ON THE QUASICONFORMAL EQUIVALENCE OF DYNAMICAL CANTOR SETS

By

HIROSHIGE SHIGA*

Abstract. The complement of a Cantor set in the complex plane is itself regarded as a Riemann surface of infinite type. The problem of this paper is the quasiconformal equivalence of such Riemann surfaces. Particularly, we are interested in Riemann surfaces given by Cantor sets which are created through dynamical methods. We discuss the quasiconformal equivalence for the complements of Cantor Julia sets of rational functions and generalized Cantor sets. We also consider the Teichmüller distance between generalized Cantor sets.

1 Introduction

Let E be a Cantor set in the Riemann sphere $\widehat{\mathbb{C}}$, that is, a totally disconnected perfect set in $\widehat{\mathbb{C}}$. The standard middle one-third Cantor set $\mathcal{C}$ is a typical example. We consider the complement $X_E := \widehat{\mathbb{C}} \setminus E$ of the Cantor set E. It is an open Riemann surface with uncountably many boundary components. We are interested in the quasiconformal equivalence of such Riemann surfaces. In the previous paper [15], we show that the complement of the limit set of a Schottky group is quasiconformally equivalent to $X_{\mathcal{C}}$, the complement of $\mathcal{C}$ ([15] Theorem 6.2). In this paper, we discuss the quasiconformal equivalence for the complements of Cantor Julia sets of hyperbolic rational functions and generalized Cantor sets (see §2 for the terminologies). We establish the following theorems.

Theorem I. *Let f be a rational function of degree $d > 1$ and $\mathcal{J}$ be the Julia set of f. Suppose that f is hyperbolic and $\mathcal{J}$ is a Cantor set. Then, the complement $X_{\mathcal{J}}$ of $\mathcal{J}$ is quasiconformally equivalent to $X_{\mathcal{C}}$.*

We should mention that Theorem I may be obtained from a result of MacManus [10] about quasi-circles on $\mathbb{C}$. In this paper, we prove the theorem by using some arguments on open Riemann surfaces and quasiconformal mappings. In fact, those arguments will be fundamental tools throughout this paper.

*The author was partially supported by the Ministry of Education, Science, Sports and Culture, Japan; Grant-in-Aid for Scientific Research (B), 16H03933, 2016–2020.

JOURNAL D'ANALYSE MATHÉMATIQUE, Vol. 147 (2022)
DOI 10.1007/s11854-022-0214-7

For a sequence $\omega = (q_n)_{n=1}^{\infty}$ with $0 < q_n < 1$, we have a generalized Cantor set $E(\omega)$ (see §2.2 for the construction). For a positive constant δ, we say that the sequence ω has a δ-lower bound if $q_n > \delta$, and it has a δ-upper bound if $q_n < 1 - \delta$ $(n = 1, 2, \dots)$. We also say that ω has lower and upper bounds if $\delta < q_n < 1 - \delta$ $(n = 1, 2, \dots)$ for some $\delta > 0$. Then, we obtain the following results.

Theorem II. *Let $\omega = (q_n)_{n=1}^{\infty}$ and $\widetilde{\omega} = (\tilde{q}_n)_{n=1}^{\infty}$ be sequences with δ-lower bound. We put*

$$d(\omega, \widetilde{\omega}) = \sup_{n \in \mathbb{N}} \max \left\{ \left| \log \frac{1 - \tilde{q}_n}{1 - q_n} \right|, |\tilde{q}_n - q_n| \right\}. \tag{1.1}$$

(1) *If $d(\omega, \widetilde{\omega}) < \infty$, then there exists an $\exp(C(\delta) d(\omega, \widetilde{\omega}))$-quasiconformal mapping φ on $\widehat{\mathbb{C}}$ such that $\varphi(E(\omega)) = E(\widetilde{\omega})$, where $C(\delta) > 0$ is a constant depending only on δ;*
(2) *if $\lim_{n \to \infty} \log \frac{1 - \tilde{q}_n}{1 - q_n} = 0$, then $E(\widetilde{\omega})$ is asymptotically conformal to $E(\omega)$, that is, there exists a quasiconformal mapping φ on $\widehat{\mathbb{C}}$ with $\varphi(E(\omega)) = E(\widetilde{\omega})$ such that for any $\varepsilon > 0$, $\varphi|_{U_\varepsilon}$ is $(1 + \varepsilon)$-quasiconformal on a neighborhood U_ε of $E(\omega)$.*

A Kleinian group G is called a **Schottky group** if there exist mutually disjoint $2g\ (\geq 4)$ closed Jordan domains $D_i, \tilde{D}_i$ and Möbius transformations γ_i $(i = 1, \dots, g)$ such that each γ_i sends D_i onto the outside of $\tilde{D}_i$ and G is generated by $\gamma_1, \dots, \gamma_g$.

From above results and a result [15] Theorem 6.2, immediately we obtain

Corollary 1.1. *Let E be a Cantor set which is a Julia set of a rational function satisfying the conditions in Theorem I. Then, the complement of the limit set of a Schottky group G is quasiconformally equivalent to X_E.*

As consequences of Theorem II (1), we obtain

Corollary 1.2. *Let $E(\omega)$ be a generalized Cantor set for $\omega = (q_n)_{n=1}^{\infty}$. Suppose that ω has lower and upper bounds. Then, $X_{E(\omega)}$ is quasiconformally equivalent to $X_{\mathcal{C}}$.*

We have also the following.

Corollary 1.3. *Let E be a Cantor set as in Corollaries 1.1 or 1.2. Then, the Cantor set E is quasiconformally removable, that is, every quasiconformal mapping on the complement of E is extended to a quasiconformal mapping on the Riemann sphere.*

The proof of Theorem II gives the following.

Corollary 1.4. *Let ω and $\widetilde{\omega}$ be sequences satisfying the same conditions as in Theorem II (2). Then, the Hausdorff dimension of $E(\widetilde{\omega})$ is the same as that of $E(\omega)$.*

It is known ([11] V. 11F. Theorem) that the generalized Cantor set $E(\omega)$ for ω is of capacity zero if and only if

$$\prod_{n=1}^{\infty}(1-q_n)^{2^{-n}}=0. \tag{1.2}$$

Hence if $\{q_n\}_{n=1}^{\infty}$ rapidly converges to one as it satisfies (1.2), then $X_{E(\omega)}$ is not quasiconformally equivalent to $X_{\mathcal{C}}$ because the positivity of the capacity of closed sets in the plane is preserved by quasiconformal mappings (cf. [11] III. Theorem 8 H). In fact, we can say more:

Theorem III. *If ω does not have an upper bound, then $X_{E(\omega)}$ is not quasiconformally equivalent to $X_{\mathcal{C}}$.*

Acknowledgements. The author thanks Prof. K. Matsuzaki for his valuable comments. This research was partly done during the author's stay in the Erwin Schrödinger institute at Vienna. He thanks the Institute for its brilliant support to his research. Finally, the author expresses his sincere gratitude to the referee for the valuable comments.

2 Preliminaries

2.1 Complex dynamics. We begin with a small and brief introduction of complex dynamics. We may refer textbooks on the topic, e.g., [6] for a general theory of complex dynamics.

Let f be a rational function of degree $d>1$ on $\mathbb{C}$. We denote by f^n the n times iterations of f. The Fatou set $\mathcal{F}$ of f is the set of points in $\widehat{\mathbb{C}}$ which have neighborhoods where $\{f^n\}_{n=1}^{\infty}$ is a normal family. The complement of $\mathcal{F}$, which is denoted by $\mathcal{J}$, is called the Julia set of f.

A rational function f is called **hyperbolic** if it is expanding near $\mathcal{J}$. More precisely, if $\mathcal{J}\not\ni\infty$, then f is hyperbolic if there exist a constant $A>1$ and a smooth metric $\sigma(z)|dz|$ in a neighborhood U of $\mathcal{J}$ such that

$$\sigma(f(z))|f'(z)|\geq A\sigma(z), \quad z\in\mathcal{J}$$

(see [6] V. 2). If $\infty\in\mathcal{J}$, the hyperbolicity of f is defined by conjugation of Möbius transformations as usual.

The hyperbolicity is also characterized in terms of the Euclidean metric and the dynamical behavior of rational functions as well.

Proposition 2.1 ([6] V. 2. Lemma 2.1 and Theorem 2.2). *A rational function f is hyperbolic if and only if every critical point belongs to $\mathcal{F}$ and is attracted to an attracting cycle. If $\infty \notin \mathcal{J}$, then the hyperbolicity of f is equivalent to the existence of $m \geq 1$ such that $|(f^m)'| > 1$ on $\mathcal{J}$.*

2.2 Generalized Cantor sets (cf. [11] I. 6). Let $\omega = (q_n)_{n=1}^{\infty} = (q_1, q_2, \dots)$ be a sequence of real numbers with $0 < q_n < 1$ for each $n \in \mathbb{N}$. We construct a Cantor set $E(\omega)$ for ω inductively as follows.

First, we remove an open interval J_1 of length q_1 from $E_0 := I = [0, 1]$ so that $I \setminus J_1$ consists of two closed intervals I_1^1, I_1^2 of the same length. We put $E_1 = \bigcup_{i=1}^{2} I_1^i$. We remove an open interval of length $|I_1^i| q_2$ from each I_1^i so that the remainder E_2 consists of four closed intervals of the same length, where $|J|$ is the length of an interval J. Inductively, we define E_{k+1} from $E_k = \bigcup_{i=1}^{2^k} I_k^i$ by removing an open interval of length $|I_k^i| q_{k+1}$ from each closed interval I_k^i of E_k so that E_{k+1} consists of 2^{k+1} closed intervals of the same length. The generalized Cantor set $E(\omega)$ for ω is defined by

$$E(\omega) = \bigcap_{k=1}^{\infty} E_k.$$

It is a generalization of the standard middle one-third Cantor set $\mathcal{C}$. In fact, $\mathcal{C} = E(\omega_0)$ for $\omega_0 = (\frac{1}{3})_{n=1}^{\infty} = (\frac{1}{3}, \frac{1}{3}, \dots)$.

We say that a sequence $\omega = (q_n)_{n=1}^{\infty}$ as above is of (δ-)lower bound if there exists a $\delta > 0$ such that $q_n \geq \delta$ for any $n \in \mathbb{N}$. We also say that a sequence ω has a (δ-)upper bound if $q_n \leq 1 - \delta$ for any $n \in \mathbb{N}$.

2.3 Hausdorff dimension. Let E be a subset of $\mathbb{C}$ and $\alpha > 0$. We consider a countable open covering $\{U_i\}_{i \in \mathbb{N}}$ of E with $\operatorname{diam}(U_i) < r$ for a given $r > 0$. Then, we set

$$\Lambda_\alpha^r(E) := \inf \left\{ \sum_{i \in \mathbb{N}} (\operatorname{diam}(U_i))^\alpha \right\},$$

where the infimum is taken over all countable open coverings $\{U_i\}_{i \in \mathbb{N}}$ with $\operatorname{diam}(U_i) < r$. We put

$$\Lambda_\alpha(E) = \lim_{r \to 0} \Lambda_\alpha^r(E)$$

and the Hausdorff dimension $\dim_H(E)$ of E by

$$\dim_H(E) = \inf\{\alpha \mid \Lambda_\alpha(E) = 0\}.$$

2.4 The quasiconformal equivalence of open Riemann surfaces. We say that two Riemann surfaces R_1, R_2 are quasiconformally equivalent if there exists a quasiconformal homeomorphism between them. We also say that they are quasiconformally equivalent near the ideal boundary if there exist compact subsets K_j of R_j (j=1, 2) and a quasiconformal homeomorphism φ from $R_1\backslash K_1$ onto $R_2\backslash K_2$.

It is obvious that if R_1, R_2 are quasiconformally equivalent, then they are quasiconformally equivalent near the ideal boundary. On the other hand, we have shown that the converse is not true in general. In fact, we have constructed two Riemann surfaces which are not quasiconformally equivalent while they are homeomorphic to each other and quasiconformally equivalent near the ideal boundary ([15] Example 3.1). We also give a sufficient condition for Riemann surfaces to be quasiconformally equivalent ([15] Theorem 5.1).

Proposition 2.2. *Let R_1, R_2 be open Riemann surfaces which are homeomorphic to each other and quasiconformally equivalent near the ideal boundary. If the genus of R_1 is finite, then R_1 and R_2 are quasiconformally equivalent.*

At the end of this section, we present a result on the removability for quasiconformal mappings.

Proposition 2.3 (cf. [9] I. Theorem 8.3)**.** *Let D be a domain or a Riemann surface and φ be a homeomorphism from D to a Riemann surface. Suppose that φ is quasiconformal on $D \setminus C$, where C is an analytic curve in D. Then, φ is a quasiconformal mapping on D.*

3 Proof of Theorem I

Let f be a hyperbolic rational function with a Cantor Julia set $\mathcal{J}$. We show that $X_{\mathcal{J}}$ is quasiconformally equivalent to $X_{\mathcal{C}}$. By Proposition 2.2, it suffices to show that there exists a compact subset K of $\mathcal{F}$ such that $\mathcal{F} \setminus K$ is quasiconformally equivalent to the complement of a compact subset of $X_{\mathcal{C}}$. Considering f^m instead of f for some $m \in \mathbb{N}$, we may assume that $|f'| > 1$ on $\mathcal{J}$ since the Julia set of f^m is the same as that of f for any $m \in \mathbb{N}$.

Considering the conjugation by Möbius transformations, we may assume that $\mathcal{J}$ does not contain ∞. Since $\mathcal{J}$ is a Cantor set, the Fatou set $\mathcal{F}$ is connected. Therefore, it follows from Proposition 2.1 that $\mathcal{F}$ itself is the attractive Fatou component and it contains the attracting fixed point z_0 of f.

It follows from the local theory of attracting fixed points (cf. [6] II. 2) that there exists a simply connected neighborhood Ω_0 of z_0 such that $f(\overline{\Omega_0}) \subset \Omega_0$. We may take Ω_0 so that the boundary $\partial\Omega_0$ is a smooth Jordan curve and it does not contain the forward orbits of critical points of f.

For $\Omega_k := f^{-k}(\Omega_0)$ ($k \in \mathbb{N}$), we have

$$z_0 \in \Omega_0 \subset \overline{\Omega_0} \subset \Omega_1 \subset \overline{\Omega_1} \subset \cdots \subset \overline{\Omega_k} \subset \Omega_{k+1} \subset \cdots \subset \mathcal{F}$$

and

$$\mathcal{F} = \bigcup_{k=0}^{\infty} \Omega_k.$$

Since $\mathcal{F}$ is connected, $\Omega_k := f^{-k}(\Omega_0)$ is connected for each $k \in \mathbb{N}$. Therefore, Ω_k is not simply connected for a sufficiently large k because $\mathcal{F}$ is not simply connected. Hence, we may assume that Ω_1 is bounded by at least two Jordan curves. Then, each Ω_k is bounded by mutually disjoint finitely many smooth Jordan curves.

Since f is hyperbolic, the Julia set $\mathcal{J}$ does not contain critical points. Moreover, it follows from Proposition 2.1 that there exists a simply connected neighborhood V_z of each $z \in \mathcal{J}$ such that $f|_{V_z}$ is injective on V_z. Hence, from compactness of $\mathcal{J}$ there exist disks $V_1, \dots, V_n$ for some $n \in \mathbb{N}$ such that $\mathcal{J} \subset \bigcup_{j=1}^{n} V_j$, $f|_{V_j}$ is injective $(1 \le j \le n)$. Then, we show

Lemma 3.1. *There exists $k_0 \in \mathbb{N}$ such that for any $k \ge k_0$, each connected component of $\Omega_{k+1} \setminus \overline{\Omega_k}$ is contained in some V_j $(1 \le j \le n)$.*

Proof. Let $\Delta_k^1, \dots, \Delta_k^{j(k)}$ be the set of connected components of $\mathbb{C} \setminus \Omega_k$. To prove the claim of this proposition, we need an observation on $\{\Delta_k^i\}$.

Since Δ_k^i is a connected component of the complement of a planar domain Ω_k bounded by finitely many Jordan curves, Δ_k^i is a closed Jordan domain, that is, a topological disk. Therefore, $\Delta_k^1, \dots, \Delta_k^{j(k)}$ are mutually disjoint closed Jordan domains in $\mathbb{C}$. If Δ_k^i is contained in V_j, then for every $l > k$, any connected component of $\mathcal{F} \setminus \Omega_l$ contained in Δ_k^i is also contained in the same V_j.

From the above observation, we see that if every connected component of $\Omega_{k_0+1} \setminus \overline{\Omega_{k_0}} \subset \Delta_{k_0}^i$ $(1 \le i \le j(k_0))$ is contained in some V_j, then it is so for $k \ge k_0$. Hence, it suffices to show that there exists $k_0 \in \mathbb{N}$ such that each $\Delta_{k_o}^i$ $(1 \le i \le j(k_0))$ is contained in some V_j.

Suppose that for any $k \in \mathbb{N}$, there exists an $i(k) \in \{1, \dots, j(k)\}$ such that $\Delta_k^{i(k)}$ is not contained in any V_j $(j = 1, 2, \dots, n)$. We may assume that $i(k) = 1$ and we put $W_k := \Delta_k^1$. By using the above observation again, we may assume that $\{W_k\}_{k=1}^{\infty}$ is nested, that is, $W_k \supset W_{k+1}$ for any k.

Noting that any relatively compact subset in $\mathcal{F}$ is eventually contained in some Ω_k, we see that W_k is in $\bigcup_{j=1}^{n} V_j$ for a sufficiently large k because $\mathcal{F} \setminus \bigcup_{j=1}^{n} V_j$ is compact in $\mathcal{F}$.

Now, we consider $\mathcal{A} := \bigcap_{k=1}^{\infty} W_k$. Since $\{W_k\}_{k=1}^{\infty}$ is a nested set of closed Jordan domains and $\mathcal{F} = \bigcup_{k=1}^{\infty} \Omega_k$, $\mathcal{A}$ has to be a connected closed subset of $\mathcal{J}$. On the other hand, the Julia set $\mathcal{J}$ is totally disconnected. Hence, we conclude that $\mathcal{A} = \{x\}$ for some $x \in \mathcal{J}$ and x is in some V_j. This means that $W_k \subset V_j$ for a sufficiently large k and we have a contradiction. □

We take $k_0 \in \mathbb{N}$ in the above lemma. Let $\omega_1, \omega_2, \ldots, \omega_\ell$ be the set of connected components of $\Omega_{k_0+1} \setminus \overline{\Omega_{k_0}}$. Each ω_j is bounded by a finite number, say $L(j) + 1$, of mutually disjoint simple closed curves. We may assume that $L(j) > 1$ $(j = 1, 2, \ldots, \ell)$ since $\{\Omega_k\}_{k=1}^{\infty}$ exhausts $\mathcal{F}$. Note that the number of connected components of $\partial\Omega_{k_0} \cap \partial\Omega_{k_0+1}$ is equal to ℓ. It is because $\partial\omega_j \cap \partial\Omega_{k_0}$ consists of one simple closed curve for each $j \in \{1, \ldots, \ell\}$.

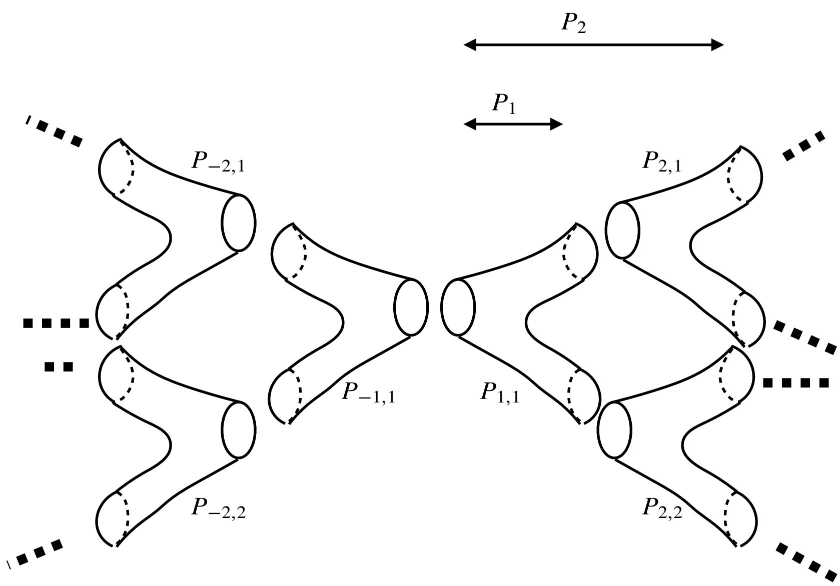

Figure 1. The middle one-third Cantor set.

For any $k > k_0$ and for a connected component ω of $\Omega_{k+1} \setminus \overline{\Omega_k}$, we have $f^{k-k_0}(\omega) \subset \Omega_{k_0+1} \setminus \overline{\Omega_{k_0}}$ and f^{k-k_0} is conformal in ω since ω is contained in some V_j. Hence, ω is conformally equivalent to ω_J for some $J \in \{1, 2, \ldots, \ell\}$. Therefore, if $k > k_0$, then $\Omega_{k+1} \setminus \overline{\Omega_k}$ contains at most ℓ conformally different connected components.

Now, we consider the middle one-third Cantor set $\mathcal{C}$ and $X_{\mathcal{C}} := \widehat{\mathbb{C}} \setminus \mathcal{C}$. It is not hard to see that $X_{\mathcal{C}}$ admits a pants decomposition $\{P_{k,j}\}_{k \in \mathbb{Z}\setminus\{0\}, j \in \{1,\ldots,2^{|k|-1}\}}$ as in Figure 1. While a construction of the pants decomposition is given in [15], we will present the construction for readers' convenience as follows.

We make the middle one-third Cantor set $\mathcal{C}$ on $I = [-1, 1]$.

First, we remove an open interval J_1 of length $2/3$ from $E_0 := I = [-1, 1]$ so that $I \setminus J_1$ consists of two closed intervals I_1^{-1}, I_1^1 of the same length, where $I_1^{-1} \subset \mathbb{R}_{<0}$ and $I_1^1 \subset \mathbb{R}_{>0}$. We put $E_1 = I_1^{-1} \cup I_1^1$. We remove an open interval of length $\frac{1}{3}|I_1^i|$ from each $I_1^{\pm 1}$ so that the remainder E_2 consists of four closed intervals of the same length, where $|J|$ is the length of an interval J. Inductively, we define E_{k+1} from $E_k = \bigcup_{i=-2^{k-1}}^{-1} I_k^i \cup \bigcup_{i=1}^{2^{k-1}} I_k^i$ by removing an open interval of length $\frac{1}{3}|I_k^i|$ from each closed interval I_k^i of E_k so that E_{k+1} consists of 2^{k+1} closed intervals of the same length. The Cantor set $\mathcal{C}$ is defined by

$$\mathcal{C} = \bigcap_{k=1}^{\infty} E_k.$$

We denote the imaginary axis by C_0^0. For any (k, i) $(k \in \mathbb{N}; i = \pm 1, \ldots, \pm 2^{k-1})$, we take a circle C_k^i which is a circle centered at the midpoint of I_k^i with radius $\frac{5}{6}|I_k^i|$. We see that all C_k^i's are mutually disjoint curves in $X_{\mathcal{C}}$ and each C_k^i contains $C_{k+1}^{\varepsilon(i)(2|i|-1)}$, C_{k+1}^{2i}, where $\varepsilon(i) = -1$ if $i < 0$ and $\varepsilon(i) = 1$ if $i > 0$. Hence, they make a pants decomposition of $X_{\mathcal{C}}$. A pair of pants bounded by C_0^0, C_2^1 (resp., C_2^{-1}) and C_2^2 (resp., C_2^{-2}) is denoted by $P_{1,1}$ (resp., $P_{-1,1}$). We also denote by $P_{\varepsilon(i)k,|i|}$ a pair of pants bounded by C_k^i, $C_{k+1}^{2i-\varepsilon(i)}$ and C_{k+1}^{2i}. Obviously, for every k with $|k| \geq 2$, $P_{k,j}$ $(j = 1, \ldots, 2^{|k|-1})$ is conformally equivalent to $P_{2,1}$. Thus, we have the pants decomposition as Figure 1. Let $P_N (N \in \mathbb{N})$ be a subdomain of $X_{\mathcal{C}}$ consisting of $P_{i,j}$ for $i = 1, \ldots, N$ and $j = 1, \ldots, 2^{i-1}$.

Let $N_0 \in \mathbb{N}$ be the largest number with $2^{N_0} + 1 \leq \ell$. We put

$$K_0 := \overline{P_{N_0} \bigcup_{j=1}^{\ell_0} P_{N_0+1,j}},$$

where $\ell_0 = \ell - 2^{N_0} - 1$. Then, K_0 is a compact subset of $X_{\mathcal{C}}$ bounded by ℓ simple closed curves. We denote them by $C_1, \ldots, C_\ell$, where $C_1 \subset \partial P_{1,1}$. We may take a subdomain G_1 of $X_{\mathcal{C}}$ so that $G_1 \setminus K_0$ is quasiconformally equivalent to $\Omega_{k_0+1} \setminus \overline{\Omega_{k_0}}$ as follows.

We take the largest number L_1 with $2^{L_1} \leq L(1)$. Then,

$$\overline{G_{1,1}} := \overline{\left(\bigcup_{i=1}^{L_1} \bigcup_{j=1,\ldots,2^{i-1}} P_{-i,j}\right) \cup \left(\bigcup_{j=1,\ldots,L(1)-2^{L_1}} P_{-L_1-1,j}\right)}$$

is a closed subdomain of $X_{\mathcal{C}}$ with $L(1) + 1$ boundary curves. Hence, $G_{1,1}$ is quasiconformally equivalent to ω_1 since both of them are planar domains bounded by the same number of closed curves.

Similarly, we may construct subdomains $G_{1,2}, \ldots, G_{1,\ell}$ such that

$$\partial G_{1,j} \cap \partial K_0 = C_j$$

and each $G_{1,j}$ is quasiconformally equivalent to ω_j ($j = 1, 2, \ldots, \ell$). Combining K_0 with $G_{1,1}, \ldots, G_{1,\ell}$, we obtain a desired subdomain G_1.

By using the same argument as above, we have a subdomain G_2 of $X_{\mathcal{C}}$ such that $G_1 \subset G_2$ and $G_2 \setminus \overline{G_1}$ is quasiconformally equivalent to $\Omega_{k_0+2} \setminus \overline{\Omega_{k_0+1}}$.

We may use this argument inductively and we obtain an exhaustion $\{G_i\}_{i=1}^{\infty}$ of $X_{\mathcal{C}}$ such that

$$K_0 \subset G_1 \subset G_2 \subset \cdots \subset G_i \subset G_{i+1} \subset \ldots, \quad X_{\mathcal{C}} = \bigcup_{i=1}^{\infty} G_i,$$

and $G_{i+1} \setminus \overline{G_i}$ are quasiconformally equivalent to $\Omega_{k_0+i+1} \setminus \overline{\Omega_{k_0+i}}$ ($i = 1, 2, \ldots$).

From this construction, we have a natural bijection $\mathcal{I}$ between the set of connected components of $G_{i+1} \setminus \overline{G_i}$ and those of $\Omega_{k_0+i+1} \setminus \overline{\Omega_{k_0+i}}$ ($i = 1, 2, \ldots$) such that

(1) if D is a connected component of $G_{i+1} \setminus \overline{G_i}$, then $\mathcal{I}(D)$ is a connected component of $\Omega_{k_0+i+1} \setminus \overline{\Omega_{k_0+i}}$;

(2) if D' is a connected component of $G_{i+2} \setminus \overline{G_{i+1}}$ which is adjacent to a connected component D of $G_{i+1} \setminus \overline{G_i}$, that is, $\partial D \cap \partial D' \neq \emptyset$, then $\mathcal{I}(D')$ is adjacent to $\mathcal{I}(D)$.

Now, we use the following proposition which is obtained from [15] Lemma 4.1 and its proof.

Proposition 3.1. *Let X, Y be Riemann surfaces. We consider simple closed curves $\alpha \subset X$ and $\beta \subset Y$ with $X \setminus \alpha = X_1 \sqcup X_2$ and $Y \setminus \beta = Y_1 \sqcup Y_2$, respectively. Suppose that there exist quasiconformal mappings $f_i : X_i \to Y_i$ ($i = 1, 2$) such that $f_1(\alpha) = f_2(\alpha) = \beta$. Then, there exist an annular neighborhood U of α and a quasiconformal mapping f on U into Y such that*

(1) $V := f(U)$ is an annular neighborhood of β;

(2) we put

$$F(p) = \begin{cases} f_i(p), & p \in X_i \setminus U \quad (i = 1, 2), \\ f(p), & p \in U. \end{cases} \tag{3.1}$$

Then F is a quasiconformal mapping from X to Y.

Proof. Since the proof is the same as that of [15, Lemma 4.1], we give a brief outline of the proof.

We take an annular neighborhood U of α so that the boundary ∂U consists of analytic Jordan curves. We put $C_i := X_i \cap \partial U$ $(i = 1, 2)$ and denote by V the annulus bounded by $f_1(C_1), f_2(C_2)$. By arranging $f_1|_{C_1}$ and $f_2|_{C_2}$, we may assume that both $f_1|_{C_1}$ and $f_2|_{C_2}$ are smooth mappings.

Take $\gamma \in \mathrm{PSL}(2, \mathbb{R})$ which represents U, that is, $\mathbb{H}/\langle\gamma\rangle$ is conformally equivalent to U. We may assume that $\gamma(z) = kz$ for some $k > 1$. Then, we verify that $f_1|_{C_1}$ together with $f_2|_{C_2}$ determines a $\langle\gamma\rangle$-compatible quasi-symmetric homeomorphism h on $\mathbb{R}$. Taking the Douady–Earle extension ([8]) of h, we obtain a $\langle\gamma\rangle$-compatible quasiconformal mapping $\tilde{f}$ on $\mathbb{H}$ with $\tilde{f}|_{\mathbb{R}} = h$. The projected mapping $f : U \to V$ is a quasiconformal mapping with $f|_{C_i} = f_i$ $(i = 1, 2)$. It follows from Proposition 2.3 that the mapping F defined by (3.1) is a quasiconformal mapping from X to Y. □

Remark 3.1. From the above construction, we see that the quasiconformal mapping f is determined by the local behaviors of f_j $(j = 1, 2)$ near α. Namely, if we have quasiconformal mappings $f_{ij} : X_i \to Y_i$ $(i, j = 1, 2)$ satisfying the above conditions and a neighborhood U_0 of α such that $f_{i1}|_{U_0 \cap X_i} = f_{i2}|_{U_0 \cap X_i}$, then we obtain quasiconformal mapping F_j as in Proposition 3.1 for f_j $(j = 1, 2)$ so that $F_1|_U = F_2|_U$ in a neighborhood U of α.

Let $\mathcal{D}'$ be a connected component of $\Omega_{k_0+i+2} \setminus \overline{\Omega_{k_o+i+1}}$ which is adjacent to a connected component $\mathcal{D}$ of $\Omega_{k_0+i+1} \setminus \overline{\Omega_{k_0+i}}$. We put $\beta := \partial\mathcal{D} \cap \partial\mathcal{D}'$ and $Y_{\mathcal{D},\mathcal{D}'} := \mathcal{D} \cup \beta \cup \mathcal{D}'$. We may find connected components D of $G_{i+1} \setminus \overline{G_i}$ and D' of $G_{i+2} \setminus \overline{G_{i+1}}$ so that $\mathcal{I}(D) = \mathcal{D}$ and $\mathcal{I}(D') = \mathcal{D}'$ and D' is adjoining D along $\alpha := \partial D \cap \partial D'$. We put

$$X_{D,D'} := D \cup \alpha \cup D'.$$

Let $X_1, X_2, \ldots, X_n$ be the set of connected components of $G_3 \setminus \overline{G_1}$ and $Y_1, \ldots, Y_n$ the set of connected components of $\Omega_{k_0+3} \setminus \overline{\Omega_{k_0+1}}$. It follows from Lemma 3.1 that the rational function f is conformal in every connected component of $\Omega_{k_0+i+1} \setminus \overline{\Omega_{k_0+i}}$ $(i \in \mathbb{N})$. Therefore, the Riemann surface $Y_{\mathcal{D},\mathcal{D}'}$ is conformally equivalent to some Y_j via f^{i-1}. We may assume that $f^{i-1}(Y_{\mathcal{D},\mathcal{D}'}) = Y_1$ and put $\beta_1 := f^{i-1}(\beta)$. Similarly, $X_{D,D'}$ is conformally equivalent to a connected component, say X_1, of $G_3 \setminus \overline{G_1}$ via a conformal map h. We put $\alpha_1 := h(\alpha)$.

Then, we have

$$X_1 \setminus \alpha_1 = X_{1,1} \sqcup X_{1,2} \quad \text{and} \quad Y_1 \setminus \beta_1 = Y_{1,1} \sqcup Y_{1,2},$$

where

$$X_{1,1} = h(D), \quad X_{1,2} = h(D'), \quad Y_{1,1} = f^{i-1}(\mathcal{D}) \quad \text{and} \quad Y_{1,2} = f^{i-1}(\mathcal{D}').$$

We see that there are quasiconformal mappings $\varphi_{1,j} : X_{1,j} \to Y_{1,j}$ $(j = 1, 2)$ such that $\varphi_{1,j}(\alpha_1) = \beta_1$. It follows from Proposition 3.1 that there exist a quasiconformal mapping $\Phi_1 : X_1 \to Y_1$ and an annular neighborhood U_1 of α_1 in X_1 such that $\Phi_1|_{X_{1,j}\setminus U_1} = \varphi_{1,j}$ $(j = 1, 2)$. Then, a mapping $\Phi_{D,D'} : X_{D,D'} \to Y_{\mathcal{D},\mathcal{D}'}$ given by

$$\Phi_{D,D'} := (f^{i-1}|_{Y_{\mathcal{D},\mathcal{D}'}})^{-1} \circ \Phi_1 \circ h$$

is a quasiconformal mapping with the same maximal dilatation as that of Φ_1 which depends only on X_1, Y_1 and $\varphi_{1,j}$ $(j = 1, 2)$.

Next, we take a connected component D'' of $G_{i+3} \setminus \overline{G_{i+2}}$ adjoining D' along $\alpha' := \partial D' \cap \partial D''$. Then, $\mathcal{D}'' := \mathcal{I}(D'')$ is a connected component of $\Omega_{k_0+i+3} \setminus \overline{\Omega_{k_0+i+2}}$ adjoining $\mathcal{D}'$ along $\beta' := \partial\mathcal{D}' \cap \partial\mathcal{D}''$. We extend Φ_1 to D''.

Put $X_{D',D''} := D' \cup \alpha' \cup D''$ and $Y_{\mathcal{D}',\mathcal{D}''} := \mathcal{D}' \cup \beta' \cup \mathcal{D}''$. Then, $X_{D',D''}$ is conformally equivalent to a connected component of $G_3 \setminus \overline{G_1}$, say X_2, via a conformal mapping g and $Y_{\mathcal{D}',\mathcal{D}''}$ is conformally equivalent to a connected component of $\Omega_{k_0+3} \setminus \overline{\Omega_{k_0+1}}$, say Y_2, via f^i. We put

$$X_{2,1} := g(D'), \quad X_{2,2} := g(D''), \quad \alpha_2 := g(\alpha'), \quad Y_{2,1} := f^i(\mathcal{D}'), \quad Y_{2,2} := f^i(\mathcal{D}'') \quad \text{and} \quad \beta_2 := f^i(\beta').$$

Note that $f|_{Y_{1,2}} : Y_{1.2} \to Y_{2,1}$ is a conformal mapping with $f|_{Y_{1,2}}(f^{i-1}(\beta')) = \beta_2$. It is also seen that

$$\widetilde{h} := g \circ (h|_{X_{1,2}})^{-1} : X_{1,2} \to X_{2,1}$$

is a conformal mapping with $\widetilde{h}(h(\alpha')) = \alpha_2$. Hence,

$$\varphi_{2,1} := f \circ \Phi_1|_{X_{1,2}} \circ \widetilde{h}^{-1} : X_{2,1} \to X_{1,2}$$

is a quasiconformal mapping with the same maximal dilatation as that of Φ_1. It is also seen that $\varphi_{2,1}(\alpha_2) = \beta_2$.

We take a quasiconformal mapping $\varphi_{2,2} : X_{2,2} \to Y_{2,2}$ with $\varphi_{2,2}(\alpha_2) = \beta_2$. It follows from Proposition 3.1 that there exist a quasiconformal mapping $\Phi_2 : X_2 \to Y_2$ and an annular neighborhood U_2 of α_2 in X_2 such that $\Phi_2|_{X_{2,j}\setminus U_2} = \varphi_{2,j}$ $(j = 1, 2)$.

Note that we may take U_2 small so that $(U_2 \cap X_{2,1}) \cap \widetilde{h}(U_1) = \emptyset$. Then, we have

$$\varphi_{2,1}|_{U_2 \cap X_{2,1}} = f \circ \varphi_{1,2}|_{\widetilde{h}^{-1}} \circ \widetilde{h}^{-1}|_{U_2 \cap X_{2,1}}$$

and we see that the maximal dilatation of $\Phi_2|U_2$ is independent of the construction of Φ_1 but depends only on $\varphi_{2,j}|_{U_2}$ $(j = 1, 2)$ (see Remark 3.1).

We define a mapping $\Phi_{D',D''}$ by

$$\Phi_{D',D''} := (f^i|_{Y_{\mathcal{D}',\mathcal{D}''}})^{-1} \circ \Phi_2 \circ g : X_{D',D''} \to Y_{\mathcal{D}',\mathcal{D}''}.$$

It is a quasiconformal mapping with the same maximal dilatation as that of Φ_2.

Since $\widetilde{h}(h(p)) \in X_{2,1} \setminus U_2$ for $p \in D' \setminus g^{-1}(U_2)$, we have

$$\begin{aligned}
\Phi_{D',D''}(p) &= (f^i|_{Y_{\mathcal{D}',\mathcal{D}''}})^{-1} \circ \Phi_2 \circ g \circ h^{-1}(h(p)) \\
&= (f^i|_{Y_{\mathcal{D}',\mathcal{D}''}})^{-1} \circ \Phi_2 \circ \widetilde{h}(h(p)) \\
&= (f^i|_{Y_{\mathcal{D}',\mathcal{D}''}})^{-1} \circ \varphi_{2,1}(\widetilde{h}(h(p)) \\
&= (f^i|_{Y_{\mathcal{D}',\mathcal{D}''}})^{-1} \circ f \circ \Phi_1|_{X_{1,2}}(h(p)) \\
&= (f^i|_{Y_{\mathcal{D}',\mathcal{D}''}})^{-1} \circ f \circ f^{i-1} \circ \Phi_{D,D'}(p) \\
&= \Phi_{D,D'}(p).
\end{aligned}$$

Thus, a mapping $\Phi_{D,D',D''}$ given by

$$\Phi_{D,D',D''}(p) = \begin{cases} \Phi_{D,D'}(p), & p \in X_{D,D'} \setminus g^{-1}(U_2) \\ \Phi_{D',D''}(p), & p \in X_{D',D''} \end{cases}$$

is a quasiconformal mapping from

$$X_{D,D',D''} := X_{D,D'} \cup X_{D',D''}$$

onto

$$Y_{\mathcal{D},\mathcal{D}',\mathcal{D}''} := Y_{\mathcal{D},\mathcal{D}'} \cup Y_{\mathcal{D}',\mathcal{D}''}$$

and the maximal dilatation depends only on X_j, Y_j and $\varphi_{i,j}$ $(i, j = 1, 2)$. In other words, we can extend a quasiconformal mapping $\Phi_{D,D'} : X_{D,D'} \to Y_{\mathcal{D},\mathcal{D}'}$ to a quasiconformal mapping $\Phi_{D,D',D''} : X_{D,D',D''} \to Y_{\mathcal{D},\mathcal{D}',\mathcal{D}''}$.

Repeating this construction inductively to cover all connected components of $G_{i+2} \setminus \overline{G_i}$ and $\Omega_{k_0+i+2} \setminus \overline{\Omega_{k_0+i}}$ $(i \in \mathbb{N})$, we obtain a homeomorphism

$$\Phi : X_{\mathcal{C}} \setminus \overline{G_1} \to \mathcal{F} \setminus \overline{\Omega_{k_0+1}}.$$

Furthermore, in each step of the argument, the maximal dilatation of the extended quasiconformal mapping depends only on a finite number of data, namely, $\{X_j\}_{j=1}^n$, $\{Y_j\}_{j=1}^n$ and prescribed quasiconformal mappings, such as $\{\varphi_{i,j}\}$, between them. Therefore, the maximal dilatations are uniformly bounded and Φ is a quasiconformal mapping.

Since $\overline{G_1}$, $\overline{\Omega_{k_0}}$ are compact subsets of planar domains, from Proposition 2.2 we verify that $X_{\mathcal{C}}$ and $\mathcal{F}$ are quasiconformally equivalent. □

4 Proof of Theorem II

Proof of (1). We divide the proof into several steps.

Step 1: Analyzing generalized Cantor sets. Let $\omega = (q_n)_{n=1}^{\infty}$ and $\widetilde{\omega} = (\tilde{q}_n)_{n=1}^{\infty}$ be sequences with δ-lower bound. We take

$$E_k = \bigcup_{i=1}^{2^k} I_k^i \quad \text{and} \quad \widetilde{E}_k = \bigcup_{i=1}^{2^k} \widetilde{I}_k^i$$

as in §2.2 for ω and $\widetilde{\omega}$, respectively. In fact, I_k^i (resp., $\widetilde{I}_k^i$) is located at the left of I_k^{i+1} (resp., $\widetilde{I}_k^{i+1}$) for $i = 1, 2, \ldots, 2^k - 1$. The set $[0, 1] \setminus E_k$ (resp., $[0, 1] \setminus \widetilde{E}_k$) consists of $2^k - 1$ open intervals $J_k^1, \ldots, J_k^{2^k-1}$ (resp., $\widetilde{J}_k^1, \ldots, \widetilde{J}_k^{2^k-1}$). Each J_k^i (resp., $\widetilde{J}_k^i$) is located between I_k^i and I_k^{i+1} (resp., $\widetilde{I}_k^i$ and $\widetilde{I}_k^{i+1}$).

Because of the construction, we have

$$|I_{k+1}^i| = \frac{1}{2}(1 - q_{k+1})|I_k^i| \quad (k = 0, 1, \ldots).$$

Therefore, we have

$$|I_k^i| = 2^{-k} \prod_{j=1}^{k} (1 - q_j). \tag{4.1}$$

Next, we estimate the length of J_k^i.

In the construction of E_{k+1} from E_k, we obtain open intervals $I_{k+1}^{2i-1}, I_{k+1}^{2i}$ and the closed interval J_{k+1}^{2i-1} such that $I_k^i = I_{k+1}^{2i-1} \cup J_{k+1}^{2i-1} \cup I_{k+1}^{2i}$ for each i, k (Figure 2).

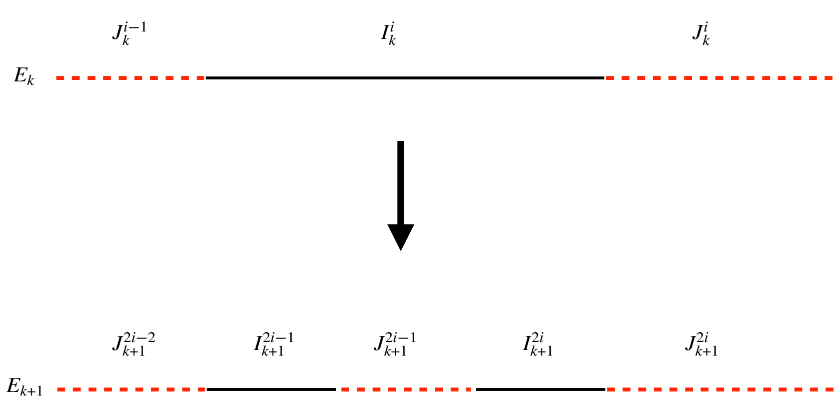

Figure 2.

If i is odd, we have

$$|J_{k+1}^i| = |I_k^i| q_{k+1} = \frac{2q_{k+1}}{1-q_{k+1}} |I_{k+1}^1| \geq 2\delta |I_{k+1}^1|, \tag{4.2}$$

as $q_{k+1} \geq \delta$.

If i is even, then $i = 2^\ell m$ for an integer ℓ with $1 \leq \ell \leq k$ and an odd number m. Since J_{k+1}^i is located between I_{k+1}^i and I_{k+1}^{i+1}, we see that $J_{k+1}^i = J_k^{i/2} = J_k^{2^{\ell-1}m}$. Repeating this argument, we have $J_{k+1}^i = J_{k-\ell+1}^m$. Since m is odd, we conclude from (4.2) that

$$\begin{aligned} |J_{k+1}^i| = |J_{k-\ell+1}^m| &= 2^{-k+\ell} q_{k-\ell+1} \prod_{j=1}^{k-\ell} (1-q_j) \\ &\geq 2^{-k+1}\delta \prod_{j=1}^{k+1}(1-q_j) \geq 4\delta |I_{k+1}^1| \end{aligned} \tag{4.3}$$

as $q_{k-\ell+1} \geq \delta$.

Thus, we obtain the following from (4.2) and (4.3).

Lemma 4.1. *Let I_k^i and J_{k+1}^i be the same ones as above for a sequence $\omega = (q_n)_{n=1}^\infty$ with δ-lower bound. Then,*

$$|J_{k+1}^i| \geq 2\delta |I_{k+1}^1| \tag{4.4}$$

hold for $i = 1, 2, \ldots, 2^{k+1} - 1$ and for $k \geq 0$.

Step 2: Constructing a pants decomposition. We draw a circle C_k^i centered at the midpoint of I_k^i with radius $\frac{1}{2}(1+\delta)|I_k^1|$ for each $k \in \mathbb{N} \cup \{0\}$ and $1 \leq i \leq 2^k$. Here, we put $I_0^1 := I(= E_0)$ and $\mathbb{D}_\delta := \{\, |z - \frac{1}{2}| \leq \frac{1}{2}(1+\delta)\}$. From (4.4), we see that $C_k^i \cap C_k^j = \emptyset$ if $i \neq j$. Since

$$\frac{1}{2} \cdot \delta |I_{k+1}^1| < \frac{1}{2} \cdot \delta |I_k^1|,$$

we also see that $C_{k+1}^i \cap C_k^j = \emptyset$. Therefore, $\bigcup_{k=1}^\infty \bigcup_{i=1}^{2^k} C_k^i$ gives a pants decomposition for $X_{E(\omega)} \setminus \mathbb{D}_\delta$.

We draw circles $\widetilde{C}_k^i$ for $\widetilde{\omega}$ by the same way. Then, we also see that $\bigcup_{k=1}^\infty \bigcup_{i=1}^{2^k} \widetilde{C}_k^i$ gives a pants decomposition for $X_{E(\widetilde{\omega})} \setminus \mathbb{D}_\delta$.

Step 3: Analyzing a pair of pants. We denote by P_k^i a pair of pants bounded by C_k^i, C_{k+1}^{2i-1} and C_{k+1}^{2i}. We consider the complex structure of P_k^i so that we may assume that the center of C_k^i is the origin with radius $\frac{1}{2}(1+\delta)|I_k^1|$. Then, the centers of C_{k+1}^{2i-1} and C_{k+1}^{2i} are

$$-\frac{1}{4}(1+q_{k+1})|I_k^1|$$

and
$$\frac{1}{4}(1+q_{k+1})|I_k^1|,$$
respectively.

Apply an affine map $z \mapsto \alpha z + \beta$ for some $\alpha > 0, \beta \in \mathbb{R}$ to P_k^i so that the circle C_k^i is sent to a circle centered at the origin with radius $1+\delta$. We denote the circle by $C_{k,1}$. Then, the circle C_{k+1}^{2i-1} is sent to a circle $C_{k,2}$ centered at
$$-x_k := -\frac{1}{2}(1+q_{k+1})$$
with radius
$$r_k := \frac{1}{2}(1+\delta)(1-q_{k+1})$$
and C_{k+1}^{2i} is sent to a circle $C_{k,3}$ centered at x_k with radius r_k. We may conformally identify P_k^i with a pair of pants $\mathcal{P}_k$ bounded by $C_{k,1}$, $C_{k,2}$ and $C_{k,3}$.

Similarly, we consider a pair of pants $\widetilde{P}_k^i$ bounded by $\widetilde{C}_k^i$, $\widetilde{C}_{k+1}^{2i-1}$ and $\widetilde{C}_{k+1}^{2i}$, and apply an affine map to the pair of pants $\widetilde{P}_k^i$ so that the circle $\widetilde{C}_k^i$ is mapped to a circle centered at the origin with radius $1+\delta$, which is the same circle as the image of C_k^i above. We denote by $\widetilde{C}_{k,i}$ the image of $\widetilde{C}_k^i$ $(i = 1, 2, 3)$. We may conformally identify $\widetilde{P}_k^i$ with a pair of pants $\widetilde{\mathcal{P}}_k$ bounded by $\widetilde{C}_{k,1}$, $\widetilde{C}_{k,2}$ and $\widetilde{C}_{k,3}$, where $\widetilde{C}_{k,1}$ is the same circle as $C_{k,1}$, $\widetilde{C}_{k,2}$ is centered at
$$-\widetilde{x}_k := -\frac{1}{2}(1+\widetilde{q}_{k+1})$$
with radius
$$\widetilde{r}_k := \frac{1}{2}(1+\delta)(1-\widetilde{q}_{k+1})$$
and $\widetilde{C}_{k,3}$ is centered at $\widetilde{x}_k$ with radius $\widetilde{r}_k$.

Step 4: Constructing intermediate pairs of pants. By applying
$$z \mapsto (x_k/\widetilde{x}_k)z$$
to $\widetilde{\mathcal{P}}_k$, we obtain a pair of pants $\widehat{P}_k$. The pair of pants $\widehat{P}_k$ is bounded by $\widehat{C}_{k,1}$, $\widehat{C}_{k,2}$ and $\widehat{C}_{k,3}$. Each $\widehat{C}_{k,i}$ is corresponding to $\widetilde{C}_{k,i}$ $(i = 1, 2, 3)$. Note that for each i, the center of $\widehat{C}_{k,i}$ is the same as that of $C_{k,i}$, and $\widehat{P}_k$ is conformally equivalent to $\widetilde{\mathcal{P}}_k$. The radius of $\widehat{C}_{k,1}$ is
$$(1+\delta)\cdot\frac{x_k}{\widetilde{x}_k} = (1+\delta)\frac{1+q_{k+1}}{1+\widetilde{q}_{k+1}},$$
and the radius of $\widehat{C}_{k,2}$, $\widehat{C}_{k,3}$ is
$$\widehat{r}_k := \frac{1}{2}(1+\delta)(1-\widetilde{q}_{k+1})\frac{1+q_{k+1}}{1+\widetilde{q}_{k+1}}.$$

Now, we take an intermediate pair of pants P'_k bounded by $\widehat{C}_{k,1}$, $C_{k,2}$ and $C_{k,3}$.

Step 5: Making quasiconformal mappings, I. In the following argument, we use a notation $d(\varphi)$ for a quasiconformal mapping φ as

$$d(\varphi) = \log K(\varphi),$$

where $K(\varphi)$ is the maximal dilatation of φ.

We suppose that $q_{k+1} \geq \widetilde{q}_{k+1}$. Then, we have

$$\widehat{r}_k \geq r_k = \frac{1}{2}(1+\delta)(1-q_{k+1}).$$

In other words, the radius of $\widehat{C}_{k,2}$, $\widehat{C}_{k,3}$ is not smaller than that of $C_{k,2}$, $C_{k,3}$.

Let $C_{k,+}$ be a circle centered at x_k with radius

$$\widetilde{R}_k := (1+\delta)\frac{x_k}{\widetilde{x}_k} - x_k,$$

so that $C_{k,+}$ is tangent with $\widehat{C}_{k,1}$.

We consider two circular annuli $A_{k,+}$ bounded by $C_{k,+}$ and $\widehat{C}_{k,3}$, $A'_{k,+}$ bounded by $C_{k,+}$ and $C_{k,3}$. We have

$$C_{k,+} \cap \mathbb{R} = \{x_k - \tilde{R}_k, x_k + \tilde{R}_k\}.$$

Since $\widetilde{q}_{k+1} \geq \delta$, we have

$$x_k - \tilde{R}_k = 2x_k\left\{1 - \frac{1+\delta}{1+\widetilde{q}_{k+1}}\right\} \geq 0.$$

Hence, $A_{k,+}, A'_{k,+} \subset \{\mathrm{Re}\, z > 0\}$.

Here, we use the following well-known fact.

Lemma 4.2. *For annuli* $A_i = \{0 < r_i < |z| < R_i < \infty\}$ $(i = 1, 2)$*, there exists a quasiconformal mapping* $\varphi : A_1 \to A_2$ *such that*

$$\varphi(r_1 e^{i\theta}) = r_2 e^{i\theta},$$
$$\varphi(R_1 E^{i\theta}) = R_2 e^{i\theta},$$

and

$$K(\varphi) = e^{d(A_1, A_2)},$$

where

$$d(A_1, A_2) = \left|\log \frac{\log R_1 - \log r_1}{\log R_2 - \log r_2}\right|.$$

Proof. The mapping φ is given by

$$\varphi : z \mapsto \frac{r_2}{r_1^k}|z|^{k-1}z$$

for $k = \frac{\log R_2 - \log r_2}{\log R_1 - \log r_1}$. Indeed, it is easy to see that $|\varphi(z)| = r_2$ when $|z| = r_1$ and $|\varphi(z)| = R_2$ when $|z| = R_1$. Moreover,

$$\varphi \circ \exp = \exp \circ f$$

holds for $f(x + iy) = k(x - \log r_1) + \log r_2 + iy$. Since exp is locally conformal, we verify that $K(\varphi) = e^{d(A_1,A_2)}$. □

It follows from Lemma 4.2 that there exists a quasiconformal mapping $\varphi_{k,+} : A_{k,+} \longrightarrow A'_{k,+}$ such that

$$d(\varphi_{k,+}) = \log \frac{\log \widetilde{R}_k - \log r_k}{\log \widetilde{R}_k - \log \widehat{r}_k},$$

$$\varphi_{k,+}(z) = z, \tag{4.5}$$

for any $z \in C_{k,+}$ and

$$\arg(\varphi_{k,+}(z) - x_k) = \arg(z - x_k) \tag{4.6}$$

for $z \in \widehat{C}_{k,3}$.

Since

$$\log \frac{c-a}{c-b} = \log \left(1 + \frac{b-a}{c-b}\right) \leq \frac{b-a}{c-b}$$

for $0 < a \leq b < c$, we obtain

$$d(\varphi_{k,+}) \leq \frac{\log \widehat{r}_k - \log r_k}{\log \widetilde{R}_k - \log \widehat{r}_k}. \tag{4.7}$$

Moreover, we have

$$\begin{aligned} \log \widetilde{R}_k - \log \widehat{r}_k &= \log \left\{ \frac{1}{1+\delta} \cdot \left(1 + \frac{2\delta}{1 - \widetilde{q}_{k+1}}\right)\right\} \\ &\geq \log \frac{1}{1-\delta} > 0, \end{aligned} \tag{4.8}$$

and

$$\log \widehat{r}_k - \log r_k = \log \frac{1 - \widetilde{q}_{k+1}}{1 - q_{k+1}} + \log \frac{1 + q_{k+1}}{1 + \widetilde{q}_{k+1}}. \tag{4.9}$$

From (4.7)–(4.9), we obtain

$$
\begin{aligned}
d(\varphi_{k,+}) &\le \Big(\log\frac{1}{1-\delta}\Big)^{-1}\Big\{\log\frac{1-\widetilde{q}_{k+1}}{1-q_{k+1}} + (q_{k+1}-\widetilde{q}_{k+1})\Big\} \\
&\le \Big(\log\frac{1}{1-\delta}\Big)^{-1} d(\omega,\widetilde{\omega}).
\end{aligned} \tag{4.10}
$$

We may do the same operation, symmetrically; we take a circle $C_{k,-}$ centered at $-x_k$ of radius $\widetilde{R}_k$ and consider two annuli $A_{k,-}$ and $A'_{k,-}$. The annulus $A_{k,-}$ is bounded by $C_{k,-}$ and $\widehat{C}_{k,2}$, and $A'_{k,-}$ is bounded by $C_{k,-}$ and $C_{k,2}$. Note that $A_{k,-}, A'_{k,-} \subset \{\mathrm{Re}\, z < 0\}$.

Then, we obtain a quasiconformal mapping $\varphi_{k,-} : A_{k,-} \to A'_{k,-}$ such that

$$
\varphi_{k,-}(z) = z \tag{4.11}
$$

for $z \in C_{k,-}$ and

$$
\arg(\varphi_{k,-}(z) + x_k) = \arg(z + x_k) \tag{4.12}
$$

for $z \in \widehat{C}_{k,2}$. Moreover, the mapping satisfies an inequality

$$
d(\varphi_{k,-}) \le \Big(\log\frac{1}{1-\delta}\Big)^{-1} d(\omega,\widetilde{\omega}). \tag{4.13}
$$

We define a mapping $\varphi_k : \widehat{P}_k \to P'_k$ by

$$
\varphi_k(z) = \begin{cases} \varphi_{k,+}(z), & z \in A_{k,+}, \\ \varphi_{k,-}(z), & z \in A_{k,-}, \\ z, & \text{otherwise.} \end{cases}
$$

As we have seen that $A_{k,+}$ is in $\{\mathrm{Re}\, z > 0\}$ and $A_{k,-}$ is in $\{\mathrm{Re}\, z < 0\}$, annuli $A_{k,+}$ and $A_{k,-}$ are mutually disjoint and the mapping φ_k is a well-defined homeomorphism. The homeomorphism φ_k is quasiconformal except circles $C_{k,+}$, $C_{k,-}$. From Proposition 2.3, it has to be quasiconformal on $\widehat{P}_k$ with

$$
d(\varphi_k) \le \Big(\log\frac{1}{1-\delta}\Big)^{-1} d(\omega,\widetilde{\omega}). \tag{4.14}
$$

Step 6: Making quasiconformal mappings, II. In this step, we make a quasiconformal mapping from P'_k to $\mathcal{P}_k$. Recall that P'_k is a pair of pants bounded by $\widehat{C}_{k,1}$, $C_{k,2}$ and $C_{k,3}$, and $\mathcal{P}_k$ is bounded by $C_{k,1}$, $C_{k,2}$ and $C_{k,3}$.

Let $C_{k,0}$ be a circle centered at the origin of radius $x_k + r_k$, so that $C_{k,0}$ is tangent with $C_{k,2}$, $C_{k,3}$. We consider circular annuli B'_k bounded by $C_{k,0}$ and $\widehat{C}_{k,1}$, and B_k bounded by $C_{k,0}$ and $C_{k,1}$. It follows from Lemma 4.2 that there exists a quasiconformal mapping $\psi_{k,0} : B'_k \to B_k$ such that

$$d(\psi_{k,0}) = \log \frac{\log(1+\delta)\frac{x_k}{\widetilde{x}_k} - \log(x_k + r_k)}{\log(1+\delta) - \log(x_k + r_k)}$$

and $\psi_{k,0}|_{C_{k,0}}$ is the identity.

As in Step 5, we have

$$d(\psi_{k,0}) \leq \frac{\log x_k - \log \widetilde{x}_k}{\log(1+\delta) - \log(x_k + r_k)}.$$

Since

$$x_k + r_k = \frac{1}{2}(1 + q_{k+1} + (1+\delta)(1 - q_{k+1})) = \frac{1}{2}(2 + \delta - \delta q_{k+1}),$$

we see that

$$\begin{aligned} \log(1+\delta) - \log(x_k + r_k) &= \log \frac{1+\delta}{1 + \frac{1}{2}\delta(1 - q_{k+1})} \\ &\geq \log \frac{1+\delta}{1 + \frac{1}{2}\delta(1-\delta)} > 0, \end{aligned} \tag{4.15}$$

and

$$\log x_k - \log \widetilde{x}_k = \log \frac{1 + q_{k+1}}{1 + \widetilde{q}_{k+1}} \leq q_{k+1} - \widetilde{q}_{k+1}. \tag{4.16}$$

From (4.15))and (4.16) we have

$$d(\psi_{k,0}) \leq \left(\log \frac{1+\delta}{1 + \frac{1}{2}\delta(1-\delta)} \right)^{-1} (q_{k+1} - \widetilde{q}_{k+1}). \tag{4.17}$$

We define a homeomorphism $\psi_k : P'_k \to \mathcal{P}_k$ by

$$\psi_k(z) = \begin{cases} \psi_{k,0}(z), & z \in B'_k, \\ z, & \text{otherwise}. \end{cases}$$

Then, as in Step 5, we see that ψ_k is quasiconformal on P'_k with

$$d(\psi_k) \leq \left(\log \frac{1+\delta}{1 + \frac{1}{2}\delta(1-\delta)} \right)^{-1} d(\omega, \widetilde{\omega}). \tag{4.18}$$

In the case where $q_{k+1} \leq \widetilde{q}_{k+1}$, the same argument still works in Steps 5 and 6; we obtain the same results.

Step 7: Making a global quasiconformal mapping. In Steps 5 and 6, we have made quasiconformal mappings $\varphi_k : \widehat{P}_k \to P'_k$ and $\psi_k : P'_k \to \mathcal{P}_k$. Thus, $\Phi_k := \psi_k \circ \varphi_k : \widehat{P}_k \to \mathcal{P}_k$ gives a quasiconformal mapping with

$$d(\Phi_k) \leq C(\delta)d(\omega, \widetilde{\omega})$$

for each $k \in \mathbb{N}$.

Because of the boundary behaviors (4.5), (4.6), (4.11) and (4.12), we see that those mappings give a homeomorphism Φ from $X_{E(\omega)} \cap \mathbb{D}_\delta$ onto $X_{E(\widetilde{\omega})} \cap \mathbb{D}_\delta$. The homeomorphism Φ is quasiconformal on $X_{E(\omega)} \cap \mathbb{D}_\delta$ except on circles which are boundaries of pairs of pants. It follows from Proposition 2.3 that Φ is quasiconformal on $X_{E(\omega)} \cap \mathbb{D}_\delta$. We define Φ for $z \in \widehat{\mathbb{C}} \setminus \mathbb{D}_\delta$ by $\Phi(z) = z$. Using Proposition 2.3 again, we verify that Φ is a quasiconformal mapping on $X_{E(\omega)}$ with

$$d(\Phi) \leq C(\delta)d(\omega, \widetilde{\omega}).$$

Furthermore, from our construction of the mapping, we see that $\Phi(\overline{z}) = \overline{\Phi(z)}$ for $z \in X_{E(\omega)}$. Therefore, Φ is extended to a homeomorphism on $\widehat{\mathbb{C}}$ to itself. Since the extended homeomorphism is quasiconformal on $\widehat{\mathbb{C}} \setminus \mathbb{R}$, it must be quasiconformal because of Proposition 2.3. Thus, we obtain a quasiconformal mapping as desired. □

Proof of (2). Take any $\varepsilon > 0$. Since $\log \frac{1-\widetilde{q}_n}{1-q_n} \to 0$ as $n \to \infty$, we also see that $q_n - \widetilde{q}_n \to 0$. Viewing (4.10) and (4.17), we verify that there exists an $N \in \mathbb{N}$ such that

$$d(\varphi_k) < \frac{1}{2}\log(1+\varepsilon) \quad \text{and} \quad d(\psi_k) < \frac{1}{2}\log(1+\varepsilon),$$

if $k > N$. Hence, if $k > N$, then

$$d(\Phi_k) = d(\psi_k \circ \phi_k) \leq d(\psi_k) + d(\varphi_k) < \log(1+\varepsilon). \tag{4.19}$$

Since the pants decompositions in Step 2 of the proof of (1) give exhaustions $X_{E(\omega)}$ and $X_{E(\widetilde{\omega})}$, (4.19) implies the maximal dilatation $K(\Phi) = e^{d(\Phi)}$ is less than $(1+\varepsilon)$ on the outside of a sufficiently large compact subset of $X_{E(\omega)}$. Therefore, $\Phi : X_{E(\omega)} \to X_{E(\widetilde{\omega})}$ is asymptotically conformal. □

5 Proof of Theorem III

In the proof of this theorem, we use Wolpert's lemma (cf. [14], [16]) for quasiconformal mappings and the hyperbolic lengths. The lemma says that if there is a K-quasiconformal mapping f from a hyperbolic Riemann surface X to another hyperbolic Riemann surface Y, then for any non-trivial closed curve α in X, we have

$$\frac{1}{K}\ell_X([\alpha]) \leq \ell_Y([f(\alpha)]) \leq K\ell_X([\alpha]),$$

where $\ell_X([\alpha])$ stands for the hyperbolic length of the geodesic homotopic to α on X.

First of all, we note that there exists a positive constant such that the hyperbolic length of any closed curve on $X_{\mathcal{C}}$ is greater than a constant.

Indeed, $X_{\mathcal{C}}$ is quasiconformally equivalent to the region of discontinuity $\Omega(G)$ of a Schottky group G (cf. [15]). The quotient space $\Omega(G)/G$ is a compact Riemann surface. Hence, there exists a positive constant such that the hyperbolic length of any non-trivial closed curve on the surface is greater than the constant. Since the quotient map is a covering map, any non-trivial closed curve in $\Omega(G)$ is projected to a non-trivial closed curve on the Riemann surface by the quotient map from $\Omega(G)$ onto $\Omega(G)/G$. Also, the covering map is isometry with respect to the hyperbolic metrics. Hence, the hyperbolic length of any non-trivial closed curve in $\Omega(G)$ is greater than some positive constant. Therefore, by Wolpert's lemma, we verify that the hyperbolic length of any non-trivial closed curve in $X_{\mathcal{C}}$ is also greater than some constant, say $d > 0$.

Suppose that there exists a K-quasiconformal map from $X_{\mathcal{C}}$ to $X_{E(\omega)}$. Then, the hyperbolic length of any closed curve in $X_{E(\omega)}$ is not less than $K^{-1}d$.

Let $\varepsilon > 0$ be an arbitrary small constant. Since $\sup\{q_n \mid n \in \mathbb{N}\} = 1$, there exist a sequence $\{n_k\}_{k=1}^{\infty}$ in $\mathbb{N}$ and $N_0 \in \mathbb{N}$ such that

$$1 - \varepsilon < q_{n_k} < 1,$$

if $k > N_0$.

Now, we look at $I^1_{n_k-1}$ of E_{n_k-1} for $k > N_0$. Then, $I^1_{n_k} \subset E_{n_k}$ is an interval of length $\frac{1}{2}(1-q_{n_k})|I^1_{n_k-1}| < \frac{1}{2}\varepsilon|I^1_{n_k-1}|$. Therefore, we may take an annulus A_k in $X_{E(\omega)}$ bounded by two concentric circles $C^1_{n_k}, C^2_{n_k}$ such that the radius of $C^1_{n_k}$ is $\frac{1}{4}\varepsilon|I^1_{n_k-1}|$ and that of $C^2_{n_k}$ is $(\frac{1}{2} - \frac{1}{4}\varepsilon)|I^1_{n_k-1}|$. If we take $\varepsilon > 0$ sufficiently small, then the length of the core curve of A_k with respect to the hyperbolic metric on A_k becomes smaller than $K^{-1}d$.

Indeed, we put

$$\Pi(z) = \exp\Big(-i\frac{\log M_\varepsilon}{\pi}\log z\Big) \quad (z \in \mathbb{H}),$$

and

$$\gamma(z) = \exp\left\{\frac{-2\pi^2}{\log M_\varepsilon}\right\}z$$

for $M_\varepsilon = \frac{\varepsilon}{(2-\varepsilon)}$, the ratio of the radius of $C^1_{n_k}$ and the radius of $C^2_{n_k}$. Then, we see that

$$\Pi(\gamma(z)) = \exp\Big\{ -i\frac{\log M_\varepsilon}{\pi}\Big(\log z - \frac{2\pi^2}{\log M_\varepsilon}\Big)\Big\}$$
$$= \exp\Big(-i\frac{\log M_\varepsilon}{\pi}\log z\Big) = \Pi(z),$$
$$\Pi\Big(\Big[1, \exp\Big(\frac{-2\pi^2}{\log M_\varepsilon}\Big)\Big]\Big) = \{\, |z| = 1\},$$
$$\Pi\Big(\Big[-\exp\Big(\frac{-2\pi^2}{\log M_\varepsilon}\Big), -1\Big]\Big) = \{\, |z| = M_\varepsilon\},$$

and

$$\Pi\Big(\Big\{z \in \mathbb{H} \mid 1 \le |z| < \exp\Big(\frac{-2\pi^2}{\log M_\varepsilon}\Big)\Big\}\Big) = \{M_\varepsilon < |z| < 1\}.$$

The domain

$$\Big\{z \in \mathbb{H} \mid 1 \le |z| < \exp\Big(\frac{-2\pi^2}{\log M_\varepsilon}\Big)\Big\}$$

is a fundamental domain for $\langle\gamma\rangle$. Hence, $\Pi : \mathbb{H} \to \{M_\varepsilon < |z| < 1\}$ is a universal covering map and $\{M_\varepsilon < |z| < 1\} = \mathbb{H}/\langle\gamma\rangle$. Since A_k is conformally equivalent to $\{M_\varepsilon < |z| < 1\}$, the hyperbolic length of the core curve A_k is $\frac{-2\pi^2}{\log M_\varepsilon}$ which is equal to the distance between i and $\gamma(i)$ with respect to the hyperbolic metric $\frac{|dz|}{\operatorname{Im} z}$ on $\mathbb{H}$. Thus, the hyperbolic length of the core curve of A_k converges to zero as $\varepsilon \to 0$.

Since $A_k \subset X_{E(\omega)}$, the length of the core curve of A_k with respect to the hyperbolic metric of $X_{E(\omega)}$ is not greater than the length with respect to the hyperbolic metric of A_k. Thus, we find a closed curve in $X_{E(\omega)}$ whose length is less that $K^{-1}d$. It is a contradiction and we complete the proof of the theorem.

6 Proofs of the Corollaries

Proof of Corollary 1.1. Let Λ be the limit set of the Schottky group G. We have shown ([15] Theorem 6.2) that X_Λ is quasiconformally equivalent to $X_{\mathcal{C}}$. Hence, it follows from Theorem I that X_E is quasiconformally equivalent to X_Λ as desired. □

Proof of Corollary 1.2. Since $\mathcal{C} = E(\omega_0)$ for $\omega_0 = (\frac{1}{3})_{n=1}^{\infty}$, the statement follows immediately from Theorem II (1). □

Proof of Corollary 1.3. Let $\varphi : X_\Lambda \to X_E$ be a quasiconformal map given by Corollary 1.1. Take any quasiconformal map ψ on X_E to $\widehat{\mathbb{C}}$. Then, $\Phi := \psi \circ \varphi$ is a quasiconformal map on X_Λ. It is known that any quasiconformal map on X_Λ is extended to a quasiconformal map on $\widehat{\mathbb{C}}$ (see [13] Theorem 1.2 (A) and the comment after the theorem). Hence, both φ and Φ are extended to $\widehat{\mathbb{C}}$ and so is $\psi = \Phi \circ \varphi^{-1}$. □

Proof of Corollary 1.4. Let $\Psi : \mathbb{C} \to \mathbb{C}$ be the quasiconformal mapping given in §4. We put $D = \dim_H(E(\omega))$ and $\widetilde{D} = \dim_H(E(\widetilde{\omega}))$. We use the argument in the proof of Theorem II (2).

For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$K(\Phi_k) < 1 + \varepsilon$$

if $k > N$, where Φ_k is the quasiconformal mapping given in §4. Therefore, $\Phi|_{U_N}$ is a $(1+\varepsilon)$-quasiconformal mapping on $U_N := E(\omega) \cup \bigcup_{k>N} \bigcup_{i=1}^{2^k} P_k^i$. Here, we use the following result by Astala [3].

Proposition 6.1. *Let Ω, Ω' be planar domains and $f : \Omega \to \Omega'$ be a K-quasiconformal mapping. Suppose that $E \subset \Omega$ is a compact subset of Ω. Then,*

$$\dim_H(f(E)) \leq \frac{2K \dim_H(E)}{2 + (K-1)\dim_H(E)}. \tag{6.1}$$

It follows from (6.1) that

$$\dim_H(E(\widetilde{\omega})) \leq \frac{2(1-\varepsilon)\dim_H(E(\omega))}{2 + \varepsilon \dim_H(E(\omega))}.$$

Since $\varepsilon > 0$ could be arbitrarily small, we obtain

$$\dim_H(E(\widetilde{\omega})) \leq \dim_H(E(\omega)).$$

By considering Φ^{-1}, we get the reverse inequality for $\dim_H(E(\omega))$ and $\dim_H(E(\widetilde{\omega}))$. Thus, we conclude that $\dim_H(E(\omega)) = \dim_H(E(\widetilde{\omega}))$ as desired. □

7 Examples

Example 7.1. Let $f_c(z) = z^2 + c$. Suppose that c is not in the Mandelbrot set. Then, it is well-known that f_c is hyperbolic and the Julia set $\mathcal{J}_{f_c}$ is a Cantor set. Thus, f_c satisfies the condition of Theorem I.

Example 7.2. Let $B_0(z)$ be a Blaschke product of degree $d > 1$. It is known ([6] III. 1. Example) that the Julia set of B_0 is either the unit circle or a Cantor set on the unit circle. Suppose that B_0 has an attracting fixed point on the unit circle. Since the attracting fixed point belongs to the Fatou set of B_0, the Julia set has to be a Cantor set. It is also easy to see that B_0 is hyperbolic. Thus, B_0 satisfies the condition on Theorem I.

In Theorem II, we have estimated the maximal dilatations for sequences with lower bound. In the next example, we may estimate the maximal dilatation for sequences without lower bound.

Example 7.3. For $0 < a < 1$ and a fixed $L \in \mathbb{N}$, we put $q_n = a^n$ and $\widetilde{q}_n = a^{n+L}$ and we consider $E(\omega)$, $E(\widetilde{\omega})$ for $\omega = (q_n)_{n=1}^{\infty}$, $\widetilde{\omega} = (\widetilde{q}_n)_{n=1}^{\infty}$. By using the same idea as in the proof of Theorem II, we claim that there exists an $\exp(Ca^{-L})$-quasiconformal mapping $\varphi : \mathbb{C} \to \mathbb{C}$ with $\varphi(E(\omega)) = E(\widetilde{\omega})$, where $C > 0$ is a constant independent of ω and $\widetilde{\omega}$.

Proof of the claim. We use the same notations for $E(\omega)$ and $E(\widetilde{\omega})$ as those in the proof of Theorem II. Then,

$$E_k = \bigcup_{i=1}^{2^k} I_k^i, \quad [0, 1] = E_k \cup \bigcup_{i=1}^{2^k-1} J_k^i$$

and for $i = 1, 2, \ldots, 2^k$,

$$|I_k^i| = \left(\frac{1}{2}\right)^k \prod_{j=1}^{k} (1 - a^j).$$

If i is odd, then

$$|J_{k+1}^i| = a^{k+1}|I_k^1| \geq 2a^{k+1}|I_{k+1}^1|.$$

If $i = 2^\ell m$ $(1 \leq \ell \leq k; m$ is odd), then we have

$$|J_{k+1}^i| = |J_{k-\ell+1}^m| \geq 4a^{k+1}|I_{k+1}^1|.$$

Thus, we conclude that

$$|J_{k+1}^i| \geq 2a^{k+1}|I_{k+1}^1|, \tag{7.1}$$

for $i = 1, 2, \ldots, 2^{k+1} - 1$.

We draw a circle C_k^i centered at the midpoint of I_k^i with radius $\frac{1}{2}(1 + a^k)|I_k^1|$ for each $k \in \mathbb{N}$ and $1 \leq i \leq 2^k$. From (7.1), we see that $C_k^i \cap C_k^j = \emptyset$ if $i \neq j$. Furthermore, $C_k^i \cap C_{k+1}^j = \emptyset$ since $a^k|I_k^1| > a^{k+1}|I_{k+1}^1|$. Therefore, $\bigcup_{k=1}^{\infty} \bigcup_{i=1}^{2^k} C_k^i$ gives a pants decomposition of $X_{E(\omega)} \cap \mathbb{D}$, where $\mathbb{D} = \{|z - \frac{1}{2}| < 1\}$. We also draw circles $\widetilde{C}_k^i$ for $\widetilde{\omega}$ in the same way. Then, $\bigcup_{k=1}^{\infty} \bigcup_{i=1}^{2^k} \widetilde{C}_k^i$ gives a pants decomposition of $X_{E(\widetilde{\omega})} \cap \mathbb{D}$.

We denote by P_k^i a pair of pants bounded by C_k^i, C_{k+1}^{2i-1} and C_{k+1}^{2i}. As in Step 3 of the proof of Theorem II, we may identify P_k^i with a pair of pants $\mathcal{P}_k$ bounded by $C_{k,1}$, $C_{k,2}$ and $C_{k,3}$, where $C_{k,1}$ is a circle centered at the origin with radius $1 + a^k$, $C_{k,2}$ is centered at

$$-x_k := -\frac{1}{2}(1 + a^{k+1})$$

with radius

$$r_k := \frac{1}{2}(1 + a^{k+1})(1 - a^{k+1})$$

and $C_{k,3}$ is centered at x_k with radius r_k.

Similarly, we take a pair of pants $\widetilde{P}_k^i$ bounded by $\widetilde{C}_k^i$, $\widetilde{C}_{k+1}^{2i-1}$ and $\widetilde{C}_{k+1}^{2i}$, which is conformally equivalent to a pair of pants $\widetilde{\mathcal{P}}_k$ bounded by $\widetilde{C}_{k,1}$, $\widetilde{C}_{k,2}$ and $\widetilde{C}_{k,3}$, where $\widetilde{C}_{k,1}$ is the same circle as $C_{k,1}$, $\widetilde{C}_{k,2}$ is centered at

$$-\widetilde{x}_k := -\frac{1}{2}(1 + a^{k+L+1})$$

with radius

$$\widetilde{r}_k := \frac{1}{2}(1 + a^{k+L+1})(1 - a^{k+L+1})$$

and $\widetilde{C}_{k,3}$ is centered at $\widetilde{x}_k$ with radius $\widetilde{r}_k$.

We also take an intermediate pair of pants, $\widehat{P}_k$, similar to that in the proof of Theorem II. Then, by using exactly the same method, we may construct a $\exp(Ca^{-L})$-quasiconformal mapping from P_k^i onto $\widetilde{P}_k^i$, where $C > 0$ is a constant independent of k and i. Since the calculation is rather long but the same as in §4, we leave it to the reader.

By gluing those quasiconformal mappings together, we get an $\exp(Ca^{-L})$-quasiconformal mapping $\varphi : \mathbb{C} \to \mathbb{C}$ with $\varphi(E(\omega)) = E(\widetilde{\omega})$ as desired. □

Cantor Julia sets of Blaschke products with parabolic fixed points. We showed ([15] Example 3.2) that a Cantor set which is the limit set of an extended Schottky group is not quasiconformally equivalent to the limit set of a Schottky group. We discuss the same thing for Cantor sets defined by non-hyperbolic rational functions.

Let $B_1(z)$ be a Blaschke product with a parabolic fixed point on the unit circle T. Suppose that there exists only one attracting petal at the parabolic fixed point. Then, we see that the Julia set $\mathcal{J}_{B_1}$ is a Cantor set on T (see [6] IV. 2. Example). However, B_1 is not hyperbolic since it has a parabolic fixed point.

It follows from Theorem I that two Riemann surfaces $X_{\mathcal{J}_{f_c}}$ for Example 7.1 and $X_{\mathcal{J}_{B_0}}$ for Example 7.2 are quasiconformally equivalent. While the Julia set $\mathcal{J}_{B_1}$ of B_1 is also a Cantor set, it is not hyperbolic. Therefore, we cannot apply Theorem I for B_1.

Now, we consider the Martin compactification of the complement. For a general theory of the Martin compactification, we may refer to [7]. Here, we note the following result.

Proposition 7.1. *Let B be a hyperbolic Blaschke product of degree $d > 1$. Suppose that the Julia set $\mathfrak{J}_B$ is a Cantor set in T. Then, the Martin compactification of $X_{\mathfrak{J}_B}$ is homeomorphic to $\widehat{\mathbb{C}}$.*

Hence, the same statements as in Proposition 7.1 hold for $X_{\mathfrak{J}_{B_0}} := \widehat{\mathbb{C}} \setminus \mathfrak{J}_{B_0}$ and the quasiconformal map φ on $X_{\mathfrak{J}_{B_0}}$ is extended to a homeomorphism of the Martin compactification of $X_{\mathfrak{J}_{B_0}}$.

Next, we consider the Martin compactification of $X_{\mathfrak{J}_{B_1}} := \widehat{\mathbb{C}} \setminus \mathfrak{J}_{B_1}$, especially the set of the Martin boundary over the parabolic fixed point of B_1. If the set contains at least two points, then it follows from Proposition 7.1 that there exists no quasiconformal map from $X_{\mathfrak{J}_{B_0}}$ to $X_{\mathfrak{J}_{B_1}}$.

Indeed, in [13] we observe the Martin compactification of the complement of the limit set of an extended Schottky group and show that the set of the Martin boundary over a parabolic fixed point consists of more than two points. It is a key fact to show that the limit set of the extended Schottky group is not quasiconformally equivalent to that of a Schottky group ([15]). However, by using an argument of Benedicks ([4]) (see also Segawa [12]) on the Martin compactification, we may show the following.

Lemma 7.1. *In the Martin compactification of $X_{\mathfrak{J}_{B_1}}$, there is exactly one minimal point over the parabolic fixed point of B_1.*

Remark 7.1. In the Martin compactification of a Riemann surface, the set corresponding to a topological boundary component of the Riemann surface is connected and the minimal points in the set are regarded as extreme points of a convex set. Thus, if the set over a boundary component on the Martin compactification contains only one minimal point, then it consists of only one point, that is, the minimal point.

Proof. To prove the lemma, we use a result by Benedicks.

Let E be a proper closed subset of $\mathbb{R} \cup \{\infty\}$. We denote by $Q(t, r)$ ($t \in \mathbb{R}$, $r > 0$) the square

$$\left\{x + iy \mid |x - t| < \frac{r}{2}, |y| < \frac{r}{2}\right\}.$$

For a fixed α with $0 < \alpha < 1$ and every $x \in \mathbb{R} \setminus \{0\}$, we consider the solution of the Dirichlet problem on $Q(x, \alpha|x|) \setminus E$ with boundary values one on $\partial Q(x, \alpha|x|)$ and zero on $E \cap Q(x, \alpha|x|)$. We denote the solution by β_x^E. Then, Benedicks [4] showed the following.

Proposition 7.2 ([4] Theorem 4)**.** *On the Martin compactification of $\widehat{\mathbb{C}} \setminus E$, there exist more than two points over ∞ if and only if*

$$\int_{|x|\geq 1} \frac{\beta_x^E(x)}{|x|} dx < \infty. \tag{7.2}$$

Let $a \in T$ be the parabolic fixed point B_1. We take a Möbius transformation γ so that $\gamma(T) = \mathbb{R} \cup \{\infty\}$ and $\gamma(a) = \infty$. For $\widehat{B}_1 := \gamma B_1 \gamma^{-1}$, we see that ∞ is a parabolic fixed point with a unique attracting petal of $\widehat{B}_1$, and $\mathfrak{J}_{\widehat{B}1} = \gamma(\mathfrak{J}_{B_1})$ is contained in $\mathbb{R} \cup \{\infty\}$.

Since $z = \infty$ is a parabolic fixed point of $\widehat{B}_1$ with only one attracting petal, there are only one attracting direction and repelling direction (cf. [6] II. 5). Because of the symmetricity of $\widehat{B}_1$, those directions are on the real line. The attracting direction is contained in the Fatou set of $\widehat{B}_1$. Hence, there exists a sufficiently large $M > 0$ such that either $\mathfrak{J}_{\widehat{B}1} \cap \{\operatorname{Re} z < -M\}$ or $\mathfrak{J}_{\widehat{B}1} \cap \{\operatorname{Re} z > M\}$ is empty. We may assume that $\mathfrak{J}_{\widehat{B}1} \cap \{\operatorname{Re} z < -M\} = \emptyset$.

Hence, $\mathfrak{J}_{\widehat{B}1} \cap Q(x, \alpha|x|) = \emptyset$ if $x < 0$ and $|x|$ is sufficiently large. Therefore, $\beta_x^{\mathfrak{J}_{\widehat{B}1}}(x) = 1$ for such x. Thus, we have

$$\int_{|x|\geq 1} \frac{\beta_x^{\mathfrak{J}_{\widehat{B}1}}(x)}{|x|} dx = \infty$$

and conclude that there exists exactly one point over ∞ from Proposition 7.2. □

Lemma 7.1 implies that we cannot use the argument used for extended Schottky groups. We exhibit the following conjecture at the end of this article.

Conjecture. $X_{\mathfrak{J}_{B1}}$ is not quasiconformally equivalent to $X_{\mathcal{C}}$.

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Hiroshige Shiga
Department of Mathematics
Kyoto Sangyo University
Motoyama, Kamigamo, Kita-ku Kyoto, Japan
email: shiga@cc.kyoto-su.ac.jp

(Received August 29, 2019 and in revised form August 25, 2020)