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Morphisms Between Aristotelian Diagrams

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Abstract. In logical geometry, Aristotelian diagrams are studied in a precise and systematic way. Although there has recently been a good amount of progress in logical geometry, it is still unknown which underlying mathematical framework is best suited for formalizing the study of these diagrams. Hence, in this paper, the main aim is to formulate such a framework, using the powerful language of *category theory*. We build multiple categories, which all have Aristotelian diagrams as their objects, while having different kinds of morphisms between these diagrams. The categories developed here are assessed according to their ability to generalize previous work from logical geometry as well as their interesting category-theoretical properties. According to these evaluations, the most promising category has as its morphisms those functions on fragments that increase in informativity on both the opposition and implication relations. Focusing on this category can significantly increase the effectiveness of further research in logical geometry.

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1. Introduction

Aristotelian diagrams have a rich history in philosophy and logic. The so-called square of opposition for Aristotle's syllogistics is undoubtedly the oldest and best-known example [19], but there exist many other, more complex types of Aristotelian diagrams as well (e.g., various types of hexagons and octagons of opposition).[1] In the last decade, these diagrams have begun to be studied in a more systematic fashion, under the heading of *logical geometry*. This research shows that despite their ancient origins, Aristotelian diagrams have natural

[1]See the introduction of [12] for bibliographic references to various historical (and contemporary) applications of Aristotelian diagrams.

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links with various areas of contemporary discrete mathematics, e.g., combinatorics, Boolean algebra and order theory [6,11–13]. A particularly promising line of research was initiated by Vignero [22], who showed that Aristotelian diagrams can be studied using the tools of *category theory* [1,18]. In particular, Vignero [22] defined a specific notion of morphism between diagrams, and then proved that the resulting category has binary products and coproducts.[2]

The present paper fits within this research line of categorifying logical geometry. However, we will not be concerned with studying Vignero's category of Aristotelian diagrams in more detail, exploring further category-theoretical constructions, etc. Rather, our aim is to take a step back and reflect on the fundamental building blocks that are required for this categorification project. After all, a category is not just determined by its *objects* (in casu: Aristotelian diagrams), but also by the *arrows* between those objects. Vignero [22] proposed one specific notion of morphism between Aristotelian diagrams (which will also be studied here), but there also exist many others. At this point, it is not obvious whether one of these qualifies as the uniquely 'correct' or canonical notion of morphism between Aristotelian diagrams. In this paper, we will therefore study these various kinds of morphisms and their corresponding categories, while keeping two theoretical desiderata in mind. On the one hand, we want to obtain a category that is richly structured and well-behaved from a categorical point of view. On the other hand, we want to achieve a conservative generalization of previous work in logical geometry (so that this previous work does not need to be revised, but can rather be seen as a special case of the newly developed categorical picture).

Pertaining to the latter of these two desiderata, we should highlight two specific notions that have been used before in logical geometry, namely Aristotelian isomorphisms and Boolean isomorphisms [12]. Informally, there exists an Aristotelian isomorphism between two Aristotelian diagrams whenever they have 'the same Aristotelian structure' (i.e., the same constellation of contradiction, (sub)contrariety and subalternation relations). Equally informally, there exists a Boolean isomorphism between two Aristotelian diagrams whenever they have 'the same underlying Boolean structure'. However, a single category only gives rise to a single notion of isomorphism. In this paper, we focus on the *Aristotelian* structure of diagrams, and thus we only consider categories that give rise to Aristotelian isomorphisms, like Vignero's category [22]. In other ongoing research, we focus on the *Boolean* structure of diagrams, and thus search for a category that gives rise to Boolean isomorphisms [3].

The paper is organized as follows. Section 2 briefly recapitulates some key notions from logical geometry. Section 3 proposes several different types of morphisms between Aristotelian diagrams, shows that each of them gives rise to a distinct category, and discusses some elementary results in category theory for each of them. Section 4 investigates which of these categories generalize

[2]Recently, Kiouvrekis et al. [16,17] have also used category theory (more specifically, institution theory) to study Aristotelian diagrams. However, their work goes in a completely different direction than what we will do in this paper.

previous work in logical geometry (by giving rise to Aristotelian isomorphisms), and which ones do not. Section 5 focuses exclusively on the former categories, and studies how they are related to each other. Section 6 wraps things up, and mentions some avenues for further research.

2. Background from Logical Geometry

Logical geometry studies (the properties of) Aristotelian diagrams, which can be formulated at different levels of generality [8]. The most general of these definitions characterizes such a diagram as a subset of a Boolean algebra.[3] In philosophy, it might seem more natural to define these diagrams as sets of statements in some logical system S, and the relations holding between them. However, this is a special case (using the Lindenbaum–Tarski algebra of S, cf. Example 2 below) of the more general Boolean algebra definition, to which we turn now.

Definition 1 (*Aristotelian diagram*). An Aristotelian diagram D is a pair $(\mathcal{F}, B)$, where B is a Boolean algebra[4] $(B, \wedge_B, \vee_B, \neg_B, 1_B, 0_B)$ and $\mathcal{F}$ is a fragment of B, i.e., $\mathcal{F} \subseteq B$. When the Boolean algebra B is clear from context, it is usually omitted as a subscript to $\wedge$, $\vee$, etc.

Before we give some examples of Aristotelian diagrams, we should first mention some of the relations that can hold between two elements in a Boolean algebra. Historically, logicians and philosophers have been interested in the four so-called Aristotelian relations, which can straightforwardly be characterized in terms of Boolean algebras [8]. These four relations are often used in visualizations of Aristotelian diagrams.

Definition 2 (*Aristotelian relations*). Given a Boolean algebra B, we say that $x, y \in B$ are:

- B-contradictory (CD_B) iff $x \wedge_B y = 0_B$ and $x \vee_B y = 1_B$, i.e. $x = \neg_B y$,
- B-contrary (C_B) iff $x \wedge_B y = 0_B$ and $x \vee_B y \neq 1_B$,
- B-subcontrary (SC_B) iff $x \wedge_B y \neq 0_B$ and $x \vee_B y = 1_B$,
- in B-subalternation (SA_B) iff $\neg_B x \vee_B y = 1_B$ and $x \vee_B \neg_B y \neq 1_B$.

These four relations are called the Aristotelian relations for B; we also write $\mathcal{AR}_B := \{CD_B, C_B, SC_B, SA_B\}$.[5] When the Boolean algebra B is clear from context, it is usually omitted as a prefix and subscript.

Figure 1 shows two basic examples of Aristotelian diagrams: $(\mathcal{F}, \{0,1\}^3)$ and $(\mathcal{F}', \{0,1\}^4)$, where $\mathcal{F} := \{100, 110, 001, 011\}$ and $\mathcal{F}' := \{1000, 1100, 0000\}$. The way in which the four relations are visualized here will be used consistently

[3] See [15] for an introduction to Boolean algebra.

[4] Following a common convention in mathematics, we tacitly identify a Boolean algebra with its underlying set in this paper.

[5] It is well-known that all Aristotelian relations can be characterized in terms of contradiction together with one of contrariety, subcontrariety and subalternation [21]. The set $\mathcal{AR}$ is thus redundant, and can be trimmed down to just two relations. However, for reasons of historical continuity, we prefer to work with all four Aristotelian relations.

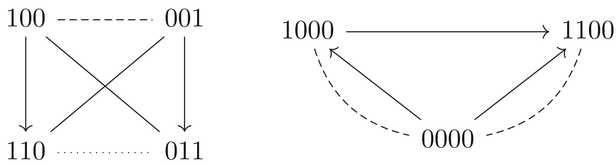

Figure 1. Two basic examples of Aristotelian diagrams. Full, dashed and dotted lines visualize contradiction, contrariety and subcontrariety, respectively; arrows visualize subalternations (cf. Definition 2)

throughout this paper. The diagram $(\mathcal{F}, \{0,1\}^3)$ is usually called a *(classical) square of opposition*, whereas the diagram $(\mathcal{F}', \{0,1\}^4)$ does not have a canonical name. The smallest Boolean algebra, B_*, has just a single element, which is equal to both 0_{B_*} and 1_{B_*}. It allows for two Aristotelian diagrams, which we state in a separate example for further reference.

Example 1. Let B_* be the Boolean algebra with a single element (namely $*$), which we call the *degenerate Boolean algebra*.[6] Since fragments are simply subsets of their ambient Boolean algebra, it is clear that we can make two Aristotelian diagrams in this context, namely $(\emptyset, B_*)$ and $(\{*\}, B_*)$. We denote the latter diagram by D_*.

As we mentioned before, philosophers and logicians are mainly interested in special kinds of Aristotelian diagrams in which the fragments are collections of statements in some logic S. Using the Lindenbaum–Tarski algebra, such kinds of diagrams naturally fit in with Definition 1, as the following example shows.

Example 2. For any logical system S that has Boolean connectives $\wedge, \vee$ and $\neg$, there exists a Boolean algebra $\mathbb{B}(\mathsf{S})$ whose underlying set is $\mathbb{B}(\mathsf{S}) := \{[\varphi] \mid \varphi \text{ is a sentence in } \mathsf{S}\}$. Here, the notation $[\varphi]$ stands for the equivalence class of φ with respect to the relation $\equiv_{\mathsf{S}}$ of logical equivalence. Conjunction $\wedge_{\mathbb{B}(\mathsf{S})}$ is given by $[\varphi] \wedge_{\mathbb{B}(\mathsf{S})} [\psi] := [\varphi \wedge \psi]$. In a similar way, disjunction and negation can be defined. Finally, we define $0_{\mathbb{B}(\mathsf{S})} := [\bot]$ and $1_{\mathbb{B}(\mathsf{S})} := [\top]$. It is not hard to check that all of this gives rise to a well-defined Boolean algebra. This algebra is usually called the Lindenbaum–Tarski algebra of S. In the remainder of the paper, we tacitly identify any logic S with its Lindenbaum–Tarski algebra $\mathbb{B}(\mathsf{S})$.

[6] Strictly speaking, there is an infinite number of degenerate Boolean algebras, since we can name the single element however we want. However, just like in other familiar categories like Set, they are all isomorphic to each other. Thus, for all intents and purposes, it is fine to identify all these degenerate algebras, and thus speak of *the* degenerate Boolean algebra B_*.

$$\begin{tikzcd}
\forall x(Sx \to Px) \arrow[r, dash, dashed] \arrow[d] \arrow[dr, dash] & \forall x(Sx \to \neg Px) \arrow[d] \arrow[dl, dash] \\
\exists x(Sx \wedge Px) \arrow[r, dash, dotted] & \exists x(Sx \wedge \neg Px).
\end{tikzcd}$$

FIGURE 2. A square of opposition $(\mathcal{F}^{cat}, \mathsf{SYL})$

Now, consider syllogistics SYL.[7] We can define the fragment $\mathcal{F}^{cat}$ in SYL as

$$\mathcal{F}^{cat} := \{[\forall x(Sx \to Px)], [\exists x(Sx \wedge Px)], [\forall x(Sx \to \neg Px)], [\exists x(Sx \wedge \neg Px)]\}.$$

This fragment consists of (the equivalence classes of) the categorical statements from syllogistics, hence the name $\mathcal{F}^{cat}$. It is now clear from Definition 1 that we have an Aristotelian diagram $(\mathcal{F}^{cat}, \mathsf{SYL})$, which is visualized in Fig. 2 (we removed the equivalence class brackets for simplicity).

Note that Definition 1 allows 1_B and 0_B to occur in $\mathcal{F}$. This is in line with recent developments in logical geometry. For example, previous research takes into account the top and bottom elements for the sake of analyzing logic-sensitivity of Aristotelian diagrams [9]. Allowing the top and bottom elements to occur in a diagram also makes algebraic sense, since they give rise to terminal objects (cf. Example 8 below). However, we will have to be careful when writing proofs, since elements can stand in several Aristotelian relations at the same time when 1_B and 0_B are involved (cf. Fig. 1 above and Remark 1 below).

Furthermore, Definition 1 does not require Aristotelian diagrams to be closed under complementation. However, most diagrams that are concretely studied in logical geometry, such as classical squares of opposition, do enjoy this property, so it makes sense to introduce a separate label for them.[8]

Definition 3 (*σ-diagram*). A σ-diagram $D = (\mathcal{F}, B)$ is an Aristotelian diagram that is closed under complementation, i.e., for every $x \in \mathcal{F}$, it holds that $\neg_B x \in \mathcal{F}$ as well.

Since they are closed under complementation, finite σ-diagrams (except for D_*) always have an even number of elements. Consequently, in logical geometry, a σ-diagram is usually not viewed as consisting of $2n$ elements, but rather of n *pairs of contradictory elements* $\{x, \neg x\}$, or PCDs for short [11]. For example, a square of opposition consists of two PCDs.

As can be seen from Definition 1, there is a clear difference between the definitions of contradiction and (sub)contrariety on the one hand, and that of

[7] The system SYL has the same language as ordinary first-order logic (FOL), but is axiomatized by adding $\exists x Sx$ as an additional axiom to FOL. This logical system is naturally interpreted on first-order models $\langle D, I\rangle$ (with domain D and interpretation function I) such that $I(S) \neq \emptyset$ [12]. It has also been called $\mathsf{FOL}_\exists$, and shown to be intertranslatable with QUARC [2,20].

[8] Visually speaking, such diagrams usually represent negation by means of central symmetry [11,13], whence the term 'σ-diagram'.

subalternation on the other. These considerations constituted the motivation to introduce and study a more extensive set of logical relations [14,21]. We will rely heavily on this work, so we restate some of its elementary findings for ease of reference.

Definition 4 (*Opposition and implication relations*). Given a Boolean algebra B, we say that $x, y \in B$ are:

- B-contradictory (CD_B) iff $x \wedge_B y = 0_B$ and $x \vee_B y = 1_B$, i.e., $x = \neg_B y$,
- B-contrary (C_B) iff $x \wedge_B y = 0_B$ and $x \vee_B y \neq 1_B$, i.e., $x <_B \neg_B y$,
- B-subcontrary (SC_B) iff $x \wedge_B y \neq 0_B$ and $x \vee_B y = 1_B$, i.e., $x >_B \neg_B y$,
- B-non-contradictory (NCD_B) iff $x \wedge_B y \neq 0_B$ and $x \vee_B y \neq 1_B$,
- in B-bi-implication (BI_B) iff $x \leq_B y$ and $x \geq_B y$, i.e., $x = y$,
- in B-left-implication (LI_B) iff $x \leq_B y$ and $x \ngeq_B y$, i.e., $x <_B y$,
- in B-right-implication (RI_B) iff $x \nleq_B y$ and $x \geq_B y$, i.e., $x >_B y$,
- in B-non-implication (NI_B) iff $x \nleq_B y$ and $y \ngeq_B x$.

The first four relations are called the opposition relations for B, while the last four are called the implication relations for B. We also write $\mathcal{OR}_B := \{CD_B, C_B, SC_B, NCD_B\}$ and $\mathcal{IR}_B := \{BI_B, LI_B, RI_B, NI_B\}$. When both $NCD_B(x,y)$ and $NI_B(x,y)$, we also say that x and y are B-unconnected (UN_B). Once again, when the Boolean algebra B is clear from context, it is usually omitted as a prefix and subscript. Note that $SA_B = LI_B$, and hence $\mathcal{AR}_B \subseteq \mathcal{OR}_B \cup \mathcal{IR}_B$.

Let us briefly explain the terminology used in Definition 4. In Aristotelian diagrams consisting of (S-equivalence classes of) formulas in some logic S, the opposition relations determine whether or not two formulas *can be true/false together* (i.e., whether or not they are 'opposed' to each other). Furthermore, in such diagrams, the implication relations determine for each pair of formulas φ and ψ whether or not φ *implies* ψ and/or vice versa.

Based on various combinatorial and conceptual considerations, Smessaert and Demey [21] have shown that there are natural orderings of informativity on the opposition as well as the implication relations.

Definition 5 (*The informativity orderings*). Let B be a Boolean algebra. The informativity ordering $\leq_{\mathcal{OR}_B}$ on the opposition relations of B is the reflexive relation on $\mathcal{OR}_B$ for which $NCD_B \leq_{\mathcal{OR}_B} C_B, SC_B \leq_{\mathcal{OR}_B} CD_B$. The informativity ordering $\leq_{\mathcal{IR}_B}$ on the implication relations of B is the reflexive relation on $\mathcal{IR}_B$ for which $NI_B \leq_{\mathcal{IR}_B} LI_B, RI_B \leq_{\mathcal{IR}_B} BI_B$. If no confusion is possible, we replace the subscripts $\mathcal{OR}_B$ and $\mathcal{IR}_B$ by B. These orderings are visualized in Fig. 3, where we leave out the subscripts altogether.

There is a natural way to combine both partial order relations of Definition 5 into a partial order relation on the Cartesian product $\mathcal{OR}_B \times \mathcal{IR}_B$, which in categorical terms boils down to the product in the category of partially ordered sets.

Definition 6 (*The combined informativity ordering* $\leq_{\mathcal{OR}_B \times \mathcal{IR}_B}$). Let B be a Boolean algebra. Let $(R_1, R_2), (R_1', R_2')$ be elements of $\mathcal{OR}_B \times \mathcal{IR}_B$. Then,

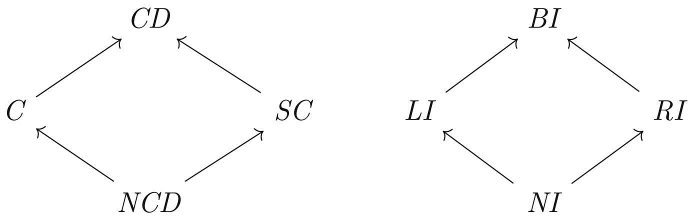

Figure 3. The informativity orderings on the opposition and implication relations; cf. [21]

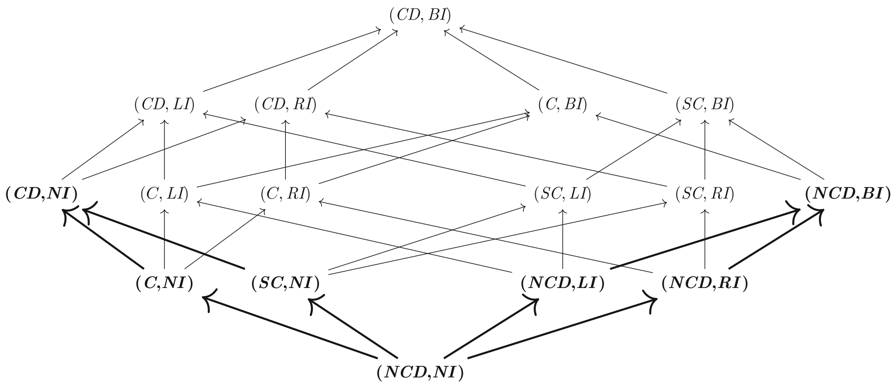

Figure 4. The combined informativity ordering; the bold part corresponds to the only possible combinations for diagrams that do not contain 0 and 1

we say that $(R_1, R_2) \leq_{\mathcal{OR}_B \times \mathcal{IR}_B} (R_1', R_2')$ if and only if $R_1 \leq_{\mathcal{OR}_B} R_1'$ and $R_2 \leq_{\mathcal{IR}_B} R_2'$. Once again, if no confusion is possible, we replace the subscript $\mathcal{OR}_B \times \mathcal{IR}_B$ by B. This ordering is visualized in Fig. 4, where we leave out the subscripts altogether.[9]

Notice from Definition 4 that both $\mathcal{OR}_B$ and $\mathcal{IR}_B$ constitute a partition[10] of the set $B \times B$, i.e., they are jointly exhaustive and mutually exclusive. In other words, every pair of elements (x, y) stands in exactly one opposition relation and exactly one implication relation. On the other hand, this is not

[9]Notice that this order relation, together with the right involution, turns $\mathcal{OR}_B \times \mathcal{IR}_B$ into a Boolean algebra isomorphic to $\{0,1\}^4$. Similarly, the order relations from Definition 5 turn $\mathcal{OR}_B$ and $\mathcal{IR}_B$ into Boolean algebras isomorphic to $\{0,1\}^2$. All of this holds except when B is the degenerate Boolean algebra B_*, in which almost all of the opposition and implication relations coincide with the empty relation, and thus with each other. In this case, $\mathcal{OR}_{B_*} \times \mathcal{IR}_{B_*}$ is isomorphic to $\{0,1\}^2$, while $\mathcal{OR}_{B_*}$ and $\mathcal{IR}_{B_*}$ are isomorphic to $\{0,1\}$.

[10]Strictly speaking, this is only true when none of the relations are empty, which is always the case unless $B = B_*$ is the degenerate Boolean algebra or $B = \{0,1\}$ is the two-element Boolean algebra. When we use the word 'partition' in this paper, we actually mean a relaxed version of this notion that is allowed to contain the empty set.

the case for $\mathcal{AR}_B$ [4]. As an example that these relations are not jointly exhaustive, think of the elements 1100 and 1010 in the Boolean algebra $\{0,1\}^4$, which stand in no Aristotelian relation whatsoever. As an example that the Aristotelian relations are not mutually exclusive either, think of the elements 0000 and 1100 in the same Boolean algebra, which are simultaneously contrary and in subalternation, see Fig. 1. Note that the latter example uses 0000, the bottom element of $\{0,1\}^4$. This is not a coincidence, as is explained in the following remark.

Remark 1. Let B be any Boolean algebra. When 0_B and 1_B are in play, elements can be in several Aristotelian relations at once. For example, if B is not degenerate (i.e., $1_B \neq 0_B$), we have that

$$\begin{aligned} CD_B(0_B, 1_B) \quad &\text{and} \quad LI_B(0_B, 1_B);\\ C_B(0_B, 0_B) \quad &\text{and} \quad BI_B(0_B, 0_B);\\ SC_B(1_B, 1_B) \quad &\text{and} \quad BI_B(1_B, 1_B), \end{aligned}$$

and for any $x \in B^- := B - \{0_B, 1_B\}$, we have that

$$\begin{aligned} SC_B(x, 1_B) \quad &\text{and} \quad LI_B(x, 1_B);\\ C_B(0_B, x) \quad &\text{and} \quad LI_B(0_B, x). \end{aligned}$$

Furthermore, if B is degenerate (i.e., $B = B_*$), then

$$CD_{B_*}(0_{B_*}, 1_{B_*}) \quad \text{and} \quad BI_{B_*}(0_{B_*}, 1_{B_*}).$$

Because of the preceding remark, fragments of Aristotelian diagrams are often assumed not to contain 0_B and 1_B. Under this assumption, it can be proven that every pair of elements stands in exactly one of the following seven pairs of opposition and implication relations [21]:

$$(CD, NI), (C, NI), (SC, NI), (NCD, NI), (NCD, LI), (NCD, RI), (NCD, BI).$$

Consequently, we can restrict ourselves to the bold part of Fig. 4. This shows that the set $(\mathcal{OR}_B \times \{NI_B\}) \cup (\{NCD_B\} \times \mathcal{IR}_B)$ constitutes a partition of $B^- \times B^-$, if we restrict all of the relations to $B^- \times B^-$. In particular, two elements $x, y \in B^-$ can be in at most one Aristotelian relation. They are in *exactly* one Aristotelian relation or they are unconnected iff $x \neq y$ (thereby eliminating $BI(x,y)$). Indeed, the only other option is (NCD, RI), but then $RI(x,y)$, and thus $LI(y,x)$. Following this perspective, the Aristotelian relations together with unconnectedness can be viewed as being jointly exhaustive and mutually exclusive on the set of unordered pairs $\{x,y\}$ of non-identical elements of B^-. Unconnectedness is pictured in Aristotelian diagrams by not connecting x and y in any way.

The following useful lemma indicates that the opposition and implication relations are closely interrelated through negation. We state it here for completeness, and to be able to use it when proving theorems later in the paper.

Lemma 1 (Lemmas 2 and 3 from [21]). *For all Boolean algebras B and elements $x, y \in B$, we have:*

1a)	$CD(x,y)$ *iff*	$CD(\neg x,\neg y)$,		*1b)*	$BI(x,y)$ *iff*	$BI(\neg x,\neg y)$,	
2a)	$C(x,y)$ *iff*	$SC(\neg x,\neg y)$,		*2b)*	$LI(x,y)$ *iff*	$RI(\neg x,\neg y)$,	
3a)	$SC(x,y)$ *iff*	$C(\neg x,\neg y)$,		*3b)*	$RI(x,y)$ *iff*	$LI(\neg x,\neg y)$,	
4a)	$NCD(x,y)$ *iff*	$NCD(\neg x,\neg y)$,		*4b)*	$NI(x,y)$ *iff*	$NI(\neg x,\neg y)$,	
5a)	$CD(x,y)$ *iff*	$BI(x,\neg y)$,		*5b)*	$BI(x,y)$ *iff*	$CD(x,\neg y)$,	
6a)	$C(x,y)$ *iff*	$LI(x,\neg y)$,		*6b)*	$LI(x,y)$ *iff*	$C(x,\neg y)$,	
7a)	$SC(x,y)$ *iff*	$RI(x,\neg y)$,		*7b)*	$RI(x,y)$ *iff*	$SC(x,\neg y)$,	
8a)	$NCD(x,y)$ *iff*	$NI(x,\neg y)$,		*8b)*	$NI(x,y)$ *iff*	$NCD(x,\neg y)$,	
9a)	$CD(x,y)$ *iff*	$BI(\neg x,y)$,		*9b)*	$BI(x,y)$ *iff*	$CD(\neg x,y)$,	
10a)	$C(x,y)$ *iff*	$RI(\neg x,y)$,		*10b)*	$LI(x,y)$ *iff*	$SC(\neg x,y)$,	
11a)	$SC(x,y)$ *iff*	$LI(\neg x,y)$,		*11b)*	$RI(x,y)$ *iff*	$C(\neg x,y)$,	
12a)	$NCD(x,y)$ *iff*	$NI(\neg x,y)$,		*12b)*	$NI(x,y)$ *iff*	$NCD(\neg x,y)$.	

3. Maps Between Aristotelian Diagrams

Now that we have covered a sufficiently broad portion of the framework of logical geometry, we can turn to category theory [1,18]. More precisely, we consider various ways to define a category that has as its objects all Aristotelian diagrams. We do this by defining several different classes of possible morphisms in the current section. Then, in the following sections, we investigate how the categories created by these classes are interrelated.

3.1. Aristotelian Morphisms

In the category of groups, the morphisms are usually taken to be group morphisms, i.e., set functions that preserve the group structure. A similar observation also holds for the categories of abelian groups, rings, vector spaces over a fixed field, etc. If the given class of objects consists of all sets with some specific added structure, we can turn it into a category by defining our morphisms to be those set functions that preserve the added structure. This suggests an obvious notion of Aristotelian morphism: they are set functions that preserve the Aristotelian relations. To formally define this, it is nice to have the following auxiliary definition, which allows us to discuss the same relations in different Boolean algebras at the same time.

Definition 7 (*Relabel functions*[11]). Let B and B' be Boolean algebras. We define the relabel function $\imath_B^{B'} : (\mathcal{OR}_B - \{\emptyset\}) \sqcup (\mathcal{IR}_B - \{\emptyset\}) \to \mathcal{OR}_{B'} \sqcup \mathcal{IR}_{B'}$ from B to B' as

$$\begin{aligned} \imath_B^{B'}(CD_B) &:= CD_{B'}; & \imath_B^{B'}(BI_B) &:= BI_{B'};\\ \imath_B^{B'}(C_B) &:= C_{B'}; & \imath_B^{B'}(LI_B) &:= LI_{B'};\\ \imath_B^{B'}(SC_B) &:= SC_{B'}; & \imath_B^{B'}(RI_B) &:= RI_{B'};\\ \imath_B^{B'}(NCD_B) &:= NCD_{B'}; & \imath_B^{B'}(NI_B) &:= NI_{B'}. \end{aligned}$$

[11]In terms of category theory, the relabel functions are isomorphisms in the category of partially ordered sets. The restrictions of these relabel functions to either the opposition or the implication relations are even isomorphisms in the category of Boolean algebras. It is clear that these functions also give rise to a similar relabeling of the sets $\mathcal{AR}_B$ and $\mathcal{OR}_B \times \mathcal{IR}_B$.

Definition 8 (*Aristotelian morphisms*). Let $D = (\mathcal{F}, B)$ and $D' = (\mathcal{F}', B')$ be Aristotelian diagrams. We say that $f : \mathcal{F} \to \mathcal{F}'$ is an Aristotelian morphism from D to D' iff it preserves all Aristotelian relations, i.e., for all Aristotelian relations $R_B \in \mathcal{AR}_B$ and $x, y \in \mathcal{F}$, we have

$$R_B(x, y) \Rightarrow \imath_B^{B'}(R_B)(f(x), f(y)).$$

In terms of category theory, f is a morphism with domain D and codomain D'. This is usually denoted as $f : D \to D'$, even though as a set function, f is a function from $\mathcal{F}$ to $\mathcal{F}'$. Let us consider some first examples of Aristotelian morphisms,[12] after which we prove that these morphisms really turn the Aristotelian diagrams into a category.

Example 3. We consider the categorical fragment $\mathcal{F}^{cat}$ from Example 2 in two Boolean algebras: the Lindenbaum–Tarski algebras of first-order logic (FOL) and syllogistics (SYL). The Aristotelian diagrams $(\mathcal{F}^{cat}, \mathsf{FOL})$ and $(\mathcal{F}^{cat}, \mathsf{SYL})$ are shown in Fig. 5. The following is clearly an Aristotelian morphism[13]

$$\varphi : (\mathcal{F}^{cat}, \mathsf{FOL}) \to (\mathcal{F}^{cat}, \mathsf{SYL}) : [\psi] \mapsto [\psi].$$

Indeed, all Aristotelian relations in the original diagram are preserved by φ. Note that while φ is a morphism from $(\mathcal{F}^{cat}, \mathsf{FOL})$ to $(\mathcal{F}^{cat}, \mathsf{SYL})$, there does not exist a morphism from $(\mathcal{F}^{cat}, \mathsf{SYL})$ back to $(\mathcal{F}^{cat}, \mathsf{FOL})$,[14] and thus a fortiori there is no isomorphism between these two diagrams either. To summarize: we are dealing here with two distinct, non-isomorphic diagrams $(\mathcal{F}^{cat}, \mathsf{FOL})$ and $(\mathcal{F}^{cat}, \mathsf{SYL})$, which are nevertheless based on one and the same fragment $\mathcal{F}^{cat}$ (but recall Footnote 13). This observation lies at the basis of the well-known phenomenon of *logic-sensitivity* of Aristotelian diagrams [5,9,10].

Example 4. Let us consider an example that does not only involve σ-diagrams. Consider the fragments $\mathcal{F} := \{\forall x Px, \forall x \neg Px, \forall x Qx, \forall x \neg Qx\}$ and $\mathcal{F}' := \{\forall x (Px \wedge Qx), \forall x(\neg Px \wedge \neg Qx)\}$, both with (the Lindenbaum-Tarski algebra of) FOL as their ambient algebra. The Aristotelian diagrams $(\mathcal{F}, \mathsf{FOL})$ and $(\mathcal{F}', \mathsf{FOL})$ are shown in Fig. 6. The map

[12] We lift the first example from Vignero [22]. Note, however, that he considered it in the context of infomorphisms instead of Aristotelian morphisms.

[13] Since we identify logics with their Lindenbaum–Tarski algebras, the fragment $\mathcal{F}^{cat}$ is strictly speaking not *identical* in the diagrams $(\mathcal{F}^{cat}, \mathsf{FOL})$ and $(\mathcal{F}^{cat}, \mathsf{SYL})$. After all, the elements of $\mathcal{F}^{cat}$ in $(\mathcal{F}^{cat}, \mathsf{FOL})$ are equivalence classes $[\psi]_{\mathsf{FOL}}$, whereas in $(\mathcal{F}^{cat}, \mathsf{SYL})$ they are equivalence classes $[\psi]_{\mathsf{SYL}}$. However, since FOL is weaker than SYL, the function φ is still well defined. In other, less algebraically oriented approaches to logical geometry, we work with the actual formulas instead of their equivalence classes, in which case two distinct diagrams really can be based on the *same* fragment after all.

[14] For example, any morphism from $(\mathcal{F}^{cat}, \mathsf{SYL})$ to $(\mathcal{F}^{cat}, \mathsf{FOL})$ would have to preserve $C_{\mathsf{SYL}}(\forall x(Sx \to Px), \forall x(Sx \to \neg Px))$, which is impossible since there are no formulas $\alpha, \beta \in \mathcal{F}^{cat}$ such that $C_{\mathsf{FOL}}(\alpha, \beta)$.

$$\begin{tikzcd}
\forall x(Sx \to Px) \arrow[dr, dash] & \forall x(Sx \to \neg Px) \arrow[dl, dash] \\
\exists x(Sx \wedge Px) & \exists x(Sx \wedge \neg Px)
\end{tikzcd}$$

(A) The diagram $(\mathcal{F}^{cat}, \mathsf{FOL})$.

$$\begin{tikzcd}
\forall x(Sx \to Px) \arrow[r, dash, dashed] \arrow[d] \arrow[dr, dash] & \forall x(Sx \to \neg Px) \arrow[d] \arrow[dl, dash] \\
\exists x(Sx \wedge Px) \arrow[r, dash, dotted] & \exists x(Sx \wedge \neg Px)
\end{tikzcd}$$

(B) The diagram $(\mathcal{F}^{cat}, \mathsf{SYL})$.

FIGURE 5. Two Aristotelian diagrams for the same fragment $\mathcal{F}^{cat}$

$$\begin{tikzcd}
\forall x Px \arrow[r, dash, dashed] & \forall x \neg Px \\
\forall x Qx \arrow[r, dash, dashed] & \forall x \neg Qx
\end{tikzcd}$$

(A) The diagram $(\mathcal{F}, \mathsf{FOL})$.

$$\begin{tikzcd}
\forall x(Px \wedge Qx) \arrow[r, dash, dashed] & \forall x(\neg Px \wedge \neg Qx)
\end{tikzcd}$$

(B) The diagram $(\mathcal{F}', \mathsf{FOL})$.

FIGURE 6. Two Aristotelian diagrams

$$\theta : (\mathcal{F}, \mathsf{FOL}) \to (\mathcal{F}', \mathsf{FOL}) : \begin{cases} \forall x Px \mapsto \forall x(Px \wedge Qx) \\ \forall x Qx \mapsto \forall x(Px \wedge Qx) \\ \forall x \neg Px \mapsto \forall x(\neg Px \wedge \neg Qx) \\ \forall x \neg Qx \mapsto \forall x(\neg Px \wedge \neg Qx) \end{cases}$$

is clearly an Aristotelian morphism.

Proposition 1. *The Aristotelian diagrams equipped with Aristotelian morphisms constitute a category. We will denote this category by* $\mathbb{D}_{\mathcal{AR}}$.

Proof. First, let us define for each diagram $D = (\mathcal{F}, B)$ the identity morphism Id_D as $Id_D : \mathcal{F} \to \mathcal{F} : x \mapsto x$, i.e., the identity function on the underlying fragment $\mathcal{F}$.[15] Next, we define composition $\circ$ of morphisms between diagrams as the usual composition of the set functions on the underlying fragments. We need to prove three things: (1) the composition of two Aristotelian morphisms is again an Aristotelian morphism, (2) for every Aristotelian morphism $f : D \to D'$, we have that $f \circ Id_D = f = Id_{D'} \circ f$, and (3) composition is associative.

1. Let $f : D \to D'$ and $g : D' \to D''$ be Aristotelian morphisms. We need to show that for all $R_B \in \mathcal{AR}_B$ and all $x, y \in D$ the following holds: $R_B(x,y) \Rightarrow \imath_B^{B''}(R_B)(gf(x), gf(y))$. Suppose $R_B(x,y)$, then $\imath_B^{B'}(R_B)(f(x), f(y))$ because f is an Aristotelian morphism. Since g is

[15]Note that the identity morphism Id_D involves the 'entire' diagram $D = (\mathcal{F}, B)$, i.e., not just its fragment $\mathcal{F}$, but also its ambient Boolean algebra B. Consequently, an identity function $\mathcal{F} \to \mathcal{F}$ can only be considered an identity morphism in $\mathbb{D}_{\mathcal{AR}}$ from $(\mathcal{F}, B)$ to $(\mathcal{F}, B')$ if $B = B'$. For example, consider the fragment $\mathcal{F} := \{\{a\}, \{b\}\}$. The identity function $Id_{\mathcal{F}}$ does not constitute an identity morphism between the diagrams $(\mathcal{F}, \wp(\{a,b\}))$ and $(\mathcal{F}, \wp(\{a,b,c\}))$ — actually, it is not even an Aristotelian morphism between these two diagrams to begin with, since $CD_{\wp(\{a,b\})}(\{a\},\{b\})$ but $C_{\wp(\{a,b,c\})}(\{a\},\{b\})$. These observations are closely related to the issue of logic-sensitivity of Aristotelian diagrams (also recall Footnote 13).

an Aristotelian morphism and since $\imath_{B'}^{B''} \circ \imath_B^{B'} = \imath_B^{B''}$, it now follows that $\imath_B^{B''}(R_B)(gf(x), gf(y))$.

2. These equalities follow immediately by definition of the identity morphism and the composition of morphisms.
3. Associativity follows directly from the associativity of functions.

$\square$

We now consider some elementary properties of this category. One of the most simple questions one can ask about a category is whether or not it has initial and terminal objects. We check this in the following example.

Example 5. Let us consider $(\emptyset, B)$, where B is an arbitrary Boolean algebra. For any other Aristotelian diagram $(\mathcal{F}', B')$, it is clear that we have exactly one Aristotelian morphism $(\emptyset, B) \to (\mathcal{F}', B')$, namely the empty set function. What this example shows in categorical terms is that the diagram $(\emptyset, B)$ is an initial object of the category $\mathbb{D}_{\mathcal{AR}}$. Now, for terminal objects, consider any diagram $D \neq D_*$. Since $*$ is the only possible element (across any Boolean algebra) for which $CD(*, *)$, it is clear that we do not have an Aristotelian morphism $D_* \to D$. Thus, D is not a terminal object. Also, D_* itself is clearly not a terminal object, since there are no elements in the fragment $\{*\}$ of D_* that are in any of the C, SC, or LI relations. In summary, $\mathbb{D}_{\mathcal{AR}}$ has initial, but no terminal objects.

It is easy to see that the σ-diagrams equipped with Aristotelian morphisms form a full subcategory of $\mathbb{D}_{\mathcal{AR}}$; we denote it with $\mathbb{D}_{\mathcal{AR}}^{\sigma}$. It will turn out that the inclusion functor from $\mathbb{D}_{\mathcal{AR}}^{\sigma}$ to $\mathbb{D}_{\mathcal{AR}}$ has a left adjoint, or, in other words, $\mathbb{D}_{\mathcal{AR}}^{\sigma}$ is a *reflective subcategory* [18, p. 91] of $\mathbb{D}_{\mathcal{AR}}$. First we prove a handy proposition.

Proposition 2. *Let $D = (\mathcal{F}, B)$ and $D' = (\mathcal{F}', B')$ be σ-diagrams, and let $f : D \to D'$ be an Aristotelian morphism. Let Q be a set containing at least one element from each PCD in D. Then f is uniquely determined by its image on Q. Equivalently, defining an Aristotelian morphism amounts to defining it on a set Q that contains at least one element from each of its PCDs.*

Proof. The proof draws directly upon the work of Smessaert and Demey [21]. Let $f_Q : Q \to \mathcal{F}'$ be a function that is a restriction of f. We can then extend f_Q to a function $f_\neg : \mathcal{F} \to \mathcal{F}'$ as follows:

$$f_\neg : \mathcal{F} \to \mathcal{F}' : x \mapsto f_\neg(x) := \begin{cases} f_Q(x), & \text{if } x \in Q \\ \neg f_Q(\neg x), & \text{if } x \in \mathcal{F} - Q. \end{cases}$$

The function $f_\neg$ is well-defined since both D and D' are σ-diagrams, and thus closed under complementation. We now need to show that $f_\neg$ really is an Aristotelian morphism. Let $R \in \mathcal{AR}_B$ and $x, y \in \mathcal{F}$ such that $R(x, y)$. We need to show that $\imath_B^{B'}(R)(f_\neg(x), f_\neg(y))$. There are four cases to consider:

1. $x, y \in Q$. In this case we are done, since f_Q preserves Aristotelian relations. Indeed, f_Q is just the restriction of f to Q.

2. $x \in Q$ and $y \in \mathcal{F} - Q$. Lemma 1 will do the heavy lifting. We use a case distinction.
   - If $CD(x, y)$, then $BI(x, \neg y)$ by Lemma 1, or in other words, $x = \neg y$. Therefore, we have that $BI(f_Q(x), f_Q(\neg y))$, and applying Lemma 1 again yields $CD(f_Q(x), \neg f_Q(\neg y))$, as required.
   - If $C(x, y)$, then $LI(x, \neg y)$ by Lemma 1. Now, since f_Q is an Aristotelian morphism, we have that $LI(f_Q(x), f_Q(\neg y))$, and applying Lemma 1 again yields $C(f_Q(x), \neg f_Q(\neg y))$, as required.
   - If $LI(x, y)$, we can repeat the previous case with all the occurrences of C and LI switched.
   - If $SC(x, y)$, then $RI(x, \neg y)$ by Lemma 1. Thus $LI(\neg y, x)$ by Definition 4. Now, since f_Q is an Aristotelian morphism, we have that $LI(f_Q(\neg y), f_Q(x))$, and applying Definition 4 again yields $RI(f_Q(x), f_Q(\neg y))$. Now Lemma 1 yields $SC(f_Q(x), \neg f_Q(\neg y))$, as required.
3. $x \in \mathcal{F} - Q$ and $y \in Q$. Analogous to the previous case.
4. $x, y \in \mathcal{F} - Q$. Suppose first that R is either CD, C or SC. Lemma 1 states that there exists an Aristotelian relation R' such that $R(x, y)$ is equivalent to $R'(\neg x, \neg y)$. Since f_Q is an Aristotelian morphism, it follows that $R'(f_Q(\neg x), f_Q(\neg y))$. Applying Lemma 1 again yields $R(\neg f_Q(\neg x), \neg f_Q(\neg y))$, which is the required result. Now, suppose that $R = LI$. Then, Lemma 1 states that $RI(\neg x, \neg y)$, which is equivalent to $LI(\neg y, \neg x)$ by Definition 4. Since f_Q is an Aristotelian morphism, it follows that $LI(f_Q(\neg y), f_Q(\neg x))$, which by Definition 4 is the same as $RI(f_Q(\neg x), f_Q(\neg y))$. A final application of Lemma 1 then gives the desired result: $LI(\neg f_Q(\neg x), \neg f_Q(\neg y))$.

If one now looks at the proof, one sees that $f_\neg$ is not just well-defined: it is (by construction) the unique extension of f_Q to $\mathcal{F}$. Indeed, since CD is preserved, for all $x \in Q$, we must have that $\neg x$ gets mapped to $\neg f_Q(x)$. Therefore, f and $f_\neg$ are equal and the statement is proven. □

Definition 9 (*Negation closure*). Let $(\mathcal{F}, B)$ be an Aristotelian diagram. The set $\mathcal{F} \cup \{\neg_B x \in B \mid x \in \mathcal{F}\}$ is called the negation closure of $\mathcal{F}$ in B, and is denoted by $Cl_\neg^B(\mathcal{F})$.

For example, the negation closure of the fragment $\{100, 110\}$ in the Boolean algebra $\{0, 1\}^3$ is given by $\{100, 110, 011, 001\}$. It is clear that every diagram $(\mathcal{F}, B)$ defines a σ-diagram of the form $(Cl_\neg^B(\mathcal{F}), B)$. Given an Aristotelian morphism $f : (\mathcal{F}, B) \to (\mathcal{F}', B')$, Proposition 2 provides us with a unique extension to a morphism between the σ-diagrams $(Cl_\neg^B(\mathcal{F}), B)$ and $(Cl_\neg^{B'}(\mathcal{F}'), B')$. We call this extension $f_\neg$. We now have all the machinery available to create our first functor: the *negation closure functor*. It is this functor that will allow us to prove that $\mathbb{D}_{\mathcal{AR}}^\sigma$ is a reflective subcategory of $\mathbb{D}_{\mathcal{AR}}$. This result justifies the longstanding practice in logical geometry of working on a fragment that contains only one element from each PCD of a given σ-diagram.

Proposition 3. *We have a functor $Cl_\neg$ from $\mathbb{D}_{\mathcal{AR}}$ to $\mathbb{D}_{\mathcal{AR}}^\sigma$, which is defined on objects as*

$$Cl_\neg(\mathcal{F}, B) := (Cl_\neg^B(\mathcal{F}), B),$$

and on morphisms as

$$Cl_\neg(f) := f_\neg.$$

Proof. Definition 9 shows that $Cl_\neg$ is well-defined on objects. Proposition 2 shows how it behaves on arrows. We only need to check functoriality. It is trivially true that $Cl_\neg(Id_D) = Id_{Cl_\neg(D)}$ for any diagram $D = (\mathcal{F}, B)$, since $Cl_\neg(Id_D)(x) = (Id_D)_\neg(x) = \neg Id_D(\neg x) = \neg\neg x = x$ for all $x \in Cl^B_\neg(\mathcal{F}) - \mathcal{F}$. Now we show that $Cl_\neg$ respects composition. Let $g : (\mathcal{F}, B) \to (\mathcal{F}', B')$ and $f : (\mathcal{F}', B') \to (\mathcal{F}'', B'')$ and let $x \in \mathcal{F}$. We then have:

$$\begin{aligned} Cl_\neg(f \circ g)(x) &= (f \circ g)(x) \\ &= f(g(x)) \\ &= Cl_\neg(f)(Cl_\neg(g)(x)) \\ &= (Cl_\neg(f) \circ Cl_\neg(g))(x). \end{aligned}$$

Since $Cl_\neg(f \circ g)$ and $Cl_\neg(f) \circ Cl_\neg(g)$ coincide on $\mathcal{F}$, Proposition 2 guarantees that they are identical. □

Theorem 1. *The inclusion functor $U : \mathbb{D}^\sigma_{\mathcal{AR}} \to \mathbb{D}_{\mathcal{AR}}$ has a left adjoint, which is given by the negation closure functor, $Cl_\neg : \mathbb{D}_{\mathcal{AR}} \to \mathbb{D}^\sigma_{\mathcal{AR}}$. Put succinctly:*

$$Cl_\neg \dashv U.$$

In other words, $\mathbb{D}^\sigma_{\mathcal{AR}}$ is a reflective subcategory of $\mathbb{D}_{\mathcal{AR}}$.

Proof. We will work with the Hom-set definition of adjoints:

$$\varphi_{D,D'} : \mathrm{Hom}_{\mathbb{D}^\sigma_{\mathcal{AR}}}(Cl_\neg D, D') \cong \mathrm{Hom}_{\mathbb{D}_{\mathcal{AR}}}(D, UD').$$

We write φ instead of $\varphi_{D,D'}$ for ease of notation, and define it as follows:

$$f \mapsto \varphi(f) := f|_{\mathcal{F}},$$

where $f|_{\mathcal{F}}$ is the restriction of f to $\mathcal{F}$. For any $g \in \mathrm{Hom}_{\mathbb{D}_{\mathcal{AR}}}(D, UD')$, it is immediately clear that $\varphi^{-1}(g)$ is the unique extension $g_\neg$ of g to $Cl^B_\neg(\mathcal{F})$, so φ really is bijective.

We now have to show that φ is natural in both D and D'. Let us first consider an Aristotelian diagram D'' and a morphism $h : D'' \to D$ in $\mathbb{D}_{\mathcal{AR}}$; we need to check if the following diagram commutes:

$$\begin{tikzcd} \mathrm{Hom}(Cl_\neg D, D') \arrow[r, "\varphi"] \arrow[d, "\mathrm{Hom}(Cl_\neg h, D')"'] & \mathrm{Hom}(D, UD') \arrow[d, "\mathrm{Hom}(h, UD')"] \\ \mathrm{Hom}(Cl_\neg D'', D') \arrow[r, "\varphi"] & \mathrm{Hom}(D'', UD'). \end{tikzcd}$$

Take any $f \in \mathrm{Hom}(Cl_\neg D, D')$. The following chain of equalities yields the required result:

$$\begin{aligned} (\varphi \circ \mathrm{Hom}(Cl_\neg h, D'))(f) &= \varphi(f \circ Cl_\neg h) \\ &= (f \circ Cl_\neg h)|_{\mathcal{F}''} \\ &= f \circ (Cl_\neg h)|_{\mathcal{F}''} \\ &= f|_{\mathcal{F}} \circ h \end{aligned}$$

$$\begin{aligned} &= \varphi(f) \circ h \\ &= (\mathrm{Hom}(h, UD') \circ \varphi)(f). \end{aligned}$$

Let us now consider a σ-diagram D''' and a morphism $h' : D' \to D'''$ in $\mathbb{D}^{\sigma}_{\mathcal{AR}}$. We show the following diagram commutes:

$$\begin{tikzcd} \mathrm{Hom}(Cl_{\neg}D, D') \arrow[r, "\varphi"] \arrow[d, "\mathrm{Hom}(Cl_{\neg}D, h')"'] & \mathrm{Hom}(D, UD') \arrow[d, "\mathrm{Hom}(D, Uh')"] \\ \mathrm{Hom}(Cl_{\neg}D, D''') \arrow[r, "\varphi"] & \mathrm{Hom}(D, UD''') \end{tikzcd}$$

Take any $f \in \mathrm{Hom}(Cl_{\neg}D, D')$. The result again follows from simple algebraic manipulation:

$$\begin{aligned} (\varphi \circ \mathrm{Hom}(Cl_{\neg}D, h'))(f) &= \varphi(h' \circ f) \\ &= (h' \circ f)|_{\mathcal{F}} \\ &= Uh' \circ f|_{\mathcal{F}} \\ &= Uh' \circ \varphi(f) \\ &= (\mathrm{Hom}(D, Uh') \circ \varphi)(f). \end{aligned}$$

We have thus shown that $Cl_{\neg} \dashv U$. □

3.2. The $\mathcal{OR}$, $\mathcal{IR}$ and $\mathcal{OR} \times \mathcal{IR}$ Morphisms

Remember from Definition 4 that the Aristotelian relations are naturally part of two other sets of relations. Therefore, it could be interesting to look at morphisms that preserve those relations, rather than the Aristotelian relations themselves. This leads to the following three kinds of morphisms.

Definition 10. ($\mathcal{OR}$, $\mathcal{IR}$ and $\mathcal{OR} \times \mathcal{IR}$ *morphisms*) Let $D = (\mathcal{F}, B)$ and $D' = (\mathcal{F}', B')$ be Aristotelian diagrams. Let $f : \mathcal{F} \to \mathcal{F}'$ be a function between fragments. We say that f is an $\mathcal{OR}$ resp. $\mathcal{IR}$ resp. $\mathcal{OR} \times \mathcal{IR}$ morphism from D to D' iff for all $R_B \in \mathcal{OR}_B$ resp. $\mathcal{IR}_B$ resp. $\mathcal{OR}_B \sqcup \mathcal{IR}_B$, and all $x, y \in \mathcal{F}$:

$$R_B(x, y) \Rightarrow \imath_B^{B'}(R_B)(f(x), f(y)).$$

Observe that every $\mathcal{OR} \times \mathcal{IR}$ morphism is by definition also an $\mathcal{OR}$ morphism and an $\mathcal{IR}$ morphism. Conversely, every function that is both an $\mathcal{OR}$ morphism and an $\mathcal{IR}$ morphism is also an $\mathcal{OR} \times \mathcal{IR}$ morphism. Furthermore, notice that Definition 10 is virtually the same as Definition 8, but with $\mathcal{AR}$ replaced by either $\mathcal{OR}, \mathcal{IR}$ or $\mathcal{OR} \sqcup \mathcal{IR}$. Let us consider some easy examples of these new kinds of morphisms.

Example 6. Given $D = (\{100, 110\}, \{0, 1\}^3)$, it is easy to check that $f : D \to D$, defined by $f(100) := 110$ and $f(110) := 100$, is an $\mathcal{OR}$ morphism, but not an $\mathcal{IR}$ morphism (and thus not an $\mathcal{OR} \times \mathcal{IR}$ morphism either) — in particular, we have $LI(100, 110)$ yet $RI(110, 100)$, i.e., $RI(f(100), f(110))$. Secondly, the map $g : (\{100, 001\}, \{0, 1\}^3) \to (\{110, 011\}, \{0, 1\}^3)$, defined by $g(100) := 110$ and $g(001) := 011$, is an $\mathcal{IR}$ morphism, but not an $\mathcal{OR}$ morphism (and thus not an $\mathcal{OR} \times \mathcal{IR}$ morphism either) — in particular, we have $C(100, 001)$ yet $SC(110, 011)$, i.e., $SC(g(100), g(001))$. Thirdly, the embedding $h : (\{10, 01\}, \{0, 1\}^2) \to (\{0, 1\}^3, \{0, 1\}^3)$, defined by $h(10) = 100$ and $h(01) = 011$, is both an $\mathcal{OR}$ morphism and an $\mathcal{IR}$ morphism, and thus also an $\mathcal{OR} \times \mathcal{IR}$ morphism.

These newly introduced kinds of morphisms are quite strong. In particular, since $\mathcal{OR}_B$ is a partition of $B \times B$, one can show that if the map f is an $\mathcal{OR}$ morphism, then its inverse f^{-1} (if it exists) is an $\mathcal{OR}$ morphism as well. (Analogous remarks apply to $\mathcal{IR}$ and $\mathcal{OR} \times \mathcal{IR}$ morphisms.) This is made precise in Proposition 4. Furthermore, and more importantly, we still need to check that the new morphisms also give rise to categories; this is done in Proposition 5.

Proposition 4. *Consider Aristotelian diagrams $D = (\mathcal{F}, B)$ and $D' = (\mathcal{F}', B')$ and a bijective set function $f : \mathcal{F} \to \mathcal{F}'$. If $f : D \to D'$ is an $\mathcal{OR}$ resp. $\mathcal{IR}$ resp. $\mathcal{OR} \times \mathcal{IR}$ morphism, then $f^{-1} : D' \to D$ is an $\mathcal{OR}$ resp. $\mathcal{IR}$ resp. $\mathcal{OR} \times \mathcal{IR}$ morphism as well.*

Proof. Suppose that $f : D \to D'$ is an $\mathcal{OR}$ morphism and consider arbitrary $x, y \in \mathcal{F}'$. Suppose, for example, that $SC_{B'}(x, y)$. Since the relations in $\mathcal{OR}_B$ are mutually exclusive, it follows that not $CD_{B'}(x, y)$ and not $C_{B'}(x, y)$ and not $NCD_{B'}(x, y)$. Now, if $CD_B(f^{-1}(x), f^{-1}(y))$, then our assumption that f is an $\mathcal{OR}$ morphism would entail that $CD_{B'}(f(f^{-1}(x)), f(f^{-1}(y)))$, i.e., $CD_{B'}(x, y)$, after all. We thus conclude that not $CD_B(f^{-1}(x), f^{-1}(y))$. In the same way, we also find that not $C_B(f^{-1}(x), f^{-1}(y))$ and not $NCD_B(f^{-1}(x), f^{-1}(y))$. Since the relations in $\mathcal{OR}_B$ are jointly exhaustive, it follows that $SC_B(f^{-1}(x), f^{-1}(y))$. This shows that if $SC_{B'}(x, y)$, then $\imath^B_{B'}(SC_{B'})(f^{-1}(x), f^{-1}(y))$. Exactly the same can be shown for $CD_{B'}$, $C_{B'}$ and $NCD_{B'}$. Taken together, this means that $f^{-1} : D' \to D$ is an $\mathcal{OR}$ morphism. In exactly the same way, we show that if f is an $\mathcal{IR}$ morphism, then f^{-1} is an $\mathcal{IR}$ morphism as well, and that if f is an $\mathcal{OR} \times \mathcal{IR}$ morphism, then f^{-1} is an $\mathcal{OR} \times \mathcal{IR}$ morphism as well. □

Proposition 5. *The Aristotelian diagrams equipped with $\mathcal{OR}$ resp. $\mathcal{IR}$ resp. $\mathcal{OR} \times \mathcal{IR}$ morphisms all give rise to categories. We will denote these categories by $\mathbb{D}_{\mathcal{OR}}$ resp. $\mathbb{D}_{\mathcal{IR}}$ resp. $\mathbb{D}_{\mathcal{OR}\times\mathcal{IR}}$.*

Proof. Completely analogous to Proposition 1. □

Using Proposition 5, the observation made immediately after Definition 10 can now be expressed as follows: $\mathrm{Hom}_{\mathbb{D}_{\mathcal{OR}\times\mathcal{IR}}}(D, D') = \mathrm{Hom}_{\mathbb{D}_{\mathcal{OR}}}(D, D') \cap \mathrm{Hom}_{\mathbb{D}_{\mathcal{IR}}}(D, D')$ for any two Aristotelian diagrams D and D'. Again, we can ask ourselves whether or not these categories have initial and/or terminal objects. It is not hard to see that the situation is exactly the same as in $\mathbb{D}_{\mathcal{AR}}$. Completely analogously to Example 5, we have that $\mathbb{D}_{\mathcal{OR}}$, $\mathbb{D}_{\mathcal{IR}}$ and $\mathbb{D}_{\mathcal{OR}\times\mathcal{IR}}$ have all diagrams of the form $(\emptyset, B)$ as initial objects, but they have no terminal objects.

As was the case for $\mathbb{D}_{\mathcal{AR}}$, the category of σ-diagrams equipped with $\mathcal{OR} \times \mathcal{IR}$ morphisms (denoted by $\mathbb{D}^{\sigma}_{\mathcal{OR}\times\mathcal{IR}}$) forms a reflective subcategory of $\mathbb{D}_{\mathcal{OR}\times\mathcal{IR}}$. We proceed in the same way as in Sect. 3.1.

Proposition 6. *Let $D = (\mathcal{F}, B)$ and $D' = (\mathcal{F}', B')$ be σ-diagrams, and let $f : D \to D'$ be an $\mathcal{OR} \times \mathcal{IR}$ morphism. Let Q be a set containing at least one element from each PCD in D. Then f is uniquely determined by its image on Q. Equivalently, defining an $\mathcal{OR} \times \mathcal{IR}$ morphism amounts to defining it on a set Q that contains at least one element from each of its PCDs.*

Proof. Entirely analogous to the proof of Proposition 2, making extensive use of Lemma 1. □

Given an $\mathcal{OR} \times \mathcal{IR}$ morphism $f : (\mathcal{F}, B) \to (\mathcal{F}', B')$, Proposition 6 provides us with a unique extension to an $\mathcal{OR} \times \mathcal{IR}$ morphism between the σ-diagrams $(Cl_{\neg}^{B}(\mathcal{F}), B)$ and $(Cl_{\neg}^{B'}(\mathcal{F}'), B')$. We call this extension $f_{\neg}$. Note that this is not an abuse of notation: if a map f is both an Aristotelian and an $\mathcal{OR} \times \mathcal{IR}$ morphism, then the map $f_{\neg}$ defined here coincides with the one from Sect. 3.1. Notice also that analogous propositions for $\mathcal{OR}$ or $\mathcal{IR}$ morphisms do *not* exist. Indeed, consider for instance the $\mathcal{OR}$ morphism f from Example 6. If we want to extend f to an $\mathcal{OR}$ morphism on the negation closure of $\mathcal{F}$ in B, we need to map the contrary pair $(100, 001)$ to some contrary pair in D. However, since f maps 100 to 110 it is clear that this is impossible. An analogous argument using g from Example 6 provides us with a similar problem in the $\mathcal{IR}$ case.

Proposition 7. *We have a functor $Cl_{\neg}$ from $\mathbb{D}_{\mathcal{OR}\times\mathcal{IR}}$ to $\mathbb{D}^{\sigma}_{\mathcal{OR}\times\mathcal{IR}}$, which is defined on objects as*

$$Cl_{\neg}(\mathcal{F}, B) := (Cl_{\neg}^{B}(\mathcal{F}), B),$$

and on morphisms as

$$Cl_{\neg}(f) := f_{\neg}.$$

Proof. Entirely analogous to the proof of Proposition 3. □

Theorem 2. *The inclusion functor $U : \mathbb{D}^{\sigma}_{\mathcal{OR}\times\mathcal{IR}} \to \mathbb{D}_{\mathcal{OR}\times\mathcal{IR}}$ has a left adjoint, which is given by the negation closure functor, $Cl_{\neg} : \mathbb{D}_{\mathcal{OR}\times\mathcal{IR}} \to \mathbb{D}^{\sigma}_{\mathcal{OR}\times\mathcal{IR}}$. Put succinctly:*

$$Cl_{\neg} \dashv U.$$

In other words, $\mathbb{D}^{\sigma}_{\mathcal{OR}\times\mathcal{IR}}$ is a reflective subcategory of $\mathbb{D}_{\mathcal{OR}\times\mathcal{IR}}$.

Proof. Entirely analogous to the proof of Theorem 1. □

We have just studied the category $\mathbb{D}^{\sigma}_{\mathcal{OR}\times\mathcal{IR}}$ of σ-diagrams equipped with $\mathcal{OR} \times \mathcal{IR}$ morphisms. In principle, we could also equip σ-diagrams with $\mathcal{OR}$ morphisms (yielding the category $\mathbb{D}^{\sigma}_{\mathcal{OR}}$) or with $\mathcal{IR}$ morphisms (yielding $\mathbb{D}^{\sigma}_{\mathcal{IR}}$). However, Lemma 2 below states that if we restrict ourselves to σ-diagrams, then every $\mathcal{OR}$ morphism is an $\mathcal{IR}$ morphism as well, and thus also an $\mathcal{OR} \times \mathcal{IR}$ morphism. Consequently, $\mathbb{D}^{\sigma}_{\mathcal{OR}}$ is the same category as $\mathbb{D}^{\sigma}_{\mathcal{OR}\times\mathcal{IR}}$.

Lemma 2. *Let $D = (\mathcal{F}, B)$ and $D' = (\mathcal{F}', B')$ be Aristotelian diagrams, and let $f : D \to D'$ be an $\mathcal{OR}$ morphism. If D is a σ-diagram, then f is also an $\mathcal{IR}$ morphism.*

Proof. Consider arbitrary $x, y \in \mathcal{F}$ and suppose that $LI_B(x, y)$; we shall prove that $\imath_B^{B'}(LI_B)(f(x), f(y))$ as well. (The proofs for BI, RI and NI are completely analogous.) First of all, note that since $y \in \mathcal{F}$ and D is a σ-diagram, we have $\neg y \in \mathcal{F}$ as well. Since $CD_B(y, \neg y)$ and f is an $\mathcal{OR}$ morphism, it follows that $\imath_B^{B'}(CD_B)(f(y), f(\neg y))$, and hence $\neg f(y) = f(\neg y)$. By Lemma 1 it follows from $LI_B(x, y)$ that $C_B(x, \neg y)$. Since f is an $\mathcal{OR}$ morphism, it follows that $\imath_B^{B'}(C_B)(f(x), f(\neg y))$, and thus also $\imath_B^{B'}(C_B)(f(x), \neg f(y))$. Another application of Lemma 1 yields $\imath_B^{B'}(LI_B)(f(x), f(y))$. □

The inverse of Lemma 2 does not hold: an $\mathcal{IR}$ morphism $f : D \to D'$ need *not* be an $\mathcal{OR}$ morphism, even if D and D' are both σ-diagrams. For a concrete counterexample, consider $D := (\{10, 01\}, \{0, 1\}^2)$ and $D' := (\{100, 001, 110, 011\}, \{0, 1\}^3)$, and $f : D \to D'$ defined by $f(10) := 100$ and $f(01) := 001$. It is easy to check that D and D' are σ-diagrams and that f is an $\mathcal{IR}$ morphism, but not an $\mathcal{OR}$ morphism — in particular, we have $CD(10, 01)$ yet $C(100, 001)$, i.e., $C(f(10), f(01))$. Because of examples like this, $\mathbb{D}^{\sigma}_{\mathcal{IR}}$ is *not* the same category as $\mathbb{D}^{\sigma}_{\mathcal{OR}\times\mathcal{IR}}$. However, we will see in Sect. 4 that there are good reasons for not studying this category in depth anyway. To summarize: we will not discuss $\mathbb{D}^{\sigma}_{\mathcal{OR}}$ as such, because it is identical with $\mathbb{D}^{\sigma}_{\mathcal{OR}\times\mathcal{IR}}$, and we will not discuss $\mathbb{D}^{\sigma}_{\mathcal{IR}}$ for reasons that will become clear later.

If we zoom out from the σ-diagrams to the entire class of Aristotelian diagrams, it is interesting to see how the morphisms defined in this subsection relate to the notion of Aristotelian morphism defined in Sect. 3.1. It follows trivially from Definitions 8 and 10 that every $\mathcal{OR} \times \mathcal{IR}$ morphism is an Aristotelian morphism, since $\mathcal{AR} \subseteq \mathcal{OR} \cup \mathcal{IR}$. However, not every Aristotelian morphism is necessarily an $\mathcal{IR}$ morphism or an $\mathcal{OR}$ morphism. A counterexample is given by θ from Example 4. Indeed, we have $NI_{\mathsf{FOL}}(\forall x Px, \forall x Qx)$ but $BI_{\mathsf{FOL}}(\forall x(Px \wedge Qx), \forall x(Px \wedge Qx))$, i.e., $BI_{\mathsf{FOL}}(\theta(\forall x Px), \theta(\forall x Qx))$, and thus θ is not an $\mathcal{IR}$ morphism. Furthermore, we have $NCD_{\mathsf{FOL}}(\forall x Qx, \forall x \neg Px)$, but $C_{\mathsf{FOL}}(\forall x(Px \wedge Qx), \forall x(\neg Px \wedge \neg Qx))$, i.e., $C_{\mathsf{FOL}}(\theta(\forall x Qx), \theta(\forall x \neg Px))$, which shows that θ is not an $\mathcal{OR}$ morphism either.

Note that, for any Aristotelian morphism f, apart from the Aristotelian relations, it also preserves BI and RI. The former is true because if $BI(x, y)$, then $x = y$, which in turn implies $f(x) = f(y)$, i.e., $BI(f(x), f(y))$ (in fact, this also proves that $\mathcal{OR}$ morphisms preserve BI as well). The latter is true because if $RI(x, y)$ then $LI(y, x)$, which in turn implies $LI(f(y), f(x))$, i.e., $RI(f(x), f(y))$. It is not true that Aristotelian morphisms preserve NCD and/or NI: a counterexample is given by θ from Example 4, as already described above. The entire situation is summarized in Table 1.

It should be noted that the four notions of morphisms we have considered thus far are for some purposes too restrictive, with few examples to work with.[16] For example, bitstring maps that delete or add bit positions occur

[16] Although, interestingly, if we restrict ourselves even further and use as morphisms all the bijective $\mathcal{OR} \times \mathcal{IR}$ morphisms, we find something peculiar. Namely, the resulting category is a groupoid with as vertex groups the automorphism groups of the Aristotelian diagrams.

Table 1. Which kinds of morphisms preserve which kinds of relations?

	CD	*C*	*SC*	*NCD*	*BI*	*LI*	*RI*	*NI*
$\mathcal{AR}$	✓	✓	✓		✓	✓	✓	
$\mathcal{OR}$	✓	✓	✓	✓	✓			
$\mathcal{IR}$					✓	✓	✓	✓
$\mathcal{OR} \times \mathcal{IR}$	✓	✓	✓	✓	✓	✓	✓	✓

quite naturally in application contexts [7], but will generally not be morphisms in any of the above categories (cf. Examples 7, 9 and 10 below). Furthermore, none of the categories considered thus far have terminal objects.

3.3. Increasing Infomorphisms

In this section, we use an interesting new idea to create morphisms, which was first explored by Vignero [22]. Notice what happens when we apply an Aristotelian morphism $f : (\mathcal{F}, B) \to (\mathcal{F}', B')$ to a pair of non-contradictory elements $\mathit{NCD}(x, y)$: depending on the concrete definition of f, we might have that $R(f(x), f(y))$, for any of the four opposition relations R. Looking at Definition 5, this can be reformulated as follows: $\mathit{NCD}(x, y) \Rightarrow R(f(x), f(y))$, for some $R \in \mathcal{OR}_{B'}$ such that $\mathit{NCD} \leq_{B'} R$. Similar remarks apply to *NI* and $\mathcal{IR}_{B'}$. We can now extend these observations to all opposition and/or implication relations and thus create morphisms that do not necessarily preserve the Aristotelian structure, but only behave well with respect to the informativity orderings from Definitions 5 and 6. This clearly leads to less restrictive notions of morphisms. First, there are Vignero's [22] infomorphisms. He characterizes an infomorphism $f : (\mathcal{F}, B) \to (\mathcal{F}', B')$ as a map that identifies a part of $\mathcal{F}'$ embedded in B' that is *at least* as informative as the fragment $\mathcal{F}$ embedded in B. Maps of this type will henceforth be called *increasing infomorphisms*, and will be studied in the present subsection. Completely analogously, we will also consider maps that identify a part of $\mathcal{F}'$ embedded in B' that is *at most* as informative as $\mathcal{F}$ embedded in B. Maps of this type will be called *decreasing infomorphisms*, and will be studied in Sect. 3.4.

We begin by formally introducing the increasing infomorphisms. Similar to Definition 10, we have three different kinds of increasing infomorphisms, corresponding to the three informativity orderings from Definitions 5 and 6.

Definition 11 (*Increasing information* $\mathcal{OR}, \mathcal{IR}$, *and* $\mathcal{OR} \times \mathcal{IR}$ *morphisms*). Let $D = (\mathcal{F}, B)$ and $D' = (\mathcal{F}', B')$ be Aristotelian diagrams. Let $f : \mathcal{F} \to \mathcal{F}'$ be a function between fragments. We say that f is an increasing information $\mathcal{OR}$ resp. $\mathcal{IR}$ resp. $\mathcal{OR} \times \mathcal{IR}$ morphism from D to D' iff for all $R_B \in \mathcal{OR}_B$ resp. $\mathcal{IR}_B$ resp. $\mathcal{OR}_B \sqcup \mathcal{IR}_B$, and all $x, y \in \mathcal{F}$, there exists some $R'_{B'} \in \mathcal{OR}_{B'}$

Footnote 16 continued
Studying this category is a possibility for further research for someone who is interested in these automorphism groups.

resp. $\mathcal{IR}_{B'}$ resp. $\mathcal{OR}_{B'} \sqcup \mathcal{IR}_{B'}$, such that

$$R_B(x, y) \Rightarrow R'_{B'}(f(x), f(y)),$$

and $\imath_B^{B'}(R_B) \leq_{B'} R'_{B'}$. These three kinds of morphisms will also be referred to as increasing infomorphisms.

Observe that every increasing information $\mathcal{OR} \times \mathcal{IR}$ morphism is by definition also an increasing information $\mathcal{OR}$ morphism and an increasing information $\mathcal{IR}$ morphism. Conversely, every function that is both an increasing information $\mathcal{OR}$ morphism and an increasing information $\mathcal{IR}$ morphism is also an increasing information $\mathcal{OR} \times \mathcal{IR}$ morphism. Notice that both Aristotelian morphisms φ and θ from Examples 3 and 4 are in fact increasing information $\mathcal{OR} \times \mathcal{IR}$ (and thus also $\mathcal{OR}$ and $\mathcal{IR}$) morphisms as well. This will turn out to be an instance of a more general fact (cf. Theorem 13 from Sect. 5). A more interesting example is given by bit deletion.

Example 7. Given $\mathcal{F}_3 := \{100, 010, 101\}$ and $\mathcal{F}_2 := \{10, 01\}$, the map $d_3 \colon \mathcal{F}_3 \to \mathcal{F}_2 \colon xyz \mapsto xy$, which deletes a bitstring's third bit position, is an increasing information $\mathcal{OR} \times \mathcal{IR}$ (and thus also $\mathcal{OR}$ and $\mathcal{IR}$) morphism from $(\mathcal{F}_3, \{0,1\}^3)$ to $(\mathcal{F}_2, \{0,1\}^2)$. However, d_3 is not an Aristotelian, $\mathcal{OR}$, $\mathcal{IR}$ or $\mathcal{OR} \times \mathcal{IR}$ morphism. For example, we have $C_{\{0,1\}^3}(100, 010)$ and $LI_{\{0,1\}^3}(100, 101)$, but $CD_{\{0,1\}^2}(10, 01)$ and $BI_{\{0,1\}^2}(10, 10)$; note that *CD* and *BI* are more informative than, but not identical to, resp. *C* and *LI*.

It is worthwhile to point out that Vignero [22] defined (increasing) infomorphisms on yet another set of relations, namely $\{CD, C, SC, BI, LI, RI, UN\}$. However, for all three of the relations *NCD*, *NI* and *UN*, the requirement of Definition 11 is vacuously satisfied. Therefore, our increasing information $\mathcal{OR} \times \mathcal{IR}$ morphisms conservatively extend Vignero's infomorphisms.

Furthermore, observe that Definition 10 can be reformulated along the lines of Definition 11, the only difference being that Definition 11 (increasing infomorphism) requires $\imath_B^{B'}(R_B) \leq_{B'} R'_{B'}$, whereas Definition 10 makes the stronger requirement that $\imath_B^{B'}(R_B) = R'_{B'}$. From the reflexivity of $\leq_{B'}$, we thus have that every $\mathcal{OR}$ resp. $\mathcal{IR}$ resp. $\mathcal{OR} \times \mathcal{IR}$ morphism is an increasing information $\mathcal{OR}$ resp. $\mathcal{IR}$ resp. $\mathcal{OR} \times \mathcal{IR}$ morphism.

Remember that both the opposition and the implication relations are jointly exhaustive and mutually exclusive on $B \times B$. Therefore, Definition 11 has the following equivalent characterization, which will be handy in proofs.

Proposition 8. *Let $D = (\mathcal{F}, B)$ and $D' = (\mathcal{F}', B')$ be Aristotelian diagrams. Let $f : \mathcal{F} \to \mathcal{F}'$ be a function between fragments. Then f is an increasing information $\mathcal{OR}$ resp. $\mathcal{IR}$ resp. $\mathcal{OR} \times \mathcal{IR}$ morphism from D to D' iff for all $x, y \in \mathcal{F}$, and for the unique $R_O, R'_O \in \mathcal{OR}$ and $R_I, R'_I \in \mathcal{IR}$ such that*

$$R_O(x, y), R_I(x, y), R'_O(f(x), f(y)) \text{ and } R'_I(f(x), f(y)),$$

we have that

$$\imath_B^{B'}(R_O) \leq_{B'} R'_O \text{ resp. } \imath_B^{B'}(R_I) \leq_{B'} R'_I \text{ resp. } (\imath_B^{B'}(R_O), \imath_B^{B'}(R_I)) \leq_{B'} (R'_O, R'_I).$$

Here, $\leq_{B'}$ is the relevant informativity ordering from Definition 5 or 6.

Proof. Uniqueness and existence of the above relations is satisfied since $\mathcal{OR}_B$ and $\mathcal{IR}_B$ are mutually exclusive and jointly exhaustive. Now, the statement follows by the pointwise definition of the combined informativity ordering. □

Proposition 9. *The Aristotelian diagrams, equipped with either kind of increasing infomorphisms, give rise to a category. We will denote these categories by* $\mathbb{D}^{Inc}_{\mathcal{OR}}$, $\mathbb{D}^{Inc}_{\mathcal{IR}}$ *and* $\mathbb{D}^{Inc}_{\mathcal{OR}\times\mathcal{IR}}$.

Proof. We prove the statement for $\mathbb{D}^{Inc}_{\mathcal{OR}}$. The other two cases are analogous. Identity morphisms and composition are defined in the same way as with Aristotelian, $\mathcal{OR}$, $\mathcal{IR}$ and $\mathcal{OR}\times\mathcal{IR}$ morphisms. Most of the proof is similar to that of Proposition 1. Therefore, we only prove that, given increasing information $\mathcal{OR}$ morphisms $f : D \to D'$ and $g : D' \to D''$, the composition $g \circ f$ is again an increasing information $\mathcal{OR}$ morphism. We use Proposition 8 to do this. Consider arbitrary $x, y \in D$ and let R_O, R'_O and R''_O be the unique opposition relations such that $R_O(x, y)$, $R'_O(f(x), f(y))$ and $R''_O((g \circ f)(x), (g \circ f)(y))$. Since f is an increasing information $\mathcal{OR}$ morphism and since the relabel functions preserve the informativity order, we have that $\imath^{B''}_{B'}(\imath^{B'}_{B}(R_O)) \leq_{B''} \imath^{B''}_{B'}(R'_O)$. Now, since g is an increasing information $\mathcal{OR}$ morphism, we have $\imath^{B''}_{B'}(\imath^{B'}_{B}(R_O)) \leq_{B''} \imath^{B''}_{B'}(R'_O) \leq_{B''} R''_O$. By transitivity of $\leq_{B''}$ and the identity $\imath^{B''}_{B'} \circ \imath^{B'}_{B} = \imath^{B''}_{B}$, we find that $g \circ f$ is indeed an increasing information $\mathcal{OR}$ morphism. □

Using this proposition, the observation made immediately after Definition 11 can now be expressed as follows: $\mathrm{Hom}_{\mathbb{D}^{Inc}_{\mathcal{OR}\times\mathcal{IR}}}(D, D') = \mathrm{Hom}_{\mathbb{D}^{Inc}_{\mathcal{OR}}}(D, D') \cap \mathrm{Hom}_{\mathbb{D}^{Inc}_{\mathcal{IR}}}(D, D')$ for any two Aristotelian diagrams D and D'. Again, we can look at initial and terminal objects. For initial objects, the situation is the same as before: they are given by all diagrams of the form $(\emptyset, B)$. However, in contrast to all four categories from the previous subsections, the categories with increasing infomorphisms *do* have terminal objects, as Example 8 shows.

Example 8. Let us consider the diagram D_* from Example 1. In this case, we have $BI(*, *)$ and $CD(*, *)$. Let $(\mathcal{F}, B)$ be any Aristotelian diagram. Then there always exists exactly one increasing information $\mathcal{OR}\times\mathcal{IR}$ morphism

$$!_{(\mathcal{F},B)} : (\mathcal{F}, B) \to D_* : x \mapsto *.$$

Let us show existence — uniqueness is trivial. If $\mathcal{F} = \emptyset$, we are immediately done. If $\mathcal{F}$ is non-empty, we can consider any $x, y \in \mathcal{F}$. Let R_O be the opposition relation such that $R_O(x, y)$ and let R_I be the implication relation such that $R_I(x, y)$. Since $CD(*, *)$ and $BI(*, *)$, we automatically find that $\imath^{B_*}_{B}(R_O) \leq CD$ and $\imath^{B_*}_{B}(R_I) \leq BI$. This means that $!_{(\mathcal{F},B)}$ is well-defined as an increasing information $\mathcal{OR}\times\mathcal{IR}$ morphism. What this example shows in categorical terms is that D_* is the terminal object in the category $\mathbb{D}^{Inc}_{\mathcal{OR}\times\mathcal{IR}}$. The same proof also works for $\mathbb{D}^{Inc}_{\mathcal{OR}}$. The category $\mathbb{D}^{Inc}_{\mathcal{IR}}$ also has D_* as a terminal object, but it is not the only one. In fact, a similar proof shows that any diagram $(\mathcal{F}, B)$ with $|\mathcal{F}| = 1$ is a terminal object in $\mathbb{D}^{Inc}_{\mathcal{IR}}$.

In the same way as for Aristotelian morphisms, we can show that the subcategory of σ-diagrams with increasing information $\mathcal{OR}\times\mathcal{IR}$ morphisms (denoted by $\mathbb{D}^{Inc,\sigma}_{\mathcal{OR}\times\mathcal{IR}}$)[17] is a reflective subcategory of $\mathbb{D}^{Inc}_{\mathcal{OR}\times\mathcal{IR}}$. Again, the morphisms f and g from Example 6 show that this will not work in case we restrict ourselves to either the $\mathcal{OR}$ or the $\mathcal{IR}$ side of the story.

Proposition 10. *Let $f : D \to D'$ be an increasing information $\mathcal{OR}\times\mathcal{IR}$ morphism. Let Q be a set containing at least one element from each PCD in D. Then f is uniquely determined by its image on Q. Equivalently, defining an increasing information $\mathcal{OR}\times\mathcal{IR}$ morphism amounts to defining it on a set Q that contains one element from each of its PCDs.*

Proof. Let f_Q be a function $Q \to \mathcal{F}'$ that is a restriction of f. We can then extend f_Q to the function $f_\neg$ just like in Proposition 2, namely:

$$f_\neg : \mathcal{F} \to \mathcal{F}' : x \mapsto f_\neg(x) := \begin{cases} f_Q(x), & \text{if } x \in Q \\ \neg f_Q(\neg x), & \text{if } x \in \mathcal{F} - Q. \end{cases}$$

The function $f_\neg$ is well-defined since both D and D' are σ-diagrams, and thus closed under complementation. We now need to show that $f_\neg$ really is an increasing information $\mathcal{OR}\times\mathcal{IR}$ morphism. The same case distinction as in Proposition 2 is required.

1. $x, y \in Q$. In this case we are trivially done, since f_Q is a restriction of an increasing information $\mathcal{OR}\times\mathcal{IR}$ morphism.
2. $x \in Q$ and $y \notin Q$. Consider items 5–8 from Lemma 1. Depending on which relations x and y are in, apply the correct item from this lemma. Then apply f_Q, which does not necessarily preserve opposition and implication the relations between x and $\neg y$, but allows for some (at least equally informative) options. However, considering all of these options and applying the correct item from Lemma 1 again shows that $f_\neg$ is an increasing information $\mathcal{OR}\times\mathcal{IR}$ morphism as well. As an example, suppose that $C(x, y)$. Then $LI(x, \neg y)$ by item 6a of Lemma 1. Since f_Q is a restriction of f, which is an increasing information $\mathcal{OR}\times\mathcal{IR}$ morphism, it follows that $LI(f_Q(x), f_Q(\neg y))$ or $BI(f_Q(x), f_Q(\neg y))$. By items 5b and 6b of Lemma 1, we then have that $C(f_\neg(x), f_\neg(y))$ or $CD(f_\neg(x), f_\neg(y))$. Notice that indeed $C \leq_{B'} C, CD$.
3. $x \notin Q$ and $y \in Q$. Analogous to the previous case, but now using items 9–12 from Lemma 1.
4. $x, y \notin Q$. Also analogous, but now using items 1–4 from Lemma 1.

From the above, one sees that $f_\neg$ is not just well-defined: it is (by construction) the unique extension of f_Q to D. Indeed, since CD is preserved, for all $x \in Q$,

[17] Recall that if we restrict ourselves to σ-diagrams, every $\mathcal{OR}$ morphism is an $\mathcal{IR}$ morphism as well, and thus also an $\mathcal{OR}\times\mathcal{IR}$ morphism (cf. Lemma 2). This result straightforwardly generalizes to the increasing information morphisms studied here: if $f : D \to D'$ is an increasing information $\mathcal{OR}$ morphism and D is a σ-diagram, then f is an increasing information $\mathcal{IR}$ morphism as well, and thus also an increasing information $\mathcal{OR}\times\mathcal{IR}$ morphism. Consequently, when turning σ-diagrams into a category, it does not matter whether we equip them with increasing $\mathcal{OR}$ morphisms or with increasing $\mathcal{OR}\times\mathcal{IR}$ morphisms, since the resulting categories are identical to each other.

we must have that $\neg x$ gets mapped to $\neg f_Q(x)$. Therefore, f and $f_\neg$ are equal and the statement is proven. $\square$

Given an increasing information $\mathcal{OR} \times \mathcal{IR}$ morphism $f : (\mathcal{F}, B) \to (\mathcal{F}', B')$, Proposition 10 provides us with a unique extension to a morphism between the σ-diagrams $(Cl^B_\neg(\mathcal{F}), B)$ and $(Cl^{B'}_\neg(\mathcal{F}'), B')$. We again call this extension $f_\neg$.

Proposition 11. *We have a functor* $Cl_\neg$ *from* $\mathbb{D}^{Inc}_{\mathcal{OR}\times\mathcal{IR}}$ *to* $\mathbb{D}^{Inc,\sigma}_{\mathcal{OR}\times\mathcal{IR}}$, *which is defined on objects as*

$$Cl_\neg(\mathcal{F}, B) := (Cl^B_\neg(\mathcal{F}), B),$$

and on morphisms as

$$Cl_\neg(f) := f_\neg.$$

Proof. Completely analogous to the proof of Proposition 3. $\square$

Theorem 3. *The inclusion functor* $U : \mathbb{D}^{Inc,\sigma}_{\mathcal{OR}\times\mathcal{IR}} \to \mathbb{D}^{Inc}_{\mathcal{OR}\times\mathcal{IR}}$ *has a left adjoint, which is given by the negation closure functor,* $Cl_\neg : \mathbb{D}^{Inc}_{\mathcal{OR}\times\mathcal{IR}} \to \mathbb{D}^{Inc,\sigma}_{\mathcal{OR}\times\mathcal{IR}}$. *Put succinctly:*

$$Cl_\neg \dashv U.$$

In other words, $\mathbb{D}^{Inc,\sigma}_{\mathcal{OR}\times\mathcal{IR}}$ *is a reflective subcategory of* $\mathbb{D}^{Inc}_{\mathcal{OR}\times\mathcal{IR}}$.

Proof. Completely analogous to the proof of Theorem 1. $\square$

3.4. Decreasing Infomorphisms

It is now time to consider one final kind of morphisms, which are in a sense dual to the increasing infomorphisms. Instead of moving to relations that are *at least* as informative, we now move to relations that are *at most* as informative. Here, too, we have three different kinds of decreasing infomorphisms, corresponding to the three informativity orderings from Definitions 5 and 6.

Definition 12 (*Decreasing information* $\mathcal{OR}$, $\mathcal{IR}$ *and* $\mathcal{OR} \times \mathcal{IR}$ *morphism*). Let $D = (\mathcal{F}, B)$ and $D' = (\mathcal{F}', B')$ be Aristotelian diagrams. Let $f : \mathcal{F} \to \mathcal{F}'$ be a function between fragments. We say that f is a decreasing information $\mathcal{OR}$ resp. $\mathcal{IR}$ resp. $\mathcal{OR} \times \mathcal{IR}$ morphism from D to D' iff for all $R_B \in \mathcal{OR}_B$ resp. $\mathcal{IR}_B$ resp. $\mathcal{OR}_B \sqcup \mathcal{IR}_B$, and all $x, y \in \mathcal{F}$, there exists some $R'_{B'} \in \mathcal{OR}_{B'}$ resp. $\mathcal{IR}_{B'}$ resp. $\mathcal{OR}_{B'} \sqcup \mathcal{IR}_{B'}$, such that

$$R_B(x, y) \Rightarrow R'_{B'}(f(x), f(y)),$$

and $\imath^{B'}_B(R_B) \geq_{B'} R'_{B'}$. These three kinds of morphisms will also be referred to as decreasing infomorphisms.

Observe that every decreasing information $\mathcal{OR} \times \mathcal{IR}$ morphism is by definition also a decreasing information $\mathcal{OR}$ morphism and a decreasing information $\mathcal{IR}$ morphism. Conversely, every function that is both a decreasing information $\mathcal{OR}$ morphism and a decreasing information $\mathcal{IR}$ morphism is also a decreasing information $\mathcal{OR} \times \mathcal{IR}$ morphism. The duality between increasing and decreasing infomorphisms can be immediately illustrated with the following examples. They concern bit additions, which can be viewed as dual to bit deletions, such as in Example 7.

Example 9. Given $\mathcal{F}_2 := \{10, 01\}$ and $\mathcal{F}_3 := \{100, 010\}$, the map $a_2 : \mathcal{F}_2 \to \mathcal{F}_3 \colon xy \mapsto xy0$, which adds a 0-bit at the end of a bitstring, is a decreasing information $\mathcal{OR}\times\mathcal{IR}$ (and thus also $\mathcal{OR}$ and $\mathcal{IR}$) morphism from $(\mathcal{F}_2, \{0,1\}^2)$ to $(\mathcal{F}_3, \{0,1\}^3)$. However, it is not an Aristotelian, $\mathcal{OR}$, or $\mathcal{OR}\times\mathcal{IR}$ morphism. For example, we have $CD_{\{0,1\}^2}(10, 01)$, but $C_{\{0,1\}^3}(100, 010)$; note that C is less informative than, but not identical to, CD.

Example 10. Given $\mathcal{F}_3 := \{100, 110\}$ and $\mathcal{F}_4 := \{1001, 1100\}$, the map $a_3 : \mathcal{F}_3 \to \mathcal{F}_4 \colon xyz \mapsto xyz\overline{y}$, which adds the negation of a bitstring's second bit at the end of that bitstring, is a decreasing information $\mathcal{OR}\times\mathcal{IR}$ morphism (and thus also $\mathcal{OR}$ and $\mathcal{IR}$) from $(\mathcal{F}_3, \{0,1\}^3)$ to $(\mathcal{F}_4, \{0,1\}^4)$. However, it is not an Aristotelian, $\mathcal{IR}$ or $\mathcal{OR}\times\mathcal{IR}$ morphism. For example, we have $LI_{\{0,1\}^3}(100, 110)$, but $NI_{\{0,1\}^4}(1001, 1100)$; note that NI is less informative than, but not identical to, LI.

Notice the similarities between Definitions 12 and 11: we have merely replaced $\leq_{B'}$ with $\geq_{B'}$. Consequently, we can observe that Definition 10 can, once again, be reformulated along the lines of Definition 12, now with the requirement that $\imath_B^{B'}(R_B) \geq_{B'} R'_{B'}$ strengthened to $\imath_B^{B'}(R_B) = R'_{B'}$. From the reflexivity of $\leq_{B'}$, we thus have that every $\mathcal{OR}$ resp. $\mathcal{IR}$ resp. $\mathcal{OR}\times\mathcal{IR}$ morphism is a decreasing information $\mathcal{OR}$ resp. $\mathcal{IR}$ resp. $\mathcal{OR}\times\mathcal{IR}$ morphism.

Just as with increasing infomorphisms, we have an alternative characterization of decreasing infomorphisms that relies on the opposition and implication relations being jointly exhaustive and mutually exclusive.

Proposition 12. *Let $D = (\mathcal{F}, B)$ and $D' = (\mathcal{F}', B')$ be Aristotelian diagrams. Let $f : \mathcal{F} \to \mathcal{F}'$ be a function between fragments. Then f is a decreasing information $\mathcal{OR}$ resp. $\mathcal{IR}$ resp. $\mathcal{OR}\times\mathcal{IR}$ morphism from D to D' iff for all $x, y \in \mathcal{F}$, and for the unique $R_O, R'_O \in \mathcal{OR}$ and $R_I, R'_I \in \mathcal{IR}$ such that*

$$R_O(x, y), R_I(x, y), R'_O(f(x), f(y)) \text{ and } R'_I(f(x), f(y)),$$

we have that

$$\imath_B^{B'}(R_O) \geq_{B'} R'_O \text{ resp. } \imath_B^{B'}(R_I) \geq_{B'} R'_I \text{ resp. } (\imath_B^{B'}(R_O), \imath_B^{B'}(R_I)) \geq_{B'} (R'_O, R'_I).$$

Here, $\leq_{B'}$ is the relevant informativity ordering from Definition 5 or 6.

Proof. Uniqueness and existence of the above relations is satisfied since $\mathcal{OR}_B$ and $\mathcal{IR}_B$ are jointly exhaustive and mutually exclusive. Now, the statement follows by the pointwise definition of the combined informativity ordering. □

Proposition 13. *The Aristotelian diagrams, equipped with either kind of decreasing infomorphisms, give rise to a category. We will denote these categories by $\mathbb{D}_{\mathcal{OR}}^{Dec}$, $\mathbb{D}_{\mathcal{IR}}^{Dec}$ and $\mathbb{D}_{\mathcal{OR}\times\mathcal{IR}}^{Dec}$.*

Proof. Entirely analogous to the proof of Proposition 9. □

Using this proposition, the observation made immediately after Definition 11 can now be expressed as follows: $\mathrm{Hom}_{\mathbb{D}^{Dec}_{\mathcal{OR}\times\mathcal{IR}}}(D, D') = \mathrm{Hom}_{\mathbb{D}^{Dec}_{\mathcal{OR}}}(D, D') \cap \mathrm{Hom}_{\mathbb{D}^{Dec}_{\mathcal{IR}}}(D, D')$ for any two Aristotelian diagrams D and D'. Beware that $\mathbb{D}^{Dec}_{\mathcal{OR}}$, $\mathbb{D}^{Dec}_{\mathcal{IR}}$ and $\mathbb{D}^{Dec}_{\mathcal{OR}\times\mathcal{IR}}$ are *not* simply the opposite categories of $\mathbb{D}^{Inc}_{\mathcal{OR}}$, $\mathbb{D}^{Inc}_{\mathcal{IR}}$ and $\mathbb{D}^{Inc}_{\mathcal{OR}\times\mathcal{IR}}$, respectively. To see this in the case of $\mathcal{OR}\times\mathcal{IR}$ (the other two cases are analogous), notice that from D_* to any diagram $(\mathcal{F}, B)$ there exist exactly $|\mathcal{F}|$ decreasing information $\mathcal{OR}\times\mathcal{IR}$ morphisms, while there exists only one such morphism in the category $(\mathbb{D}^{Inc}_{\mathcal{OR}\times\mathcal{IR}})^{op}$. Thus, even though the formulation of Definitions 11 and 12 is dual in terms of the informativity ordering, it is not dual in terms of category theory. This becomes even more apparent when we try to define initial and terminal objects in $\mathbb{D}^{Dec}_{\mathcal{OR}}$, $\mathbb{D}^{Dec}_{\mathcal{IR}}$ and $\mathbb{D}^{Dec}_{\mathcal{OR}\times\mathcal{IR}}$. The initial objects are the same as for all the previous categories (see Example 5). However, in stark contrast to their increasing counterparts, terminal objects do not exist in $\mathbb{D}^{Dec}_{\mathcal{IR}}$ and $\mathbb{D}^{Dec}_{\mathcal{OR}\times\mathcal{IR}}$, as the following example shows.

Example 11. Suppose that $(\mathcal{F}, B)$ is a terminal object in $\mathbb{D}^{Dec}_{\mathcal{IR}}$. Then there exists a unique decreasing information $\mathcal{IR}$ morphism f from $(\{1100, 1010\}, \{0,1\}^4)$ to $(\mathcal{F}, B)$. Since $NI(1100, 1010)$ and NI is the least informative relation of $\mathcal{IR}$, we have that $NI(f(1100), f(1010))$ as well. In particular, $f(1100)$ and $f(1010)$ should be distinct elements in $\mathcal{F}$. It is furthermore clear that $BI(f(1100), f(1100))$ and $BI(f(1010), f(1010))$ since the same holds for 1100 and 1010. Putting this information together, we can make the following picture:

$$\begin{tikzcd} f(1100) \arrow[r, no head, "NI"] \arrow[loop below, no head, "BI"] & f(1010). \arrow[loop below, no head, "BI"] \end{tikzcd}$$

However, the symmetry of this diagram clearly shows that we can define another decreasing information $\mathcal{IR}$ morphism g from $(\{1100, 1010\}, \{0,1\}^4)$ to $(\mathcal{F}, B)$ as follows:

$$g(1100) := f(1010),$$
$$g(1010) := f(1100).$$

This contradicts $(\mathcal{F}, B)$ being a terminal object. Therefore, $\mathbb{D}^{Dec}_{\mathcal{IR}}$ does not have any terminal objects. Adding opposition relations to the story for a decreasing information $\mathcal{OR}\times\mathcal{IR}$ morphism f alters the picture to

$$\begin{tikzcd} f(1100) \arrow[r, no head, "(NCD{,}NI)"] \arrow[loop below, no head, "(NCD{,}BI)"] & f(1010). \arrow[loop below, no head, "(NCD{,}BI)"] \end{tikzcd}$$

Therefore, a similar argument shows that $\mathbb{D}^{Dec}_{\mathcal{OR}\times\mathcal{IR}}$ does not have terminal objects. This line of reasoning does not work for $\mathbb{D}^{Dec}_{\mathcal{OR}}$. In fact, it is not hard to see that every diagram $(\mathcal{F}, B)$ such that $\mathcal{F}$ consists of exactly one element $x \notin \{0_B, 1_B\}$ is a terminal object in this category, since we have $NCD(x, x)$.

Again, for the case of $\mathcal{OR} \times \mathcal{IR}$, the category of σ-diagrams forms a reflective subcategory of $\mathbb{D}^{Dec}_{\mathcal{OR}\times\mathcal{IR}}$.[18] We proceed in the same way as before. Here, too, the morphisms f and g from Example 6 show that this will not work in case we restrict ourselves to either the $\mathcal{OR}$ or the $\mathcal{IR}$ side of the story.

Proposition 14. *Let $D = (\mathcal{F}, B)$ and $D' = (\mathcal{F}', B')$ be σ-diagrams, and let $f : D \to D'$ be a decreasing information $\mathcal{OR} \times \mathcal{IR}$ morphism. Let Q be a set containing at least one element from each PCD in D. Then f is uniquely determined by its image on Q. Equivalently, defining a decreasing information $\mathcal{OR} \times \mathcal{IR}$ morphism amounts to defining it on a set Q that contains at least one element from each of its PCDs.*

Proof. Entirely analogous to the proof of Proposition 10. □

Given a decreasing information $\mathcal{OR}\times\mathcal{IR}$ morphism $f : (\mathcal{F}, B) \to (\mathcal{F}', B')$, Proposition 14 provides us with a unique extension to a morphism between the σ-diagrams $(Cl^B_\neg(\mathcal{F}), B)$ and $(Cl^{B'}_\neg(\mathcal{F}'), B')$. We call this extension $f_\neg$ once again.

Proposition 15. *We have a functor $Cl_\neg$ from $\mathbb{D}^{Dec}_{\mathcal{OR}\times\mathcal{IR}}$ to $\mathbb{D}^{Dec,\sigma}_{\mathcal{OR}\times\mathcal{IR}}$, which is defined on objects as*

$$Cl_\neg(\mathcal{F}, B) := (Cl^B_\neg(\mathcal{F}), B),$$

and on morphisms as

$$Cl_\neg(f) := f_\neg.$$

Proof. Entirely analogous to the proof of Proposition 11. □

Theorem 4. *The inclusion functor $U : \mathbb{D}^{Dec,\sigma}_{\mathcal{OR}\times\mathcal{IR}} \to \mathbb{D}^{Dec}_{\mathcal{OR}\times\mathcal{IR}}$ has a left adjoint, which is given by the negation closure functor, $Cl_\neg : \mathbb{D}^{Dec}_{\mathcal{OR}\times\mathcal{IR}} \to \mathbb{D}^{Dec,\sigma}_{\mathcal{OR}\times\mathcal{IR}}$. Put succinctly:*

$$Cl_\neg \dashv U.$$

In other words, $\mathbb{D}^{Dec,\sigma}_{\mathcal{OR}\times\mathcal{IR}}$ is a reflective subcategory of $\mathbb{D}^{Dec}_{\mathcal{OR}\times\mathcal{IR}}$.

Proof. Entirely analogous to the proof of Theorem 1. □

4. Isomorphisms

In the previous section, we described ten different categories, and investigated their properties on a basic level of category theory. More concretely, we checked whether or not they have initial and/or terminal objects, and proved that some of them have the σ-diagrams as a reflective subcategory. The present section is devoted to the isomorphisms that arise in each of these categories. Remember that we are searching for a category whose isomorphisms are exactly the Aristotelian isomorphisms that have already been studied in logical geometry, such

[18] Again, decreasing information $\mathcal{OR}$ and $\mathcal{OR} \times \mathcal{IR}$ morphisms are exactly the same when restricted to σ-diagrams.

that this category generalizes previous work in this field. For our purposes, this is actually even the most important requirement we will impose on our various categories: satisfying this criterion can be considered mandatory for a category to be a viable candidate for further research. Therefore, the present section can be regarded as a kind of 'filter' where all ten categories must pass through. After this section, we will continue solely with those categories that did not get 'filtered out' because they give rise to a wrong notion of isomorphism.

First of all, it should be clear that if $f : D \to D'$ is an isomorphism in any of the above categories, then f is a bijection. Indeed, if $g : D' \to D$ is such that $g \circ f = Id_D$ and $f \circ g = Id_{D'}$, then f and g, viewed as set functions on the underlying fragments, are isomorphisms in the category of sets, hence bijections. Of course, the converse is not true in any of the above categories: not every bijection on fragments is an isomorphism.

Let us first focus on the category $\mathbb{D}_{\mathcal{AR}}$. Since morphisms in this category are called 'Aristotelian morphisms', the isomorphisms in this category are naturally called 'Aristotelian isomorphisms'. This does not violate previous work in logical geometry, as the following theorem shows.

Theorem 5. *The categorical notion of Aristotelian isomorphism corresponds with the original notion of Aristotelian isomorphism that was defined in Definition 4 of [12]. More precisely: $f : D \to D'$ is an Aristotelian isomorphism iff f is bijective and for all $R \in \mathcal{AR}$ and for all $x, y \in \mathcal{F}$, it holds that $R(x, y) \Leftrightarrow \imath_B^{B'}(R)(f(x), f(y))$.*

Proof. $\Rightarrow$ Suppose that $f : D \to D'$ is an Aristotelian isomorphism. This means that f is an Aristotelian morphism and that there exists an Aristotelian morphism $g : D' \to D$ such that $g \circ f = Id_D$ and $f \circ g = Id_{D'}$. As already mentioned above, f is clearly a bijection as a set function on fragments. Now let $R \in \mathcal{AR}$ and $x, y \in \mathcal{F}$ arbitrary; we show that $R(x, y) \Leftrightarrow \imath_B^{B'}(R)(f(x), f(y))$. Since f is an Aristotelian morphism, it follows that $R(x, y) \Rightarrow \imath_B^{B'}(R)(f(x), f(y))$. Since g is an Aristotelian morphism and $\imath_{B'}^{B}(\imath_B^{B'}(R)) = R$, it follows that $\imath_B^{B'}(R)(f(x), f(y)) \Rightarrow \imath_{B'}^{B}(\imath_B^{B'}(R))(gf(x), gf(y)) \Leftrightarrow R(gf(x), gf(y))$. Given that $gf = Id_D$, we obtain that $R(f(x), f(y)) \Rightarrow R(x, y)$.

$\Leftarrow$ Suppose that f is bijective and for all $R \in \mathcal{AR}$ and for all $x, y \in \mathcal{F}$, it holds that $R(x, y)$ iff $\imath_B^{B'}(R)(f(x), f(y))$. We have to show that f is an Aristotelian morphism and that there exists an Aristotelian morphism $g : D' \to D$ such that $g \circ f = Id_D$ and $f \circ g = Id_{D'}$. For all $x, y \in \mathcal{F}$, we have $R(x, y) \Rightarrow \imath_B^{B'}(R)(f(x), f(y))$, which precisely says that f is an Aristotelian morphism. The function $f : \mathcal{F} \to \mathcal{F}'$ is bijective, so $f^{-1} : \mathcal{F}' \to \mathcal{F}$ exists and is well-defined. By definition of f^{-1}, we have $f^{-1} \circ f = Id_D$ and $f \circ f^{-1} = Id_{D'}$. We now show that f^{-1} is an Aristotelian morphism. Let $a, b \in \mathcal{F}'$ arbitrary and suppose $R(a, b)$. Since $f \circ f^{-1} = Id_{D'}$, it follows that $R(ff^{-1}(a), ff^{-1}(b))$. We have $\imath_{B'}^{B}(R)(x, y) \Leftrightarrow R(f(x), f(y))$ for all $x, y \in \mathcal{F}$, and thus in particular for $f^{-1}(a)$ and $f^{-1}(b)$. Hence it follows that $\imath_{B'}^{B}(R)(f^{-1}(a), f^{-1}(b))$, as desired. $\square$

Next, we focus on the categories $\mathbb{D}_{\mathcal{OR}\times\mathcal{IR}}^{Inc}$ and $\mathbb{D}_{\mathcal{OR}\times\mathcal{IR}}^{Dec}$. After that, we look at the isomorphisms in the category $\mathbb{D}_{\mathcal{OR}\times\mathcal{IR}}$. Instead of proving analogues of Theorem 5, we investigate how isomorphisms in these categories are related to each other and to the isomorphisms in $\mathbb{D}_{\mathcal{AR}}$. To be able to do this, we first need the following two lemmas on the level of morphisms.

Lemma 3. *Suppose that f is a bijective increasing resp. decreasing information $\mathcal{OR} \times \mathcal{IR}$ morphism from D to D'. Then f^{-1} is a bijective decreasing resp. increasing information $\mathcal{OR} \times \mathcal{IR}$ morphism from D' to D.*[19]

Proof. This follows immediately from the dual nature of Definitions 11 and 12 (recall that this duality concerns the underlying informativity ordering, rather than the resulting categories). □

Lemma 4. *Every Aristotelian morphism is an increasing information $\mathcal{OR}$, $\mathcal{IR}$, and $\mathcal{OR} \times \mathcal{IR}$ morphism.*

Proof. Let $f : D \to D'$ be an Aristotelian morphism. Let $x, y \in \mathcal{F}$ be arbitrary. Let R_O, R'_O be the unique opposition relations such that $R_O(x, y)$ and $R'_O(f(x), f(y))$. If $R_O \in \mathcal{AR}$, it follows that $\imath_B^{B'}(R_O) = R'_O$ since f is an Aristotelian morphism. Otherwise, $R_O = NCD$, which is the least informative element of $\mathcal{OR}$. In both cases, it is thus true that $\imath_B^{B'}(R_O) \leq_{B'} R'_O$. Therefore, f is an increasing information $\mathcal{OR}$ morphism. Since Aristotelian morphisms also preserve *LI*, *RI* and *BI* relations, the same proof also holds for the implication relations, which shows that f is an increasing information $\mathcal{IR}$ morphism as well. Combining these two results, we find that f is an increasing information $\mathcal{OR} \times \mathcal{IR}$ morphism. □

Theorem 6. *Every increasing information $\mathcal{OR} \times \mathcal{IR}$ isomorphism is a decreasing information $\mathcal{OR} \times \mathcal{IR}$ isomorphism, and vice versa.*

Proof. Suppose that $f : D \to D'$ is an increasing information $\mathcal{OR} \times \mathcal{IR}$ isomorphism. Then there exists an increasing information $\mathcal{OR} \times \mathcal{IR}$ isomorphism $g : D' \to D$ such that $g \circ f = Id_D$ and $f \circ g = Id_{D'}$. Since f, g are bijections, $g = f^{-1}$ and $f = g^{-1}$, Lemma 3 tells us that g, f are also decreasing information $\mathcal{OR} \times \mathcal{IR}$ morphisms. Therefore, f is a decreasing information $\mathcal{OR} \times \mathcal{IR}$ isomorphism. The same reasoning holds for the converse. □

Theorem 7. *Every Aristotelian isomorphism is an increasing information $\mathcal{OR} \times \mathcal{IR}$ isomorphism.*

Proof. Suppose that $f : D \to D'$ is Aristotelian isomorphism. Then, f is an Aristotelian morphism, and there exists an Aristotelian morphism $g : D' \to D$ such that $g \circ f = Id_D$ and $f \circ g = Id_{D'}$. By Lemma 4, f and g are increasing information $\mathcal{OR} \times \mathcal{IR}$ morphisms. Hence, f is an increasing information $\mathcal{OR} \times \mathcal{IR}$ isomorphism. □

[19] Together with the reflexivity and antisymmetry of the informativity ordering, this lemma immediately proves the $\mathcal{OR} \times \mathcal{IR}$ part of Proposition 4 as a corollary. Indeed, a bijective $\mathcal{OR} \times \mathcal{IR}$ morphism is both increasing and decreasing, by the reflexivity of $\leq$. The current lemma says that its inverse is also both increasing and decreasing, which by the antisymmetry of $\leq$ implies that this inverse is also an $\mathcal{OR} \times \mathcal{IR}$ morphism.

Theorem 8. *Every increasing information $\mathcal{OR} \times \mathcal{IR}$ isomorphism is an Aristotelian isomorphism.*

Proof. Suppose that $f : D \to D'$ is an increasing information $\mathcal{OR} \times \mathcal{IR}$ isomorphism. Then, f is an increasing information $\mathcal{OR} \times \mathcal{IR}$ morphism, and there exists an increasing information $\mathcal{OR} \times \mathcal{IR}$ morphism $g : D' \to D$ such that $g \circ f = Id_D$ and $f \circ g = Id_{D'}$. We will now show that f is an Aristotelian isomorphism by using Theorem 5. As usual, it is clear that f is bijective. Now, let $x, y \in \mathcal{F}$ be arbitrary, and let $R_O, R'_O \in \mathcal{OR}$ be the unique opposition relations such that $R_O(x, y)$ and $R'_O(f(x), f(y))$. Since f is an increasing information $\mathcal{OR}\times\mathcal{IR}$ morphism, it follows that $\imath_B^{B'}(R_O) \leq_{B'} R'_O$. Let R''_O be the unique opposition relation such that $R''_O(gf(x), gf(y))$. Recall that $g\circ f = Id_{\mathcal{F}}$, and hence, $R''_O = R_O$. Since $R'_O(f(x), f(y))$ and g is an increasing information $\mathcal{OR}\times\mathcal{IR}$ morphism, it follows that $\imath_{B'}^{B}(R'_O) \leq_B R''_O = R_O$, which is equivalent to saying that $\imath_B^{B'}(R_O) \geq_{B'} R'_O$. By the antisymmetry of $\leq_{B'}$, we have that $\imath_B^{B'}(R_O) = R'_O$. The same proof also holds for the implication relations. Since $\mathcal{AR} \subset \mathcal{OR} \cup \mathcal{IR}$, Theorem 5 allows us to conclude that f is an Aristotelian isomorphism. $\square$

Theorem 9. *Every Aristotelian isomorphism is a decreasing information $\mathcal{OR}\times\mathcal{IR}$ isomorphism, and vice versa.*

Proof. This follows directly from Theorems 6, 7, and 8. $\square$

Theorem 10. *Every $\mathcal{OR} \times \mathcal{IR}$ isomorphism is an Aristotelian/increasing information $\mathcal{OR}\times\mathcal{IR}$/decreasing information $\mathcal{OR}\times\mathcal{IR}$ isomorphism, and vice versa.*

Proof. In light of the previous theorems, it suffices to prove the statement for the increasing information $\mathcal{OR}\times\mathcal{IR}$ isomorphisms. It is clear that every $\mathcal{OR}\times\mathcal{IR}$ isomorphism is an increasing information $\mathcal{OR}\times\mathcal{IR}$ isomorphism, simply by definition. For the other direction, suppose that f is an increasing information $\mathcal{OR}\times\mathcal{IR}$ isomorphism. Then, by Theorem 6, f is also a decreasing information $\mathcal{OR} \times \mathcal{IR}$ isomorphism. Since f is now both increasing and decreasing, it is by definition (and the antisymmetry of the informativity ordering) also an $\mathcal{OR} \times \mathcal{IR}$ isomorphism. $\square$

Together, Theorems 5-10 imply that the isomorphisms in the categories $\mathbb{D}_{\mathcal{AR}}$, $\mathbb{D}_{\mathcal{OR}\times\mathcal{IR}}$, $\mathbb{D}^{Inc}_{\mathcal{OR}\times\mathcal{IR}}$ and $\mathbb{D}^{Dec}_{\mathcal{OR}\times\mathcal{IR}}$ are all equivalent to the Aristotelian isomorphisms that were already studied in logical geometry. These four categories thus satisfy the mandatory requirement, as we were hoping for. As for the six other candidates, the following two examples show that their isomorphisms do *not* satisfy this requirement.

Example 12. Let f be the $\mathcal{OR}$ morphism from Example 6. Since f is its own inverse, f is an isomorphism in $\mathbb{D}_{\mathcal{OR}}$, $\mathbb{D}^{Inc}_{\mathcal{OR}}$ and $\mathbb{D}^{Dec}_{\mathcal{OR}}$. On the other hand, remember that we have $LI(100, 110)$ and $RI(110, 100)$, or in other words $RI(f(100), f(110))$. Thus, f is not an Aristotelian (iso)morphism.

Table 2. Comparison of the different kinds of morphisms

	Terminal objects	Initial objects	σ-diagrams as reflective subcategory	Aristotelian isomorphisms
$\mathbb{D}_{\mathcal{AR}}$		✓	✓	✓
$\mathbb{D}_{\mathcal{OR}}$		✓		
$\mathbb{D}_{\mathcal{IR}}$		✓		
$\mathbb{D}_{\mathcal{OR}\times\mathcal{IR}}$		✓	✓	✓
$\mathbb{D}^{Inc}_{\mathcal{OR}}$	✓	✓		
$\mathbb{D}^{Inc}_{\mathcal{IR}}$	✓	✓		
$\mathbb{D}^{Inc}_{\mathcal{OR}\times\mathcal{IR}}$	✓	✓	✓	✓
$\mathbb{D}^{Dec}_{\mathcal{OR}}$	✓	✓		
$\mathbb{D}^{Dec}_{\mathcal{IR}}$		✓		
$\mathbb{D}^{Dec}_{\mathcal{OR}\times\mathcal{IR}}$		✓	✓	✓

Example 13. Let g be the $\mathcal{IR}$ morphism from Example 6. It is clear that both g and g^{-1} preserve all implication relations. For example, $NI(100, 001)$ and $NI(110, 011)$, or in other words $NI(f(100), f(001))$. Therefore, g is an isomorphism in $\mathbb{D}_{\mathcal{IR}}$, $\mathbb{D}^{Inc}_{\mathcal{IR}}$ and $\mathbb{D}^{Dec}_{\mathcal{IR}}$. On the other hand, remember that we have $C(100, 001)$ and $SC(110, 011)$, or in other words $SC(f(100), f(001))$. Thus, g is not an Aristotelian (iso)morphism.

Our results from this section and the previous one are summarized in Table 2. From this table, of which the final column is the most important one, it is clear that we can henceforth restrict our attention to the categories involving $\mathcal{OR} \times \mathcal{IR}$ and $\mathcal{AR}$. These four categories are further examined in the following section.

5. Relations Between the Different Types of Morphisms

The considerations of the previous section on isomorphisms allow us to narrow our focus to just the Aristotelian morphisms and the (increasing/decreasing information) $\mathcal{OR} \times \mathcal{IR}$ morphisms. We start by investigating how these four notions relate to each other. Then, we will use this information, together with the category-theoretical results from Sect. 3, to evaluate the usefulness of the four remaining categories. It is easiest to start by comparing the $\mathcal{OR} \times \mathcal{IR}$ morphisms to the other ones.

Theorem 11. *Every $\mathcal{OR} \times \mathcal{IR}$ morphism is an Aristotelian, an increasing information $\mathcal{OR} \times \mathcal{IR}$ and a decreasing information $\mathcal{OR} \times \mathcal{IR}$ morphism. None of the three converse statements hold.*

Proof. The first part of the theorem is trivial by the definitions of all the morphisms involved, the fact that $\mathcal{AR} \subset \mathcal{OR} \cup \mathcal{IR}$ and the reflexivity of the informativity ordering. For the converse statements, concrete counterexamples can be found in Examples 3, 7 and 9, respectively. $\square$

Theorem 12. *If a map f is both an increasing and a decreasing information $\mathcal{OR} \times \mathcal{IR}$ morphism, then f is an $\mathcal{OR} \times \mathcal{IR}$ morphism.*

Proof. Suppose that f is both an increasing and a decreasing information $\mathcal{OR} \times \mathcal{IR}$ morphism. Take any $x, y \in \mathcal{F}$. Let R_O, R'_O, R_I and R'_I be the unique opposition and implication relations such that $R_O(x, y)$, $R'_O(f(x), f(y))$, $R_I(x, y)$ and $R'_I(f(x), f(y))$. Since f is an increasing infomorphism, we have that $\imath_B^{B'}(R_O) \leq_{B'} R'_O$ and $\imath_B^{B'}(R_I) \leq_{B'} R'_I$. But since f is also a decreasing infomorphism, we also have that $R'_O \leq_{B'} \imath_B^{B'}(R_O)$ and $R'_I \leq_{B'} \imath_B^{B'}(R_I)$. By the antisymmetry of $\leq_{B'}$, we have that $R'_O = \imath_B^{B'}(R_O)$ and $R'_I = \imath_B^{B'}(R_I)$. This means that f is an $\mathcal{OR} \times \mathcal{IR}$ morphism. □

Now, let us compare each of the three remaining kinds of morphisms two by two.

Theorem 13. *Every Aristotelian morphism is an increasing information $\mathcal{OR} \times \mathcal{IR}$ morphism, but not the other way around.*

Proof. For the positive statement, see Lemma 4. A concrete counterexample for the converse statement is given by the bit deletion map d_3 of Example 7, as is already explained there. □

Theorem 14. *Not every increasing information $\mathcal{OR} \times \mathcal{IR}$ morphism is a decreasing information $\mathcal{OR} \times \mathcal{IR}$ morphism, and also not the other way around.*

Proof. Suppose, toward a contradiction, that every increasing information $\mathcal{OR} \times \mathcal{IR}$ morphism is a decreasing information $\mathcal{OR} \times \mathcal{IR}$ morphism. Then by Theorem 12, every increasing information $\mathcal{OR} \times \mathcal{IR}$ morphism is an $\mathcal{OR} \times \mathcal{IR}$ morphism, which contradicts Theorem 11. For the other direction, an analogous argument holds. □

Theorem 15. *Not every Aristotelian morphism is a decreasing information $\mathcal{OR} \times \mathcal{IR}$ morphism, and also not the other way around.*

Proof. A concrete example of an Aristotelian morphism that is not a decreasing information $\mathcal{OR} \times \mathcal{IR}$ morphism is given by the map φ from Example 3. As explained there, this map is an Aristotelian morphism. However, we have $NI_{\mathsf{FOL}}(\forall x(Sx \rightarrow Px), \exists x(Sx \wedge Px))$ and also $LI_{\mathsf{SYL}}(\forall x(Sx \rightarrow Px), \exists x(Sx \wedge Px))$, or in other words, $LI_{\mathsf{SYL}}(\varphi(\forall x(Sx \rightarrow Px)), \varphi(\exists x(Sx \wedge Px)))$. Since $\imath_{\mathsf{FOL}}^{\mathsf{SYL}}(NI) \not\geq_{\mathsf{SYL}} LI$, this means that φ is not a decreasing information $\mathcal{OR} \times \mathcal{IR}$ morphism.

A concrete example of a decreasing information $\mathcal{OR} \times \mathcal{IR}$ morphism that is not an Aristotelian morphism is given by the bit addition map a_2 from Example 9, as was already explained there. □

Combining all these theorems allows us to make the Venn diagram in Fig. 7. A colored area means that there exist morphisms in this part of the Venn diagram, while a white area means that there do not exist such morphisms. Because of Theorem 11, the middle area is where the $\mathcal{OR} \times \mathcal{IR}$ morphisms are located. Together with Theorem 12, it also creates the white 'triangle' at

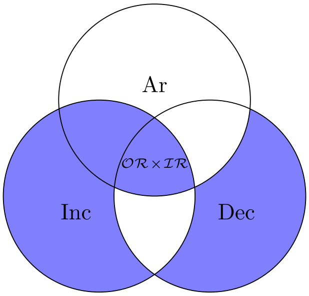

Figure 7. How the different kinds of morphisms relate to each other

the bottom. Theorem 13 accounts for the other two white areas and, together with the white triangle at the bottom, it also implies that the lower left area is colored. Similarly, Theorem 14 and the white triangle at the bottom together imply that the lower right area is colored. Finally, Theorem 15 and the white area at the top together imply that the final remaining area (between Ar and Inc) should be colored as well.

Now, what does such a Venn diagram tell us? For starters, it tells us that, even though these four kinds of morphisms give rise to the same notion of isomorphism, they have varying levels of restrictiveness. The most restrictive morphisms, which give us the least amount of examples, are the $\mathcal{OR}\times\mathcal{IR}$ morphism. They are more restrictive than the Aristotelian morphisms, which are in turn more restrictive than the increasing information $\mathcal{OR}\times\mathcal{IR}$ morphisms. Therefore, from this point of view, the category $\mathbb{D}^{Inc}_{\mathcal{OR}\times\mathcal{IR}}$ is more interesting to study than either $\mathbb{D}_{\mathcal{AR}}$ or $\mathbb{D}_{\mathcal{OR}\times\mathcal{IR}}$. Indeed, this category is just as powerful as these other categories on the level of isomorphisms, while simultaneously allowing for more maps (like bit deletions) to be studied. The category $\mathbb{D}^{Dec}_{\mathcal{OR}\times\mathcal{IR}}$ is a bit strange from this perspective. On the one hand, it seems to be defeated by the category $\mathbb{D}^{Inc}_{\mathcal{OR}\times\mathcal{IR}}$, since it has slightly fewer colored regions in the Venn diagram. More concretely, the increasing information $\mathcal{OR}\times\mathcal{IR}$ morphisms have the advantage of containing all the Aristotelian morphisms, whereas some of these are left out in the category $\mathbb{D}^{Dec}_{\mathcal{OR}\times\mathcal{IR}}$. However, there are certain maps (like bit additions) that have proven to be interesting in logical geometry, which are decreasing rather than increasing. This suggests that we cannot dismiss this category altogether.

Looking back at Table 2, we see that there is not much that we can use from there to differentiate between the four remaining categories. Indeed, all four of them have the σ-diagrams as a reflective subcategory and all of them have the same initial objects. The only categorical property that stands out is the existence of terminal objects: they are only present in the category $\mathbb{D}^{Inc}_{\mathcal{OR}\times\mathcal{IR}}$. Additionaly, this is a conservative extension of the category that was studied by Vignero [22] (using slightly different terminology), where it was proven that this category also has binary products and coproducts. All these considerations together point towards $\mathbb{D}^{Inc}_{\mathcal{OR}\times\mathcal{IR}}$ as the most promising candidate for a category-theoretic framework for logical geometry. However, as has become clear throughout the current section, we will have to continue paying attention to the decreasing information $\mathcal{OR} \times \mathcal{IR}$ morphisms as well.

6. Outlook and Conclusion

In this paper, we defined ten different categories that have Aristotelian diagrams as their objects, in the hope that one of them turns out to be a nice mathematical framework for studying these diagrams. These categories were analyzed and compared on two accounts. On the one hand, we checked which of them are capable of generalizing important previous work from logical geometry. More concretely, we found that four of these ten categories give rise to the notion of Aristotelian isomorphism. Also, we found that bit deletions and additions can be considered as morphisms in some of these categories. On the other hand, the ten categories were investigated from an elementary category-theoretical perspective. In particular, we considered initial and terminal objects, and we proved that in some cases, the σ-diagrams form a reflective subcategory.

Combining all these results, the category $\mathbb{D}^{Inc}_{\mathcal{OR}\times\mathcal{IR}}$, which has increasing information $\mathcal{OR} \times \mathcal{IR}$ morphisms as its arrows, comes out on top. Two main avenues for further research now present themselves. The first is to incorporate this framework in both past and future work in logical geometry, for instance by investigating the precise category-theoretical relation between Aristotelian diagrams and involution posets [6]. The second is to study this category on a deeper level of category theory, i.e., investigating other limits, colimits, adjunctions, etc.

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