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How Many Structure Constants do Exist in Riemannian Geometry?

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Abstract After reading such a question, any mathematician or physicist will say that, according to a well known result of L.P. Eisenhart found in 1926, the answer is surely "One", namely the constant allowing to describe the so-called " *constant Riemannian curvature* " condition. The purpose of this paper is to prove the contrary by studying the case of two dimensional Riemannian geometry in the light of an old work of E. Vessiot published in 1903 but *still totally unknown today* after more than a century. In fact, we shall compute locally the *Vessiot structure equations* and prove that there are indeed " Two " *Vessiot structure constants* satisfying a single *linear Jacobi condition* showing that one of them must vanish while the other one must be equal to the known one or that both must be equal. This result depends on deep mathematical reasons in the formal theory of Lie pseudogroups, involving both the Spencer δ-cohomology and diagram chasing in homological algebra. Another similar example will illustrate and justify this comment out of the classical tensorial framework of the famous " *equivalence problem* ". The case of contact transformations will also be studied. Though it is quite unexpected, we shall reach the conclusion that the mathematical foundations of both classical and conformal Riemannian geometry must be revisited. We have treated the case of conformal geometry and its application in recent papers (Pommaret in J Mod Phys 12:829–858, 2021. https://doi.org/10.4236/jmp.2020.1110104; The conformal group revisited. arxiv:2006.03449; Nonlinear conformal electromagnetism. arxiv:2007.01710).

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1 Introduction

When X is a manifold of dimension n with local coordinates $(x^1, ..., x^n)$, we first sketch the discovery of Vessiot ([8,22]) *still not known today after more than a century* for reasons which are not scientific at all ([10]). Roughly, using standard notations of jet theory ([8–11,21]), a Lie pseudogroup $\Gamma \subset aut(X)$ is made by finite invertible transformations $y = f(x)$ solutions of a (nonlinear in general) system of OD or PD equations $\mathcal{R}_q \subset \Pi_q$ while, using vertical bundles, the infinitesimal transformations $\xi \in \Theta$ are solutions of the linearized system $R_q =$

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$id_q^{-1}(V(\mathcal{R}_q)) \subset J_q(T)$ where $T = id^{-1}(V(X \times X)$ is the tangent bundle of X, $id : X \to X \times X : x \to (x, x)$ the identity map and $id_q = j_q(id)$ the q-jet of the identity. When Γ is transitive, there is a canonical epimorphism $\pi_0^q : R_q \to T$. Also, as changes of source x commute with changes of target y, they exchange between themselves any generating set of *differential invariants* $\{\Phi^\tau(y_q)\}$ of order q. Then one can introduce a *natural bundle* $\mathcal{F}$ over X, also called *bundle of geomeric objects*, by patching changes of coordinates of the form $\bar{x} = \varphi(x)$, $\bar{u} = \lambda(u, j_q(\varphi(x))$ thus obtained (see examples below). A section ω of $\mathcal{F}$ is called a *geometric object* or *structure* on X and transforms like $\bar{\omega}(f(x)) = \lambda(\omega(x), j_q(f)(x))$ or simply $\bar{\omega} = j_q(f)^{-1}(\omega)$. This is a way to generalize vectors and tensors ($q = 1$) or even connections ($q = 2$). As a byproduct, we have $\Gamma = \{f \in aut(X) | j_q(f)^{-1}(\omega) = \omega\}$ and we may say that Γ *preserves* ω. Replacing $j_q(f)$ by f_q, we also obtain the *Lie form* $\mathcal{R}_q = \{f_q \in \Pi_q | f_q^{-1}(\omega) = \omega\}$. Coming back to the infinitesimal point of view and setting $f_t = exp(t\xi) \in aut(X), \forall \xi \in T$, we may define the *ordinary Lie derivative* with value in the vector bundle $F_0 = \omega^{-1}(V(\mathcal{F}))$ by the formula:

$$\mathcal{D}\xi = \mathcal{L}(\xi)\omega = \frac{d}{dt} j_q(f_t)^{-1}(\omega)|_{t=0} \Rightarrow \Theta = \{\xi \in T | \mathcal{L}(\xi)\omega = 0\}$$

and we say that $\mathcal{D}$ is a *Lie operator* because $\mathcal{D}\xi = 0, \mathcal{D}\eta = 0 \Rightarrow \mathcal{D}[\xi, \eta] = 0$ as we already saw.

In the jet framework at any order q, we shall introduce by linearity as in ([8,11,14,15]) the *formal Lie derivative* $L(\xi_q)$ in such a way that $\mathcal{L}(\xi) = L(j_q(\xi))$. It follows that the infinitesimal Lie equations defining R_q can be written in the so-called *Medolaghi form* $L(\xi_q)\omega = 0$ with coefficients depending on $j_1(\omega)$ in a very specific fashion ([6,8]):

$$\Omega^\tau \equiv (L(\xi_q)\omega)^\tau = -L_k^{\tau\mu}(\omega(x))\xi_\mu^k + \xi^r \partial_r \omega^\tau(x) = 0$$

Let us suppose that the symbol $g_q \subset S_q T^* \otimes T$ is involutive, in such a way that this system becomes formally integrable and thus involutive, that is all the equations of order $q + r$ could be obtained by differentiating r times *only*, $\forall r \geq 0$. Then, as we shall see in the following examples, ω must satisfy certain (non-linear in general) integrability conditions of the form:

$$I(j_1(\omega)) = c(\omega)$$

called *Vessiot structure equations*, linearly depending on a certain number of *Vessiot structure constants* c eventually satisfying algebraic *Jacobi conditions* $J(c) = 0$ expressed by polynomials of degree ≤ 2. With more details, the degree is exactly 2 when $q = 1$ with homogeneous polynomials but can be equal to 1 and we let the reader compare this situation to the Riemannian or contact cases ([8], Theorem 4.8, p 325 and Example 4.15, p 329, [14]). Contrary to the structure constants of Cartan, the structure constants of Vessiot have NOTHING TO DO with Lie algebras and the constant Riemannian curvature is indeed a fine example. In the second section, we shall treat a specific example when $n = 2$ while in the third section, we shall treat the Riemannian case with full details when $n = 2$. The most striking result of this paper is that, though at first sight there does not seem to be any link between these two examples, we shall discover at the end of the paper that they are in fact... identical !. We want to point out that these structure equations were perfectly known by E. Cartan (1869–1951) who *never* said that these results were at least competing with or even superseding the corresponding *Cartan structure equations* that he has developed about at the same time for similar purposes ([1]). The underlying reason is of a purely personal origin related to the *differential Galois Theory* within a kind of "*mathematical affair* " involving the best french mathematicians of that time ([9]). The original letters, given to the author of this paper by M. Janet, a friend of E. Vessiot, have ben published in ([10]) and have been put as a deposit in the main library of Ecole Normale Supérieure in Paris for future historical studies.

Finally, we can choose for the generating *compatibility conditions* (CC) $\mathcal{D}_1$ of $\mathcal{D}$ the first order linearization of a non-linear version described by the Vessiot structure equations:

$$\frac{\partial I}{\partial j_1(\omega)}(j_1(\omega))j_1(\Omega) = \frac{\partial c}{\partial \omega}(\omega)\Omega$$

that is *exactly* what is usually done for the flat Minkowski metric in general relativity ([12,16,19]).

2 Motivating Examples

We show that the Vessiot structure equations may even exist when $n = 1$ ([12]). In the remaining of this paper, the reader may refer to ([7,20] or [11]) for the elements of homological algebra allowing to chase in the commutative diagrams that we shall present.

Example 2.1 When $m = n = 1$, the *affine* transformations $y = ax + b$ are solutions of the second order linear system $y_{xx} = 0$, the sections of the corresponding linearized systems are respectively satisfying $\xi_{xx} = 0$. The only generating differential invariant $\Phi \equiv y_{xx}/y_x$ of the affine case transforms like $u = \bar{u}\partial_x\varphi + (\partial_{xx}\varphi/\partial_x\varphi)$ when $\bar{x} = \varphi(x)$. The corresponding geometric object defined by the section $u = \gamma(x)$ does transform like the Christoffel symbols, namely:

$$\gamma(x) = \bar{\gamma}(f(x))\partial_x\varphi + (\partial_{xx}\varphi/\partial_x\varphi)$$

For this, if γ is the geometric object of the affine group $y = ax + b$ and $0 \neq \alpha = \alpha(x)dx \in T^*$ is a 1-form, we consider the geometric object $\omega = (\alpha, \gamma)$ and get at once the two *Medolaghi equations*:

$$\mathcal{L}(\xi)\alpha \equiv \alpha\partial_x\xi + \xi\partial_x\alpha = 0, \qquad \mathcal{L}(\xi)\gamma \equiv \partial_{xx}\xi + \gamma\partial_x\xi + \xi\partial_x\gamma = 0$$

Differentiating the first equation and substituting the second, we get the zero order equation:

$$\xi(\alpha\partial_{xx}\alpha - 2(\partial_x\alpha)^2 + \alpha\gamma\partial_x\alpha - \alpha^2\partial_x\gamma) = 0 \quad \Leftrightarrow \quad \xi\partial_x\left(\frac{\partial_x\alpha}{\alpha^2} - \frac{\gamma}{\alpha}\right) = 0$$

and the *Vessiot structure equation* $\partial_x\alpha - \gamma\alpha = c\alpha^2$ with $c = cst$. Alternatively, setting $\beta = -1/\alpha \in T$, we get $\partial_x\beta + \gamma\beta = c$. With $\alpha = 1, \beta = -1, \gamma = 0 \Rightarrow c = 0$ we get the translation subgroup $y = x + b$ while, with $\alpha = 1/x, \beta = -x, \gamma = 0 \Rightarrow c = -1$ we get the dilatation (sometime called dilation) subgroup $y = ax$.

Working now with isometries, we just need to set $\omega = \alpha^2$ in order to obtain the Killing equation $2\omega\xi_x + \xi\partial_x\omega = 0$ and the corresponding Vessiot structure equation $\partial_x\omega - 2\omega\gamma = c'\omega^{\frac{3}{2}}$ with $c' = 2c$. Similarly, if ν is the geometric object of the projective group $y = (ax + b)/(cx + d)$, transforming like the well known *Schwarzian differential invariant* $\Psi = (y_{xxx}/y_x) - \frac{3}{2}(y_{xx}/y_x)^2 = d_x\Phi - \frac{1}{2}\Phi^2$, we may consider the new geometric object $\omega = (\gamma, \nu)$ and get the sole *Vessiot structure equation* $\partial_x\gamma - \frac{1}{2}\gamma^2 - \nu = 0$ in a coherent way, without any structure constant.

Example 2.2 With $m = n = 2$, let us now consider the Lie group of transformations $\{y^1 = ax^1 + b, y^2 = cx^2 + d \mid a, b, c, d = cst, ac = 1\}$ as an algebraic Lie pseudogroup Γ. It is easy to exhibit the corresponding first order system $\mathcal{R}_1$ of finite Lie equations in Lie form by introducing the three generating differential invariants and the corresponding Lie form:

$$\Phi^1 \equiv \frac{y^1_2}{y^1_1} = 0, \quad \Phi^2 \equiv \frac{y^2_1}{y^2_2} = 0, \quad \Phi^3 \equiv y^1_1 y^2_2 = 1$$

The details of the corresponding tricky computations, first done in 1978 ([8]), have been improved in 2016 ([14]) and are again revisited in this paper. As we shall see, its major interest is to work out the *two* Vessiot structure constants existing like for the Riemannian structure but without having *any* tensorial framework ([5]).

First of all, we notice that this system is of finite type with a vanishing second order symbol and is thus formally integrable but not involutive. For this, we may introduce the six generating differential invariants obtained after one prolongation, exactly like the six Christoffel symbols in 2-dimensional Riemannian geometry:

$$\Phi^4 \equiv \frac{y^1_{11}}{y^1_1} = 0, \Phi^5 \equiv \frac{y^1_{12}}{y^1_1} = 0, \Phi^6 \equiv \frac{y^1_{22}}{y^1_1} = 0, \Phi^7 \equiv \frac{y^2_{22}}{y^2_2} = 0, \Phi^8 \equiv \frac{y^2_{12}}{y^2_2} = 0, \Phi^9 \equiv \frac{y^2_{11}}{y^2_2} = 0$$

while noticing that $\Phi^3(1 - \Phi^1\Phi^2) \equiv y^1_1 y^2_2 - y^1_2 y^2_1 \neq 0$ is changing like the Jacobian of the change of coordinates $\bar{x} = \varphi(x)$.

Looking to the way these invariants are transformed under an arbitrary change $\bar{x} = \varphi(x)$, we obtain for example $u^1 = (\partial_2\varphi^1 + \bar{u}^1\partial_2\varphi^2)/(\partial_1\varphi^1 + \bar{u}^1\partial_1\varphi^2)$ and so on for describing the natural fiber bundle $\mathcal{F}$ with local coordinates $(x^1, x^2; u^1, u^2, u^3)$ and section $\omega = (\omega^1, \omega^2, \omega^3)$ becoming $(0, 0, 1)$ with our choice of the above Lie form for $\mathcal{R}_1$ and $\omega^3(1-\omega^1\omega^2) = 1 \neq 0$. Passing to the infinitesimal point of view, we obtain the first order system $R_1 \subset J_1(T)$ in the Medolaghi form with jet notation $\Omega \equiv L(\xi_1)\omega = 0$ as in ([8]) and may compare it to the three equations of the Killing system in dimension $n = 2$:

$$\left\{\begin{array}{l} \Omega^1 \equiv \xi^1_2 + \omega^1\xi^2_2 - \omega^1\xi^1_1 - (\omega^1)^2\xi^2_1 + \xi^r\partial_r\omega^1 = 0 \\ \Omega^2 \equiv \xi^2_1 + \omega^2\xi^1_1 - \omega^2\xi^2_2 - (\omega^2)^2\xi^1_2 + \xi^r\partial_r\omega^2 = 0 \\ \Omega^3 \equiv \omega^3(\xi^1_1 + \xi^2_2) + \omega^1\omega^3\xi^2_1 + \omega^2\omega^3\xi^1_2 + \xi^r\partial_r\omega^3 = 0 \end{array}\right.$$

that we can extend to six *intermediate* new equations like in ([14]), including in particular:

$$\Omega^4 \equiv \xi^1_{11} + \omega^1\xi^2_{11} + \omega^4\xi^1_1 + 2(\omega^5 - \omega^1\omega^4)\xi^2_1 + \xi^r\partial_r\omega^4 = 0$$

$$\Omega^5 \equiv \xi^1_{12} + \omega^1\xi^2_{12} + \omega^4\xi^1_2 = (\omega^6 - \omega^1\omega^5)\xi^2_1 + \omega^5\xi^2_2 + \xi^r\partial_r\omega^5 = 0$$

$$\Omega^6 \equiv \xi^1_{22} + \omega^1\xi^2_{22} + +2\omega^6\xi^2_2 + 2\omega^5\xi^1_2 - \omega^6\xi^1_1 - \omega^1\omega^6\xi^2_1 + \xi^r\partial_r\omega^6 = 0$$

Taking into account these new invariants bringing for example six relations like:

$$\partial_1\omega^1 - \omega^5 + \omega^1\omega^4 = 0, \qquad \partial_2\omega^1 - \omega^6 + \omega^1\omega^5 = 0$$

$$\partial_1\omega^2 - \omega^9 + \omega^2\omega^8 = 0, \qquad \partial_2\omega^2 - \omega^8 + \omega^2\omega^7 = 0$$

$$\partial_1\omega^3 - \omega^3(\omega^4 + \omega^8) = 0, \qquad \partial_2\omega^3 - \omega^3(\omega^5 + \omega^7) = 0$$

The determinant of the 6×6 matrix with respect to $(\omega^4, ..., \omega^9)$ is $\omega^3(1 - \omega^1\omega^2) \neq 0$.

After tedious but elementary substitutions, we obtain for example:

$$d_2\Omega^4 - d_1\Omega^5 \equiv (\partial_2\omega^4 - \partial_1\omega^5)(\xi^1_1 + \xi^2_2) + \xi^r\partial_r(\partial_2\omega^4 - \partial_1\omega^5) = 0$$

However, we have also:

$$\omega^3(1 - \omega^1\omega^2)(\xi^1_1 + \xi^2_2) + \xi^r\partial_r(\omega^3(1 - \omega^1\omega^2)) = 0$$

Replacing $(\omega^4, ..., \omega^9)$ by their rational expressions in $j_1(\omega)$, the quotient of $\partial_2\omega^4 - \partial_1\omega^5$ by $\omega^3(1 - \omega^1\omega^2)$ is well defined, say equal to $c(x)$, and we obtain the new zero order equation $\xi^r\partial_r c(x) = 0$ contradicting the formal integrability of the given system unless we obtain the two Vessiot structure equations with the two Vessiot structure constants c', c'', namely:

$$\partial_2\omega^4 - \partial_1\omega^5 = c'\ \omega^3(1 - \omega^1\omega^2), \qquad \partial_1\omega^7 - \partial_2\omega^8 = c''\ \omega^3(1 - \omega^1\omega^2)$$

It remains to prove that there is only one Vessiot structure equation of order two with a single structure constant. For this, first of all we notice that:

$$\partial_2(\omega^4 + \omega^8) - \partial_1(\omega^5 + \omega^7) = 0 \ \Rightarrow\ c' - c'' = 0 \ \Rightarrow\ c' = c'' = c$$

Now, if $R_q \subset J_q(T)$, let us define $J^0_q(T)$ by the short exact sequence:

$$0 \rightarrow J^0_q(T) \rightarrow J_q(T) \xrightarrow{\pi^q_0} T \rightarrow 0$$

and set $R^0_q = R_q \cap J^0_q(T) \subset J_q(T)$. We have the commutative and exact diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \rightarrow & g_{q+1} & = & g_{q+1} & \rightarrow & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & R^0_{q+1} & \rightarrow & R_{q+1} & \rightarrow & T & & \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \rightarrow & R^0_q & \rightarrow & R_q & \rightarrow & T & \rightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

It follows that $R_{q+1}\rightarrow R_q$ is an epimorphism if and only if $R^0_{q+1}\rightarrow R^0_q$ and $R_{q+1}\rightarrow T$ are both epimorphisms. It just remains to use successively $q=1$ and $q=2$.

In the present situation, constructing the same diagram as the one used in the study of the Killing system (see [8] or [11] for the details), we have the commutative and exact diagram allowing to construct the second order CC when $g_2=0$, $g_3=0$, namely:

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \rightarrow & S_3T^*\otimes T & \rightarrow & S_2T^*\otimes F_1 & \rightarrow F_2\rightarrow 0 & \\
 & & \downarrow & & \downarrow & & \\
0 & \rightarrow & T^*\otimes S_2T^*\otimes T & \rightarrow & T^*\otimes T^*\otimes F_1 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
0\rightarrow\wedge^2T^*\otimes g_1 & \rightarrow & \wedge^2T^*\otimes T^*\otimes T & \rightarrow & \wedge^2T^*\otimes F_1 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}
$$

We obtain the isomorphism $F_2\simeq\wedge^2T^*\otimes g_1$ by a snake chase and deduce thus the relation $dim(F_2)=dim(\wedge^2T^*\otimes g_1)=dim(g_1)=1$ because $n=2$. Of course, as $dim(F_1)=3$, we also obtain $dim(F_2)=dim(S_2T^*\otimes F_1)-dim(S_3T^*\otimes T)=9-8=1$ in a coherent way.

For the sake of completeness, we provide the only component of the second order CC, namely:

$\Omega^1\equiv\xi^1_2=0, \Omega^2\equiv\xi^2_1=0, \Omega^3\equiv\xi^1_1+\xi^2_2=0 \;\;\Rightarrow\;\; d_{11}\Omega^1+d_{22}\Omega^2-d_{12}\Omega^3=0$

that must be compared to the linearized Riemann operator for the Euclidean metric leading to:

$\Omega_{11}\equiv 2\xi^1_1=0, \Omega_{12}\equiv\xi^1_2+\xi^2_1=0, \Omega^{22}\equiv 2\xi^2_2=0 \;\;\Rightarrow\;\; d_{11}\Omega_{22}+d_{22}\Omega_{11}-2d_{12}\Omega_{12}=0$

In a more contructive way, we have $\omega^5=\partial_1\omega^1+\omega^1\;\omega^4$ where ω^4 is given in a rational way by the formula:

$\omega^3\partial_2\omega^2-\partial_1\omega^3+\omega^2\partial_2\omega^3-\omega^2\omega^3\partial_1\omega^1+\omega^3(1-\omega^1\omega^2)\omega^4=0$

Accordingly, if we do want to solve the equivalence problem $\Phi^1=\bar{\omega}^1$, $\Phi^2=\bar{\omega}^2$, $\Phi^3=\bar{\omega}^3$, *we must know the Vessiot structure equations*. As for the Vessiot structure constant c, it must be the same at first sight because the Vessiot structure equations are invariant under any diffeomorphism. However, we have to take into account the fact that two sections ω and $\bar{\omega}$ of $\mathcal{F}$ may give the same infinitesimal Lie equations. For example, in the present situation, we must have:

$$\bar{\omega}^1=\omega^1, \qquad \bar{\omega}^2=\omega^2, \qquad \bar{\omega}^3=a\omega^3 \qquad \Rightarrow \qquad \bar{c}=c/a$$

where $a\neq 0$ is the parameter of the multiplicative group of the real line. It follows that we have $c=0\Rightarrow\bar{c}=0$ both with $c\neq 0\Rightarrow\bar{c}\neq 0$ and it just remains to exhibit such situations.

In the present situation, we have $\omega^1=0, \omega^2=0, \omega^3=1\Rightarrow c=0$

However, the new pseudogroup:

$$\bar{\Gamma}=\{y^1=\frac{ax^1+b}{cx^1+d}, y^2=\frac{ax^2+b}{cx^2+d}\mid a,b,c,d=cst\}$$

is easily seen to be provided by the new specialization:

$$\bar{\omega}^1=0, \quad \bar{\omega}^2=0, \quad .\bar{\omega}^3=1/(x^2-x^1)^2\Rightarrow\bar{c}=-2$$

leading to the new Lie form:

$$y^1_2=0, \quad y^2_1=0, \quad \frac{1}{(y^2-y^1)^2}y^1_1y^2_2=\frac{1}{(x^2-x^1)^2}$$

It follows that the equivalence problem $y^1_2/y^1_1=0$, $y^2_1/y^2_2=0$, $y^1_1y^2_2=1/(x^2-x^1)^2$ cannot be solved.

Indeed, we should get $y^1=f(x^1)$, $y^2=g(x^2)$ with $\partial_1f(x^1)\partial_2g(x^2)=1/(x^2-x^1)^2$. Inverting the formula and setting $x^1=x^2=x$, we should conclude that $\partial_xf(x)=0$ or $\partial_xg(x)=0$ but this is impossible.

We could also notice that $0=ac=\bar{c}=-2$ for a certain $a\neq 0$ which yelds a contradiction.

Needless to say that no classical tool can produce anyone of these results.

3 Riemann Structure

In such a way to emphasize that this section could be even more striking, we shall copy almost "word by word" the procedure of the preceding section.

With $m = n = 2$, let us consider the Lie group of isometries $y = Ax + B$ where A is an orthogonal matrix for the Euclidean metric $\omega = (dx^1)^2 + (dx^2)^2$ as an algebraic Lie pseudogroup Γ. It is easy to exhibit the corresponding first order system $\mathcal{R}_1$ of finite Lie equations in Lie form by introducing the three generating differential invariants $\Phi_{ij} \equiv \omega_{kl}(y) y_i^k y_j^l$ and the corresponding Lie form:

$$\Phi_{11} \equiv (y_1^1)^2 + (y_1^2)^2 = 1, \quad \Phi_{22} \equiv (y_2^1)^2 + (y_2^2)^2 = 1, \quad \Phi_{12} \equiv y_1^1 y_2^1 + y_1^2 y_2^2 = 0$$

The details of the following tricky computations, first done in 1978 ([8]), have been improved in 2016 ([14]) and are again revisited in this paper. Its major interest is to work out the *two* Vessiot structure constants existing but within a tensorial framework now. By this way, we prove that the well-known formal integrability result found by L. P. Eisenhart in 1926 ([2]) on the constant Riemannian curvature condition is only a very particular case of the Vessiot structure equations found by E. Vessiot more than twenty years before ([22]).

First of all, we notice that this system is finite type with a vanishing second order symbol and is thus formally integrable but not involutive. For this, we may introduce the six generating differential invariants Φ_{ij}^k obtained after one prolongation and transforming like the six Christoffel symbols of Riemannian geometry:

$$\gamma_{ij}^k = \frac{1}{2}\omega^{kr}(\partial_i \omega_{rj} + \partial_j \omega_{ir} - \partial_r \omega_{ij}) = \gamma_{ji}^k \quad \Rightarrow \quad 2\Phi_{kr}\Phi_{ij}^k \equiv d_i \Phi_{rj} + d_j \Phi_{ir} - d_r \Phi_{ij}$$

by introducing the inverse matrix of ω. We notice that:

$$det(\Phi_{ij}) = \Phi_{11}\Phi_{22} - (\Phi_{12})^2 \equiv (y_1^1 y_2^2 - y_2^1 y_1^2)^2 \neq 0$$

is changing like the square of the Jacobian Δ and we have $\gamma_{ri}^r = \frac{1}{2}\omega^{rs}\partial_i \omega_{rs} = \partial_i (det(\omega))^{\frac{1}{2}}$.

Looking to the way these invariants are transformed under an arbitrary change $\bar{x} = \varphi(x)$ of local coordinates, we obtain $u_{ij} = \partial_i \varphi^k \partial_j \varphi^l \bar{u}_{kl}$ for describing the natural fiber bundle $\mathcal{F} = S_2 T^*$ with local coordinates $(x^1, x^2; u_{11}, u_{22}, u_{12})$ and section $\omega = (\omega_{11}, \omega_{22}, \omega_{12})$ becoming $(1, 1, 0)$ with our choice and $det(\omega) \neq 0$. Passing to the infinitesimal point of view we obtain the first order system $R_1 \subset J_1(T)$ in the Medolaghi form, also called Killing system, with jet notation $\Omega \equiv L(\xi_1)\omega = 0$ as in ([8,11]) in dimension $n = 2$, using capital letters for the linearization:

$$\Omega_{ij} \equiv \omega_{rj}(x)\xi_i^r + \omega_{ir}(x)\xi_j^r + \xi^r \partial_r \omega_{ij}(x) = 0$$

that we can extend to six *intermediate* new second order equations, namely:

$$\Gamma_{ij}^k \equiv \xi_{ij}^k + \gamma_{rj}^k(x)\xi_i^r + \gamma_{ir}^k(x)\xi_j^r - \gamma_{ij}^r(x)\xi_r^k + \xi^r \partial_r \gamma_{ij}^k(x) = 0 \Rightarrow \Gamma_{ri}^r \equiv \xi_{ri}^r + \gamma_{sr}^s \xi_i^r + \xi^r \partial_r \gamma_{si}^s = 0$$

It is only now that we have to use specific concepts of Riemannian geometry, namely the *Riemann* and *Ricci* tensors:

$$\rho_{l,ij}^k \equiv \partial_i \gamma_{lj}^k - \partial_j \gamma_{li}^k + \gamma_{lj}^r \gamma_{ri}^k - \gamma_{li}^r \gamma_{rj}^k \quad \Rightarrow \quad \rho_{ij} = \rho_{i,rj}^r \quad \Rightarrow \quad \varphi_{ij} = \rho_{ij} - \rho_{ji} = \rho_{r,ij}^r = \partial_i \gamma_{rj}^r - \partial_j \gamma_{ri}^r$$

Accordingly, *if we use the Christoffel symbols* γ "per se", that is *independently of* ω, we may have $\rho_{ij} \neq \rho_{ji}$ (*care*) and we have the following lemma:

Lemma 3.1 *The γ alone are the geometric objects for the affine group $y = Ax + B$ with $n(n+1)$ parameters, which are a section of an affine natural bundle modeled on $S_2T^* \otimes T$. Introducing the Spencer δ-map, the $n^2(n+1)/2$ generating differential invariants (Φ_{ij}^k) allow to describe a vanishing Riemann tensor in the Janet sequence for the Lie operator $T \to S_2T^* \otimes T : \xi \to \Gamma \equiv \mathcal{L}(\xi)\gamma$:*

$$0 \to \Theta \to T \to S_2T^* \otimes T \to F_1 \to \ldots$$

where F_1 is defined by the short exact sequence $0 \to F_1 \to \wedge^2 T^* \otimes T^* \otimes T \xrightarrow{\delta} \wedge^3 T^* \otimes T \to 0$.

Proof As $g_2 = 0$ and thus $g_3 = 0$, we have the commutative and exact diagram:

$$
\begin{array}{cccccccc}
 & 0 & & 0 & & & & \\
 & \downarrow & & \downarrow & & & & \\
0 \rightarrow & S_3T^* \otimes T & \rightarrow & T^* \otimes S_2T^* \otimes T & \rightarrow & F_1 & \rightarrow & 0 \\
 & \downarrow \delta & & \| & & & & \\
0 \rightarrow & T^* \otimes S_2T^* \otimes T & = & T^* \otimes S_2T^* \otimes T & \rightarrow & 0 & & \\
 & \downarrow \delta & & \downarrow & & & & \\
 & \underline{\wedge^2T^* \otimes T^* \otimes T} & & 0 & & & & \\
 & \downarrow \delta & & & & & & \\
 & \wedge^3T^* \otimes T & & & & & & \\
 & \downarrow & & & & & & \\
 & 0 & & & & & &
\end{array}
$$

and deduce from a snake chase that $F_1 \simeq \delta(T^* \otimes S_2T^* \otimes T)$.

Now, from the transformation rules of γ, we deduce that $y^k_r \Phi^r_{ij} = y^k_{ij}$. Differentiating formally with respect to x^i and substituting, we obtain:

$$d_i \Phi^k_{lj} - d_j \Phi^k_{li} + \Phi^r_{lj}\Phi^k_{ri} - \Phi^r_{li}\Phi^k_{rj} = 0$$

In dimension 2, we have $F_1 \simeq \wedge^2T^* \otimes T^* \otimes T$ with $dim(F_1) = 4$ like in the previous section. $\square$

The linearization provides:

$$R^k_{l,ij} \equiv -\rho^r_{l,ij}\xi^k_r + \rho^k_{r,ij}\xi^r_l + \rho^k_{l,rj}\xi^r_i + \rho^k_{l,ir}\xi^r_j + \xi^r\partial_r\rho^k_{l,ij} = 0 \;\Rightarrow\; R_{ij} \equiv \rho_{rj}\xi^r_i + \rho_{ir}\xi^r_j + \xi^r\partial_r\rho_{ij} = 0$$
$$F_{ij} \equiv R^r_{r,ij} \equiv \varphi_{rj}\xi^r_i + \varphi_{ir}\xi^r_j + \xi^r\partial_r\varphi_{ij} = 0$$

In the specific dimension $n = 2$ considered, we have by chance the simplified formulas:

$$\rho_{11} = \rho^r_{1,r1} = \rho^2_{1,21},\;\; \rho_{12} = \rho^r_{1,r2} = \rho^1_{1,12},\;\; \rho_{21} = \rho^r_{2,r1} = \rho^2_{2,21},\;\; \rho_{22} = \rho^r_{2,r2} = \rho^1_{2,12}$$

and thus the *specific isomorphism* $\wedge^2T^* \otimes T^* \otimes T \simeq \wedge^2T^* \oplus S_2T^*$ *only for* $n = 2$, defined by:

$$(\rho^k_{l,ij}) \rightarrow (\rho_{ij}) \rightarrow (\frac{1}{2}(\rho_{ij} - \rho_{ji}), \frac{1}{2}(\rho_{ij} + \rho_{ji}))$$

by counting the dimensions with $1 \times 2 \times 2 = 4 = 1 + 3$ while using the canonical splitting of the short exact δ-sequence:

$$0 \rightarrow S_2T^* \xrightarrow{\delta} T^* \otimes T^* \xrightarrow{\delta} \wedge^2T^* \rightarrow 0$$

Now, we have proved in many books ([8,11,14,15]) or papers that, for any dimension n, two sections ω and $\bar{\omega}$ provide the same system of infinitesimal Lie equations if and only if $\bar{\omega} = a\omega$ for $a \neq 0$ the parameter of the multiplicative group of the real line. It follows that we have necessarily a first Vessiot constant c_1 in such a way that:

$$\frac{1}{2}(\rho_{ij} + \rho_{ji}) = c_1\omega_{ij}$$

However, after linearization, we also obtain $\omega^{ij}\Omega_{ij} = 2\xi^r_r + \xi^r\partial_r(log(det(\omega)) = 0$ for any n, thus

$$2det(\omega)(\xi^1_1 + \xi^2_2) + \xi^r\partial_r det(\omega) = 0$$

for $n = 2$ and obtain therefore a *second* Vessiot structure constant c_2 such that we have (Compare [8] to the computer algebra result found in [5]) $R_{12} - R_{21} = 0 \Rightarrow \varphi_{12}\xi^r_r + \xi^r\partial_r\varphi_{12} = 0$ both with $2det(\omega)\xi^r_r + \xi^r\partial_r det(\omega) = 0$, a result leading to:

$$\frac{1}{2}(\rho_{12} - \rho_{21}) = \frac{1}{2}\varphi_{12} = c_2(det(\omega))^{\frac{1}{2}}$$

Finally, we do want second order *integrability conditions* for the metric ω, that is, we must eliminate γ by using the Levi-Civita isomorphism $(\omega, \gamma) \simeq j_1(\omega)$. Then, it is well known that $\varphi_{ij} = 0$ and we must thus have $c_2 = 0$. Also, using the same diagram as in the previous section, we must have *only one second order integrability condition* which is indeed the standard ([2]) constant curvature condition expressed by means of the Ricci tensor which is now symmetric.

Remark 3.2 Let us prove that there is almost no difference with the example 2.2 presented in the preceding section, even though the background group is quite different. For this, let us introduce the different generating differential invariants and the new Lie form:

$$\Phi_{11} = 2\Phi^2\Phi^3 \equiv 2y_1^1 y_1^2 = 0,\ \ \Phi_{22} = 2\Phi^1\Phi^3 \equiv 2y_2^1 y_2^2 = 0,\ \ \Phi_{12} = \Phi^3 + \Phi^1\Phi^2\Phi^3 \equiv y_1^1 y_2^2 + y_2^1 y_1^2 = 1$$

in such a way that:

$$\omega_{11} = 2\omega_2\omega_3, \quad \omega_{22} = 2\omega_1\omega_3, \quad \omega_{12} = \omega_3(1 + \omega_1\omega_2) \quad \Rightarrow \quad d_{11}\Omega_{22} + d_{22}\Omega_{11} - 2d_{12}\Omega_{12} = 0$$

We now notice that $\Phi_{11}\Phi_{22} - (\Phi_{12})^2 = -\Delta^2$ and obtain again $\Phi_{ij} \equiv \omega_{kl} y_i^k y_j^l$ as before but now with the strange metric $\omega_{11} = 0, \omega_{22} = 0, \omega_{12} = 1$ in such a way that $det(\omega) = -1 < 0$ contrary to the previous example where $\bar{\omega}_{11} = 1, \bar{\omega}_{22} = 1, \bar{\omega}_{12} = 0$ leading to $det(\bar{\omega}) = 1 > 0$. However, the equivalence problem between these two structures cannot be solved. Indeed, taking the determinants of the equations $\omega_{kl}(x)\partial_i f^k(x)\partial_j f^l(x) = \bar{\omega}_{ij}(x)$, we should obtain $det(\omega)\Delta^2 = det(\bar{\omega}) = 1$ and thus the contradiction $\Delta^2 = -1$.

Remark 3.3 When $n = 3$, the Lie pseudogroup of transformations preserving the so-called *contact* 1-form $\alpha = dx^1 - x^3 dx^2$ is unimodular because it also preserves $\beta = d\alpha = dx^2 \wedge dx^3$ and thus the volume form $\alpha \wedge \beta = dx^1 \wedge dx^2 \wedge dx^3$. Hence, we can consider the geometric object $\omega = (\alpha, \beta)$ with $\alpha \wedge \beta \neq 0$ which is a meaningful condition like $det(\omega) \neq 0$ previously. In this case, the Vessiot structure equations are $d\alpha = c'\beta, d\beta = c''\alpha \wedge \beta$ with two Vessiot structure constants (see [14] for details). Closing this exterior system, we obtain the quadratic integrability condition:

$$0 = d(d\alpha) = c'd\beta = c'c''\alpha \wedge \beta \qquad \Rightarrow \qquad c'c'' = 0$$

which is quite unusual. Hence, one of the two constants must vanish like before but for a completely different reason. More generally, we refer the reader to ([16]) for the study of how the extension modules in homological algebra may depend on the Vessiot structure constants.

4 Conclusion

According to the Italian mathematician U. Amaldi in 1907 ([10], Introduction), at the beginning of the last century, only two frenchmen, namely E. Cartan and E. Vessiot, were knowing and understanding the work of S. Lie on the infinite groups of transformations, now called Lie pseudogroups. However, the respective *Cartan structure equations* and *Vessiot structure equations* that they developed about at the same time were so different that Amaldi said that “only the future should say which of the two should be the most important one”. Unhappily and mainly for private reasons that we have explained, Cartan and followers never told that there could be another way superseding their approach. Also Cartan never told to A. Einstein in his letters of 1930 on absolute parallelism about the work of M. Janet in 1920 on systems of partial differential equations ([3]). Then, D.C. Spencer, largely ignoring these tentatives, created around 1970 new tools for the formal study of systems of partial differential equations ([21]), in particular the ones allowing to define Lie pseudogroups ([4]). As a matter of fact, the nonlinear Spencer sequences are superseding the Cartan structure equations because they are able to quotient the results down to the base manifold while Cartan was doing exterior calculus on jet bundles but this result is largely unknown today. As a byproduct, the work of Vessiot is still almost totally unknown after more than a century. It is only in 2008 that an interesting PhD thesis has been done by a German student who wanted to use computer algebra for exhibiting the Vessiot structure constants in the Riemannian case ([5]). However, it is clear that the student was lucky to be able to use the corresponding tensorial framework. The purpose of the present paper was to illustrate and justify these results by means of a general constructive procedure, proving that things are not so simple for Riemannian or contact structures. We have already proved that the case of a conformal structure is even more intricate since it highly depends on the dimension of the base manifold, for example space-time ([13,14,16,17]). Finally, the author wants to thank an anonymous referee for his many constructive comments that have been taken into account.

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