omega$ as a graph in some coordinate direction (where $2B^{(i)}$ is the concentric ball of $B^{(i)}$ with twice the radius of $B^{(i)}$). For $\delta > 0$ sufficiently small, we can suppose that

$$\partial\Omega_\delta \subset B^{(1)} \cup \cdots \cup B^{(N)}.$$

In this setting, it suffices to check that $B^{(i)} \cap \partial\Omega_\delta$ can locally be written as a graph of a convex function for all $i \in \{1, \ldots, N\}$. Without loss of generality, we argue for $i = 1$ and assume that

$$2B^{(1)} \cap \partial\Omega = \{x_n = \gamma(x')\}$$

for some $\gamma \in C^2(\mathbb{R}^{n-1})$ satisfying

$$D^2\gamma \geq a_0 \mathrm{Id} \quad (\text{for some } a_0 > 0). \tag{3.25}$$

Thus, by (3.24), in this chart $\partial\Omega_\delta$ can be locally parameterized by

$$\begin{aligned}(x', \gamma(x')) &- t(x', \gamma(x'), \delta)\nu(x', \gamma(x')) \\ &= (x' - t(x', \gamma(x'), \delta)\nu'(x', \gamma(x')), \gamma(x') - t(x', \gamma(x'), \delta)\nu_n(x', \gamma(x'))),\end{aligned} \tag{3.26}$$

where we used the notation $\nu = (\nu', \nu_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and x' belongs to a domain of $\mathbb{R}^{n-1}$. We now introduce the function

$$(x', t) \longmapsto G(x', \delta, t) := u((x', \gamma(x')) - t\nu(x', \gamma(x'))) - \delta.$$

Let also $g(x', \delta) := t(x', \gamma(x'), \delta)$. We notice that, as in (3.18), for $|t|$ small enough,

$$\partial_t G(x', \delta, t) = -\nabla u((x', \gamma(x')) - t\nu(x', \gamma(x'))) \cdot \nu(x', \gamma(x')) \geq \frac{a}{2}.$$

Also G is locally a C^2 function and $G(x', \delta, g(x', \delta)) = 0$. As a consequence, we find that g is locally a C^2 function. We also observe that, by (3.26), the set $\partial\Omega_\delta$ can be locally parameterized by

$$(x' - g(x', \delta)\nu'(x', \gamma(x')), \gamma(x') - g(x', \delta)\nu_n(x', \gamma(x'))). \tag{3.27}$$

Now we set

$$H(x', \delta) := (x' - g(x', \delta)\nu'(x', \gamma(x')), \delta).$$

Observe that H is locally a C^2 function. Moreover, by (3.22) we have

$$g(x', 0) = t(x', \gamma(x'), 0) = 0 \tag{3.28}$$

and, up to a rotation, we can focus our analysis at a point x'_0 for which

$$\nu(x'_0, \gamma(x'_0)) = -e_n. \tag{3.29}$$

Therefore

(3.30) the Jacobian matrix of H at $(x'_0, 0)$ is the identity

and we can exploit the Inverse Function Theorem, denote by I the inverse function of H in the vicinity of $(x'_0, 0)$ and have that I is also a C^2 function. Using the notation $H = (H', H_n)$ and $I = (I', I_n)$, we have that $H(I(y', \varepsilon)) = (y', \varepsilon)$ and thus

$$I_n(y', \varepsilon) = H_n(I(y', \varepsilon)) = \varepsilon.$$

Additionally,

$$x' - g(x',\delta)\nu'(x',\gamma(x')) = H'(I(y',\varepsilon)) = y'$$

and

$$I'(y',\delta) = I'(x' - g(x',\delta)\nu'(x',\gamma(x')),\delta) = I'(H(x',\delta)) = x'.$$

Accordingly, recalling (3.27), we can locally parameterize $\partial\Omega_\delta$ as

$$\begin{aligned}(y',\gamma(I'(y',\delta)) - g(I'(y',\delta),\delta)\nu_n(I'(y',\delta),\gamma(I'(y',\delta))))\\ = (y',\gamma(I'(y',\delta)) - \eta(y',\delta)),\end{aligned}$$

with

$$\eta(y',\delta) := g(I'(y',\delta),\delta)\ \nu_n(I'(y',\delta),\gamma(I'(y',\delta))).$$

For this reason, to complete the proof of the convexity property of $\partial\Omega_\delta$, it suffices to check the convexity of the function

$$y' \longmapsto \Gamma(y',\delta) := \gamma(I'(y',\delta)) - \eta(y',\delta) \tag{3.31}$$

that describes its graph. To this end, we remark that, for each $i,j \in \{1,\dots,n-1\}$,

$$\delta_{ij} = \frac{\partial}{\partial y'_i} y'_j = \frac{\partial}{\partial y'_i} H_j(I'(y',\varepsilon),\varepsilon) = \sum_{\ell=1}^{n-1} \frac{\partial H_j}{\partial x'_\ell}(I'(x',\varepsilon),\varepsilon)\frac{\partial I_\ell}{\partial y'_i}(y',\varepsilon),$$

that is

$$\mathrm{Id} = \frac{\partial H'}{\partial x'}(I'(y',\varepsilon),\varepsilon)\frac{\partial I'}{\partial y'}(y',\varepsilon).$$

Setting $y'_0 := H'(x'_0,0)$, we thus deduce from (3.30) that

$$\frac{\partial I'}{\partial y'}(y'_0,0) = \mathrm{Id}.$$

Also, by (3.29), we have that the gradient of γ at x'_0 vanishes. With these items of information, we find that

$$\frac{\partial}{\partial y'_i}\gamma(I'(y',\delta)) = \sum_{\ell=1}^{n-1} \frac{\partial \gamma}{\partial x'_\ell}(I'(y',\delta))\frac{\partial I_\ell}{\partial y'_i}(y',\delta)$$

and

$$\begin{aligned}\frac{\partial^2}{\partial y'_i\partial y'_j}\gamma(I'(y',\delta))\Big|_{(y',\delta)=(y'_0,0)} &= \sum_{\ell,m=1}^{n-1} \frac{\partial^2\gamma}{\partial x'_\ell \partial x'_m}(I'(y'_0,0))\frac{\partial I_\ell}{\partial y'_i}(y'_0,0)\frac{\partial I_m}{\partial y'_j}(y'_0,0)\\ &= \frac{\partial^2\gamma}{\partial y'_i\partial y'_j}(I'(y'_0,0)).\end{aligned}$$

Besides, owing to (3.28), we have

$$\eta(y', 0) = g(I'(y', 0), 0)\, \nu_n(I'(y', 0), \gamma(I'(y', 0))) = 0$$

and therefore

$$\frac{\partial^2 \eta}{\partial y'_i \partial y'_j}(y', 0) = 0.$$

As a result, recalling the definition of Γ in (3.31),

$$\frac{\partial^2 \Gamma}{\partial y'_i \partial y'_j}(y'_0, 0) = \frac{\partial^2}{\partial y'_i \partial y'_j}\gamma(I'(y', \delta))\Big|_{(y',\delta)=(y'_0,0)} = \frac{\partial^2 \gamma}{\partial y'_i \partial y'_j}(I'(y'_0, 0)).$$

By the uniform convexity of the domain Ω in (3.25), we obtain that

$$\frac{\partial^2 \Gamma}{\partial y'_i \partial y'_j}(y'_0, 0) \geq a_0 \,\mathrm{Id}.$$

This, together with the fact that Γ is a C^2 function in (y', δ), gives

$$\frac{\partial^2 \Gamma}{\partial y'_i \partial y'_j} \geq \frac{a_0}{2}\,\mathrm{Id}$$

in a neighborhood of $(y'_0,0)$, and proves that Ω_δ is uniformly convex, as desired.□

4 A Faber–Krahn inequality and proofs of Theorems 1.1 and 1.2

This section is devoted to showing a quantitative Faber–Krahn inequality for $\lambda_{\mathcal{L}}(\Omega)$ and in particular to proving Theorems 1.1 and 1.2. In what follows, $\lambda_{\mathcal{L}}(\Omega)$ denotes the principal Dirichlet eigenvalue of $\mathcal{L}$ (in Ω), as given by Definition 3.1, and u_0 the corresponding principal eigenfunction (according to Theorem 3.3).

We begin by proving Theorem 1.1.

Proof of Theorem 1.1. Let $\widehat{B}^{(m)}$ be the (unique) Euclidean ball with centre 0 and volume m. If $u_0 \in \mathbb{X}(\Omega) \setminus \{0\}$ is the principal eigenfunction of $\mathcal{L}$ in Ω, we define

$$u_0^* : \mathbb{R}^n \to \mathbb{R}$$

as the (decreasing) Schwarz symmetrization of u_0. Now, since $u_0 \in \mathbb{X}(\Omega)$, it follows from a well-known theorem by Polya–Szegö that

$$u_0^* \in \mathbb{X}(\widehat{B}^{(m)}) \quad \text{and} \quad \int_{\widehat{B}^{(m)}} |\nabla u_0^*|^2\, dx \leq \int_\Omega |\nabla u|^2\, dx; \tag{4.1}$$

furthermore, by [34, Theorem A.1] we also have

$$\iint_{\mathbb{R}^{2n}} \frac{|u_0^*(x) - u_0^*(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \le \iint_{\mathbb{R}^{2n}} \frac{|u_0(x) - u_0(y)|^2}{|x-y|^{n+2s}}\,dx\,dy. \tag{4.2}$$

Gathering together all these facts and recalling (1) in Theorem 3.3, we then get

$$\begin{aligned} \lambda_{\mathcal{L}}(\Omega) = \mathcal{D}_{\Omega,s}(u_0) &= \int_\Omega |\nabla u_0|^2\,dx + \iint_{\mathbb{R}^{2n}} \frac{|u_0(x) - u_0(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ &\ge \int_{\widehat{B}^{(m)}} |\nabla u_0^*|^2\,dx + \iint_{\mathbb{R}^{2n}} \frac{|u_0^*(x) - u_0^*(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ &= \mathcal{D}_{\widehat{B}^{(m)},s}(u_0^*) \ge \lambda_{\mathcal{L}}(\widehat{B}^{(m)}). \end{aligned} \tag{4.3}$$

From this, recalling that $\lambda_{\mathcal{L}}(\cdot)$ is translation-invariant (see Remark 3.2-(1)), we derive the validity of (1.1) for every Euclidean ball $B^{(m)}$ with volume m.

To complete the proof of Theorem 1.1, let us suppose that

$$\lambda_{\mathcal{L}}(\Omega) = \lambda_{\mathcal{L}}(B^{(m)})$$

for some (and hence, for every) ball $B^{(m)}$ with $|B^{(m)}| = m$. By (4.3) we have

$$\begin{aligned} \int_\Omega |\nabla u_0|^2\,dx + \iint_{\mathbb{R}^{2n}} \frac{|u_0(x) - u_0(y)|^2}{|x-y|^{n+2s}}\,dx\,dy = \lambda_{\mathcal{L}}(\Omega) = \lambda_{\mathcal{L}}(\widehat{B}^{(m)}) \\ = \int_{\widehat{B}^{(m)}} |\nabla (u_0)^*|^2\,dx + \iint_{\mathbb{R}^{2n}} \frac{|u_0^*(x) - u_0^*(y)|^2}{|x-y|^{n+2s}}\,dx\,dy. \end{aligned}$$

In particular, by (4.1)–(4.2) we get

$$\iint_{\mathbb{R}^{2n}} \frac{|u_0(x) - u_0(y)|^2}{|x-y|^{n+2s}}\,dx\,dy = \iint_{\mathbb{R}^{2n}} \frac{|u_0^*(x) - u_0^*(y)|^2}{|x-y|^{n+2s}}\,dx\,dy.$$

We are then entitled to apply once again [34, Theorem A.1], which ensures that u_0 must be proportional to a translate of a symmetric decreasing function. As a consequence of this fact, we immediately deduce that

$$\Omega = \{x \in \mathbb{R}^n : u_0(x) > 0\}$$

must be a ball (up to a set of zero Lebesgue measure). □

Now that we have established Theorem 1.1, we can finally prove the main result of this paper, namely, the quantitative version of inequality (1.1) presented in Theorem 1.2.

The proof of Theorem 1.2 is based on the following technical lemma.

Lemma 4.1. *Let* $s \in (0, 1/2)$*. Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set with boundary* $\partial\Omega$ *of class* C^3*. Let* $m := |\Omega|$*, and let* $B^{(m)}$ *be any Euclidean ball with volume* m*. Moreover, let* u_0 *be the principal eigenfunction of* $\mathcal{L}$ *in* Ω*, let* $\delta_0 > 0$ *be as in Lemma 3.7, and let*

$$0 < \delta < \min\left\{\frac{1}{2}m^{-1/2}, \delta_0\right\}. \tag{4.4}$$

Let also $\Omega_\delta := \{x \in \Omega : u_0(x) > \delta\}$*.*

Then, there exists a small $\widetilde{\varepsilon} > 0$*, only depending on* n *and* s*, with the following property: if* $0 < \varepsilon < \widetilde{\varepsilon}$ *is such that*

$$\lambda_{\mathcal{L}}(\Omega) \leq (1+\varepsilon)\lambda_{\mathcal{L}}(B^{(m)}), \tag{4.5}$$

then we have the estimate

$$|\Omega_\delta| \geq \left[1 - \frac{2n}{s} \cdot \max\{\delta|\Omega|^{1/2}, \varepsilon\}\right] \cdot |\Omega|.$$

Proof. First of all we observe that, since $\partial\Omega$ is of class C^3, from Theorem 3.6 we derive that the principal eigenfunction u_0 of $\mathcal{L}$ in Ω satisfies

$$u_0 \in C_b(\mathbb{R}^n) \cap C^2(\overline{\Omega})$$

(actually, $u_0 \in C^{2,\alpha}(\overline{\Omega})$ if $\alpha + 2s < 1$); moreover, since $\delta < \delta_0$, by Lemma 3.7 we know that $\partial\Omega_\delta$ is of class C^1. We then consider the function

$$v := (u_0 - \delta)_+.$$

Since $u_0 \in \mathbb{X}(\Omega)$ and $u_0 \leq \delta$ on $\mathbb{R}^n \setminus \Omega_\delta$, it is readily seen that $v \in \mathbb{X}(\Omega_\delta) \subseteq \mathbb{X}(\Omega)$; as a consequence, since u_0 is a weak solution of (3.2), we have

$$\begin{aligned}
\mathcal{D}_{\Omega_\delta,s}(v) &= \int_{\Omega_\delta} |\nabla v|^2\,dx + \iint_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&= \int_{\Omega} \langle \nabla u_0, \nabla v\rangle\,dx + \iint_{\mathbb{R}^{2n}} \frac{(u_0(x)-u_0(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy \\
&= \lambda_{\mathcal{L}}(\Omega)\int_{\Omega} u_0 v\,dx \\
&= \lambda_{\mathcal{L}}(\Omega)\int_{\Omega_\delta} (u_0-\delta)u_0\,dx \\
&\leq \lambda_{\mathcal{L}}(\Omega)\left(\int_{\Omega_\delta} (u_0-\delta)^2\,dx\right)^{1/2} \\
&= \lambda_{\mathcal{L}}(\Omega)\|v\|_{L^2(\Omega_\delta)}
\end{aligned}$$

where we have used Hölder's inequality and the fact that $\|u_0\|_{L^2(\Omega)} = 1$.

On the other hand, since $u_0 \leq \delta$ in $\Omega \setminus \Omega_\delta$ and $\delta|\Omega|^{1/2} < 1/2$ (by the choice of δ and the definition of Ω_δ), an application of Minkowski's inequality gives

$$\|v\|_{L^2(\Omega_\delta)} = \left(\int_{\Omega_\delta} (u_0 - \delta)^2\, dx\right)^{1/2} \geq \left(\int_{\Omega_\delta} u_0^2\, dx\right)^{1/2} - \left(\int_{\Omega_\delta} \delta^2\, dx\right)^{1/2}$$
$$\geq \left(1 - \int_{\Omega\setminus\Omega_\delta} \delta^2\right)^{1/2} - \delta|\Omega|^{1/2} \geq 1 - 2\delta|\Omega|^{1/2}.$$

Gathering together all these estimates, and recalling (4.5), we obtain

$$\tag{4.6} \begin{aligned} \lambda_{\mathcal{L}}(\Omega_\delta) &\leq \frac{\mathcal{D}_{\Omega_\delta,s}(v)}{\|v\|^2_{L^2(\Omega_\delta)}} \leq \frac{\lambda_{\mathcal{L}}(\Omega)}{\|v\|_{L^2(\Omega_\delta)}} \leq \lambda_{\mathcal{L}}(\Omega)(1 - 2\delta|\Omega|^{1/2})^{-1} \\ &\leq (1+\varepsilon)\lambda_{\mathcal{L}}(B^{(m)})(1 - 2\delta|\Omega|^{1/2})^{-1}. \end{aligned}$$

Using Remark 3.2-(2) and the Faber–Krahn inequality in Theorem 1.1 (notice that we are in the position of applying Theorem 1.1 for the set $t\Omega_\delta$ in light of the regularity result in Lemma 3.7, and notice also that $|t\Omega_\delta| = |\Omega| = m$), we get

$$\tag{4.7} t^{-2s}\lambda_{\mathcal{L}}(\Omega_\delta) \geq \lambda_{\mathcal{L}}(t\Omega_\delta) \geq \lambda_{\mathcal{L}}(B^{(m)}), \quad \text{where } t := \frac{|\Omega|^{1/n}}{|\Omega_\delta|^{1/n}} > 1.$$

By combining (4.6) with (4.7), we then get

$$\tag{4.8} \frac{|\Omega_\delta|}{|\Omega|} \geq \left(\frac{\lambda_{\mathcal{L}}(B^{(m)})}{\lambda_{\mathcal{L}}(\Omega_\delta)}\right)^{n/(2s)} \geq \left[\frac{1 - 2\delta|\Omega|^{1/2}}{1+\varepsilon}\right]^{n/(2s)}.$$

Finally, if $\widetilde{\varepsilon} > 0$ is sufficiently small and $0 < \varepsilon < \widetilde{\varepsilon}$, we obtain

$$\left[\frac{1 - 2\delta|\Omega|^{1/2}}{1+\varepsilon}\right]^{n/(2s)} \geq 1 - \frac{2n}{s} \cdot \max\{\delta|\Omega|^{1/2}, \varepsilon\}.$$

From this and (4.8), we obtain the desired result. □

Now we provide a convexity result that turns out to be useful for the proof of Theorem 1.2:

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and convex. Then, there exists $\widehat{\varepsilon} > 0$ with the following property: if there exists a ball $B \subseteq \Omega$ such that*

$$\tag{4.9} |B| \geq (1-\varepsilon)|\Omega| \quad \textit{for some } 0 < \varepsilon < \widehat{\varepsilon},$$

then there exists a ball B_*, which is concentric to B, such that $\Omega \subseteq B_*$ and

$$\tag{4.10} |\Omega| \geq (1 - C\varepsilon^{\frac{2}{n+1}})|B_*|.$$

Here, the positive constant C depends only on n.

Proof. Up to a translation, we can assume that B is centered at the origin. We assume that B has radius R and we take $P \in \overline{\Omega}$ maximizing the distance to the origin among points in Ω. Let also $\delta := |P| - R$. By construction, we have that $\delta \geq 0$ (since Ω contains B) and that Ω is contained in the ball B_* of radius $R+\delta$. Since $|P| = R + \delta$, up to a rotation we can suppose that $P = (0, \dots, 0, R + \delta)$. We also consider the convex hull H of P and B. By construction, H lies in the closure of Ω. Let K be the right circular cone obtained by intersecting H and the halfspace $\{x_n \geq R\}$. Notice that the height of the cone K is equal to δ and we denote by r the radius of its basis. By triangular similitude (see the triangles $\overset{\triangle}{PTO}$ and $\overset{\triangle}{PRQ}$ in Figure 2 on page 438), we see that

$$\frac{r}{\delta} = \frac{R}{\sqrt{(R+\delta)^2 - R^2}}.$$

As a consequence,

$$\begin{aligned} |\Omega \setminus B| \geq |K| \geq c r^{n-1}\delta &= c\delta\Big(\frac{\delta R}{\sqrt{(R+\delta)^2 - R^2}}\Big)^{n-1} \\ &= c\delta\Big(\frac{\delta R}{\sqrt{2\delta R + \delta^2}}\Big)^{n-1} = c\delta^{\frac{n+1}{2}}\Big(\frac{R}{\sqrt{2R+\delta}}\Big)^{n-1}, \end{aligned} \tag{4.11}$$

for some $c > 0$ depending only on n.

On the other hand, in view of (4.9),

$$|\Omega \setminus B| = |\Omega| - |B| \leq \varepsilon|\Omega|.$$

Combining this and (4.11), we have that

$$c\delta^{\frac{n+1}{2}}\Big(\frac{R}{\sqrt{2R+\delta}}\Big)^{n-1} \leq \varepsilon|\Omega|. \tag{4.12}$$

Also, by (4.9) we know that, for ε sufficiently small,

$$|\Omega| \leq 2(1-\varepsilon)|\Omega| \leq 2|B| = CR^n$$

for some $C > 0$ depending only on n. From this and (4.12), up to renaming c we conclude that

$$\frac{c\delta^{\frac{n+1}{2}}}{(2R+\delta)^{\frac{n-1}{2}}} \leq \varepsilon R. \tag{4.13}$$

Now we claim that

$$\delta \leq \widetilde{C}R, \tag{4.14}$$

where $\widetilde{C} := 2 + \frac{2^{\frac{n+1}{2}}}{c}$. Indeed, suppose by contradiction that $\delta > \widetilde{C}R$. Then, by (4.13),

$$1 \geq \varepsilon \geq \frac{c\delta^{\frac{n+1}{2}}}{R(2R+\delta)^{\frac{n-1}{2}}} = \frac{c\delta}{R(\frac{2R}{\delta}+1)^{\frac{n-1}{2}}} \geq \frac{c\delta}{R(\frac{2}{\widetilde{C}}+1)^{\frac{n-1}{2}}}$$
$$\geq \frac{c\delta}{R(1+1)^{\frac{n-1}{2}}} \geq \frac{c\widetilde{C}}{2^{\frac{n-1}{2}}} \geq 2.$$

This is a contradiction and thus (4.14) is established.

Now, combining (4.13) and (4.14) we find that

$$\frac{\widetilde{c}\delta^{\frac{n+1}{2}}}{R^{\frac{n-1}{2}}} \leq \varepsilon R,$$

with $\widetilde{c} := \frac{c}{(2+\widetilde{C})^{\frac{n-1}{2}}}$, and therefore

$$\delta \leq C_\star \varepsilon^{\frac{2}{n+1}} R \tag{4.15}$$

with $C_\star > 0$ depending only on n.

We also remark that

$$|B_*| = |B| \frac{(R+\delta)^n}{R^n} \leq |\Omega| \frac{(R+\delta)^n}{R^n}.$$

This and (4.15) entail that

$$|B_*| \leq (1 + C_\star \varepsilon^{\frac{2}{n+1}})|\Omega|. \tag{4.16}$$

It is also useful to point out that, for every $t \geq 0$,

$$(1+t)(1-t) = 1 - t^2 \leq 1,$$

thus we deduce from (4.16) that

$$|B_*| \leq \frac{(1 + C_\star \varepsilon^{\frac{2}{n+1}})(1 - C_\star \varepsilon^{\frac{2}{n+1}})|\Omega|}{(1 - C_\star \varepsilon^{\frac{2}{n+1}})} \leq \frac{|\Omega|}{(1 - C_\star \varepsilon^{\frac{2}{n+1}})},$$

as desired. □

In spite of some comments the have appeared in the literature, we believe that the exponent in formula (4.10) of Lemma 4.2 is optimal, as remarked in Appendix A.

Proof of Theorem 1.2. Along the proof, constants depending only on n, s and Ω may change passing from a line to another. Nevertheless, to avoid a cumbersome notation, we will keep the same symbol C for all of them.

Let u_0^* be the decreasing Schwarz symmetrization of the first eigenfunction u_0 (given by Theorem 3.3). We recall from Theorem 3.3 that we can assume $\|u_0\|_{L^2} = 1$, and hence $\|u_0^*\|_{L^2} = 1$ as well. We define the sets

$$\Gamma(t) := \{x \in \Omega : u_0(x) = t\},$$

and

$$\Gamma^*(t) := \{x \in \Omega : u_0^*(x) = t\}.$$

We further define $T := \sup_\Omega u_0 \in (0, +\infty)$ and the function

$$\psi(t) := \int_{\Gamma(t)} \frac{1}{|\nabla u_0|} d\mathcal{H}^{n-1}, \quad \text{for all } t \in (0, T).$$

We recall that

$$\mathcal{H}^{n-1}(\Gamma(t)) \leq \psi(t) \int_{\Gamma(t)} |\nabla u_0| \, d\mathcal{H}^{n-1}, \tag{4.17}$$

and, thanks to the classical isoperimetric inequality,

$$\mathcal{H}^{n-1}(\Gamma^*(t)) \leq \mathcal{H}^{n-1}(\Gamma(t)). \tag{4.18}$$

We also recall that $\Gamma(t)$ is the boundary of the set $\Omega_t := \{x \in \Omega : u_0(x) > t\}$ and $\Gamma^*(t)$ is the boundary of a ball with volume equal to $|\Omega_t|$. Therefore,

$$\mathcal{H}^{n-1}(\Gamma^*(t)) = n|B_1|^{1/n}|\Omega_t|^{1-1/n}, \quad \text{for every } t \in [0, T), \tag{4.19}$$

where, as usual, $|B_1|$ denotes the n-dimensional Lebesgue measure of the unit ball.

Now we take $\widetilde{\varepsilon}$ to be as in Lemma 4.1 and $\widehat{\varepsilon}$ as in Lemma 4.2. We also define $\overline{\varepsilon} := \min\{\frac{1}{4|\Omega|}, \delta_0^2\}$, where δ_0 is as in Lemma 3.7. We stress that $\widetilde{\varepsilon}$, $\widehat{\varepsilon}$ and $\overline{\varepsilon}$ are small quantities depending only on the structural parameters of the problem and we suppose that the parameter ε_0 in the statement of Theorem 1.2 satisfies

$$\varepsilon_0 < \min\left\{\widetilde{\varepsilon}, \widehat{\varepsilon}^{2n}, \overline{\varepsilon}, |\Omega|, \frac{s^2}{16n^2|\Omega|}\right\}. \tag{4.20}$$

Step I. We first prove that, if $\varepsilon \in (0, \varepsilon_0)$ and (1.2) is satisfied, then

$$\int_0^T [\mathcal{H}^{n-1}(\Gamma(t))^2 - \mathcal{H}^{n-1}(\Gamma^*(t))^2] \frac{1}{\psi(t)} \, dt \leq \lambda_{\mathcal{L}}(B)\varepsilon. \tag{4.21}$$

To this aim, we use (4.17) and (4.18) to observe that

$$\begin{aligned} \int_\Omega |\nabla u_0|^2 \, dx &= \int_0^T \int_{\Gamma(t)} |\nabla u_0| \, d\mathcal{H}^{n-1} \, dt = \int_0^T \mathcal{H}^{n-1}(\Gamma(t))^2 \frac{dt}{\psi(t)} \\ &\geq \int_0^T \mathcal{H}^{n-1}(\Gamma^*(t))^2 \frac{dt}{\psi(t)} = \int_B |\nabla u_0^*|^2 \, dx. \end{aligned}$$

Consequently, using (1.2) and (4.2), we get that

$$
\begin{aligned}
&\int_0^T [\mathcal{H}^{n-1}(\Gamma(t))^2 - \mathcal{H}^{n-1}(\Gamma^*(t))^2] \frac{1}{\psi(t)}\,dt \\
&\qquad = \int_\Omega |\nabla u_0|^2\,dx - \int_B |\nabla u_0^*|^2\,dx \\
&\qquad =\lambda_{\mathcal{L}}(\Omega) - \lambda_{\mathcal{L}}(B) + \iint_{\mathbb{R}^{2n}} \frac{|u_0^*(x) - u_0^*(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\qquad\quad - \iint_{\mathbb{R}^{2n}} \frac{|u_0(x) - u_0(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\qquad \le\lambda_{\mathcal{L}}(\Omega) - \lambda_{\mathcal{L}}(B) \le \lambda_{\mathcal{L}}(B)\varepsilon,
\end{aligned}
$$

which is precisely (4.21).

Step II. We prove that if $\varepsilon \in (0, \varepsilon_0)$ and (1.2) is satisfied, then there exists a structural constant $C > 0$ and $\delta > 0$ sufficiently small such that

$$
\inf_{0\le t\le\delta} [\mathcal{H}^{n-1}(\Gamma(t))^2 - \mathcal{H}^{n-1}(\Gamma^*(t))^2] \le C\varepsilon^{1/2}. \tag{4.22}
$$

For this we take

$$
\delta := \varepsilon^{1/2} \tag{4.23}
$$

and we point out that, as a consequence of (4.20), the condition in (4.4) is satisfied. This allows us to use Lemma 4.1, yielding that

$$
|\Omega \setminus \Omega_\delta| \le \frac{2n|\Omega|}{s} \max\{\varepsilon^{1/2}|\Omega|^{1/2}, \varepsilon\} = \frac{2n|\Omega|^{3/2}\varepsilon^{1/2}}{s}.
$$

Therefore, because of the above choice for δ and exploiting the Cauchy–Schwarz inequality, we also find that

$$
\begin{aligned}
\varepsilon = \delta^2 = \left(\int_0^\delta dt\right)^2 &\le \left(\int_0^\delta \frac{dt}{\psi(t)}\right)\left(\int_0^\delta \psi(t)\,dt\right) \\
&= \left(\int_0^\delta \tfrac{dt}{\psi(t)}\right)|\Omega \setminus \Omega_\delta| \\
&\le \frac{2n|\Omega|^{3/2}\varepsilon^{1/2}}{s}\left(\int_0^\delta \tfrac{dt}{\psi(t)}\right),
\end{aligned}
$$

and thus there exists some $C > 0$ such that

$$
\int_0^\delta \frac{dt}{\psi(t)} \ge C\varepsilon^{1/2}. \tag{4.24}
$$

Hence, using (4.21) and (4.24), and recalling (4.18), we deduce that

$$
\begin{aligned}
&\inf_{0\le t\le\delta}[\mathcal{H}^{n-1}(\Gamma(t))^2-\mathcal{H}^{n-1}(\Gamma^*(t))^2]\\
&\qquad\le \inf_{0\le t\le\delta}[\mathcal{H}^{n-1}(\Gamma(t))^2-\mathcal{H}^{n-1}(\Gamma^*(t))^2]C\varepsilon^{-1/2}\int_0^\delta \frac{dt}{\psi(t)}\\
&\qquad\le C\varepsilon^{-1/2}\int_0^\delta [\mathcal{H}^{n-1}(\Gamma(t))^2-\mathcal{H}^{n-1}(\Gamma^*(t))^2]\frac{1}{\psi(t)}\,dt\\
&\qquad\le C\varepsilon^{1/2}\lambda_{\mathcal{L}}(B),
\end{aligned}
$$

which gives (4.22).

We notice that, by the very definition of infimum, and recalling the choice of δ in (4.23), there exists $\tau\in[0,\delta]=[0,\varepsilon^{1/2}]$ such that

$$
\mathcal{H}^{n-1}(\Gamma(\tau))^2-\mathcal{H}^{n-1}(\Gamma^*(\tau))^2\le 2C\varepsilon^{1/2}, \tag{4.25}
$$

for $\varepsilon\in(0,\varepsilon_0)$ such that (1.2) is satisfied.

Step III. We let $\varepsilon\in(0,\varepsilon_0)$ such that (1.2) is satisfied, and we take $\tau\in[0,\delta]$ such that (4.25) holds true. We prove that there exists a structural constant $C>0$ such that

$$
\mathcal{H}^{n-1}(\partial\Omega_\tau)=\mathcal{H}^{n-1}(\Gamma(\tau))\le n|B_1|^{1/n}|\Omega_\tau|^{1-1/n}+C\varepsilon^{1/2}. \tag{4.26}
$$

For this, recalling that (4.18) and (4.19) hold for every $t\in[0,T)$, and using (4.25), we have that

$$
\begin{aligned}
\mathcal{H}^{n-1}(\Gamma(\tau))-\mathcal{H}^{n-1}(\Gamma^*(\tau))&=\frac{\mathcal{H}^{n-1}(\Gamma(\tau))^2-\mathcal{H}^{n-1}(\Gamma^*(\tau))^2}{\mathcal{H}^{n-1}(\Gamma(\tau))+\mathcal{H}^{n-1}(\Gamma^*(\tau))}\\
&\le\frac{2C\varepsilon^{1/2}}{\mathcal{H}^{n-1}(\Gamma(\tau))+\mathcal{H}^{n-1}(\Gamma^*(\tau))}\le\frac{2\varepsilon^{1/2}}{\mathcal{H}^{n-1}(\Gamma^*(\tau))}\\
&=\frac{2\varepsilon^{1/2}}{n|B_1|^{1/n}|\Omega_\tau|^{1-1/n}}.
\end{aligned}
$$

From this, eventually modifying the constant $C>0$, we further obtain that

$$
\mathcal{H}^{n-1}(\Gamma(\tau))\le n|B_1|^{1/n}|\Omega_\tau|^{1-1/n}+\frac{C\varepsilon^{1/2}}{|\Omega_\tau|^{1-1/n}}. \tag{4.27}
$$

Furthermore, since $\tau\le\delta=\varepsilon^{1/2}$, recalling (4.20), we have that the condition in (4.4) is satisfied, and therefore we can exploit Lemma 4.1. In this way, we obtain that

$$
|\Omega_\tau|^{1-1/n}\ge\left[1-\frac{2n}{s}\cdot\varepsilon^{1/2}|\Omega|^{1/2}\right]^{1-1/n}|\Omega|^{1-1/n}\ge\left(\frac{|\Omega|}{2}\right)^{1-1/n},
$$

thanks to (4.20). Plugging this information into (4.27), we obtain (4.26).

Step IV. We are now ready to finish the proof of (1.3). Once this is established, the existence of a ball B_2 for which (1.4) holds follows from the convexity of Ω and Lemma 4.2 (notice that we are in the position of exploiting Lemma 4.2 thanks to (4.20)).

Hence we focus on the proof of (1.3). We let $\rho > 0$ be the inradius of Ω_τ and let us consider the ball $B^{(1)}$ whose radius is ρ and such that $B^{(1)} \subseteq \Omega_\tau$. We recall that the convexity of the set Ω_τ ensures that the following Bonnesen-type inequality holds, see, e.g. [27,40,50]:

$$\left(\frac{\mathcal{H}^{n-1}(\partial\Omega_\tau)}{\mathcal{H}^{n-1}(\partial B^{(1)})}\right)^{n/(n-1)} - \frac{|\Omega_\tau|}{|B^{(1)}|} \geq \left[\left(\frac{\mathcal{H}^{n-1}(\partial\Omega_\tau)}{\mathcal{H}^{n-1}(\partial B^{(1)})}\right)^{1/(n-1)} - 1\right]^n. \tag{4.28}$$

Also, from (4.26), we have that

$$\mathcal{H}^{n-1}(\partial\Omega_\tau) \leq n|B_1|^{1/n}|\Omega_\tau|^{1-1/n} + C\varepsilon^{1/2}.$$

Therefore,

$$\begin{aligned}\left(\frac{\mathcal{H}^{n-1}(\partial\Omega_\tau)}{\mathcal{H}^{n-1}(\partial B^{(1)})}\right)^{n/(n-1)} &- \frac{|\Omega_\tau|}{|B^{(1)}|} \\ &\leq \left(\frac{n|B_1|^{1/n}|\Omega_\tau|^{1-1/n} + C\varepsilon^{1/2}}{\mathcal{H}^{n-1}(\partial B^{(1)})}\right)^{n/(n-1)} - \frac{|\Omega_\tau|}{|B^{(1)}|}.\end{aligned} \tag{4.29}$$

Furthermore, using the isoperimetric inequality we see that

$$\frac{|\Omega_\tau|^{1-1/n}}{\mathcal{H}^{n-1}(\partial\Omega_\tau)} \leq \frac{|B_1|^{1-1/n}}{\mathcal{H}^{n-1}(\partial B_1)} = \frac{|B_1|^{1-1/n}}{n|B_1|} = \frac{1}{n|B_1|^{1/n}}.$$

As a result, we obtain that

$$\begin{aligned}&\left[\left(\frac{\mathcal{H}^{n-1}(\partial\Omega_\tau)}{\mathcal{H}^{n-1}(\partial B^{(1)})}\right)^{1/(n-1)} - 1\right]^n \\ &\qquad\geq \left[\left(\frac{|\Omega_\tau|^{1-1/n}}{|B^{(1)}|^{1-1/n}}\right)^{1/(n-1)} - 1\right]^n \\ &\qquad= \left[\left(\frac{|\Omega_\tau|}{|B^{(1)}|}\right)^{1/n} - 1\right]^n = \frac{[|\Omega_\tau|^{1/n} - |B^{(1)}|^{1/n}]^n}{|B^{(1)}|}.\end{aligned} \tag{4.30}$$

Now, combining (4.28), (4.29) and (4.30), we find that

$$\left(\frac{n|B_1|^{1/n}|\Omega_\tau|^{1-1/n} + C\varepsilon^{1/2}}{\mathcal{H}^{n-1}(\partial B^{(1)})}\right)^{n/(n-1)} - \frac{|\Omega_\tau|}{|B^{(1)}|} \geq \frac{[|\Omega_\tau|^{1/n} - |B^{(1)}|^{1/n}]^n}{|B^{(1)}|}.$$

Recalling that $B^{(1)}$ has radius ρ, hence $|B^{(1)}| = |B_1|\rho^n$ and

$$\mathcal{H}^{n-1}(\partial B^{(1)}) = \mathcal{H}^{n-1}(\partial B_1)\rho^{n-1} = n|B_1|\rho^{n-1},$$

it thus follows that

$$(|\Omega_\tau|^{1-1/n} + C\varepsilon^{1/2})^{n/(n-1)} - |\Omega_\tau| \geq [|\Omega_\tau|^{1/n} - |B^{(1)}|^{1/n}]^n. \tag{4.31}$$

We also point out that

$$\frac{|\Omega|}{2} \leq |\Omega_\tau| \leq |\Omega|, \tag{4.32}$$

for sufficiently small τ, and therefore

$$\begin{aligned}(|\Omega_\tau|^{1-1/n} + C\varepsilon^{1/2})^{n/(n-1)} &= |\Omega_\tau|\Big(1 + \frac{C\varepsilon^{1/2}}{|\Omega_\tau|^{1-1/n}}\Big)^{n/(n-1)} \\ &\leq |\Omega_\tau|\Big(1 + \frac{2^{1-1/n}C\varepsilon^{1/2}}{|\Omega|^{1-1/n}}\Big)^{n/(n-1)} \leq |\Omega_\tau|(1 + C\varepsilon^{1/2}),\end{aligned}$$

up to renaming C in the last inequality.

Combining this and (4.31) we gather that

$$[|\Omega_\tau|^{1/n} - |B^{(1)}|^{1/n}]^n \leq C|\Omega_\tau|\varepsilon^{1/2}.$$

This, the Bernoulli inequality and (4.32) entail that

$$|\Omega_\tau|^{1/n} - |B^{(1)}|^{1/n} \leq C|\Omega_\tau|^{1/n}\varepsilon^{1/(2n)},$$

which gives (1.3). □

Appendix A Convex sets, remarks on the literature, and optimality of Lemma 4.2

In the literature, it seems to be suggested (see, e.g., the end of Section 2 in [49]) that the result in Lemma 4.2 could be improved, for instance by posing the following natural question:

Problem A.1. *Let $\varepsilon > 0$. If $\Omega \subset \mathbb{R}^n$ is bounded and convex, contains the unit ball B_1 and*

$$|B_1| \geq (1-\varepsilon)|\Omega|, \tag{A.1}$$

is it true that there exists a ball B_* such that $\Omega \subseteq B_*$ and

$$|\Omega| \geq (1 - C\varepsilon)|B_*| \tag{A.2}$$

for some constant $C > 0$?

Here we show that the answer to Problem A.1 is negative. We provide two counterexamples, one closely related to the proof of Lemma 4.2, and one in which we additionally assume that the set Ω is uniformly convex and its boundary is of class C^∞.

Counterexample 1. Let $n \geq 2$. Let $\varepsilon > 0$ be small and

$$\delta := \varepsilon^{\frac{2}{n+1}}.$$

Let $P := (0, \dots, 0, 1+\delta)$ and Ω be the convex hull of $B_1 \cup P$; see Figure 2.

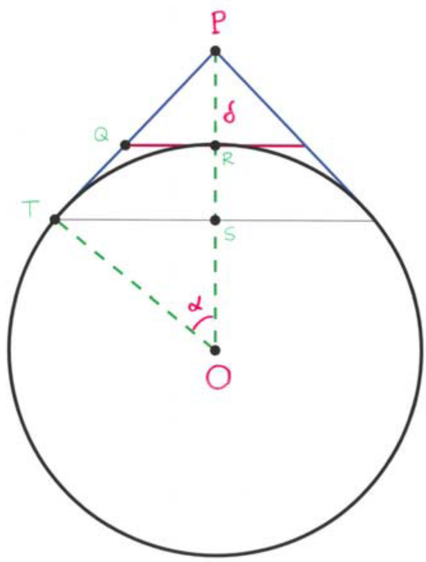

Figure 2. The convex hull of $B_1 \cup P$.

We claim that (A.1) holds true. Indeed, considering Figure 2, we have that $\overline{OR} = 1 = \overline{OT}$ and $\overline{PR} = \delta$. As a result, we see that $\overline{OP} = \overline{OR} + \overline{PR} = 1 + \delta$ and

$$\overline{PT} = \sqrt{\overline{OP}^2 - \overline{RO}^2} = \sqrt{(1+\delta)^2 - 1} = \sqrt{2\delta + \delta^2} \in [\sqrt{\delta}, 2\sqrt{\delta}].$$

We also remark that the triangles $\overset{\triangle}{PTO}$, and $\overset{\triangle}{PST}$ are similar and accordingly

$$\frac{\overline{ST}}{\overline{PT}} = \frac{\overline{OT}}{\overline{OP}} \quad \text{and} \quad \frac{\overline{PS}}{\overline{PT}} = \frac{\overline{PT}}{\overline{OP}}.$$

These identities entail that

$$\overline{ST} = \frac{\overline{OT}\ \overline{PT}}{\overline{OP}} \leq \frac{2\sqrt{\delta}}{1+\delta} \leq 4\sqrt{\delta}$$

and

$$\overline{PS} = \frac{\overline{PT}^2}{\overline{OP}} = \frac{2\delta + \delta^2}{1+\delta} \leq 4\delta.$$

As a result, if

$$\mathcal{Q} := P + [-4\sqrt{\delta}, 4\sqrt{\delta}]^{n-1} \times [-4\delta, 4\delta],$$

we find that $\Omega \setminus B_1 \subseteq \mathcal{Q}$.

From this, it follows that

$$\begin{aligned}|\Omega| &\leq |B_1| + |\Omega \setminus B_1| \leq |B_1| + |\mathcal{Q}| = |B_1| + (8\sqrt{\delta})^{n-1}(8\delta)\\ &\leq |B_1|(1 + C\delta^{\frac{n+1}{2}}) = |B_1|(1 + C\varepsilon)\end{aligned}$$

for some $C > 0$, which gives (A.1) (up to renaming ε).

Now, consider a ball B_* such that $\Omega \subseteq B_*$. Since

$$(0, \dots, 0, 1 + \delta) \in \overline{\Omega} \quad \text{and} \quad (0, \dots, 0, -1) \in \overline{\Omega},$$

we have that the diameter of Ω is at least $2+\delta$, hence the radius of B_* is at least $1+\frac{\delta}{2}$ and thus, using (A.1),

$$\begin{aligned}|B_*| \geq |B_1|\Big(1 + \frac{\delta}{2}\Big)^n \geq |B_1|\Big(1 + \frac{\delta}{2}\Big) &\geq \Big(1 + \frac{\delta}{2}\Big)\Big(1 - \varepsilon\Big)|\Omega|\\ &= \Big(1 + \frac{\delta}{2}\Big)(1 - \delta^{\frac{n+1}{2}})|\Omega| = \Big(1 + \frac{\delta}{2} + O(\delta^{\frac{n+1}{2}})\Big)|\Omega|\\ &= \Big(1 + \frac{\delta}{2} + O(\delta^{\frac{3}{2}})\Big)|\Omega| \geq \Big(1 + \frac{\delta}{4}\Big)|\Omega|.\end{aligned}$$

This yields that (A.2) is not satisfied in this case, since otherwise

$$\begin{aligned}1 = \frac{|\Omega|}{|\Omega|} &\geq \frac{(1-\varepsilon)|B_*|}{|B_*|/(1 + \frac{\delta}{4})} = \Big(1 + \frac{\delta}{4}\Big)(1 - \delta^{\frac{n+1}{2}})\\ &= \Big(1 + \frac{\delta}{4}\Big)(1 + O(\delta^{\frac{3}{2}})) = 1 + \frac{\delta}{4} + O(\delta^{\frac{3}{2}}) \geq 1 + \frac{\delta}{8} > 1,\end{aligned}$$

which is a contradiction.

Counterexample 2. For simplicity, we take here $n = 2$ (the case $n > 2$ can be obtained by rotations of the example constructed in a given plane). Let $\varepsilon > 0$ be small and

$$\delta := \varepsilon^{\frac{2}{3}}.$$

We construct here a counterexample of Problem A.1 even under the additional assumption that Ω is uniformly convex with boundary of class C^∞. For this, let $f \in C_0^\infty([-1, 1], [0, 1])$ with $f(0) = 1$ and

$$(-\pi, \pi] \ni \vartheta \mapsto g(\vartheta) := 1 + c\delta f\Big(\frac{\vartheta}{\sqrt{\delta}}\Big),$$

with

$$c := \frac{1}{4(1 + \|f\|_{C^2(\mathbb{R})})}.$$

We consider the set

$$\Omega := \{(\rho\cos\vartheta, \rho\sin\vartheta),\ \vartheta \in (-\pi, \pi], \rho \in [0, g(\vartheta))\}.$$

We observe that $g(\vartheta) \in [1, 1+\delta]$ for all $\vartheta \in (-\pi, \pi]$, hence $B_1 \subseteq \Omega$, and

$$g(\vartheta) = 1 \quad \text{whenever } |\vartheta| > \sqrt{\delta}. \tag{A.3}$$

We let $\kappa(\vartheta)$ be the curvature of $\partial\Omega$ at the point $(g(\vartheta)\cos\vartheta, g(\vartheta)\sin\vartheta)$. Then,

$$\begin{aligned}\kappa(\vartheta) &= \frac{2\dot{g}^2(\vartheta) - g(\vartheta)\ddot{g}(\vartheta) + g^2(\vartheta)}{(\dot{g}^2(\vartheta) + g^2(\vartheta))^{\frac{3}{2}}} \\ &= \frac{2c^2\delta\dot{f}^2\left(\frac{\vartheta}{\sqrt{\delta}}\right) - cg(\vartheta)\ddot{f}(\frac{\vartheta}{\sqrt{\delta}}) + g^2(\vartheta)}{(c^2\delta\dot{f}^2(\frac{\vartheta}{\sqrt{\delta}}) + g^2(\vartheta))^{\frac{3}{2}}}.\end{aligned}$$

Hence, if

$$\xi(\vartheta) := cg(\vartheta)\ddot{f}\Big(\frac{\vartheta}{\sqrt{\delta}}\Big),$$

we have that

$$|\xi(\vartheta)| \le c(1+\delta)\|f\|_{C^2(\mathbb{R})} \le 2c\|f\|_{C^2(\mathbb{R})} \le \frac{1}{2} \tag{A.4}$$

and

$$\kappa(\vartheta) = \frac{O(\delta) - \xi(\vartheta) + (1 + O(\delta))^2}{(O(\delta) + (1+O(\delta))^2)^{\frac{3}{2}}} = \frac{1 - \xi(\vartheta) + O(\delta)}{(1 + O(\delta))^{\frac{3}{2}}} = 1 - \xi(\vartheta) + O(\delta).$$

From this relation and (A.4), we conclude that

$$\kappa(\vartheta) \in \Big[\frac{1}{4}, 2\Big],$$

hence Ω is uniformly convex.

Moreover, using polar coordinates and (A.3),

$$|\Omega \setminus B_1| = \int_{-\pi}^{\pi} \left[\int_1^{g(\vartheta)} \rho d\rho\right] d\vartheta \le \int_{-\sqrt{\delta}}^{\sqrt{\delta}} \left[\int_1^{1+\delta} 2\, d\rho\right] d\vartheta \le 4\delta^{\frac{3}{2}} = 4\varepsilon,$$

and therefore

$$|B_1| \ge |\Omega| - 4\varepsilon = \Big(1 - \frac{4\varepsilon}{|\Omega|}\Big)|\Omega|,$$

which shows that (A.1) holds true (up to renaming ε).

Now, consider a ball B_* such that $\Omega \subseteq B_*$. Since

$$(1 + c\delta, 0) = (g(0)\cos 0, g(0)\sin 0) \in \overline{\Omega}$$

and $(-1, 0) = (g(\pi)\cos\pi, g(\pi)\sin\pi) \in \overline{\Omega}$, we have that the diameter of Ω is at least $2 + c\delta$, hence the radius of B_* is at least $1 + \frac{c\delta}{2}$. Thus, using (A.1),

$$\begin{aligned}|B_*| &\geq |B_1|\Big(1 + \frac{c\delta}{2}\Big)^n \geq |B_1|\Big(1 + \frac{c\delta}{2}\Big) \geq \Big(1 + \frac{c\delta}{2}\Big)(1-\varepsilon)|\Omega| \\ &= \Big(1 + \frac{c\delta}{2}\Big)(1 - \delta^{\frac{n+1}{2}})|\Omega| = \Big(1 + \frac{c\delta}{2} + O(\delta^{\frac{n+1}{2}})\Big)|\Omega| \\ &= \Big(1 + \frac{c\delta}{2} + O(\delta^{\frac{3}{2}})\Big)|\Omega| \geq \Big(1 + \frac{c\delta}{4}\Big)|\Omega|.\end{aligned}$$

This yields that (A.2) is not satisfied in this case, since otherwise

$$\begin{aligned}1 = \frac{|\Omega|}{|\Omega|} &\geq \frac{(1-\varepsilon)|B_*|}{|B_*|/(1 + \frac{c\delta}{4})} = \Big(1 + \frac{c\delta}{4}\Big)(1 - \delta^{\frac{n+1}{2}}) \\ &= \Big(1 + \frac{c\delta}{4}\Big)(1 + O(\delta^{\frac{3}{2}})) = 1 + \frac{c\delta}{4} + O(\delta^{\frac{3}{2}}) \geq 1 + \frac{c\delta}{8} > 1,\end{aligned}$$

which is a contradiction.

Appendix B $C^{2,\alpha}$-regularity for $s < 1/2$

For completeness, we present here an explicit proof of Theorem 2.8 in the case $s \in (0, 1/2)$. In this situation the action of the fractional operator is better behaved since it does not produce boundary singularities on functions that are smooth (or even just Lipschitz) up to the boundary and with zero external datum. This fact makes the proof technically easier since it allows one to "reabsorb" the fractional operator into the source term of the equation. For this reason, we thought it could be of some interest, at least for some readers, to find here a self-contained result with its own proof. The precise statement is the following:

Theorem B.1. *Let $s \in (0, 1/2)$ and $\alpha \in (0, 1)$ be such that*

$$\alpha + 2s < 1. \tag{B.1}$$

Suppose that $\partial\Omega$ is of class $C^{2,\alpha}$. If $f \in C^\alpha(\overline{\Omega})$ and if $u_f \in \mathbb{X}(\Omega)$ denotes the unique weak solution of $(\mathrm{D})_f$ *(according to Theorem 2.4), then*

$$u_f \in C_b(\mathbb{R}^n) \cap C^{2,\alpha}(\overline{\Omega}).$$

In particular, u_f is a classical solution of $(\mathrm{D})_f$.

Proof. We split the proof into different steps.

Step I. We consider the function space $\mathbb{B}(\Omega)$ defined as follows:

$$\mathbb{B}(\Omega) := \{u \in C(\mathbb{R}^n) : u \equiv 0 \text{ in } \mathbb{R}^n \setminus \Omega \text{ and } u|_{\overline{\Omega}} \in C^{2,\alpha}(\overline{\Omega})\} \subseteq C_b(\mathbb{R}^n).$$

Then, we claim that there exists a constant $\mathbf{c} = \mathbf{c}_{n,s,\alpha,\Omega} > 0$ such that

$$\|(-\Delta)^s u\|_{C^\alpha(\overline{\Omega})} \leq \mathbf{c}\|u\|_{C^1(\overline{\Omega})} < \infty \quad \forall u \in \mathbb{B}(\Omega). \tag{B.2}$$

In fact, since $u \in \mathbb{B}(\Omega)$ and since, by assumption (B.1), $\beta := \alpha + 2s \in (0, 1)$, it is not difficult to recognize that $u \in C^\beta(\mathbb{R}^n)$, and

$$[u]_{C^\beta(\mathbb{R}^n)} \leq \operatorname{diam}(\Omega)^{1-\beta}\|u\|_{C^1(\overline{\Omega})}.$$

As a consequence, since one obviously has $\beta = \alpha + 2s > 2s$, we are entitled to apply the result in [56, Prop. 2.1.7]: this gives $(-\Delta)^s u \in C^\alpha(\mathbb{R}^n)$ and

$$[(-\Delta)^s u]_{C^\alpha(\mathbb{R}^n)} \leq c[u]_{C^\beta(\mathbb{R}^n)} \leq c\operatorname{diam}(\Omega)^{1-\beta}\|u\|_{C^1(\overline{\Omega})}, \tag{B.3}$$

where $c = c_{n,s,\alpha} > 0$ is a constant independent of u. To complete the proof of (B.2), we then turn to estimate the L^∞-norm of $(-\Delta)^s u$ in terms of $\|u\|_{C^1(\overline{\Omega})}$.

First of all we observe that, on account of (B.3), for every $x \in \overline{\Omega}$ one has

$$\begin{aligned}
|(-\Delta)^s u(x)| &\leq [(-\Delta)^s u]_{C^\alpha(\mathbb{R}^n)} \cdot |x - x_0|^\alpha + |(-\Delta)^s u(x_0)| \\
&\leq c\operatorname{diam}(\Omega)^{1+\alpha-\beta}\|u\|_{C^1(\overline{\Omega})} + |(-\Delta)^s u(x_0)| \\
&\qquad\qquad (\text{setting } \rho := c\operatorname{diam}(\Omega)^{1+\alpha-\beta}) \\
&= \rho\|u\|_{C^1(\overline{\Omega})} + |(-\Delta)^s u(x_0)|,
\end{aligned} \tag{B.4}$$

where $x_0 \in \Omega$ is chosen in such a way that

$$d_0 := \operatorname{dist}(x_0, \partial\Omega) = \sup_{x\in\Omega}(\operatorname{dist}(x, \partial\Omega)).$$

On the other hand, since $u \in \mathbb{B}(\Omega)$ and $s < 1/2$, we have the estimate

$$\begin{aligned}
|(-\Delta)^s u(x_0)| &\leq \int_{\mathbb{R}^n} \frac{|u(x_0) - u(y)|}{|x_0 - y|^{n+2s}}\, dy = \int_{\mathbb{R}^n} \frac{|u(x_0) - u(x_0 - z)|}{|z|^{n+2s}}\, dz \\
&\leq \int_{\{|z|\leq d_0\}} \frac{1}{|z|^{n+2s}} |\langle \nabla u(x_0 - \tau z), z\rangle|\, dz \\
&\quad + 2\|u\|_{L^\infty(\mathbb{R}^n)} \int_{\{|z|>d_0\}} \frac{1}{|z|^{n+2s}}\, dz \\
&\leq \sqrt{n}\|u\|_{C^1(\overline{\Omega})} \int_{\{|z|\leq d_0\}} \frac{1}{|z|^{n+2s-1}}\, dz \\
&\quad + 2\|u\|_{L^\infty(\mathbb{R}^n)} \int_{\{|z|>d_0\}} \frac{1}{|z|^{n+2s}}\, dz \\
&= \kappa(d_0^{1-2s} + d_0^{-2s})\|u\|_{C^1(\overline{\Omega})},
\end{aligned} \tag{B.5}$$

where $\kappa = \kappa_{n,s} > 0$ is another constant which does not depend on u. Gathering together (B.3), (B.4) and (B.5), we finally obtain

$$\begin{aligned}\|(-\Delta)^s u\|_{C^\alpha(\overline{\Omega})} &= \|(-\Delta)^s u\|_{L^\infty(\Omega)} + [(-\Delta)^s u]_{C^\alpha(\overline{\Omega})} \\ &\leq (c\,\mathrm{diam}(\Omega)^{1-\beta} + \rho)\|u\|_{C^1(\overline{\Omega})} + \kappa(d_0^{2-2s} + d_0^{-2s})\|u\|_{C^1(\overline{\Omega})} \\ &\leq \mathbf{c}\|u\|_{C^1(\overline{\Omega})}\end{aligned}$$

which is exactly the claimed (B.2). We explicitly point out that the constant $\mathbf{c}$ only depends on n, s, α and Ω (as the same is true of c).

Step II. In this second step, we establish the following facts:

(1) $\mathcal{L}u \in C^\alpha(\overline{\Omega})$ for every $u \in \mathbb{B}(\Omega)$;

(2) there exists a constant $C = C(n, \alpha, s) > 0$ such that

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C(\|\mathcal{L}u\|_{C^\alpha(\overline{\Omega})} + \sup_\Omega |u|) \quad \text{for all } u \in \mathbb{B}(\Omega). \tag{B.6}$$

As regards assertion (1), it is a direct consequence of (B.2), together with the fact that $\Delta u \in C^\alpha(\overline{\Omega})$ if $u \in \mathbb{B}(\Omega) \subseteq C^{2,\alpha}(\overline{\Omega})$. We now turn to prove assertion (2).

To this end we first notice that, since $\partial\Omega$ is of class $C^{2,\alpha}$, we are entitled to apply [37, Thm. 6.14]: there exists a constant $C = C(n, \alpha) > 0$ such that

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C(\|\Delta u\|_{C^\alpha(\overline{\Omega})} + \sup_\Omega |u|), \tag{B.7}$$

for every function $u \in C^{2,\alpha}(\overline{\Omega})$ satisfying $u \equiv 0$ on $\partial\Omega$. Then, by combining (B.2) with (B.7), we obtain the following chain of inequalities:

$$\begin{aligned}\|u\|_{C^{2,\alpha}(\overline{\Omega})} &\leq C(\|\Delta u\|_{C^\alpha(\overline{\Omega})} + \sup_\Omega |u|) \\ &= C(\|\mathcal{L}u - (-\Delta)^s u\|_{C^\alpha(\overline{\Omega})} + \sup_\Omega |u|) \\ &\leq C(\|\mathcal{L}u\|_{C^\alpha(\overline{\Omega})} + \|(-\Delta)^s u\|_{C^\alpha(\overline{\Omega})} + \sup_\Omega |u|) \\ &\leq C'(\|\mathcal{L}u\|_{C^\alpha(\overline{\Omega})} + \|u\|_{C^1(\overline{\Omega})} + \sup_\Omega |u|),\end{aligned} \tag{B.8}$$

holding true for every $u \in \mathbb{B}(\Omega)$. Now, owing again to the regularity of $\partial\Omega$, we can invoke the global interpolation inequality contained in, e.g., [37, Chap. 6]: there exists a constant $\theta > 0$, independent of u, such that

$$\|u\|_{C^1(\overline{\Omega})} \leq \frac{1}{2C'}\|u\|_{C^{2,\alpha}(\overline{\Omega})} + \theta \sup_\Omega |u|.$$

This, together with (B.8), immediately gives the desired (B.6).

Step III. In this last step, we complete the proof of the theorem by using the so-called **method of continuity** (see, e.g., [37, Thm. 5.2]). To this end, we first notice that $\mathbb{B}(\Omega)$ is endowed with a structure of Banach space by the norm

$$\|u\|_{\mathbb{B}(\Omega)} := \|u\|_{C^{2,\alpha}(\overline{\Omega})} \quad \forall\, u \in \mathbb{B}(\Omega).$$

Moreover, for every $0 \leq t \leq 1$ we define

$$\mathcal{L}_t := (1-t)\Delta + t\mathcal{L} = \Delta + t(-\Delta)^s.$$

Owing to (1)–(2) in Step II, we derive that $\mathcal{L}_t$ maps $\mathbb{B}(\Omega)$ into $C^\alpha(\overline{\Omega})$, and

$$\text{(B.9)} \qquad \|u\|_{\mathbb{B}(\Omega)} = \|u\|_{C^{2\alpha}(\overline{\Omega})} \leq C(\|\mathcal{L}_t u\|_{C^\alpha(\overline{\Omega})} + \sup_\Omega |u|), \quad \forall\, u \in \mathbb{B}(\Omega),$$

where $C > 0$ is a suitable constant independent of u and t. On the other hand, by carefully scrutinizing the proof of [7, Thm. 4.7], it is easy to see that

$$\text{(B.10)} \qquad \sup_\Omega |u| = \sup_{\mathbb{R}^n} |u| \leq \kappa \|\mathcal{L}_t u\|_{L^\infty(\Omega)} \leq \kappa \|\mathcal{L}_t u\|_{C^\alpha(\overline{\Omega})},$$

where $\kappa > 0$ is another constant independent of u and t. Thanks to (B.9)–(B.10), we are then entitled to apply the method of continuity in this setting: indeed,

- $\mathbb{B}(\Omega)$ and $C^\alpha(\overline{\Omega})$ are Banach spaces;
- $\mathcal{L}_0, \mathcal{L}_1$ are linear and bounded from $\mathbb{B}(\Omega)$ into $C^\alpha(\overline{\Omega})$ (see (B.2));
- there exists a constant $\hat{C} > 0$ such that

$$\|u\|_{\mathbb{B}(\Omega)} \leq \hat{C} \|\mathcal{L}_t\|_{C^\alpha(\overline{\Omega})} \quad \text{for every } u \in \mathbb{B}(\Omega) \text{ and } t \in [0,1].$$

As a consequence, since $\mathcal{L}_0 = \Delta$ is surjective, we deduce that also $\mathcal{L}_1 = \mathcal{L}$ is surjective: for every $f \in C^\alpha(\overline{\Omega})$ there exists a (unique) $\hat{u}_f \in \mathbb{B}(\Omega)$ such that

$$\text{(B.11)} \qquad \mathcal{L}\hat{u}_f = f \quad \text{pointwise in } \Omega.$$

We explicitly notice that, since $\hat{u}_f \in \mathbb{B}(\Omega)$, one has $\hat{u}_f \in C_b(\mathbb{R}^n) \cap C^{2,\alpha}(\overline{\Omega})$ and $\hat{u}_f \equiv 0$ on $\mathbb{R}^n \setminus \Omega$; thus, by (B.11) we derive that $\hat{u}_f$ is a classical solution of $(\mathrm{D})_f$. In view of these facts, to complete the proof we are left to show that

$$\hat{u}_f = u_f \quad \text{a.e. in } \mathbb{R}^n.$$

To this end we first notice that, since $\hat{u}_f \in C^{2,\alpha}(\overline{\Omega})$ and since $\hat{u}_f \equiv 0$ on $\partial\Omega$, we surely have $\hat{u}_f \in H_0^1(\Omega)$; this, together with (2.1), implies that

$$\text{(B.12)} \qquad \hat{u}_f \in \mathbb{X}(\Omega).$$

Since $\hat{u}_f$ is a classical solution of $(\mathrm{D})_f$, from (B.12) and Remark 2.6 we infer that $\hat{u}_f$ is also a weak solution of problem $(\mathrm{D})_f$; on the other hand, since u_f is the unique weak solution of this problem, we conclude that $\hat{u}_f \equiv u_f$ a.e. in $\mathbb{R}^n$, as desired.

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