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Finite Tree-Countermodels via Refutation Systems in Extensions of Positive Logic with Strong Negation

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Abstract. A sufficient condition for an extension of positive logic with strong negation to be characterized by a class of finite trees is given.

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1. Introduction

In this paper, extensions of positive logic with strong negation are studied. We give a sufficient condition (called the reduction property) for the completeness of such a logic with respect to the class of finite trees. The condition (implicitly) involves a certain refutation system employing Mints-style normal forms, and it generalizes some results in [5,6]. Our completeness proof is constructive. As an example, the method is applied to the connexive logic **C** (introduced in [7]), an important non-classical logic.

2. Preliminaries

Let FOR be the set of all formulas generated from the set $\mathsf{VAR} = \{p, q, p_1, p_2, \ldots\}$ of propositional variables by the connectives:

$\sim$ (strong negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication).

Greek capital letters $(\Phi, \Psi, \ldots)$ stand for *finite sets* of formulas. A *literal* is either a or $\sim a$, where $a \in \mathsf{VAR}$. We define: $\varphi \leftrightarrow \psi = (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ and $\varphi \Leftrightarrow \psi = (\varphi \leftrightarrow \psi) \wedge (\sim\psi \leftrightarrow \sim\varphi)$. Also $\Phi \longrightarrow \Psi = \bigwedge \Phi \rightarrow \bigvee \Psi$, where $\Psi \neq \emptyset$. (If $\Phi = \emptyset$ then $\bigwedge \Phi = p \rightarrow p$.) We write: "$\Phi; \Psi$" for "$\Phi \cup \Psi$" and "$\Phi; \psi$" for "$\Phi; \{\psi\}$".

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Positive (Intuitionistic) Logic (**Lp**) is the set of all $\sim$-free formulas that are derivable from the axioms below by *modus ponens* ($\frac{\varphi \quad \varphi\to\psi}{\psi}$).

(Ax1) $\varphi \to (\psi \to \varphi)$
(Ax2) $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$
(Ax3) $(\varphi \wedge \psi) \to \varphi$
(Ax4) $(\varphi \wedge \psi) \to \psi$
(Ax5) $(\varphi \to \psi) \to ((\varphi \to \chi) \to (\varphi \to (\psi \wedge \chi)))$
(Ax6) $\varphi \to (\varphi \vee \psi)$
(Ax7) $\psi \to (\varphi \vee \psi)$
(Ax5) $(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi))$

By an *extension of* **Lp** we mean a set $\mathbf{L} \subseteq \mathsf{FOR}$ closed *under subsitution* and *modus ponens* and such that $\mathbf{Lp} \subseteq \mathbf{L}$. Extensions of **Lp** will also be called *logics*. We say that φ is **L**-*equivalent* to ψ iff $\varphi \leftrightarrow \psi \in \mathbf{L}$.

A *frame* is a pair $\mathcal{W} = (W, R)$, where W is a non-empty set of points (worlds) and R is a reflexive, transitive relation on W. A *model* is a triple $\mathcal{M} = (\mathcal{W}, V^+, V^-)$, where $\mathcal{W}$ is a frame and V^+ (V^-) is a verification (falsification) valuation (that is, a function assigning to a propositional variable a the set of points at which a is true (false)) that satisfies the *persistency condition*: If $w \in V^+(a)$ ($w \in V^-(a)$) and wRx, then $x \in V^+(a)$ ($x \in V^-(a)$). The verification and falsification relations ($\models^+$ and $\models^-$) between $\mathcal{M}$ worlds and formulas are defined as follows.

$\mathcal{M}, w \models^+ a$ iff $w \in V^+(a)$ $\quad \mathcal{M}, w \models^- a$ iff $w \in V^-(a)$ ($a \in \mathsf{VAR}$)
$\mathcal{M}, w \models^+ \varphi \wedge \psi$ iff $\mathcal{M}, w \models^+ \varphi$ and $\mathcal{M}, w \models^+ \psi$
$\mathcal{M}, w \models^- \varphi \wedge \psi$ iff $\mathcal{M}, w \models^- \varphi$ or $\mathcal{M}, w \models^- \psi$
$\mathcal{M}, w \models^+ \varphi \vee \psi$ iff $\mathcal{M}, w \models^+ \varphi$ or $\mathcal{M}, w \models^+ \psi$
$\mathcal{M}, w \models^- \varphi \vee \psi$ iff $\mathcal{M}, w \models^- \varphi$ and $\mathcal{M}, w \models^- \psi$
$\mathcal{M}, w \models^+ \varphi \to \psi$ iff for all x with wRx ($\mathcal{M}, x \models^+ \varphi$ implies $\mathcal{M}, x \models^+ \psi$)
$\mathcal{M}, w \models^- \varphi \to \psi$ iff for all x with wRx ($\mathcal{M}, x \models^+ \varphi$ implies $\mathcal{M}, x \models^- \psi$)
$\mathcal{M}, w \models^+ \sim \varphi$ iff $\mathcal{M}, w \models^- \varphi$
$\mathcal{M}, w \models^- \sim \varphi$ iff $\mathcal{M}, w \models^+ \varphi$

We say that a formula φ is *valid* in a frame $\mathcal{W}$ (in symbols $\varphi \in \mathsf{VAL}(\mathcal{W})$) iff for every model $\mathcal{M} = (\mathcal{W}, V^+, V^-)$ and $w \in W$, we have: $\mathcal{M}, w \models^+ \varphi$.

The condition for "$\mathcal{M}, w \models^- \varphi \to \psi$" is that for the logic **C**. However, it can be replaced with some other standard condition because it is not really used in our general result.

For convenience, we say that a set Φ of formulas is (not) true at w iff so is every $\varphi \in \Phi$.

The one-point frame $\circ = (\{x_0\}, (x_0, x_0))$ will be especially important. We also write "$m = (\circ, v^+, v^-)$" instead of "$(\circ, V^+, V^-)$", and "$m \models \varphi$" instead of "$m, x_0 \models \varphi$", where $\models \in \{\models^+, \models^-\}$.

We say that a logic **L** is *complete with respect to* a class **W** of frames (or **L** is *characterized by* **W**) iff $\mathbf{L} = \bigcap\{\mathsf{VAL}(\mathcal{W}) : \mathcal{W} \in \mathbf{W}\}$.

We assume that every logic defined by an axiom system has a general characterization by a class $\mathbf{W_L}$ of (possibly infinite) frames obtained by using canonical models (see e.g. [3]). The frames in $\mathbf{W_L}$ are also referred to as **L**-*frames*.

By a *generated frame* we mean a frame $\mathcal{W}$ with a least point x (that is, xRy for all $y \in W$).

Let $\mathcal{W}_i$ be an x_i-generated frame ($1 \leq i \leq k$). Then $x_0(\mathcal{W}_1, \ldots, \mathcal{W}_k)$ is the x_0-generated frame shown below. (It is obtained from $\mathcal{W}_1, \ldots, \mathcal{W}_k$ by adding a new root, and it resembles Jaśkowski's construction from S. McCall, *Polish Logic: 1920–1939.*)

$$\begin{array}{ccc} \mathcal{W}_1 & \cdots & \mathcal{W}_k \\ & \nwarrow \quad \nearrow & \\ & x_0 & \end{array}$$

In this paper, we focus on the logics satisfying the following condition.

If generated frames $\mathcal{W}_1, \ldots, \mathcal{W}_k$ *are* **L**-*frames, then so is* $x_0(\mathcal{W}_1, \ldots, \mathcal{W}_k)$.

In that case, we also say that $\mathbf{W_L}$ is root-closed.

3. Normal Forms

Definition 3.1. A general form (cf. [1,5]) is a formula

$$\alpha = \Delta; \Gamma \longrightarrow \Theta$$

where

$$\Delta = \{(a_i \to b_i) \to c_i : 1 \leq i \leq k\} \quad (\text{or } \Delta = \emptyset)$$

all a_i, b_i, c_i are literals, Γ is a finite set of formulas of the kind:

a or $a \to b$ or $a \to (b \to c)$ or $a \to b \vee c$, where a, b, c are literals, and Θ is a finite, nonempty set of literals.

The rank $r(\alpha)$ of α is k, which is the size of Δ. (If $\Delta = \emptyset$ then $k = 0$.)

Definition 3.2. A *normal form* is a general form such that:

If $a \to \gamma \in \Gamma$ then $a \in \Theta$.

Definition 3.3. Let α be a normal form. An **L**-model $(\circ, v^+, v^-)$ is called *determined by* α (symbolically, $m(\alpha) = (\circ, v^+, v^-)$) iff

$x_0 \in v^+(\varphi)$ iff $\varphi \in \Gamma$, and $x_0 \in v^-(\varphi)$ iff $\sim\varphi \in \Gamma$ ($\varphi \in \mathsf{VAR}$).

Definition 3.4. An **L** *special normal form* is a normal form α (so $\alpha = \Delta; \Gamma \longrightarrow \Theta$) with the property that $m(\alpha) \not\models^+ \alpha_0$, where $\alpha_0 = \Gamma \longrightarrow \Theta$. (Recall that $m(\alpha) \models^+ \varphi$ stands for $m(\alpha), x_0 \models^+ \varphi$).

4. L Refutation System

The refutation system $\mathbf{R_L}$ is defined as follows.

- *Refutation axioms*: All **L** special normal forms of rank 0.

- *Refutation rules*:
  Normalization rules: (Here $\Psi \neq \emptyset$).

$$\frac{\varphi; \Phi \longrightarrow \Psi}{\varphi \vee \psi; \Phi \longrightarrow \Psi} \qquad \frac{\psi; \Phi \longrightarrow \Psi}{\varphi \vee \psi; \Phi \longrightarrow \Psi}(R^{\vee})$$

$$\frac{\psi; \Phi \longrightarrow \Psi}{\varphi \to \psi; \Phi \longrightarrow \Psi} \qquad \frac{\varphi \to \psi; \Phi \longrightarrow \Psi; \varphi}{\varphi \to \psi; \Phi \longrightarrow \Psi}(R^{\to})$$

  Normal-form rules:

$$\frac{\alpha_1 \ldots \alpha_k}{\alpha}(R_{\mathbf{L}}) \qquad \frac{\beta_i}{\alpha}(R_i)$$

  where $\alpha(= \Delta; \Gamma \longrightarrow \Theta)$ is an **L** special normal form of rank $k > 0$, and $\alpha_i = \Delta_i^-; b_i \to c_i; \Gamma; a_i \longrightarrow b_i$, $\beta_i = \Delta_i^-; c_i; \Gamma; (a_i \to b_i) \longrightarrow \Theta$, $\Delta_i^- = \Delta - \{(a_i \to b_i) \to c_i\} \quad (1 \leq i \leq k)$.

We say that a formula φ is *refutable* (in symbols $\dashv \varphi$) iff φ is derivable from refutation axioms by refutation rules. And we say that a logic **L** is $\mathbf{R_L}$-*complete* iff for every normal form α, we have: Either $\alpha \in \mathbf{L}$ or $\dashv \alpha$.

Remark 4.1. Since $\mathbf{Lp} \subseteq \mathbf{L}$, we have:

(i) The rule R_i and all normalization rules have the property that $\varphi \to \psi \in \mathbf{L}$, where ψ is the premise and φ is the conclusion.
(ii) Every formula α_i is **L**-equivalent to $\Delta; \Gamma \longrightarrow (a_i \to b_i)$.
(iii) Every formula β_i is **L**-equivalent to $\Delta; \Gamma; (a_i \to b_i) \longrightarrow \Theta$.

Definition 4.2. A formula φ is **L**-*reducible* to a finite set Ψ of formulas iff

(i) $\Psi \longrightarrow \varphi \in \mathbf{L}$.
(ii) φ is derivable from each $\psi \in \Psi$ by the normalization rules.

Corollary 4.3. *If φ is* **L**-*reducible to Ψ, then $\varphi \to \psi \in \mathbf{L}$ for each $\psi \in \Psi$.*

5. The Reduction Property

Definition 5.1. A logic **L** has *the reduction property* iff we have:

1. For any formula φ, there is a general form α_φ such that: $\varphi \to \alpha_\varphi \in \mathbf{L}$, and if $\alpha_\varphi \in \mathbf{L}$ then $\varphi \in \mathbf{L}$.
2. Every general form α of rank k is **L**-reducible to some normal forms $\alpha_1, \ldots, \alpha_n$ of rank k.
3. Let α be a normal form. If $m(\alpha) \models^+ \alpha_0$, then $\alpha_0 \in \mathbf{L}$.

Theorem 5.2. *If a logic* **L** *has the reduction property, then* **L** *is* $\mathbf{R_L}$-*complete.*

Proof. Assume that **L** has the reduction property. We show, by induction on the rank of a normal form α, that either $\alpha \in \mathbf{L}$ or $\dashv \alpha$.

(1) $k = 0$. Then $\alpha = \alpha_0 = \Gamma \longrightarrow \Theta$.

Consider the model $m(\alpha)$. Either $m(\alpha) \models^+ \alpha_0$ or $m(\alpha) \not\models^+ \alpha_0$. If $m(\alpha) \models^+ \alpha_0$ then $\alpha_0 \in \mathbf{L}$ (by Definition 5.1(3)), so $\alpha \in \mathbf{L}$. And if $m(\alpha) \not\models^+ \alpha_0$ then α is an **L** special normal form of rank 0, which is a refutation axiom, so $\dashv \alpha$. Hence, $\alpha \in \mathbf{L}$ or $\dashv \alpha$.

(2) $k > 0$ and we assume that the theorem is true for normal forms of rank $< k$. Then $\alpha = \Delta; \Gamma \longrightarrow \Theta$ and $\Delta \neq \emptyset$.

Consider the general forms α_i, β_i $(1 \leq i \leq k)$. They are of rank $< k$. By Definition 5.1(2), each of them is **L**-reducible to some normal forms of rank $< k$, which (by the induction hypothesis) are in **L** or refutable. So, all α_i, β_i are also in **L** or refutable (by Definition 4.2).

If some β_i is refutable, then so is α (by R_i), so we assume that every $\beta_i \in \mathbf{L}$. Then, the formula $\Delta; \Gamma; (a_i \to b_i) \longrightarrow \Theta$ is in **L** as well (by Remark 4.1(iii)). Now, if some $\alpha_i \in \mathbf{L}$, then the formula $\Delta; \Gamma \longrightarrow (a_i \to b_i)$ is in **L**, so it follows that $\alpha \in \mathbf{L}$, and so we may assume that each α_i is refutable. Also, either $m(\alpha) \models^{+} \alpha_0$ or $m(\alpha) \not\models^{+} \alpha_0$. If $m(\alpha) \models^{+} \alpha_0$ then $\alpha_0 \in \mathbf{L}$ (by Definition 5.1(3)), so $\alpha \in \mathbf{L}$. And if $m(\alpha) \not\models^{+} \alpha_0$ then α is an **L** special normal form, so $\dashv \alpha$ (by R).

Therefore, either $\alpha \in \mathbf{L}$ or $\dashv \alpha$, as required. □

6. $\mathbf{R_L}$ Refutation Trees

The derivations in $\mathbf{R_L}$ can be presented as refutation trees. By an $\mathbf{R_L}$ *refutation tree* for a formula φ we mean a finite immediate-successor tree $\mathcal{RT}$ whose nodes are labelled with formulas and which satisfies the following conditions. (For any node x in $\mathcal{RT}$, $\varphi(x)$ is the label of x).

- φ is the label of the origin x_0.
- If x is an end node, then $\varphi(x)$ is an $\mathbf{R_L}$ axiom.
- If $x_1, \ldots, x_n$ are the immediate successors of a node x, then $\varphi(x)$ is obtained from $\varphi(x_1), \ldots, \varphi(x_n)$ by an $\mathbf{R_L}$ rule.

7. Finite Tree-Countermodels

Recall that a logic is an extension of **Lp** such that its general characterization $\mathbf{W_L}$ is root-closed. By modifying some results in [4], we now transform refutation trees into countermodels.

Let $\mathcal{RT}$ be an $\mathbf{R_L}$ refutation tree for a normal form α. We construct a finite tree-countermodel $(\mathcal{T}, V^{+}, V^{-})$ as follows.

- First, we define the finite, reflexive, transitive tree $\mathcal{RT}^{\uparrow}_{\circ}$ by taking the reflexive, transitive closure of the irreflexive, intransitive relation in $\mathcal{RT}$.
- Second, we delete all nodes in $\mathcal{RT}^{\uparrow}_{\circ}$ that are obtained by R_i or a normalization rule, getting the subtree $\mathcal{T}$ of $\mathcal{RT}^{\uparrow}_{\circ}$. Note that every node in $\mathcal{T}$ is either an end node or a node obtained by $R_{\mathbf{L}}$, so the label of each node in $\mathcal{T}$ is an **L** special normal form.
- Third, we define valuations V^{+}, V^{-} thus.
  $x \in V^{+}(\varphi)$ iff $\varphi \in \Gamma$, and $x \in V^{-}(\varphi)$ iff $\sim\varphi \in \Gamma$ $(\varphi \in \mathsf{VAR})$.
  By inspecting the refutation rules, we can see that if y is a successor of x in $\mathcal{T}$, then a literal is in Φ' whenever it is in Φ, where $\varphi(x) = \Phi \longrightarrow \Psi$ and $\varphi(y) = \Phi' \longrightarrow \Psi'$. So, the persitency condition is satisfied. Let

$\mathcal{M} = (\mathcal{T}, V^+, V^-)$.
Note that for any literal a, we have: $\mathcal{M}, x_0 \models^+ a$ iff $m(\alpha) \models^+ a$.

- Finally, for any node x in $\mathcal{RT}$, we define its world $x^\star$ in $\mathcal{T}$ as follows. If x is an end node or obtained by $R_{\mathbf{L}}$, then $x^\star = x$; and if x is obtained from x_1 by R_i or a normalization rule, then $x^\star = x_1^\star$.

Lemma 7.1. *If x is a node in $\mathcal{RT}$, then $\varphi(x)$ is not true at $x^\star$.*

Proof. (by induction on the number n_x of nodes in the subtree of $\mathcal{RT}$ generated by x).

(1) $n_x = 1$. Then $\varphi(x)$ is a refutation axiom, which is an $\mathbf{L}$ special normal form $\Gamma \longrightarrow \Theta$ (so $m(\alpha) \not\models^+ \alpha_0$). Since $x^\star = x$, we have: $\varphi(x^\star) = \varphi(x)$. Hence, $\mathcal{M}, x^\star \models^+ a$ for every literal $a \in \Gamma$. Note that if $a \to \gamma \in \Gamma$ then $a \notin \Gamma$ (otherwise, $a \in \Theta$ (because $\varphi(x)$ is a normal form) and $m(\alpha) \models^+ \alpha_0$, which is impossible). Thus, $\mathcal{M}, x^\star \models^+ a \to \gamma$ for all $a \to \gamma \in \Gamma$. Therefore $\mathcal{M}, x^\star \not\models^+ \varphi(x)$.

(2) $n_x > 1$ and we assume that the theorem holds for the subtrees with fewer elements than n_x.

  (2.1) $\varphi(x)$ is obtained from $\varphi(x_1)$ by R_i or a normalization rule. Then $x^\star = x_1^\star$. Since $n_{x_1} < n_x$, by the induction hypothesis, $\varphi(x_1)$ is not true at $x_1^\star$. Hence $\varphi(x_1)$ is not true at $x^\star$. So, by Remark 4.1(i), $\varphi(x)$ is not true at $x^\star$ (because $\varphi(x) \to \varphi(x_1)$ is true everywhere).

  (2.2) $\varphi(x)$ is obtained from $\varphi(x_1), \ldots, \varphi(x_k)$ by $R_{\mathbf{L}}$. Then $\varphi(x) = \Delta; \Gamma \longrightarrow \Theta$ is an $\mathbf{L}$ special normal form of rank $k > 0$, and $\varphi(x_i) = \Delta_i^-; b_i \to c_i; \Gamma; a_i \longrightarrow b_i$ $(1 \le i \le k)$. Also

$x^\star = x$ (so $\varphi(x^\star) = \varphi(x)$) and $n_{x_i} < n_x$ for all i. By the induction hypothesis, $\varphi(x_i)$ is not true at $x_i^\star$ $(1 \le i \le k)$, so $\Delta; \Gamma \longrightarrow (a_i \to b_i)$ is not true at $x_i^\star$ $(1 \le i \le k)$ (by Remark 4.1(ii)). Hence, every $a_i \to b_i$ is not true at $x_i^\star$ and $\Delta; \Gamma$ is true at each $x_i^\star$. So, every $a_i \to b_i$ is not true at $x^\star$ (because $x^\star$ precedes every $x_i^\star$). Also, $\Delta; \Gamma$ is true at all $x_i^\star$, so Δ is true at $x^\star$ and every $a \to \gamma \in \Gamma$ is true at $x^\star$ (see (1) above). Of course, every literal in Γ is true and Θ is not true at $x^\star$.

Therefore $\varphi(x)$ is not true at $x^\star$, as required. □

Theorem 7.2. *If a logic* $\mathbf{L}$ *has the reduction property, then* $\mathbf{L}$ *is characterized by the class of finite, reflexive, transitive trees.*

Proof. Assume that $\mathbf{L}$ has the reduction property, and assume that $\varphi \notin \mathbf{L}$. Then, (by Definition 5.1(1)) its general form $\alpha_\varphi \notin \mathbf{L}$, and (by Definition 5.1(2)) $\alpha \notin \mathbf{L}$ for some normal form α. So, by Theorem 5.2, α is refutable. Hence, by Lemma 7.1, α is not true at some point in some model based on a finite, reflexive, transitive tree. Thus, φ is not true there either (because, by Corollary 4.3, $\varphi \to \alpha$ is true everywhere). Therefore, φ is not valid in some finite, reflexive, transitive tree, which gives the result. □

8. Example

The connexive logic **C** was introduced in [7] as the extension of **Lp** by the following axioms.

(Ax 9) $\sim\sim\varphi \leftrightarrow \varphi$
(Ax10) $\sim(\varphi \wedge \psi) \leftrightarrow (\sim\varphi \vee \sim\psi)$
(Ax11) $\sim(\varphi \vee \psi) \leftrightarrow (\sim\varphi \wedge \sim\psi)$
(Ax12) $\sim(\varphi \rightarrow \psi) \leftrightarrow (\varphi \rightarrow\sim\psi)$

Our refutation system $\mathbf{R_C}$ is obtained from [5,6] by the following straightforward modifications.

C Special Normal Forms

Proposition 8.1. *$m(\alpha) \not\models^+ \alpha_0$ iff $\Gamma \cap \Theta = \emptyset$.*

Proof. ($\Rightarrow$) If $\Gamma \cap \Theta \neq \emptyset$, then $\alpha_0 \in \mathbf{C}$, so α_0 is true everywhere, so there is no model on $\circ$ such that the formula α_0 is not true.

($\Leftarrow$) Assume that $\Gamma \cap \Theta = \emptyset$. Then $m(\alpha) \not\models^+ \alpha_0$. $\square$

Corollary 8.2. *A* **C** *special normal form is a normal form α such that $\Gamma \cap \Theta = \emptyset$.*

The Reduction Property

Theorem 8.3. *The logic* **C** *has the reduction property.*

Proof. 1. For any formula φ, the general form α_φ is constructed as follows. First, for every subformula ψ of φ, we define a unique corresponding variable a_ψ thus. If $\psi \in \mathsf{VAR}$ then $a_\psi = \psi$, and if $\psi \notin \mathsf{VAR}$ then a_ψ is a new variable.
Second, we define the set Σ_φ as follows.

$$\Sigma_\varphi = \{(a_{\psi_1} \otimes a_{\psi_2}) \Leftrightarrow a_{\psi_1 \otimes \psi_2} : \psi_1 \otimes \psi_2 \in \mathsf{SUB}(\varphi), \otimes \in \{\rightarrow, \wedge, \vee\}\}$$
$$\cup \{\sim\psi_1 \Leftrightarrow a_{\sim\psi_1} :\sim\psi_1 \in \mathsf{SUB}(\varphi)\}$$

Third, we define: $\mathsf{N}(\varphi) = \Sigma_\varphi \longrightarrow a_\varphi$.
Note that the Deduction Theorem and the Replacement Theorem below hold for **C**.
$(\psi \Leftrightarrow \chi) \rightarrow (\varphi \Leftrightarrow \varphi(\psi/\chi))$, where $\varphi(\psi/\chi)$ results from φ by replacing some occurrences of ψ with χ.
Thus (see [5]), it can be proved that

$$\Sigma_\varphi \longrightarrow (\psi \Leftrightarrow a_\psi) \in \mathbf{C} \text{ for any } \psi \in \mathsf{SUB}(\varphi).$$

Hence, in particular, $\Sigma_\varphi \longrightarrow (\varphi \Leftrightarrow a_\varphi) \in \mathbf{C}$, so $\Sigma_\varphi \longrightarrow (\varphi \rightarrow a_\varphi) \in \mathbf{C}$.
Therefore, $\varphi \longrightarrow (\Sigma_\varphi \rightarrow a_\varphi) \in \mathbf{C}$, that is, $\varphi \longrightarrow \mathsf{N}(\varphi) \in \mathbf{C}$.
Also, if $\mathsf{N}(\varphi) \in \mathbf{C}$ then $\varphi \in \mathbf{C}$ (substitute ψ for a_ψ).
Finally, by using the axioms for $\sim$, we get the the general form α_φ **C**-equivalent to $\mathsf{N}(\varphi)$, which gives the result.

2. Take any general form α of rank k. We have to show that α is **C**-reducible to some normal forms such that:
($*$) If $a \rightarrow \gamma \in \Gamma$ (so a is a literal), then $a \in \Theta$.

Let $a \to \gamma \in \Gamma$, and let $\Gamma^- = \Gamma - \{a \to \gamma\}$.
Consider the **C** general forms

$$\alpha^1 = \Delta; \Gamma^-; \gamma \longrightarrow \Theta \text{ and } \alpha^2 = \Delta; \Gamma \longrightarrow \Theta; a$$

Of course, $\alpha^1 \wedge \alpha^2 \to \alpha \in \mathbf{C}$. Also, α is dervable from both (so from some) α^1, α^2 by $R^{\to}$. Now, in α^1, we have γ instead of $a \to \gamma$. If γ is a literal, then the condition $(*)$ is satisfied (because $a \to \gamma \notin \Gamma^-$). So assume that γ is not a literal. Then γ is either $b \vee c$ or $b \to c$, where b, c are literals.
If $\gamma = b \vee c$ then we consider the general forms

$$\alpha^{1.1} = \Delta; \Gamma^-; b \longrightarrow \Theta \text{ and } \alpha^{1.2} = \Delta; \Gamma^-; c \longrightarrow \Theta$$

in which the $\to$-formula $a \to \gamma$ is eliminated. We have: $\alpha^1 \wedge \alpha^2 \to \alpha^1 \in \mathbf{C}$, and α^1 is dervable from both (so from some) $\alpha^{1.1}, \alpha^{1.2}$ by $R^{\vee}$. Hence, α is **C**-reducible to $\alpha^{1.1}, \alpha^{1.2}, \alpha^2$, and $a \to \gamma$ satisfies the condition $(*)$.
And if $\gamma = b \to c$ then proceed as above, obtaining general forms in which $a \to \gamma$ satisfies the condition $(*)$.
Note that these transformations do not affect the rank k. Applying this procedure to the other $\to$-formulas in Γ, we, finally, get the result.

3. To prove that $m(\alpha) \models^+ \alpha_0$ implies $\alpha_0 \in \mathbf{C}$, suppose that $m(\alpha) \models^+ \alpha_0$ but $\alpha_0 \notin \mathbf{C}$. Note that $\Gamma \cap \Theta = \emptyset$. (Otherwise $\alpha_0 \in \mathbf{C}$). Also, if $a \to \gamma \in \Gamma$ then $a \notin \Gamma$ (because α is a normal form and $\alpha_0 \notin \mathbf{C}$). Then $m(\alpha) \not\models^+ \alpha_0$, which is a contradiction.

□

Corollary 8.4. (i) **C** *is* $\mathbf{R_C}$*-complete.*
(ii) **C** *is characterized by the class of finite, reflexive, transitive trees.*

Remark 8.5. By straightforward modifications, similar results can be obtained for the connexive logic **C3** (introduced in [2]).

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