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On Pairs of Dual Consequence Operations

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Abstract. In the paper, the authors discuss two kinds of consequence operations characterized axiomatically. The first one are consequence operations of the type Cn^+ that, in the intuitive sense, are *infallible* operations, always leading from accepted (true) sentences of a deductive system to accepted (true) sentences of the deductive system (see Tarski in Monatshefte für Mathematik und Physik 37:361–404, 1930, Comptes Rendus des Séances De la Société des Sciences et des Lettres de Varsovie 23:22–29, 1930; Pogorzelski and Słupecki in Stud Logic 9:163–176, 1960, Stud Logic 10:77–95, 1960). The second kind are dual consequence operations of the type Cn^- that can be regarded as *anti-infallible* operations leading from non-accepted (rejected, false) sentences of a deductive system to non-accepted (rejected, false) sentences of the system (see Słupecki in Funkcja Łukasiewicza, 33–40, 1959; Wybraniec-Skardowska in Teoria zdań odrzuconych, 5–131, Zeszyty Naukowe Wyższej Szkoły Inżynierskiej w Opolu, Seria Matematyka 4(81):35–61, 1983, Ann Pure Appl Logic 127:243–266, 2004, in On the notion and function of rejected propositions, 179–202, 2005). The operations of the types Cn^+ and Cn^- can be ordinary finitistic consequence operations or unit consequence operations. A deductive system can be characterized in two ways by the following triple:

by the triple: $(+,-)$ $< S, Cn^+, Cn^- >$
or by the triple: $(-,+)$ $< S, Cn^-, Cn^+ >$.

We compare axiom systems for operations of the types Cn^+ and Cn^-, give some methodological properties of deductive systems defined by means of these operations (e.g. consistency, completeness, decidability in Łukasiewicz's sense), as well as formulate different metatheorems concerning them.

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1. Introduction

The notion of logical consequence is one of the most important syntactic notions of the syntactic theory of deductive systems, in particular - logical systems. The universal concept of the consequence operation was formalized in the so-called Tarski's general theory **T** of deductive systems in 1930 ([33, see Section 2]). The consequence operation was also characterized axiomatically by Tarski [34] for the so-called enriched theories of deductive systems based on classical logic (describing properties of the *classical consequence operation*; see Sects. 8.1, 8.2) and theories of deductive systems based on some non-classical logic (describing properties of *non-classic consequence operation*; see Pogorzelski [14], Słupecki [15] and Pogorzelski and Wojtylak [16]).

The method of characterizing the syntax of specific logical deductive systems, in the spirit Tarski's ideas, is based on a consideration of finite axiom systems for the consequence operation Cn as systems with respect to *assertion*, i.e. systems for which, in the intuitive sense, Cn consequence is an *infallible* operation, always leading from accepted (true) sentences of a system to accepted (true) sentences of the system.

There is another, dual method of axiomatic characterizing of deductive systems by means of a dual consequence operation which can be regarded as an *anti-infallible* operation leading from non-accepted (rejected, false) sentences of the system to non-accepted (rejected, false) sentences of the system. The notion of the dual (rejection) consequence operation Cn^{-1} was introduced by Słupecki [26] on the basis of the theory **T** in connection with Łukasiewicz's idea of dual axiomatic characterization of deductive systems, both with respect to assertion (determined by the consequence Cn) and *rejection* (determined by the consequence Cn^{-1}) and investigations for their *saturation* (*decidability in Łukasiewicz's sense*).

The very notion of *rejection* was introduced into formal logic by Jan Łukasiewcz already in 1921 [8]. The concept of rejection of some sentences by means of earlier rejected sentences was used by Łukasiewicz in his studies concerning Aristotelian syllogistic and some sentential calculi [9–12]. As Łukasiewicz states in the article *On Aristotle's Syllogistic*, in 1939 [9], the idea of demolishing some sentences on the ground of others comes from Aristotle. Namely, Aristotle not only used examples to reject false syllogistic forms, but also reduced some false syllogistic forms to other ones, the erroneousness of which was already shown. As was pointed out by Łukasiewicz in his post-war reproduced research on Aristotle's syllogistic in 1951 ([10, pp. 74, 71]), the modern formal logic should use the notion of 'rejection' as an operation opposed to Frege's concept of 'assertion' and a rule of rejection could be stated corresponding to the rule of detachment by assertion. In the research Łukasiewicz used an axiomatic method of rejection, accepting some sentences as *rejected axioms* and some others of them as rejected by means of two rules of rejection. One of them, which was anticipated by Aristotle and corresponds to the rule of detachment by assertion, is as follows:

a. *The rule of detachment by rejection*: *A sentence α is rejected if the conditional sentence built of α as its antecedent and a sentence β as its consequent is asserted and β is rejected*.

The second rejection rule used by Łukasiewicz is as follows:

b. *The rejection rule by substitution*: *A sentence α is rejected if a substitution of α is rejected*.

A rejected sentence in Łukasiewicz's sense is an axiom or a sentence derivable from rejected axioms by means of his rejection rules.

It turns out that if all rejected axioms are false, so are all rejected sentences.

Łukasiewicz's idea of the rejected sentence was adopted by Słupecki in his important continuation of Łukasiewicz's investigations of Aristotle's syllogistic [25]. Słupecki used the following definition which is equivalent to the Łukasiewicz's definition of the rejected sentence:

A rejected sentence in Słupecki's sense is such a sentence for which there exists a rejected axiom which is derivable from the sentence (and theses of the system) by means of inference rules.

The above definition was closer to Aristotle's idea of rejection (refutation) of syllogism by reducing them to syllogisms rejected earlier. Słupecki adapted it for any deductive system and generalized the idea of the rejected sentences. Apart from Tarski's *infallible* finitistic consequence operation Cn, Słupecki [26] introduced the *anti-infallible* rejection operation Cn^{-1} defined for any set X of sentences of the deductive system by means of Cn as follows:

$$(Cn, Cn^{-1}) \qquad \alpha \in Cn^{-1}(X) \text{ if and only if there is in } X \text{ a sentence } \beta \text{ such that } \beta \in Cn(\{\alpha\}),$$

and proved that this operation is also a finitistic consequence operation in Tarski's sense. Słupecki also showed that the operation Cn^{-1} satisfies topological axioms for the closure operation in Kuratowski's sense [7].

In the semantic stylization, it follows from $\alpha \in Cn^{-1}(X)$ that if all sentences of X are false, then so is the rejection sentence α; in the pragmatic stylization, $\alpha \in Cn^{-1}(X)$ means that, under definite circumstances, if all sentences of X are not accepted, then neither is α.

The above definition and remarks allow characterizing a deductive system in a two-side manner by means of the pair of consequence operations $<Cn, Cn^{-1}>$, not only as an *asserted system* (based on the consequence operation Cn) but also as a *refutation system* (based on the rejected consequence operation Cn^{-1}). The dual axiomatic method characterization of deductive systems in accordance with Łukasiewicz's idea of investigations of their *saturation* (*decidability in a sense*) is formalized in Sect. 4.

The investigations initiated by Słupecki [26] have been continued by the first co-author of this paper [38,39], later also by Bryll [4–6] and Słupecki's circle of researchers (see [29,30]). The initial studies concerning the rejection consequence Cn^{-1} and refutation issues were connected with their applications in methodology of empirical sciences. Namely, there is a strict relation between the notion of the rejected consequence Cn^{-1}, Łukasiewicz's rule a. (the rejected rule by detachment), and the methodological procedure of falsification, rejection (refutation) of a hypothesis. It is easy to see that if X is a set of sentences, each of which is the negation of an empirical ascertained sentence and α is a hypothesis, then, in accordance with the definition $(Cn, Cn^{-1}), \alpha$ is a rejected hypothesis on the ground of the set X if at least one sentence β of the set X is a consequence of α, i.e. if it entails at least one β in X, the sentence contradictory to experience, rejected on the basis of the experience. So, the set $Cn^{-1}(X)$ includes all the hypotheses which are rejected on the ground of the negations of empirical sentences, i.e. those which are refuted on the basis of sentences that are not in agreement with the empirical data.[1]

Refutation of a hypothesis is connected with its rejection on the basis of only one sentence that is not in agreement with empirical data. If, in a deductive system, the theorem of deduction is valid, then the right side of the definition (Cn, Cn^{-1}) states that:

there exists in X a sentence β such that the implication '$\alpha \rightarrow \beta$' $\in Cn(\emptyset)$.

So, if β is rejected as a sentence of X, which is a negation of an empirical ascertained sentence, then α is refuted on the basis of only one sentence in X.

This statement justifies the fact that investigations concerning Słupecki's consequence operation Cn^{-1} have been based on the so-called enriched theory of deductive systems $\mathbf{T}^{+}$ built by Tarski [34] (see Sects. 8.2). $\mathbf{T}^{+}$ is the theory based on the theory $\mathbf{T}$ and its two new primitive terms are symbols 'c' and 'n' that are the names of classical functors of implication and negation, respectively.

The rejection consequence Cn^{-1} is not an ordinary finitistic consequence operation but, as we see, it is a *unit operation*: any sentence that is rejected is rejected on the basis of only one sentence, while a sentence deducible by means of the ordinary operation Cn is derivable on the basis a finite set of sentences. So, the consequences of the pair $< Cn, Cn^{-1} >$ are not quite dual. Dual to the rejection consequence operation Cn^{-1} is an *infallible consequence* Cn^{+1} induced by the ordinary "positive" consequence operation $Cn(Cn = Cn^{+}$; see Sect. 5).[2]

All pairs of dual unit consequences operations $< Cn^{1}, dCn^{1} >$ can be characterized axiomatically (see Sect. 6). In contrast to the ordinary consequences in Tarski's sense they are the closure topological operations in Kuratowski's sense [7]; in particular they are additive operations.

[1] Let us mention in this place that the refutation of hypothesis is also used in other domains of knowledge, e.g. in jurisdiction. Let as also mention here that Karl Poper was known for his opposition to the classical justification account of knowledge and advancing empirical falsification instead. Jan Woleński applied in [36] the idea of Poper of comparing theories from the point of view of their content and for comparing the true content and the false content of a theory; beside the consequence Cn he applied also the consequence Cn^{-1}, as well as its generalization.

[2] These two mutual dual operations were examined already in [39].

The presence of dual consequence operations of the pair $< Cn^{+1}, Cn^{-1} >$ in the two-side formal syntactic characterization of deductive systems is satisfactory if these systems possess in their language the conjunction functor because a deduction from a finite set of sentences is usually reduced to the deduction from a single conjunction of sentences of the set. However, this fact is based on the assumption that the finitistic consequence operation $Cn = Cn^{+}$ is conjunctive. If it also satisfies the condition $Cn(\emptyset) \neq \emptyset$, then dual to it is the finitistic consequence operation dCn introduced by Wójcicki [37].

The consequence operation dCn defined by Cn on the basis of the theory **T** is the generalization of the rejection consequence operation Cn^{-1} (see Sect. 7). The pair of dual finitistic consequence operations $< Cn, dCn >$ may serve the purpose of bi-level formalization of deductive systems but dCn is an **anti-***infallible* operation only if we overimpose on the Cn operation some additional conditions. In particular, if Cn is a classical consequence operation, then Cn and dCn are mutual dual consequence operations, similarly like the operations Cn^{+1} and Cn^{-1}. So, two-side formalizations of the pairs of notions $<Cn^{+1}, Cn^{-1}>$ and $<Cn, dCn>$ run in two different directions:

$$< +, - > \text{ or } < -, + >,$$

where "+" symbolizes an *infallible* consequence operation, while "−" − an *anti-infallible* one.

In Sect. 8 we show two-sided formalization of deductive systems by means of two kinds of dual consequence operations.

The aim of the paper is not only to recollect, systematize, enlighten, explicate and bring closer to the reader certain results relating to dual consequence operations, results obtained earlier but which were included in publications that are hard to access nowadays and frequently rendered solely in the Polish language. The paper contains also new results concerning some properties of pairs of consequence and notions connected with them, as well as formulates different metatheorems concerning them.

Dual consequence operations play not only a significant role in syntactic metalogical studies. Due to their application in empirical sciences methodology they are vital both in the semantic and pragmatic aspects. The studies commenced by Łukasiewicz and Słupecki on *saturation* (Ł-*decidability*) of logical systems, thus the possibility of characterizing a deductive system in a two-side manner, not only as an asserted system (based on the notion of assertion consequence), but also as a refutation system (based on the notion of *rejection consequence* or *dual consequence*), became a standard procedure applied in metalogical studies on concrete sentential calculi (see Sect. 9).

The exposing, in the present work, of the role of unit consequence operations as individual unit operations is also significant for another reason. Unit operations have found their application, among others, in studies on certain algebras (see [42,43]) and in computer science studies (see [1,2]), especially in questions concerning approximation of information.

2. General Characterization of the Consequence Operation Cn

The notion of a consequence operation is the fundamental notion of the theory of deductive systems. The general notion of consequence was formalized by Tarski [33] in his consequence theory, the theory denoted by **T**. The theory **T** is based on two primitive notions: *the set* S *of all propositions of an arbitrary, but fixed, language and the consequence operation* Cn on the power set $P(S)$ of all subsets of the set S, i.e. a function

$$Cn : P(S) \to P(S),$$

which to any set of propositions X assigns the set $Cn(X)$ of all propositions deducible from the propositions in X (i.e. the consequences of the propositions in X).

In the axioms of theory **T**, the variables $x, y, z, \ldots$ run over the elements of the set S, while the variables $X, Y, Z, \ldots$ take values in $P(S)$.

The original axiom system for **T** is the following:

A1. $\mathrm{card}(S) \leq \aleph_o$ – denumerability of S,
A2. $X \subseteq Cn(X) \subseteq S$ – the consequence Cn is compact,
A3. $CnCn(X) = Cn(X)$ – the consequence Cn is idempotent,
A4'. $Cn(X) = \bigcup \{Cn(Y) \mid Y \in Fin\,(X)\}$ – the consequence Cn is finitistic.

It is worth observing that there are other axiom systems for **T**, but the most frequently used axiom system for **T** consists of the axioms A1-A3 and the following axioms:

A4. $X \subseteq Y \Rightarrow Cn(X) \subseteq Cn(Y)$ – the consequence Cn is monotonic,
A5. $Cn(X) \subseteq \bigcup \{Cn(Y) \mid Y \in Fin\,(X)\}$.

The method of characterizing the syntax of specific logical deductive systems, in the spirit of Tarski's ideas, is based on a consideration of axiom systems for the consequence operation as systems in regard to *acceptance*, i.e. systems for which, in the intuitive sense, consequence is an *infallible* operation, always leading from accepted (true) sentences of a system to accepted (true) sentences of the system.

There is another, dual, method of axiomatic characterizing of deductive systems by means of a rejection consequence operation or a dual consequence operation, which can be regarded as an *anti-infallible* operation leading from non-accepted (rejected, false) sentences of the system to non-accepted (rejected, false) sentences of the system. The rejection consequence operation can be formally introduced on the basis of the theory **T**.

3. Rejection Consequence Operation Cn^{-1}

The notion of rejection (dual) consequence was introduced and formalized by Słupecki [26] as the rejection function Cn^{-1} which is a generalization of the notion of *rejection* introduced into metalogical investigations by Łukasiewicz [8–13] for a bi-level dual characterization of deductive systems.

Łukasiewicz's idea of the dual characterization of a deductive system consists in the system existing on two levels, both with respect to *acceptance* (determined by the consequence operation Cn) and with respect to *rejection* (determined by the rejection function Cn^{-1}), and investigation of its *saturation, decidability in Łukasiewicz's sense* (see [9–12]).

Formally, the rejection function Cn^{-1} was defined by means of the consequence operation Cn by adding the following definition to the theory **T**:

Definition 3.1. (rejection consequence operation Cn^{-1} induced by the consequence operation Cn)

$$Cn^{-1}(X) = \{y \mid \exists x \in X(x \in Cn(\{y\}))\}.$$

A proposition belongs to the set $Cn^{-1}(X)-$ *the set of propositions rejected on the basis of propositions in* X if and only if it has a proposition in X among its consequences.

The names consequence applied to Cn^{-1} is justified by the following result of Słupecki:

Metatheorem 3.2.

(i) Cn^{-1} *satisfies the general axioms* A1–A5 *of the consequence theory* **T**,
(ii) $Cn^{-1}(X \cup Y) = Cn^{-1}(X) \cup Cn^{-1}(Y)-$ *it is additive,*
(iii) $Cn^{-1}(\emptyset) = \emptyset-$ *it is normal,*
moreover,
(iv) $y \in Cn^{-1}(X) \Rightarrow \exists x \in X(y \in Cn^{-1}(\{x\}))-$ *it is a unit operation.*

So, Cn^{-1} is the so-called *unit consequence operation.*

In **T**, with Definition 3.1, the following theorem justifies the name *rejection consequence* given to the consequence operation Cn^{-1} (cf. [3]), p. 356):

Theorem 3.3.

$$Cn(X) \subseteq X \Rightarrow Cn^{-1}(S\backslash X) \subseteq S\backslash X.$$

Proof. (*ad absurdum*).

Let us assume that $Cn(X) \subseteq X, x \in Cn^{-1}(S\backslash X)$ and $x \notin S\backslash X$. Then, from Definition 3.1. it follows that in $S\backslash X$ there is a sentence y such that $y \in Cn(\{x\})$. Thus, because $x \in X, Cn(\{x\}) \subseteq Cn(X)$ and $y \in X$. Hence, we have a contradiction.

Theorem 3.3 helps to understand the intuitive sense of this kind of consequence. Indeed, if the consequence operation Cn determines a deductive system as a set closed with respect to some inference rules, or more generally some rules of logical entailment, i.e. if Cn is the ordinary consequence operation, an *infallible consequence* which yields a true (or accepted as true) conclusion for true (or accepted as true, respectively) premises, then taking X to be the set of true (or accepted as true) propositions, $S\backslash X$ is the set of false (or not accepted as true, respectively) propositions and by Theorem 3.3, the expressions rejected on the basis of false propositions (or not accepted as true) are also false (or not accepted as true, respectively) and the rejection consequence operation Cn^{-1} always leads from false (or not accepted) propositions to false (or not accepted) propositions and can be regarded as an *anti-infallible* consequence operation.

The notion of rejection consequence operation introduced by Słupecki [26] according to Łukasiewicz's idea of rejection of propositions was examined by Wybraniec-Skardowska [39–41, 44], as well as other researchers belonging to Słupecki's circle of scientific research. □

4. Decidability in Łukasiewicz's Sense of Deductive Systems

Having at our disposal two types of consequences Cn and Cn^{-1}, we can speak of two kinds of systems: the system with respect to acceptance (A-*system*) and the system with respect to rejection, refutation (R-*system*), and provide complete syntactic characterization of a deductive system asking if it is *saturated*, decidable in Łukasiewicz's sense (Ł-*decidable system*, Ł-*saturated system*). If Cn is understood as an *infallible*, ordinary, derivable consequence operation, then we put $Cn = Cn^{+}$ and define these systems as follows:

Definition 4.1. (A-*syst.*) X is A-*system* iff $Cn^{+}(X) = X$,

Definition 4.2. (R^{-1}-*syst.*) Y is R^{-1}-*system* iff $Cn^{-1}(Y) = Y$.

Thus, every deductive system with the bi-level formalization $< X, Y >$, with the acceptance A-system X and the R^{-1}-refutation system Y can be characterized by the following triple:

$$< +, -1 > \qquad\qquad < S, Cn^{+}, Cn^{-1} > .$$

The extension of the theory **T** with the definition Definition 3.1 describes each system of this kind. Tarski's theory **T**, enriched by the definition of rejection consequence, can be seen as a *generalized consequence theory* providing a two-sided characterization of deductive systems.

In **T**, we can define basic syntactic notions, in particular, the following ones: *consistency* and *consistency with respect to rejection*, *completeness* and *completeness with respect to rejection.*

Moreover, we can define the notion of Ł-*decidability* corresponding to the notion of *saturation* or *decidability used by Łukasiewicz*. The notion was introduced by Słupecki (see [29]) by means of the notions of Ł-*consistence* and Ł-*completeness.*[3]

$< X, Y >$ is Ł-*consistent* iff $Cn^{+}(X) \cap Cn^{-1}(Y) = \emptyset$,
$< X, Y >$ is Ł-*complete* iff $Cn^{+}(X) \cup Cn^{-1}(Y) = S$,
$< X, Y >$ is Ł-*decidable* iff $< X, Y >$ is Ł-*consistent* & $< X, Y >$ is Ł-*complete*.

A deductive system with the two-sided finite axiomatization is Ł-*decidable* if and only if its A-assertion system and its R^{-1}-refutation system are disjoint and their union yields the set of all its propositions.

[3] The term 'Ł-*decidability*' could be replaced with a more adequate term 'Ł- *saturatibility*' but the former has functioned in the logical literature for a few decades (see also [27]).

5. Unit Consequence Operation Cn^{+1} Induced by Cn

The bi-level formalization of a deductive system in the form $< +, -1 >$ by means of consequences operations Cn^{+} and Cn^{-1} gives rise to the following question: Are these operations reverse dually in relation to each other? If the answer were: Yes, then for $Cn = Cn^{+}$ we have $Cn^{-1} = (Cn^{+})^{-1}$ and the following equation should be valid:

$$Cn^{+} = (Cn^{-1})^{-1}.$$

But we can only prove

Corollary 5.1. $$(Cn^{-1})^{-1} \leq Cn^{+}$$

and observe that the reverse implication to Theorem 3.3. is only valid if Cn^{+} is a unit consequence operation Cn^{+1}, so a specific finitistic operation which can be defined on the basis of **T**, looks as follows:

Definition 5.2. (the unit consequence operation Cn^{+1} induced by the consequence operation Cn)

$$Cn^{+1}(X) = \{y \mid \exists x \in X(y \in Cn(\{x\}))\}.$$

Let us note that

Corollary 5.3. $Cn^{+1}(\{x\}) = Cn(\{x\})$, *for every* $x \in S$.

It is easy to see that

Metatheorem 5.4.

(i) Cn^{+1} *satisfies the general axioms A1-A5 of the consequence theory* **T**,
(ii) $Cn^{+1}(X \cup Y) = Cn^{+1}(X) \cup Cn^{+1}(Y)-$ *it is additive,*
(iii) $Cn^{+1}(\emptyset) = \emptyset-$ *it is normal,*
moreover,
(iv) $y \in Cn^{+1}(X) \Rightarrow \exists x \in X(y \in Cn^{+1}(\{x\}))-$ *it is a unit operation.*

So, Cn^{+1} is also the so-called *unit consequence operation*. It is weaker than the operation $Cn = Cn^{+}$ in the sense of lattice theory, i.e.

$$Cn^{+1} \leq Cn^{+}.$$

Let us also observe that using Definition 3.1 for $Cn = Cn^{+1}$ and $Cn = Cn^{-1}$ we obtain:

Corollary 5.5. $Cn^{-1} = (Cn^{+1})^{-1}$ *and* $Cn^{+1} = (Cn^{-1})^{-1}$

and state that the consequence operations Cn^{+1} and Cn^{-1} are mutually dual.

They are also mutually definable because

Corollary 5.6.

a. $Cn^{+1}(X) = \{y \mid \exists x \in X(x \in Cn^{-1}(\{y\}))\}.$
b. $Cn^{-1}(X) = \{y \mid \exists x \in X(x \in Cn^{+1}(\{y\}))\}.$

On the basis of **T**, with definitions Definitions 3.1 and 5.2, we obtain the counterpart of Theorem 3.3:

Theorem 5.7.

$$Cn^{+1}(X) \subseteq X \Leftrightarrow Cn^{-1}(S \backslash X) \subseteq S \backslash X.$$

which allows understanding Cn^{+1} as the unit *infallible* consequence operation while Cn^{-1} as the reverse dual unit *anti-infallible* consequence operation.

The dual operations Cn^{+1} and Cn^{-1}, defined on the ground **T**, satisfy the general axiom for unit consequence operations given in the next section.

6. The Dual Unit Consequence Operations Cn^1 and dCn^1

Let Cn^1 be a function

$$Cn^1 : P(S) \to P(S)$$

that satisfies the following axiom:

A^1. $$Cn^1(X) = \{y \mid \exists x \in X(Cn^1(\{y\}) \subseteq Cn^1(\{x\}))\}.$$

The axiom states that a proposition belongs to the set of unit consequences of the set X if and only if the set of unit consequences of the sentence is included in the set of unit consequences of a single proposition in X.

It was proved (see [39]) that, in fact, the unit consequence operation Cn^1 defined in this way is a consequence operation and unit operation. It is also an additive and normal operation. So, the following metaheorem holds:

Metatheorem 6.1. *The operation* Cn^1 *satisfies the general axioms* A1–A5 *of* **T** *and the following conditions*:

(i) $Cn^1(X \cup Y) = Cn^1(X) \cup Cn^1(Y)$*—it is additive,*
(ii) $Cn^1(\emptyset) = \emptyset$*—it is normal,*
(iii) $y \in Cn^1(X) \Rightarrow \exists x \in X(y \in Cn^1(\{x\}))$*—it is a finitistic unit operation.*

It was also proved that the following methateorem is true:

Metatheorem 6.2. *The general axioms* A1–A5 *of* **T** *for any operation* $Cn : P(S) \to P(S)$*, together with the additivity of* Cn *and the condition* $Cn(\emptyset) = \emptyset$*, define* Cn *as a unit consequence operation* (*i.e. an operation satisfying* A^1).

The above theorem explains the fact why we call operations Cn^{+1} and Cn^{-1} unit consequences (see Metatheorems 3.2, 5.4). We also see that these operations are topological closure operators in the sense of Kuratowski [7]. We also stated earlier that they are dual in relation to each other.

Formally, the definition of the *dual operation* to the unit consequence Cn^1 is as follows:

Definition 6.3. (dual consequence corresponding to the unit consequence Cn^1)

$$dCn^1(X) = \{y \mid \exists x \in X(Cn^1(\{x\}) \subseteq Cn^1(\{y\}))\}.$$

Thus we have

Corollary 6.4. $$dCn^1(X) = \{y \mid \exists x \in X(x \in Cn^1(\{y\}))\}.$$

Corollary 6.5. $dCn^1(X) = \{y \mid \exists x \in X(dCn^1(\{y\}) \subseteq dCn^1(\{x\}))\}.$

Thus, by replacing Cn^1 by dCn^1 in A^1 we can state:

Metatheorem 6.6. *The dual operation dCn^1 to the operation Cn^1 satisfies the axiom A^1 and thus it is a unit consequence operation satisfying all conditions of Metatheorem 6.1.*

It can be proved that the unit operations Cn^1 and dCn^1 are mutually dual because the following theorem holds:

Theorem 6.7. $ddCn^1 = Cn^1.$

These unit operations are also mutually dual in the intuitive sense because the following theorem holds:

Theorem 6.8.

$$Cn^1(X) \subseteq X \Leftrightarrow dCn^1(S \backslash X) \subseteq S \backslash X$$

which, in the intuitive sense, says that if the consequence operation Cn^1 leads from true sentences to true sentences, then the dual consequence operation dCn^1 leads from false (or not accepted) sentences to false (or not accepted) sentences, and reversely.

Let us put in $\mathrm{A}^1$and in Definition 6.3 instead of Cn^1: Cn^{+1} and Cn^{-1}. Then we can state formally that the operations Cn^{+1} and Cn^{-1} are unit consequence operations and that they are also mutually dual:

$$dCn^{+1} = Cn^{-1} \quad \text{and} \quad dCn^{-1} = Cn^{+1},$$

which, on the basis of the theory **T**, justifies Corollary 5.5 and Theorem 5.7.

The introduction of unit dual consequence operations in two-level formalization of a deductive system is justified by the fact that rejection of some proposition (hypothesis) is always rejection from a single proposition and deduction from a finite set of propositions can be reduced to a deduction from a single proposition that is a conjunction of the propositions in that set.

Nevertheless, in the next part of this paper we generalize investigations on dual consequence operations to finitistic consequence operations.

7. Consequence Operation Cn and its Dual Counterpart dCn

Since the bi-level formalization of deductive systems on the basis of unit consequence operations limits the scope of applicability of the notion of Ł-*decidability* to systems founded on unit axioms with respect to acceptance (although the axioms can be treated as a conjunction of a finite number of axioms), it is convenient to consider—instead of the unit rejection consequence operation Cn^{-1}—a stronger than the latter consequence operation $dCn = Cn^{-}$, dual to the finitistic consequence operation Cn. Intuitive refutation of propositions on the basis of a finite number of unacceptable propositions consists in rejecting the proposition on the basis of one that is the disjunction of the propositions rejected earlier. Then, the theory **T** can be strengthened by adding to it Wójcicki's definition [37] of the consequence operation Cn^- dual to the ordinary finite consequence Cn:

Definition 7.1. (the dual consequence operation Cn^- corresponding to consequence operation Cn)

$$dCn(X) = Cn^-(X) = \{y \mid \exists Y \in FinX(\bigcap\{Cn(\{x\}) \mid x \in Y\} \subseteq Cn(\{y\}))\}.$$

The operation Cn^- is a finitistic consequence operation because from the above definition the following metatheorem follows:

Metatheorem 7.2. *The operation Cn^- satisfies the general axioms* A1–A5 *of the consequence theory* **T**.

Let us notice that as long as it concerns singletons, the operation Cn^- does not differ from the operation Cn^{-1}:

Corollary 7.3.

$$Cn^-(\{x\}) = dCn^{+1}(\{x\}) = dCn(\{x\}) = Cn^{-1}(\{x\}) \quad for\, any\, x \in S.$$

The operation Cn^- is not a unit operation consequence. It is stronger than Cn^{-1} in the sense of the lattice theory, i.e.

$$Cn^{-1} \leq Cn^-.$$

Nevertheless, it satisfies intuitions related to the rejected consequence Cn^{-1} if we put some new conditions for the consequence operation Cn.

Definition 7.4. (*the conjunctive operation Cn*) Let the language of a given deductive system include k as the symbol of conjunction of its sentences. The operation Cn is *conjunctive* iff for every finite set of sentences $x_1, \ldots, x_n$ of the system

$$Cn(\{k(x_1, \ldots, x_n)\}) = Cn(\{x_1, \ldots, x_n\}),$$

where $k(x_1, \ldots, x_n)$ is the conjunction of sentences $x_1, \ldots, x_n$, that can be defined inductively (for $n = 1, k(x_1) = x_1$).

Corollary 7.5. *If Cn is the conjunctive consequence operation, then for any finite set $X = \{x_1, \ldots, x_n\}$*

$$\left\{\bigcap Cn^{-1}\{x_i\} \mid x_i \in X\right\} = Cn^{-1}\{k(x_1, \ldots, x_n)\}.$$

Theorem 7.6. *If Cn is the conjunctive consequence operation and $Cn(\emptyset) \neq \emptyset$ (it is a non-normal operation), then*

$$(Cn^-)^- = Cn.$$

Proof. Let us assume that

(1) $y \in (Cn^{-})^{-}(X)$.

First, let us consider a case when the existing finite set Y' satisfying the Definition 7.1 for $Cn = Cn^{-}$ is nonempty. Then on the basis of (1) we have:

$$\mathrm{Y'} = \{x_1,\ldots,x_n\} \subseteq X \;\&\; \left\{\bigcap\{Cn^{-}(\{x_i\}) \mid x_i \in \mathrm{Y'}\}\right\} \subseteq Cn^{-}(\{y\}) \quad (1.1)$$

$$\Leftrightarrow \mathrm{Y'} = \{x_1,\ldots,x_n\} \subseteq X \;\&\; \left\{\bigcap\{Cn^{-1}(\{x_i\}) \mid x_i \in \mathrm{Y'}\}\right\} \subseteq Cn^{-1}(\{y\})$$

(Corollary 7.3.)

$$\Leftrightarrow \mathrm{Y'} = \{x_1,\ldots,x_n\} \subseteq X \;\&\; Cn^{-1}(\{k(x_1,\ldots,x_n)\}) \subseteq Cn^{-1}(\{y\})$$

(Corollary 7.5.)

$$\Leftrightarrow \mathrm{Y'} = \{x_1,\ldots,x_n\} \subseteq X \;\&\; Cn(\{y\}) \subseteq Cn(\{k(x_1,\ldots,x_n)\})$$

(Definition 3.1.)

$$\Leftrightarrow \mathrm{Y'} = \{x_1,\ldots,x_n\} \subseteq X \;\&\; y \in Cn(\{x_1,\ldots,x_n\})$$

(Definition 7.4, assumption)

$$\Rightarrow y \in Cn(X).$$

If the existing set Y' is empty, then from (1) we can obtain formula

$$\mathrm{Y'} = \emptyset \,\&\, \left\{\bigcap\{Cn^{-}(\{x_i\}) \mid x_i \in Y'\right\} \subseteq Cn^{-}(\{y\}) \Rightarrow y \in Cn(X). \quad (1.2)$$

For the proof (1.2) we note that if Y' $= \emptyset$ & $\bigcap\{Cn^{-}(\{x_i\}) \mid x_i \in \mathrm{Y'}\} \subseteq Cn^{-}(\{y\})$, then $Cn^{-}(\{y\}) = S$, and we have to make use of our assumption: $Cn(\emptyset) \neq \emptyset$.

From (1), (1.1) and (1.2) we have

(2) $(Cn^{-})^{-}(X) \subseteq Cn(X)$ for any $X \subseteq S$.

In the proof of the reverse inclusion to (2):

(3) $Cn(X) \subseteq (Cn^{-})^{-}(X)$ for any $X \subseteq S$

we consider a case when $X = \emptyset$ and a case when $X \neq \emptyset$. In the first case, it is convenient to make use of the property: $(Cn^{-})^{-}(\emptyset) = \bigcap\{Cn(\{x_i\}) \mid x_i \in S\}$ (see [6]). In the second case, we can get (3), starting from the end of the proof (1.1) and moving upward. □

Theorem 7.6 is a generalization of the Theorem 6.7 holding for unit consequence operations, in particular, for operations Cn^{+1} and Cn^{-1}.

In accordance to Theorem 7.6, the consequence operation Cn^{-} is not only dual to the ordinary finitistic consequence operation *Cn*, but the consequence operation *Cn* is also dual to the consequence operation Cn^{-} provided that it is a non-normal conjunctive consequence operation. Thus, these two finitistic operations are mutually dual only when some conditions are satisfied.

The next theorem describes the above-mentioned relation between the rejection consequence operations Cn^{-1} and the dual consequence operation Cn^{-}. It requires introducing the notion of a disjunctive operation.

Definition 7.7. (*disjunctive operation* Cn) Let the language of a deductive system include a as the symbol of a disjunction of its sentences. The operation Cn is *disjunctive* iff for every finite set Y of sentences $x_1,\ldots,x_n$ of the system

$$Cn\{a(x_1,\ldots,x_n)\} = \left\{\bigcap Cn(\{x_i\})\,|\,x_i \in Y\right\},$$

where $a(x_1,\ldots,x_n)$ is the disjunction of sentences $x_1,\ldots,x_n$ defined inductively (for $n = 1, a(x_1) = x_1$).

Theorem 7.8. *If the operation Cn is disjunctive, then* (*see* [6,32])

$$X \neq \emptyset \Rightarrow Cn^-(X) = Cn^{-1}(AX),$$

where AX is the set of all finite disjunctions of propositions of the set X.

So, we justify our intuition on refutation of a proposition on the basis of a finite set of propositions: it is a rejection of the proposition on the basis only one proposition that is a finite disjunction of the propositions of the set.

Let as note that

Theorem 7.9. $$Cn^-(\emptyset) = \{x \mid Cn(\{x\}) = S\}.$$

The counterpart of the theorem Theorem 5.7 is not generally true for the consequence operation Cn and the dual consequence operation Cn^-.

Theorem 7.10. *If the consequence operation is disjunctive and a set $X \neq S$ satisfies the disjunction condition:*

$$(a) \qquad a(x_1,\ldots,x_n) \in X \Rightarrow \text{ there exist } x_i \in X$$

then

$$Cn(X) \subseteq X \Rightarrow Cn^-(S\backslash X) \subseteq S\backslash X.$$

Proof. (*ad absurdum*). Let us assume that $X \neq S, X$ satisfies the property (a) and:

(1) $Cn(X) \subseteq X$, (2) $y \in Cn^-(S\backslash X)$ and (3) $y \notin S\backslash X$.

It follows from formula (2) and Definition 7.1 that there exists a finite set $X' \subseteq S\backslash X$ such that

(4) $\{\bigcap\{Cn(\{x_i\}) \mid x_i \in X'\} \subseteq Cn(\{y\})).$

If $X' = \emptyset$, then from (4) we have $Cn(\{y\}) = S$, and from (3) and (1) it follows that $Cn(\{y\}) \subseteq X$. Hence, we get the equation $X = S$ that is in contradiction with our assumption that $X \neq S$. So, let us assume that $X' = \{x_1,\ldots,x_n\} \subseteq S\backslash X$. It follows from (3) that $\{y\} \subseteq X$. Thus, we obtain

(5) $Cn(\{y) \subseteq Cn(X).$

Formulas (4), (5) and (1) yield

(6) $\{\bigcap\{Cn(\{x_i\})\,|\,x_i \in X'\} \subseteq X$

and on the basis of (6) and the assumption of our theorem that Cn is disjunctive we can state that $Cn\{a(x_1,\ldots,x_n)\} \subseteq X$ and $a(x_1,\ldots,x_n) \in X$. Applying the assumption (a), we state that there exists $x_i \in X$. However, x_i as an element of X' does not belong to X and we have a contradiction. □

Theorem 7.11. *If the consequence operation is conjunctive and the set X satisfies the condition*:

$$(k) \qquad x_1,\ldots,x_n \in X \Rightarrow k(x_1,\ldots,x_n) \in X$$

then

$$Cn^-(S\backslash X) \subseteq S\backslash X \Rightarrow Cn(X) \subseteq X.$$

Proof. (*ad absurdum*). Let us assume that

(1) $Cn^-(S\backslash X) \subseteq S\backslash X$, (2) $y \in Cn(X)$ and (3) $y \notin X$.

From (3) we have: $\{y\} \subseteq S\backslash X$. Thus, from (1) $Cn^-(\{y\}) \subseteq S\backslash X$ and applying Definition 7.1. we have:

(4) $\forall z(Cn(\{y\}) \subseteq Cn(\{z\}) \Rightarrow z \notin X)$.

Formula (4) is equivalent to formula

(5) $\forall z(y \in Cn(\{z\}) \Rightarrow z \notin X)$.

Formula (5) is equivalent to formula

(6) $\forall z(z \in X \Rightarrow y \notin Cn(\{z\})$.

The consequence operation Cn is finitistic and conjunctive. Thus, from (2) and (k) we obtain

(7) $y \in Cn(\{x_1,\ldots,x_n\}) = Cn(\{k(x_1,\ldots,x_n)\}), x_1,\ldots,x_n \in X$ and $k(x_1,\ldots,x_n) \in X$.

Applying $k(x_1,\ldots,x_n)$ to (6) we have

(8) $y \notin Cn(\{k(x_1,\ldots,x_n)\})$.

Thus, formulas (7) and (8) are contradictory. □

In accordance to Theorem 7.10, if Cn is disjunctive and an *infallible* consequence operation, always leading from true (accepted) sentences of a set satisfying the condition (a) to true (accepted) sentences of the set, then the dual consequence Cn^- is *anti-infallible* and always leads from untrue (non-accepted as true) sentences of the complement of the set to untrue (non-accepted as true) sentences of this complement of the given set. Theorem 7.11 is, in some sense, a reverse theorem and holds if Cn is conjunctive and the condition (k) is satisfied.

The method of formal-theoretical characterizing of syntax of deductive systems, in the spirit Tarski's ideas, is based on consideration of axiom systems of the general notion of consequence operation $Cn = Cn^+$ for the deductive systems investigated in regard to *acceptance*, i.e. the systems for which, in the intuitive sense, the Cn^+ consequence operation is an *infallible* operation. By putting: $Cn^- = Cn^{-1}$ or, with the above-mentioned assumptions, by putting: $Cn^- = dCn^+$, we can also characterize formally some deductive systems by means of the *anti-infallible*, in the intuitive sense, consequence operation Cn^{-} and consider the two-level syntactic characterization of the deductive systems (as A-*assertion systems* and R-*refutation systems*) by means of the following triple of notions:

$$(+,-) \qquad\qquad < S, Cn^{+}, Cn^{-} > .$$

Then, we can also formally define the notion of Ł-*decidability*, *saturation*, i.e. *decidability in Łukasiewicz's sense*, for the deductive systems, replacing the symbol 'Cn^{-1}' by the symbol 'Cn^{-}' in the definitions of notions of Ł-*consistence* and Ł-*completeness*, given in Sect. 4.

It is worth mentioning, in this place, that there is a reverse method of a formal-theoretical characterization of syntax of some deductive systems by means of the following triple of notions:

$$(-,+) \qquad\qquad < S, Cn^{-}, Cn^{+} > .$$

In order to characterize deductive systems by means of the triple $(-,+)$, by putting $Cn = Cn^{-}$, we first give the axiom system for the operation Cn^{-}, postulating that it is a finitistic *anti-infallible*, in the intuitive sense, consequence operation, and then we define *infallible* consequence operation Cn^{+} by the formula: $Cn^{+} = dCn^{-}$. Then, the triple of notions (–, +) should be characterized by a theory dual to the theory **T**.

The theory **T**, enriched by some definitions, or the theory dual to **T**, characterized so far, apply to the deductive systems based on a logic. An axiom system for the theory of deductive systems, extended over **T**, based on the classical sentential logic was built by Tarski in his so-called enriched theory $\mathbf{T}^{+}$ of deductive systems [34] and axiom systems for many richer than **T** theories of deductive systems based on non-classical sentential logics were given in papers of Pogorzelski [14] and Słupecki [15] and Pogorzelski and Wojtylak [16]. On the basis of the theory $\mathbf{T}^{+}$, extended by definitions of the operations Cn^{-1} or dCn, and characterized in the next Sect. 8, we can formalize properties of deductive systems based on the classical sentential logic and considered on two above-mentioned levels by means of the triple of notions $(+,-)$. A problem appears: How, in a similar way, to define adequately certain counterparts of rejection or dual consequence operations on the ground of theories of deductive systems based on non-classical sentential logics by means of a triple of notions like $(+,-)$ (cf. [24]).

8. The Classical Consequence Operation Cn^{+} and the Dual Consequences Operations $(Cn^{+})^{-1}$ and $(Cn^{+})^{-}$

8.1. On Adequate Axiom Systems for Theories of Deductive Systems Richer than T

Tarski's theory **T** [33], interpreted as a general consequence operation theory, applies to arbitrary deductive systems. Richer deductive-system theories, extended over **T**, apply only to deductive systems based on a logic (a propositional calculus). The symbol '$Cn^{+}(X)$' should then be intuitively understood as the consequence set, i.e., the set of consequences of X with respect to the set A (which usually is the axiom system for the propositional calculus) and logical proof rules. When a deductive-system theory is based on some propositional calculus, it contains – besides notions corresponding to the system $< S, Cn^+ >$ or $< S, Cn^+, (Cn^+)^- >$ - as many new primitive notions as there are in the logical calculus on which the deductive system described by the richer theory is based.

If the primitive notions of a propositional calculus are the corresponding symbols of the system:

$$\Rightarrow, \sim, \wedge, \vee$$

representing the propositional connectives, respectively: implication, negation, conjunction, disjunction, then the primitive notions of the deductive-system theory based on this calculus are their metalogical counterparts:

$$c, n, k, a$$

characterized by *specific axioms* of the theory.

Let L_C denote the set of all counterparts (called *S-substitutions*) of the laws of the propositional calculus C, which the theory is based on. We shall then say that

Definition 8.1. A system of specific axioms for a theory $\mathbf{T}^C$ based on some propositional calculus C is *adequate* with respect to this calculus if and only if the following conditions hold:

- The expression $L_C \subseteq Cn^+(\emptyset)$ is a theorem of the theory $\mathbf{T}^C$,
- If the expression $\alpha \in Cn^+(\emptyset)$ is a theorem of the theory $\mathbf{T}^C$, then $\alpha \in L_C$.

Theories of deductive systems based on classical propositional calculi with different axiom systems are called *classical deductive-system theories.* They are *theories of the classical consequence operation* Cn^+.

8.2. Adequate Axiom Systems for the Classical Consequence Operation Cn^+

In this part we present the axiom system of the original so-called *Tarski's enriched deductive systems theory* [34] and one of its equivalent variants. These axiom systems characterize formally the classical consequence operation Cn^+.

Tarski's classical consequence operation theory, denoted here by $\mathbf{T}_0^+$, is the theory of deductive systems based on the implicational-negational propositional Łukasiewicz's calculus $C^{\Rightarrow,\sim}$. Its adequate specific axiom system characterizes the classical consequence operation Cn^+ using the counterparts of the classical connectives of implication $\Rightarrow$ and negation $\sim$, i.e. using metalinguistic connectives c and n. The primitive terms of $\mathbf{T}_0^+$ are

PT: S, Cn^+, c, n.

They are characterized by means of the axiom system consisting of axioms A1–A5 of $\mathbf{T}$ and some specific axioms for c and n:

A1. $\mathrm{card}(S) \leq \aleph_o$
A2. $X \subseteq Cn^+(X) \subseteq S$

A3. $Cn^+Cn^+(X) = Cn^+(X)$
A4. $X \subseteq Y \Rightarrow Cn^+(X) \subseteq Cn^+(Y)$
A5. $Cn^+(X) \subseteq \bigcup\{Cn^+(Y) \mid Y \in Fin\,(X)\}$
A'6$^+$. $cxy, nx \in S$
A7$^+$. $cxy \in Cn^+(X) \Leftrightarrow y \in Cn^+(X \cup \{x\})$
A8$^+$. $Cn^+(\{x, nx\}) = S$
A9$^+$. $Cn^+(\{x\}) \cap Cn^+(\{nx\}) = Cn^+(\emptyset)$.

In $\mathbf{T}_0^+$ we define connectives k and a as follows:

$$\mathrm{D}k.\, k(x,y) =_{df}\; ncxny \qquad \text{and} \qquad \mathrm{D}a.\, a(x,y) =_{df}\; cnxy.$$

The theory $\mathbf{T}_0^+$ of the classical consequence operation Cn^+ can be replaced by the equivalent theory $\mathbf{T}^+$ characterizing deductive systems based on the classical propositional calculus $C^{\Rightarrow,\sim,\wedge,\vee}$ with implication, negation, conjunction and disjunction as primitive notions. Then, the primitive terms of $\mathbf{T}^+$ are the notions of the following tuple:

$$< S, Cn^+, n, c, k, a >$$

and its axiom system consists of axioms: A1–A5, A6$^+$: $cxy, nx, k(x,y)$, $a(x,y) \in S$; axioms A7$^+$–A9$^+$ and the following two additional axioms:

A10$^+$. $Cn^+(\{k(x,y)\}) = Cn^+(\{x,y\})$
A11$^+$. $Cn^+(X \cup \{a(x,y)\}) = Cn^+(X \cup \{x\}) \cap Cn^+(X \cup \{y\})$.

Let us note that according to A10$^+$ the classical consequence operation Cn^+ is conjunctive and with accordance to A11$^+$ it is a disjunctive operation.

In $\mathbf{T}^+$ we accept the Definition 5.2 of the unit operation Cn^{+1} (for $Cn = Cn^+$).

8.3. A Two-sided Characterization of Deductive-Systems by Means of the Operations Cn^+ and $(Cn^+)^{-1}$

A theoretical two-sided syntactic characterization of deductive systems based on the classical propositional calculus $C^{\Rightarrow,\sim,\wedge,\vee}$ is possible by extended the theory $\mathbf{T}^+$ to the theory $\mathbf{T}^{<+,-1>}$ axiomatizing the tuple of notions of the type $< +, -1 >$, exactly – the following tuple of primitive notions:

$$< +, -1 > \qquad\qquad < S, Cn^+, (Cn^+)^{-1}, n, c, k, a >,$$

where $(Cn^+)^{-1}$ is the rejection consequence operation defined by means of the Definition 3.1 (for $Cn = Cn^+$) added to the theory $\mathbf{T}^+$ (see [29,39]).

The theory $\mathbf{T}^{<+,-1>}$ allows characterizing every deductive system based on the classical propositional calculus $C^{\Rightarrow,\sim,\wedge,\vee}$ on two levels: first as an A-asserted system and then – as a R^{-1}- refutation system.

In $\mathbf{T}^{<+,-1>}$ the definitions of syntactic notions, in particular the notion of Ł-*decidabilty* (decidability in Łukasiewicz's sense) for a complete syntactic characterization of deductive systems, are the same as in $\mathbf{T}$. Of course, all the theorems formulated on the basis of the theory $\mathbf{T}$ enriched by the Definition 3.1 are valid in $\mathbf{T}^{<+,-1>}$.

The classical consequence operation Cn^+ can be characterized in the theory $\mathbf{T}^{<+,-1>}$ by means of the rejection operation $(Cn^+)^{-1}$ in the following way (see [29,30,39]):

Theorem 8.2.

a. $X \neq \emptyset \Rightarrow (Cn^+(X) = N(Cn^+)^{-1}(ANX) = N(Cn^+)^{-1}(NKX),$
b. $Cn^+(X) = N(Cn^+)^{-1}(ANX \cup \{ncxx\}) = N(Cn^+)^{-1}(N(KX \cup \{cxx\})),$

where NX is the set of all propositions contrary to the propositions of the set X, AX is the set of all finite disjunctions of the set X and KX is the set of all finite conjunctions of propositions of X.

Theorem 8.2 suggests a possibility of a theoretical description of syntax of deductive systems based on the classical calculus by reversing the levels of considerations and their description first as R^{-1}-refutation systems and then as A-asserted systems. The starting point is then an axiomatizing of the rejection operation Cn^{-1} as a primitive notion and defining the classical consequence operation Cn^+ by means of the operation.

8.4. A Two-sided Characterization of Deductive Systems by means of the Operations Cn^{-1} and Cn^+

The aforementioned reverse direction of the theoretical bi-level description of the syntax of deductive systems consists in axiomatizing of the following tuple of notions:

$$< -1, + > \qquad\qquad < S, Cn^{-1}, Cn^+, n, c, k, a > .$$

For this purpose, we first formalize the following tuple of primitive notions of the so-called theory $\mathbf{T}^{-1}$ of rejected propositions:

$$< S, Cn^{-1}, n, c, k > .$$

The axiom system for $\mathbf{T}^{-1}$, in which $Cn^{-1} : P(S) \to P(S)$ is the primitive notion, is compounded from the following axioms (see [39,41]):

A1^{-1}. $\mathrm{card}(S) \leq \aleph_o$
A2^{-1}. $nx, cxy, k(x,y) \in S$
A3^{-1}. $y \in Cn^{-1}(X) \Leftrightarrow \exists x \in X(Cn^{-1}(\{\mathrm{y}\}) \subseteq Cn^{-1}(\{x\}))$
A4^{-1}. $y \in Cn^{-1}(cxy)$
A5^{-1}. $x \in Cn^{-1}(\{y\}) \Leftrightarrow Cn^{-1}(cxy) = S$
A6^{-1}. $x_1 \in Cn^{-1}(\{y_1\}) \wedge x_2 \in Cn^{-1}(\{y_2\}) \Rightarrow k(x_1,x_2) \in Cn^{-1}(\{k(y_1,y_2)\})$
A7^{-1}. $Cn^{-1}(\{ck(x,y)z\}) = Cn^{-1}(\{cxcyz\}$
A8^{-1}. $k(x,nx) \in Cn^{-1}(\{y\})$
A9^{-1}. $x, nx \in Cn^{-1}(\{y\}) \Rightarrow Cn^{-1}(\{y\}) = S$
A10^{-1}a. $k(x,y) \sim k(y,x)$
A10^{-1}b. $k(k(x,y),z) \sim k(x,k(y,z))$
A10^{-1}c. $y \sim z \Rightarrow k(x,y) \sim k(x,z)$
A10^{-1}d. $k(x,x) \sim x$
D$_\sim$. $x \sim y \Leftrightarrow \forall z(x \in Cn^{-1}(\{z\}) \Leftrightarrow y \in Cn^{-1}(\{z\}))$
Da. $a(x,y) =_{df} cnxy.$

Let us notice that the axiom $A3^{-1}$ states that the operation Cn^{-1} is a unit consequence (see Sect. 6, axiom A^1).

By expanding $\mathbf{T}^{-1}$ with the following additional definition of the finitistic operation Cn^+:

DCn^+. $Cn^+(X) = NCn^{-1}(ANX \cup \{ncxx\}) = NCn^{-1}(N(KX \cup \{cxx\}))$

we obtain the theory $\mathbf{T}^{<-1,+>}$ characterizing the above-given tuple of notions $< -1, + >$ and in which all axioms of the theory $\mathbf{T}^{<+,-1>}$ are theorems. Since all axioms of $\mathbf{T}^{<-1,+>}$ (for $Cn^{-1} = (Cn^+)^{-1}$) and the definition DCn^+ are theorems of $\mathbf{T}^{<+,-1>}$ these two theories are equivalent and dual. So,

(*) $\mathbf{T}^{<+,-1>}$ is equivalent to $\mathbf{T}^{<-1,+>}$.

8.5. A Two-sided Characterization of Deductive Systems By Means of Unit Consequence Operations

A theoretical bi-aspectual description of syntax of deductive systems based on the classical propositional calculus can be made by a suitable axiomatic characterization of the following tuples of notions:

$$< +1, +, -1 > \qquad < S, Cn^{+1}, Cn^+, Cn^{-1}, n, c, k, a >$$

or

$$< -1, +1, + > \qquad < S, Cn^{-1}, Cn^{+1}, Cn^+, n, c, k, a >,$$

where Cn^{+1} is intuitively understood as the unit consequence operation induced by the classical consequence operation Cn^+ and $Cn^{-1}-$ as the unit rejection consequence operation (see Sects. 5, 4).

Justification of the above statement follows from the fact that the classical consequence operation theory $\mathbf{T}^+$ is equivalent to the theory $\mathbf{T}^{+1}$ characterizing the tuple of primitive notions

$$< S, Cn^{+1}, n, c, k >$$

by means of the following axioms (cf. [28]):

$A1^{+1}$. $\mathrm{card}(S) \leq \aleph_o$
$A2^{+1}$. $nx, cxy, k(x,y) \in S$
$A3^{+1}$. $y \in Cn^{+1}(X) \Leftrightarrow \exists x \in X(Cn^{+1}(\{y\}) \subseteq Cn^{+1}(\{x\}))$
$-Cn^{+1}$ is a unit consequence operation
$A4^{+1}$. $KCn^{+1}(KX) \subseteq Cn^{+1}(KX)$
$A5^{+1}$. $cxy \in Cn^{+1}(X) \Rightarrow y \in Cn^{+1}(K(X \cup \{x\}))$
$A7^{+1}$. $Cn^{+1}(\{x\}) \cap Cn^{+1}(\{nx\}) = Cn^{+1}(\{cxx\})$
$A8^{+1}$. $Cn^{+1}(\{k(x,nx)\}) = S$,

where KX is the set of all finite conjunctions of propositions in X and the finite consequence operation Cn^+ is defined as follows:

DCn^+. $Cn^+(X) = Cn^{+1}(KX \cup \{cxx\})$;

the connective a is defined by Da.

It is obvious that in $\mathbf{T}^+$ we define the unit operation Cn^{+1} as in Sect. 5:

DCn^{+1}. $Cn^{+1}(X) = \{y \mid \exists x \in X(y \in Cn^+(\{x\}))\}$.

Adding to the theory $\mathbf{T}^{+1}$ the Definition 3.1 of the rejection operation Cn^{-1} (for $Cn = Cn^{+}$) we obtain the theory $\mathbf{T}^{<+1,+,-1>}$ and we can state that

(**) $\mathbf{T}^{<+1,+,-1>}$ is equivalent to $\mathbf{T}^{<+,-1>}$.

Adding to the theory $\mathbf{T}^{-1}$ the following definition of the operation Cn^{+1}:
DCn^{+1}. $y \in Cn^{+1}(X) \Leftrightarrow \exists x \in X(y \in Cn^{-1}(\{x\}))$
and the definition DCn^{+}, we obtain the theory $\mathbf{T}^{<-1,+1,+>}$ and we can state that

(***) $\mathbf{T}^{<-1,+1,+>}$ is equivalent to $\mathbf{T}^{<-1,+>}$.

From (*), (**), (***) it follows that

$$\mathbf{T}^{<+1,+,-1>} \text{ is equivalent to } \mathbf{T}^{<-1,+1,+>}$$

and deductive systems based on the classical propositional calculus can be characterized on two opposite levels: by means of unit consequence operations and the notions of the type $< +1, +, -1 >$ or of the notions of the type $< -1, +1, + >$.

8.6. A Two-sided Characterization of Deductive Systems by means of the Dual Operations Cn^{+} and Cn^{-}

A theoretical bi-aspectual description of the syntax of deductive systems based on the classical propositional calculus can be made by a suitable axiomatic characterization of the following tuple of notions:

$< +, - >$ $\quad < S, Cn^{+}, Cn^{-}, n, c, k, a >$

or the dual tuple of notions:

$< -, + >$ $\quad < S, Cn^{-}, Cn^{+}, n, c, a, k >$,

where the first tuple is characterized by the theory $\mathbf{T}^{<+,->}$, while the second one by the theory $\mathbf{T}^{<-,+>}$. These theories are characterized by the axiom systems and definitions given below (cf. Spasowski [32]):

Theory $\mathbf{T}^{<+,->}$:

A1. $\text{card}(S) \leq \aleph_o$
A2. $X \subseteq Cn^{+}(X) \subseteq S$
A3. $Cn^{+}Cn^{+}(X) = Cn^{+}(X)$
A4. $X \subseteq Y \Rightarrow Cn^{+}(X) \subseteq Cn^{+}(Y)$
A5. $Cn^{+}(X) \subseteq \bigcup\{Cn^{+}(Y) \mid Y \in Fin\,(X)\}$
A'6^{+}. $cxy, nx, kxy, axy \in S$
A7^{+}. $cxy \in Cn^{+}(X) \Leftrightarrow y \in Cn^{+}(X \cup \{x\})$
A8^{+}. $Cn^{+}(\{x, nx\}) = S$
A9^{+}. $Cn^{+}(\{x\}) \cap Cn^{+}(\{nx\}) = Cn^{+}(\emptyset)$
A10^{+}. $Cn^{+}(\{k(x, y)\}) = Cn^{+}(\{x, y\})$
A11^{+}. $Cn^{+}(X \cup \{a(x, y)\}) = Cn^{+}(X \cup \{x\}) \cap Cn^{+}(X \cup \{y\})$
Dc^{-}. $c^{-}xy =_{df} ncyx$
DCn^{-}. Wójcicki's Definition 7.1 of dual consequence operation induced by the conseuence operation $Cn^{+}(Cn^{-} = dCn^{+})$.

The implication $c^{-}xy$ bears the following intuition: it is true in one and only one case, when both nx and y are true propositions, i.e., when x is a false proposition and y is a true proposition.

Theory $\mathbf{T}^{<-,+>}$:

A1. $\mathrm{card}(S) \leq \aleph_o$
A2. $X \subseteq Cn^{-}(X) \subseteq S$
A3. $Cn^{-}Cn^{-}(X) = Cn^{-}(X)$
A4. $X \subseteq Y \Rightarrow Cn^{-}(X) \subseteq Cn^{-}(Y)$
A5. $Cn^{-}(X) \subseteq \bigcup\{Cn^{-}(Y) \mid Y \in Fin\,(X)\}$
A'6^{-}. $c^{-}xy, nx, kxy, axy \in S$
A7^{-}. $c^{-}xy \in Cn^{-}(X) \Leftrightarrow y \in Cn^{-}(X \cup \{x\})$
A8^{-}. $Cn^{-}(\{x, nx\}) = S$
A9^{-}. $Cn^{-}(\{x\}) \cap Cn^{-}(\{nx\}) = Cn^{-}(\emptyset)$
A10^{-}. $Cn^{-}(X \cup \{k(x,y)\}) = Cn^{-}(X \cup \{x\}) \cap Cn^{-}(X \cup \{y\})$
A11^{-}. $Cn^{-}(\{a(x,y)\}) = Cn^{-}(\{x,y\})$
Dc. $cxy =_{df} nc^{-}yx$
DCn^{-}. Wójcicki's Definition 7.1 of dual consequence operation induced by the conseuence operation Cn^{-} ($Cn^{+} = dCn^{-}$).

Let us also note that

$$\mathbf{T}^{+} \text{ is equivalent to } \mathbf{T}^{-}$$

and deductive systems based on the classical propositional calculus can be characterized twofold by means of the notions of the type $< +, - >$ or of the type $< -, + >$.

Let us note that that direct implication following from the axiom A7^{+} of the theory $\mathbf{T}^{+}$ corresponds to the usual *modus ponens rule* r^{+} and the direct implication following from axiom A7^{-} of the theory $\mathbf{T}^{-}$ is a counterpart of the *refutation modus ponens rule* r^{-} corresponding to Łukasiewicz's rejection rule r_0^{-}:

Modus ponens	*Refutation modus ponens*	*Łukasiewiczi's rejection rule*
$r^{+}\!: \dfrac{\vdash cxy \quad \vdash x}{\vdash y}$	$r^{-}\!: \dfrac{\dashv c^{-}xy \quad \dashv x}{\dashv y}$	$r_0^{-}\!: \dfrac{\vdash cxy \quad \dashv y}{\dashv x}$

The symbol '$\vdash$' denotes here the symbol of assertion while the symbol '$\dashv$' – the symbol of refutation.

These rules are sufficient to carry out an examination of the complete two-sided syntactic characterization of deductive systems based on classical deductive systems and an examination of Ł-*decidability* of some other deductive systems based on non-classical propositional logics (in many such systems that are not based on metalanguage characterization of these logics, apart from the Łukasiewicz's rule r_0^{-} also his rejection rule by substitution that states: if a substitution of a formula is rejected then the formula is rejected too).

It is worth mentioning here that it is difficult to build some axiomatic theories of deductive systems based a non-classical logic and formalizing not only the notion of usual non-classical consequence operations of the type Cn but also comprising properties of dual—to them—consequence operations of the type Cn^-, because in many such deductive systems, beside refutation rules given above, there oblige additional specific refutation rules.

9. More Important Findings Concerning Ł-Decidability of Deductive Systems

Let **S** be any given deductive system and *For* be the set of all of its formulas. Let the bi-aspectual axiomatic method characterization of **S** be based on its $S^+ \in$ A-system and its $S^- \in$ R-system characterized, respectively, by the couples:

$$< A, R > \quad \text{and} \quad < A^-, R^- >,$$

where A is the countable set of all its asserted accepted axioms, A^- the countable set of its refutation axioms, R the set of all its inference rules and R the set of all its refutation rules.

The asserted system $S^+ = C^+(A, R)$ is the set of all formulas of *For* being consequences obtained from axioms of A by means of inference rules of R, i.e. it is the smallest set including the set A and closed under the inference rules of R.

The refutation system $S^- = C^-(A^-, R^-)$ is the set of all formulas of *For* being consequences obtained from refutation axioms of A^- by means of inference rules of R^-, i.e. it is the smallest set including the set A^- and closed under the refutation rules of R^-.

A general theory of refutation systems is presented by Skura [24].

The deductive system **S** is Ł-*decidable* (*decidable in Łukasiewicz's sense*) if and only if the following conditions are satisfied:

(i) $S^+ \cap S^- = \emptyset$ – **S** is Ł-*consistent*,
(ii) $S^+ \cup S^- = For$ – **S** is Ł-*complete*.

The relation between Ł-*decidability* and decidability in the ordinary sense describes the following metatheorem formulated by Słupecki [27]:

Metatheorem 9.1. *If the deductive system* **S** *is Ł-decidable and the set* S^+ *of all its accepted formulas and the set* S^- *of all its rejected formulas are recursively enumerable sets, then the deductive system* **S** *is decidable.*

Nowadays it is a common methodological principle to consider Ł-*decidability* of concrete logical deductive systems. Some results of research on Ł-*decidability* of such systems are presented in papers of Bryll [6], Skura [17–23], Sochacki [31], Wybraniec-Skardowska [40, 44]. Let us note here only that all well-known propositional calculi are Ł-*decidable* (Ł-*saturated*).

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