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Localizations of a Ring at Localizable Sets, their Groups of Units and Saturations

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Abstract We continue to develop the most general theory of one-sided fractions started in Bavula (Localizable sets and the localization of a ring at a localizable set. arXiv:2112.13447). The aim of the paper is to introduce 10 types of saturations of a set in a ring and using them to study localizations of a ring at localizable sets, their groups of units and various maximal localizable sets satisfying some natural conditions. The results are obtained for denominator sets (the classical situation), Ore sets and localizable sets.

Keywords Localizable set · Localization of a ring at a localizable set · Denominator set · Localization · Left ore set · Localization at a left ore set · The group of units · Saturation

Mathematics Subject Classification Primary: 16U20 · 16P50 · Secondary: 16P60

Contents

1 Introduction

In the paper, all rings and their homomorphisms are unital.

In [2], Ore's method of localization was extended to *localizable left Ore sets*, a criterion was given of when a left Ore set is localizable, and prove that *all left and right* Ore sets of an arbitrary ring are localizable (not just denominator sets as in Ore's method of localization). Applications are given for certain classes of rings (semi-prime Goldie rings, Noetherian commutative rings, the algebras of polynomial integro-differential operators).

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In [3], some basic results of the most general theory of *one-sided fractions* was developed and the following new concepts were introduced and studied: the almost Ore set, the localizable set, the perfect localizable set, the localization of a ring and a module at a localizable set. Their relations are given by the chain of inclusions:

$$\{\text{Denominator sets}\} \subseteq \{\text{Ore sets}\} \subseteq \{\text{almost Ore sets}\} \subseteq \{\text{perfect localizable sets}\} \subseteq \{\text{localizable sets}\}.$$

Localizable sets are generalization of Ore sets and denominator sets, and the localization of a ring/module at a localizable set is a generalization of localization of a ring/module at a denominator set.

In this paper, for a subset S of a ring R, the following concepts are introduced: the *left saturation* S_l^{sat}, the *right saturation* S_r^{sat}, the *weak saturation* S^{ws}, the *left weak saturation* S_l^{ws}, and the *right weak saturation* S_r^{ws}. If the set S is a left or right localizable set then so are some of its saturations (Thereom 1.2).

In Sect. 2, for a left denominator set $S \in \mathrm{Den}_l(R)$ of a ring R, explicit descriptions of the group of units $(SR)^\times$ of the ring $S^{-1}R$, and the monoids $(SR)_l^\times$ and $(SR)_r^\times$ of left and right invertible elements of $S^{-1}R$ are obtained (Theorem 2.1).

The largest element $S_l(R, \mathfrak{a}, S^{-1}R)$ in $(\mathrm{Den}_l(R, \mathfrak{a}, S^{-1}R) \subseteq)$ and its characterizations where $S \in \mathrm{Den}_l(R)$. Let $S, T \in \mathrm{Den}_l(R)$. The denominator set T is called S-*saturated* if $sr \in T$, for some $s \in S$ and $r \in R$, then $r \in T$, and if $r's' \in T$, for some $s' \in S$ and $r' \in R$, then $r' \in T$, [2].

Let $S \in \mathrm{Den}_l(R)$, $\mathfrak{a} = \mathrm{ass}_l(S) := \{r \in R \mid sr = 0 \text{ for some } s \in S\}$, $\mathrm{Den}_l(R, \mathfrak{a}) := \{T \in \mathrm{Den}_l(R) \mid \mathrm{ass}_l(T) = \mathfrak{a}\}$, and

$$\mathrm{Den}_l(R, \mathfrak{a}, S^{-1}A) = \{T \in \mathrm{Den}_l(R, \mathfrak{a}) \mid T^{-1}R \simeq S^{-1}R, \text{ an } R-\text{isomorphism}\}.$$

[2, Proposition 3.1] describes the largest element $S_l(R, \mathfrak{a}, S^{-1}R)$ of the partially ordered set $(\mathrm{Den}_l(R, \mathfrak{a}, S^{-1}R), \subseteq)$. [2, Lemma3.3.(1)] gives another description of $S_l(R, \mathfrak{a}, S^{-1}R)$ in terms of the group of units of the ring $S^{-1}R$.

Lemma 1.1 *([2, Lemma 3.3.(1)]) Let R be a ring, $S \in \mathrm{Den}_l(R, \mathfrak{a})$, and $\sigma : R \to S^{-1}R$, $r \mapsto \frac{r}{1}$. Then the set $S_l(R, \mathfrak{a}, S^{-1}R) = \sigma^{-1}((S^{-1}R)^\times)$ is the largest element of the partially ordered set $(\mathrm{Den}_l(R, \mathfrak{a}, S^{-1}R), \subseteq)$. The set $S_l(R, \mathfrak{a}, S^{-1}R)$ is S-saturated.*

Similarly, for a *right* denominator set $S \in \mathrm{Den}_r(R, \mathfrak{a})$ (resp., $S \in \mathrm{Den}(R, \mathfrak{a})$) , we denote by $S_r(R, \mathfrak{a}, RS^{-1})$ (resp. $S(R, \mathfrak{a}, S^{-1}R)$) the largest element of the poset $(\mathrm{Den}_r(R, \mathfrak{a}), \subseteq)$ (resp., $(\mathrm{Den}(R, \mathfrak{a}), \subseteq)$).

Definition. Let R be a ring and $S \subseteq R$. The sets

$$S_l^{sat} := \{a \in R \mid ba, cb \in S \ \text{ for some } b, c \in R\},$$
$$S_r^{sat} := \{a \in R \mid ab, bc \in S \ \text{ for some } b, c \in R\},$$
$$S^{ws} := \{a \in R \mid ba, ac \in S \ \text{ for some } b, c \in R\},$$
$$S_l^{ws} := \{a \in R \mid ba \in S \ \text{ for some } b \in R\},$$
$$S_r^{ws} := \{a \in R \mid ac \in S \ \text{ for some } c \in R\}$$

are called the *left saturation*, the *right saturation*, the *weak saturation*, the *left weak saturation*, and the *right weak saturation* of S, respectively. By the very definition,

$$S^{ws} = S_l^{ws} \cap S_r^{ws} \supseteq S_l^{sat} \cap S_r^{sat}, \ \ S_l^{sat} \subseteq S_l^{ws}, \ \text{ and } \ S_r^{sat} \subseteq S_r^{ws}. \tag{1.1}$$

A ring R is called a *finite ring* if $yx = 1$ implies $xy = 1$ (every one-sided inverse is the inverse).

Theorem 1.2 is another characterization of the sets $S_*(R, \mathfrak{a}, S^{-1}R)$ in terms of the five saturations above where $* \in \{l, r, \emptyset\}$. Its proof is given in Sect. 2.

Theorem 1.2 *Let R be a ring.*

1. *If $S \in \mathrm{Den}_l(R, \mathfrak{a})$ then $S_l(R, \mathfrak{a}, S^{-1}R) = S_l^{sat}$.*

2. *If* $S \in \mathrm{Den}_r(R, \mathfrak{a})$ *then* $S_r(R, \mathfrak{a}, RS^{-1}) = S_r^{sat}$.
3. *If* $S \in \mathrm{Den}(R, \mathfrak{a})$ *then* $S(R, \mathfrak{a}, S^{-1}R) = S_l(R, \mathfrak{a}, S^{-1}R) = S_r(R, \mathfrak{a}, RS^{-1}) = S_l^{sat} = S_r^{sat} = S^{ws}$ *and* $S^{ws} = S_l^{sat} \cap S_r^{sat}$.
4. *If* $S \in \mathrm{Den}_l(R, \mathfrak{a})$ *and the ring* $S^{-1}R$ *is a finite ring then* $S_l(R, \mathfrak{a}, S^{-1}R) = S_l^{sat} = S_l^{ws}$.
5. *If and* $S \in \mathrm{Den}_r(R, \mathfrak{a})$ *and the ring* RS^{-1} *is a finite ring iff* $S_r(R, \mathfrak{a}, RS^{-1}) = S_r^{sat} = S_r^{ws}$.
6. *If* $S \in \mathrm{Den}(R, \mathfrak{a})$ *and the ring* $S^{-1}R$ *is a finite ring iff*
   $S(R, \mathfrak{a}, S^{-1}R) = S_l(R, \mathfrak{a}, S^{-1}R) = S_r(R, \mathfrak{a}, RS^{-1}) = S_l^{sat} = S_r^{sat} = S^{ws} = S_l^{ws} = S_r^{ws}$.

Theorem 1.2 shows that saturations of $*$ denominators sets are also $*$ denominators sets.

Corollary 2.2 is a strengthening of Theorem 1.2 in the case when the ring $S^{-1}R$ is either a domain or a one-sided Noetherian ring or does not contain an infinite direct sum of one-sided ideals. Corollary 2.3 shows that saturations are idempotent functors in the case of denominator sets.

Applications are given for the algebra $\mathbb{S}_n$ of one-sided inverses (Proposition 2.5) where saturations are explicitly described (the algebra $\mathbb{S}_n$ is not a finite ring. It is neither left nor right Noetherian, not a domain and contains infinite direct sums of left and right ideals).

Finiteness criterion for a localization of a ring via its saturations. Theorem 1.3 is a finiteness criterion for a localization of a ring at a localizable set which is given in terms saturations.

Theorem 1.3 *Let R be a ring.*

1. *If* $S \in \mathrm{Den}_l(R, \mathfrak{a})$ *then the ring* $S^{-1}R$ *is a finite ring iff* $S_l(R, \mathfrak{a}, S^{-1}R) = S_l^{sat} = S_l^{ws}$.
2. *If* $S \in \mathrm{Den}_r(R, \mathfrak{a})$ *then the ring* RS^{-1} *is a finite ring iff* $S_r(R, \mathfrak{a}, RS^{-1}) = S_r^{sat} = S_r^{ws}$.
3. *If* $S \in \mathrm{Den}(R, \mathfrak{a})$ *then the ring* $S^{-1}R$ *is a finite ring iff*
   $S(R, \mathfrak{a}, S^{-1}R) = S_l(R, \mathfrak{a}, S^{-1}R) = S_r(R, \mathfrak{a}, RS^{-1}) = S_l^{sat} = S_r^{sat} = S^{ws} = S_l^{ws} = S_r^{ws}$.

The largest element $\mathcal{S}_*(R, \mathfrak{a}, S^{-1}R)$ in $(\mathbb{L}_*(R, \mathfrak{a}, S^{-1}R) \subseteq)$ and its characterizations where $S \in \mathbb{L}_*(R, \mathfrak{a})$. In Sect. 3, the results of Sect. 2 for the denominator sets are generalized for localizable sets. At the beginning of Sect. 3, some results are collected from [3] on localizable sets and localizations of rings at localizable sets. Proposition 3.8.(2), is an explicit description of the largest element $\mathcal{S}_*(R, \mathfrak{a}, S^{-1}A)$ of the partially ordered set $(\mathbb{L}_*(R, \mathfrak{a}, \mathcal{R}), \subseteq)$ of all $*$ localizable sets S in R with $\mathrm{ass}_R(S) = \mathfrak{a}$ and $R\langle S^{-1}\rangle \simeq \mathcal{R}$ where $* \in \{l, r, \emptyset\}$. Theorem 3.10 is another characterization of the set $\mathcal{S}_*(R, \mathfrak{a}, R\langle S^{-1}\rangle)$ which is given in terms of the five saturations (it is an analogue of Theorem 1.2 but for localizable sets). In the case of Ore sets, we can strengthen Theorem 3.10, see Theorem 1.4.

Definition. Let R be a ring and $S \subseteq R$. The sets

$S_l^{bsat} := \{a \in R \mid s_1bas_2, t_1cbt_2 \in S \text{ for some } s_1, s_2, t_1, t_2 \in S \text{ and } b, c \in R\}$,
$S_r^{bsat} := \{a \in R \mid s_1abs_2, t_1bct_2 \in S \text{ for some } s_1, s_2, t_1, t_2 \in S \text{ and } b, c \in R\}$,
$S^{wbs} := \{a \in R \mid s_1bas_2, t_1act_2 \in S \text{ for some } s_1, s_2, t_1, t_2 \in S \text{ and } b, c \in R\}$,
$S_l^{wbs} := \{a \in R \mid s_1bas_2 \in S \text{ for some } s_1, s_2 \in S \text{ and } b, \in R\}$,
$S_r^{wbs} := \{a \in R \mid s_1abs_2 \in S \text{ for some } s_1, s_2 \in S \text{ and } b \in R\}$

are called the *left bi-saturation*, the *right bi-saturation*, the *weak bi-saturation*, the *left weak bi-saturation*, and the *right weak bi-saturation* of S, respectively. By the very definition,

$$S^{wbs} = S_l^{wbs} \cap S_r^{wbs} \supseteq S_l^{bsat} \cap S_r^{bsat}, \ \ S_l^{bsat} \subseteq S_l^{wbs}, \ \text{ and } S_r^{bsat} \subseteq S_r^{wbs}. \tag{1.2}$$

Theorem 1.4 *We keep the notation of Theorem 3.10. Suppose that* $S \in \mathrm{Ore}(R, \mathfrak{a})$ *where* $\mathfrak{a} = \mathrm{ass}_R(S)$.

1. $\mathcal{S}(R, \mathfrak{a}, S^{-1}R) = S_l^{bsat} = S_r^{bsat} = S^{wbs}$.
2. *Suppose, in addition, that the ring* $S^{-1}R$ *is either a domain or a one-sided Noetherian ring or does not contain an infinite direct sum of one-sided ideals then* $\mathcal{S}(R, \mathfrak{a}, S^{-1}R) = S_l^{wbs} = S_r^{wbs}$.

The proof of Theorem 1.4 is given at the end of Sect. 3.

2 Localizations of a Ring at Denominator Sets, their Groups of Units and Saturations

Let R be a ring. A multiplicative subset S of R is called a *left Ore set* if it satisfies the *left Ore condition*: for each $r \in R$ and $s \in S$, $Sr \bigcap Rs \neq \emptyset$. Let $\mathrm{Ore}_l(R)$ be the set of all left Ore sets of R. For $S \in \mathrm{Ore}_l(R)$, $\mathrm{ass}_l(S) := \{r \in R \mid sr = 0 \text{ for some } s \in S\}$ is an ideal of the ring R.

A left Ore set S is called a *left denominator set* of the ring R if $rs = 0$ for some elements $r \in R$ and $s \in S$ implies $tr = 0$ for some element $t \in S$, i.e., $r \in \mathrm{ass}_l(S)$. Let $\mathrm{Den}_l(R)$ be the set of all left denominator sets of R. For $S \in \mathrm{Den}_l(R)$, let $S^{-1}R = \{s^{-1}r \mid s \in S, r \in R\}$ be the *left localization* of the ring R at S (the *left quotient ring* of R at S). Let us stress that in Ore's method of localization one can localize *precisely* at left denominator sets. In a similar way, right Ore and right denominator sets are defined. Let $\mathrm{Ore}_r(R)$ and $\mathrm{Den}_r(R)$ be the set of all right Ore and right denominator sets of R, respectively. For $S \in \mathrm{Ore}_r(R)$, the set $\mathrm{ass}_r(S) := \{r \in R \mid rs = 0 \text{ for some } s \in S\}$ is an ideal of R. For $S \in \mathrm{Den}_r(R)$, $RS^{-1} = \{rs^{-1} \mid s \in S, r \in R\}$ is the *right localization* of the ring R at S.

Given ring homomorphisms $\nu_A : R \to A$ and $\nu_B : R \to B$. A ring homomorphism $f : A \to B$ is called an *R-homomorphism* if $\nu_B = f\nu_A$. A left and right set is called an *Ore set*. Let $\mathrm{Ore}(R)$ and $\mathrm{Den}(R)$ be the set of all Ore and denominator sets of R, respectively. For $S \in \mathrm{Den}(R)$,

$$S^{-1}R \simeq RS^{-1}$$

(an R-isomorphism) is the *localization* of the ring R at S, and $\mathrm{ass}(R) := \mathrm{ass}_l(R) = \mathrm{ass}_r(R)$.

For a ring R and $* \in \{l, r, \emptyset\}$, $\mathrm{Den}_*(R, 0)$ be the set of $*$ denominator sets T of R such that $T \subseteq \mathcal{C}_R$, i.e., the multiplicative set T is a $*$ Ore set of R that consists of regular elements of the ring R.

The group of units $(S^{-1}R)^\times$ and monoids of one-sided inverses of a localization $S^{-1}R$ where $S \in \mathrm{Den}_l(R)$. For a ring R, we denote by $R^\times$ its group of units. Let $R_l^\times := \{a \in R \mid ba = 1 \text{ for some } b \in R\}$ and $R_r^\times := \{a \in R \mid ab = 1 \text{ for some } b \in R\}$, the sets of left and right invertible elements of the ring R, respectively. The sets $R_l^\times$ and $R_r^\times$ are multiplicative monoids that contain the group $R^\times$ and $R^\times = R_l^\times \cap R_r^\times$. The ring R is called a **finite ring** if $ab = 1$ implies $ba = 1$ (every one-sided inverse is the inverse). The ring R is a finite ring iff $R^\times = R_l^\times = R_r^\times$ iff $R^\times = R_l^\times$ iff $R^\times = R_r^\times$. Every domain or a one-sided Noetherian ring is a finite ring. It is well-known that the algebra of one-sided inverses, $\mathbb{S}_1 = K\langle x, y \mid yx = 1\rangle$, is not a finite ring (see [1] for generalizations and their properties). Let K be a field of characteristic zero and I_1 be the subalgebra of the algebra $\mathrm{End}_K(K[x])$ which is generated by the K-derivation $\partial = \frac{d}{dx}$ and the integration $\int : K[x] \to K[x]$, $x^n \mapsto \frac{x^n}{n!}$ where $n \geq 0$. Then the K-algebra homomorphism

$$\mathbb{S}_1 \to I_1, \quad x \mapsto \int, \quad y \mapsto \partial$$

is an isomorphism (since $\partial \int = 1$, $\mathbb{S}_1 = \bigoplus_{i,j\geq 0} Kx^iy^j$, and $I_1 = \bigoplus_{i,j\geq 0} K \int^i \partial^j$). Clearly, the elements ∂ and $\int$ are not invertible (since $\ker(\partial) = K \neq 0$). Hence, the algebras $\mathbb{S}_1$ and I_1 are not finite.

For a ring R and its left denominator set S, Theorem 2.1 gives an explicit descriptions of the set $(S^{-1}R)^\times$, $(S^{-1}R)_l^\times$ and $(S^{-1}R)_r^\times$.

Theorem 2.1 *Let R be a ring, $S \in \mathrm{Den}_l(R)$, and $T \in \mathrm{Den}_r(T)$. Then*

1. $(S^{-1}R)^\times = \{s^{-1}a \mid ba, cb \in S \text{ for some } b, c \in R, s \in S\}$.
2. $(S^{-1}R)_l^\times = \{s^{-1}a \mid ba \in S \text{ for some } b \in R, s \in S\}$.
3. $(S^{-1}R)_r^\times = \{s^{-1}a \mid t_1a = a_1t,\ a_1b = t_1 \text{ for some elements } t, t_1 \in S \text{ and } a_1, b \in R, s \in S\}$.
4. $(AT^{-1})_r^\times = \{as^{-1} \mid ac \in S \text{ for some } c \in R, s \in S\}$.
5. *If, in addition,* $S \in \mathrm{Den}(R)$ *then* $(S^{-1}R)^\times = \{s^{-1}a \mid Ra \cap S \neq \emptyset, aA \cap S \neq \emptyset\} = \{s^{-1}a \mid ba, ac \in S \text{ for some } b, c \in R, s \in S\}$.
6. *If, in addition, the ring R is finite then* $(S^{-1}R)^\times = (S^{-1}R)_l^\times = \{s^{-1}a \mid ba \in S \text{ for some } b \in R, s \in S\}$.
7. *If, in addition, the ring R is finite and* $S \in \mathrm{Den}(R)$ *then* $(S^{-1}R)^\times = \{s^{-1}a \mid ba \in S \text{ for some } b \in R, s \in S\} = \{s^{-1}a \mid ab \in S \text{ for some } b \in R, s \in S\}$.

Proof 2. An element $s^{-1}a \in S^{-1}R$ (where $s \in S$ and $a \in R$) belongs to the monoid $(S^{-1}R)_l^\times$ iff $a \in (S^{-1}R)_l^\times$ iff $t^{-1}b_1a = 1$ for some elements $t \in S$ and $b_1 \in R$ iff $ba \in S$ for some element $b \in R$ (the equality $b_1a = t$ that holds in the ring $S^{-1}R$ is equivalent to the equality $t_1(b_1a - t) = 0$ in R for some element $t_1 \in S$, then put $b = t_1b_1$).

1. Let $B = \{s^{-1}a \mid ba, cb \in S$ for some $b, c \in R, s \in S\}$.

(i) $B \subseteq (S^{-1}R)^\times$: If $ba = s$ and $cb = t$ for some elements $s, t \in S$ and $b, c \in R$ then

$$b \cdot as^{-1} = 1 \ \text{ and } \ t^{-1}c \cdot b = 1,$$

and so $b \in (S^{-1}R)^\times$ and so $a = b^{-1}s \in (S^{-1}R)^\times$.

(ii) $B \supseteq (S^{-1}R)^\times$: Given an element $a \in R$ such that $a \in (S^{-1}R)^\times$. Then $a \in (S^{-1}R)_l^\times$, and so $ba = s$ for some elements $s \in S$ and $b \in R$, by statement 2. Then $a^{-1} = s^{-1}b$, and so

$$as^{-1} \cdot b = 1.$$

Hence, $b \in (S^{-1}R)_l^\times$, and so $cb \in S$, by statement 2.

By the statements (i) and (ii), $B = (S^{-1}R)^\times$.

3. An element $s^{-1}a \in S^{-1}R$ (where $s \in S$ and $a \in R$) belongs to the monoid $(S^{-1}R)_r^\times$ iff $a \in (S^{-1}R)_r^\times$ iff

$$at^{-1}b = 1$$

for some elements $t \in S$ and $b \in R$. The set S is a left Ore set, hence $\tau_1a = a_1't$ for some elements $\tau_1 \in S$ and $a_1' \in R$. Now, $at^{-1}b = 1$ and $\tau_1a = a_1't$ iff

$$\tau_1at^{-1}b = a_1'b = \frac{\tau_1}{1} \ \text{ and } \ \tau_1a = a_1't$$

iff $\tau_2a_1'b = \tau_2\tau_1$ for some element $\tau_2 \in S$ and $\tau_1a = a_1't$ iff $\tau_2a_1'b = \tau_2\tau_1$ for some element $\tau_2 \in S$ and $\tau_2\tau_1a = \tau_2a_1't$ iff

$$t_1a = a_1t \ \text{ and } \ a_1b = t_1$$

where $t_1 = \tau_2\tau_1 \in S$ and $a_1 = \tau_2a_1' \in R$ (the operations are reversible).

4. By statement 2, $(RT^{-1})_r^\times = \{as^{-1} \mid ab \in T, s \in T\}$ (apply statement 2 to the opposite ring

$$(RT^{-1})^{op} = (T^{op})^{-1}R^{op}$$

of the ring TR^{-1}).

5. The second equality is obvious. By statement 4,

$$(RS^{-1})_r^\times = \{as^{-1} \mid ac \in S \ \text{ for some } \ c \in R, s \in S\}.$$

Since S is a denominator set, $S^{-1}R \simeq RS^{-1}$. In particular,

$$(S^{-1}R)_r^\times = (RS^{-1})_r^\times = \{as^{-1} \mid ac \in S \text{ for some } c \in R, s \in S\}.$$

By statement 2, $(S^{-1}R)^\times \subseteq (S^{-1}R)_l^\times = \{s^{-1}a \mid ba \in S$ for some $b \in R, s \in S\}$. Now, statement 5 follows from the fact that $R^\times = R_l^\times \cap R_r^\times$.

6. The ring R is a finite ring. Hence,

$$(S^{-1}R)^\times = (S^{-1}R)_l^\times = \{s^{-1}a \mid ba \in S \ \text{ for some } \ b \in R, s \in S\},$$

by statement 2.

7. The ring R is a finite ring and $S \in \mathrm{Den}(R)$. Hence,

$$(S^{-1}R)^\times = (S^{-1}R)_l^\times = \{s^{-1}a \mid ba \in S \text{ for some } b \in R, s \in S\}, \ \text{by statement 2},$$
$$(S^{-1}R)^\times = (AS^{-1})_r^\times = \{s^{-1}a \mid ab \in S \text{ for some } b \in R, s \in S\}, \ \text{by statement 4},$$

and statement 7 follows. □

Proof of Theorem 1.2 1. Statement 1 follows from Theorem 2.1.(1) and Lemma 1.1: $a \in S_l(R, \mathfrak{a}, S^{-1}R)$ iff $\frac{a}{1} \in (S^{-1}R)^\times$ (Lemma 1.1) iff $s^{-1}b \cdot a = 1$ and $t^{-1}c \cdot b = 1$ for some elements $s, t \in S$ and $b, c \in R$ (since $b \cdot as^{-1} = 1$) iff $sb \cdot a, tc \cdot b \in S$ for some elements $s, t \in S$ iff $a \in S_l^{sat}$.

2. Statement 2 follows from statement 1 (by applying statement 1 to the opposite ring).

3. By Lemma 1.1 and its right analogue, we have that

$$S_l(R, \mathfrak{a}, S^{-1}R) = \sigma^{-1}((S^{-1}R)^\times) = S_r(R, \mathfrak{a}, RS^{-1}).$$

Therefore,

$$S(R, \mathfrak{a}, S^{-1}R) = S_l(R, \mathfrak{a}, S^{-1}R) = S_r(R, \mathfrak{a}, S^{-1}R) = S_l^{sat} = S_r^{sat},$$

by statements 1 and 2. By (1.1), $S^{ws} \supseteq S_l^{sat} \cap S_r^{sat}$. Given an element $a \in S^{ws}$. Then $ba, ac \in S$ for some elements $b, c \in R$. It follows that $\frac{a}{1} \in (S^{-1}R)^\times$. By Lemma 1.1, we have that $a \in S_l(R, \mathfrak{a}, S^{-1}R)$, and so

$$a \in S_l^{sat} = S_r^{sat} = S_l^{sat} \cap S_r^{sat}.$$

Therefore, $S^{ws} = S_l^{sat} \cap S_r^{sat}$.

4. The first equality in statement 4 follows from statement 1. By the very definition, $S_l^{sat} \subseteq S_l^{ws}$. Given an element $a \in S_l^{ws}$. Then $s := ba \in S$ for some element $b \in R$. Then $s^{-1}b \cdot a = 1$. By the assumption, the ring $S^{-1}R$ is a finite ring. Hence, $\frac{a}{1} \in (S^{-1}R)^\times$. By Lemma 1.1 and statement 1, $a \in S_l(R, \mathfrak{a}, S^{-1}R) = S_l^{sat}$. Therefore, $S_l^{sat} = S_l^{ws}$.

5. Statement 5 follows from statement 4 (by using the opposite rings).

6. Statement 6 follows from statements 3–5. □

Corollary 2.2 *1. Suppose that $S \in \mathrm{Den}_l(R)$ and the ring $S^{-1}R$ is either a domain or a one-sided Noetherian ring or does not contain an infinite direct sum of one-sided ideals then $S_l(R, \mathfrak{a}, S^{-1}R) = S_l^{sat} = S_l^{ws}$.*

2. Suppose that $S \in \mathrm{Den}_r(R)$ and the ring RS^{-1} is either a domain or a one-sided Noetherian ring or does not contain an infinite direct sum of one-sided ideals then $S_r(R, \mathfrak{a}, RS^{-1}) = S_r^{sat} = S_r^{ws}$.

3. Suppose that $S \in \mathrm{Den}(R)$ and the ring $S^{-1}R$ is either a domain or a one-sided Noetherian ring or does not contain an infinite direct sum of one-sided ideals then

$$S(R, \mathfrak{a}, S^{-1}R) = S_l(R, \mathfrak{a}, S^{-1}R) = S_r(R, \mathfrak{a}, RS^{-1}) = S_l^{sat} = S_r^{sat} = S^{ws} = S_l^{ws} = S_r^{ws}.$$

Proof The ring is a finite ring provided it is either a domain or a one-sised Noetherian ring or does not contain an infinite direct sum of one-sided ideals. Now, statements 1-3 follow from Theorem 1.2.(4–6). □

If the ring R is a *domain* the fact that $\sigma^{-1}((S^{-1}R)^\times) = \{a \in R \mid ba \in S\}$ was proven in [4, Proposition 10].

Corollary 2.3 *Let R be a ring.*

1. *If $S \in \mathrm{Den}_l(R, \mathfrak{a})$ then $\left(S_l^{sat}\right)_l^{sat} = S_l^{sat}$.*
2. *If $S \in \mathrm{Den}_r(R, \mathfrak{a})$ then $\left(S_r^{sat}\right)_r^{sat} = S_r^{sat}$.*
3. *If $S \in \mathrm{Den}(R, \mathfrak{a})$ then $\left(S_l^{sat}\right)_l^{sat} = \left(S_r^{sat}\right)_r^{sat} = \left(S^{ws}\right)^{ws} = S_l^{sat} = S_r^{sat} = S^{ws}$.*
4. *If $S \in \mathrm{Den}_l(R, \mathfrak{a})$ and the ring $S^{-1}R$ is a finite ring then $\left(S_l^{sat}\right)_l^{sat} = \left(S_l^{ws}\right)_l^{ws} = S_l^{sat} = S_l^{ws}$.*
5. *If and $S \in \mathrm{Den}_r(R, \mathfrak{a})$ and the ring RS^{-1} is a finite ring iff $\left(S_r^{sat}\right)_r^{sat} = \left(S_r^{ws}\right)_r^{ws} = S_r^{sat} = S_r^{ws}$.*
6. *If $S \in \mathrm{Den}(R, \mathfrak{a})$ and the ring $S^{-1}R$ is a finite ring iff*

$$\left(S_l^{sat}\right)_l^{sat} = \left(S_r^{sat}\right)_r^{sat} = \left(S^{ws}\right)^{ws} = \left(S_l^{ws}\right)_l^{ws} = \left(S_r^{ws}\right)_r^{ws} = S_l^{sat} = S_r^{sat} = S^{ws} = S_l^{ws} = S_r^{ws}.$$

Proof The corollary follows from Theorem 1.2. □

Proof of Theorem 1.3 . 1. ($\Rightarrow$) Theorem 1.2.(4).

($\Leftarrow$) Suppose that $S_l(R, \mathfrak{a}, S^{-1}R) = S_l^{sat} = S_l^{ws}$. We have to show that $(S^{-1}R)^\times = (S^{-1}R)_l^\times = (S^{-1}R)_r^\times$. Notice that

$$(S^{-1}R)^\times = (S^{-1}R)_l^\times = (S^{-1}R)_r^\times \Leftrightarrow (S^{-1}R)^\times = (S^{-1}R)_l^\times \Leftrightarrow (S^{-1}R)^\times = (S^{-1}R)_r^\times.$$

So, it suffices to show that $(S^{-1}R)^\times = (S^{-1}R)_l^\times$. An element $s^{-1}a \in S^{-1}R$ belongs to the set $(S^{-1}R)_l^\times$ where $s \in S$ and $a \in R$ iff $\frac{a}{1} \in (S^{-1}R)^\times$ iff $t^{-1}ba = 1$ for some elements $t \in S$ and $b \in R$ iff $ba \in S$ iff

$$a \in S_l^{ws}.$$

Similarly, an element $s^{-1}a \in S^{-1}R$ belongs to the set $(S^{-1}R)^\times$ where $s \in S$ and $a \in R$ iff $\frac{a}{1} \in (S^{-1}R)^\times$ iff $t^{-1}ba = 1$ for some elements $t \in S$ and $b \in R$ such that $t^{-1}b \in (S^{-1}R)_l^\times$ iff $ba \in S$ and $cb \in S$ iff $a \in S_l^{sat}$. Therefore, $(S^{-1}R)^\times = (S^{-1}R)_l^\times$ iff

$$S_l^{sat} = S_l^{ws},$$

and we are done.

2. ($\Rightarrow$) Theorem 1.2.(5).

($\Leftarrow$) Repeat the proof of the implication ($\Leftarrow$) of statement 1 by making obvious modifications (changing 'l' to 'r').

3. Statement 3 follows from statements 1 and 2. $\square$

The algebra $\mathbb{S}_n$ of one-sided inverses. Let K be a field and $K^\times$ be its group of units, and $P_n := K[x_1, \ldots, x_n]$ be a polynomial algebra over K.

Definition, [1]. The *algebra $\mathbb{S}_n$ of one-sided inverses* of P_n is an algebra generated over a field K by $2n$ elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ that satisfy the defining relations:

$$y_1x_1 = \cdots = y_nx_n = 1, \quad [x_i, y_j] = [x_i, x_j] = [y_i, y_j] = 0 \ \text{ for all } i \neq j,$$

where $[a, b] := ab - ba$, the commutator of elements a and b.

By the very definition, the algebra $\mathbb{S}_n$ is obtained from the polynomial algebra P_n by adding commuting, left (or right) inverses of its canonical generators. The algebra $\mathbb{S}_1$ is a well-known primitive algebra [5], p. 35, Example 2. Over the field $\mathbb{C}$ of complex numbers, the completion of the algebra $\mathbb{S}_1$ is the *Toeplitz algebra* which is the C^*-algebra generated by a unilateral shift on the Hilbert space $l^2(\mathbb{N})$ (note that $y_1 = x_1^*$). The Toeplitz algebra is the universal C^*-algebra generated by a proper isometry.

Clearly, $\mathbb{S}_n = \mathbb{S}_1^{\otimes n}$ and $\mathbb{S}_1 = K\langle x, y \,|\, yx = 1\rangle = \bigoplus_{i,j\geq 0} Kx^iy^j$. For each natural number $d \geq 1$, let $M_d(K) := \bigoplus_{i,j=0}^{d-1} KE_{ij}$ be the algebra of d-dimensional matrices where $\{E_{ij}\}$ are the matrix units, and $M_\infty(K) := \varinjlim M_d(K) = \bigoplus_{i,j\in\mathbb{N}} KE_{ij}$ be the algebra (without 1) of infinite dimensional matrices. The algebra $\mathbb{S}_1$ contains the ideal $F := \bigoplus_{i,j\in\mathbb{N}} KE_{ij}$, where

$$E_{ij} := x^iy^j - x^{i+1}y^{j+1}, \quad i, j \geq 0. \tag{2.1}$$

For all natural numbers i, j, k, and l, $E_{ij}E_{kl} = \delta_{jk}E_{il}$ where δ_{jk} is the Kronecker delta function. The ideal F is an algebra (without 1) isomorphic to the algebra $M_\infty(K)$ via $E_{ij} \mapsto E_{ij}$. For all $i, j \geq 0$,

$$xE_{ij} = E_{i+1,j}, \quad yE_{ij} = E_{i-1,j} \quad (E_{-1,j} := 0), \tag{2.2}$$

$$E_{ij}x = E_{i,j-1}, \quad E_{ij}y = E_{i,j+1} \quad (E_{i,-1} := 0). \tag{2.3}$$

$$\mathbb{S}_1 = K \oplus xK[x] \oplus yK[y] \oplus F, \tag{2.4}$$

the direct sum of vector spaces. Then

$$\mathbb{S}_1/F \simeq K[x, x^{-1}] =: L_1, \quad x \mapsto x, \quad y \mapsto x^{-1}, \tag{2.5}$$

since $yx = 1$, $xy = 1 - E_{00}$ and $E_{00} \in F$.

Lemma 2.4 is used in the proof of Proposition 2.5.

Lemma 2.4 *Let R be a ring, $\mathfrak{a}$ be an ideal of R, and $\pi : R \to \overline{R} := R/\mathfrak{a}$, $r \mapsto r + \mathfrak{a}$. Suppose that S is a multiplicative set in R such that $\overline{S} := \pi(S) \in \mathrm{Den}_*(\overline{R}, \overline{\mathfrak{b}})$ and $\mathfrak{a} \subseteq \mathrm{ass}_*(S)$ where $* \in \{l, r, \emptyset\}$. Then $S \in \mathrm{Den}_*(R, \mathfrak{b})$ where $\mathfrak{b} = \pi^{-1}(\overline{\mathfrak{b}})$.*

Proof We prove the lemma for $* = l$. The other two cases can be proven in a similar way. For each element $r \in R$, let $\overline{r} = \pi(r)$.

(i) $S \in \mathrm{Ore}_l(R)$: Given elements $s \in S$ and $r \in R$. Then $\overline{s} \in \overline{S}$ and $\overline{r} \in \overline{R}$. Since $\overline{S}$ is a left Ore set in $\overline{R}$, $\overline{s}_1\overline{r} = \overline{r}_1\overline{s}$ for some elements $s_1 \in S$ and $r_1 \in R$. Hence,

$$a := s_1 r - r_1 s \in \mathfrak{a}.$$

Since $\mathfrak{a} \subseteq \mathrm{ass}_l(S)$, we can choose an element, say $s_2 \in S$, such that $0 = s_2 a = s_2 s_1 r - s_2 r_1 s$, and the statement (i) follows.

(ii) $\mathrm{ass}_l(S) = \mathfrak{b}$: Given an element $b \in \mathfrak{b}$. Then $\overline{b} \in \overline{\mathfrak{b}}$, and so $\overline{s}\overline{b} = 0$ for some element $s \in S$ (since $\overline{S} \in \mathrm{Den}_l(\overline{R}, \overline{\mathfrak{b}})$). Hence, $sb \in \mathfrak{a}$, and so $tsb = 0$ for some element $t \in S$ (since $\mathfrak{a} \subseteq \mathrm{ass}_l(S)$). Therefore, $b \in \mathrm{ass}_l(S)$ and $\mathfrak{b} \subseteq \mathrm{ass}_l(S)$.

Conversely, given an element $a \in \mathrm{ass}_l(S)$. Then $sa = 0$ for some element $s \in S$. Then $\overline{sa} = 0$, and so $\overline{a} \in \overline{\mathfrak{b}}$ and $a \in \mathfrak{b}$. Therefore, $\mathfrak{b} \supseteq \mathrm{ass}_l(S)$, and the statement (ii) follows.

(iii) $S \in \mathrm{Den}_l(R, \mathfrak{b})$: In view of the statements (i) and (ii), we have to show that if $as = 0$ for some elements $a \in R$ and $s \in S$ then $a \in \mathfrak{b}$. Clearly, $\overline{as} = 0$, and so $\overline{a} \in \overline{\mathfrak{b}}$. Hence, $a \in \pi^{-1}(\overline{\mathfrak{b}}) = \mathfrak{b}$, as required. $\square$

The algebra $\mathbb{S}_n$ admits the *involution*

$$\eta : \mathbb{S}_n \to \mathbb{S}_n, \quad x_i \mapsto y_i, \quad y_i \mapsto x_i, \quad i = 1, \ldots, n,$$

i.e. it is a K-algebra anti-isomorphism ($\eta(ab) = \eta(b)\eta(a)$ for all $a, b \in \mathbb{S}_n$) such that $\eta^2 = \mathrm{id}_{\mathbb{S}_n}$, the identity map on $\mathbb{S}_n$. So, the algebra $\mathbb{S}_n$ is *self-dual* (i.e. it is isomorphic to its opposite algebra, $\eta : \mathbb{S}_n \simeq \mathbb{S}_n^{op}$). This means that left and right algebraic properties of the algebra $\mathbb{S}_n$ are the same.

Let $\mathfrak{a}_n := (x_1y_1 - 1, \ldots, x_ny_n - 1)$, an ideal of $\mathbb{S}_n$. By [1, Eq. (19)], the factor algebra

$$\mathbb{S}_n/\mathfrak{a}_n = L_n = K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

is the Laurent polynomial algebra. Clearly, $L_n^\times = \{\lambda x^\alpha \mid \lambda \in K^\times, \alpha \in \mathbb{Z}^n\}$ where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Let

$$\sigma : \mathbb{S}_n \to L_n, \quad a \mapsto a + \mathfrak{a}_n.$$

Then $\mathcal{L}_n := \sigma^{-1}(L_n^\times) = \{\lambda x^\alpha + a \mid \lambda \in K^\times, \alpha \in \mathbb{Z}^n, a \in \mathfrak{a}_n\}$.

Proposition 2.5 *Let $X = \langle x_1, \ldots, x_n \rangle$ and $Y = \langle y_1, \ldots, y_n \rangle$ be multiplicative submonoids of $(\mathbb{S}_n, \cdot)$ that are generated by the elements in the brackets. Then*

1. *$Y \in \mathrm{Den}_l(\mathbb{S}_n, \mathfrak{a}_n)$, $Y^{-1}\mathbb{S}_n = L_n$, $S_l(\mathbb{S}_n, \mathfrak{a}_n, L_n) = Y_l^{sat} = Y_l^{ws} = \mathcal{L}_n$.*
2. *$X \in \mathrm{Den}_r(\mathbb{S}_n, \mathfrak{a}_n)$, $\mathbb{S}_nX^{-1} = L_n$, $S_r(\mathbb{S}_n, \mathfrak{a}_n, L_n) = X_r^{sat} = X_r^{ws} = \mathcal{L}_n$.*

Proof 1. Recall that $\mathbb{S}_n = \mathbb{S}_1^{\otimes n}$. By [1, Eq. (19)], $\mathfrak{a}_n = \mathfrak{p}_1 + \cdots + \mathfrak{p}_i + \cdots + \mathfrak{p}_n$ where

$$\mathfrak{p}_1 = F \otimes \mathbb{S}_{n-1}, \ldots, \mathfrak{p}_i = \mathbb{S}_{i-1} \otimes F \otimes \mathbb{S}_{n-i}, \ldots, \mathfrak{p}_n = \mathbb{S}_{n-1} \otimes F.$$

By (2.2), $\mathfrak{p}_i \subseteq \mathrm{ass}_l(S_i)$ where $S_i = \{y_i^j \mid j \geq 0\} \subseteq Y$. Hence, $\mathfrak{a}_n \subseteq \mathrm{ass}_l(Y)$. Notice that $Y \in \mathrm{Den}_l(L_n, 0)$. By Lemma 2.4, $Y \in \mathrm{Den}_l(\mathbb{S}_n, \mathfrak{a}_n)$. Now,

$$Y^{-1}\mathbb{S}_n \simeq Y^{-1}(\mathbb{S}_n/\mathfrak{a}_n) = Y^{-1}L_n = L_n.$$

The algebra $Y^{-1}\mathbb{S}_n \simeq L_n$ is a Noetherian algebra. Hence,

$$S_l(\mathbb{S}_n, \mathfrak{a}, L_n) = Y_l^{sat} = Y_l^{ws} = \mathcal{L}_n,$$

by Corollary 2.2.(1) and Lemma 1.1.

2. By applying the involution η of the algebra $\mathbb{S}_n$ to statement 1 we obtain statement 2 (since $\eta(\mathfrak{a}_n) = \mathfrak{a}_n$, $\eta(Y) = X$ and $\eta(X) = Y$). $\square$

3 Localizations of a Ring at Localizable Sets, their Groups of Units and Saturations

The goal of Sect. 3 is to generalize results of Sect. 2 for localizable sets. At the beginning of the section, we collect some results on localizable sets and localizations of rings at localizable sets from [3] that are used in the section.

The ring $R\langle S^{-1}\rangle$. Let R be a ring and S be a multiplicative set in R (that is $SS \subseteq S$, $1 \in S$ and $0 \notin S$). Let $R\langle X_S\rangle$ be a ring freely generated by the ring R and a set $X_S = \{x_s \mid s \in S\}$ of free noncommutative indeterminates (indexed by the elements of the set S). Let us consider the factor ring

$$R\langle S^{-1}\rangle := R\langle X_S\rangle / I_S \tag{3.1}$$

of the ring $R\langle X_S\rangle$ at the ideal I_S generated by the set of elements $\{sx_s - 1, x_s s - 1 \mid s \in S\}$.

The kernel of the ring homomorphism

$$R \to R\langle S^{-1}\rangle, \quad r \mapsto r + I_S \tag{3.2}$$

is denoted by $\text{ass}(S) = \text{ass}_R(S)$. The ideal $\text{ass}_R(S)$ of R has a complex structure, its description is given in [3, Proposition 2.12] when S is a left localizable set.

Localizable sets.

Definition, [3]. A multiplicative set S of a ring R is called a *left localizable set* of R if

$R\langle S^{-1}\rangle = \{\overline{s}^{-1}\overline{r} \mid \overline{s} \in \overline{S}, \overline{r} \in \overline{R}\} \neq \{0\}$

where $\overline{R} = R/\mathfrak{a}$, $\mathfrak{a} = \text{ass}_R(S)$ and $\overline{S} = (S + \mathfrak{a})/\mathfrak{a}$, i.e., every element of the ring $R\langle S^{-1}\rangle$ is a left fraction $\overline{s}^{-1}\overline{r}$ for some elements $\overline{s} \in \overline{S}$ and $\overline{r} \in \overline{R}$. Similarly, a multiplicative set S of a ring R is called a *right localizable set* of R if

$R\langle S^{-1}\rangle = \{\overline{rs}^{-1} \mid \overline{s} \in \overline{S}, \overline{r} \in \overline{R}\} \neq \{0\}$,

i.e., every element of the ring $R\langle S^{-1}\rangle$ is a right fraction $\overline{rs}^{-1}$ for some elements $\overline{s} \in \overline{S}$ and $\overline{r} \in \overline{R}$. A right and left localizable set of R is called a *localizable set* of R.

The sets of left localizable, right localizable and localizable sets of R are denoted by $\mathbb{L}_l(R)$, $\mathbb{L}_r(R)$ and $\mathbb{L}(R)$, respectively. Clearly, $\mathbb{L}(R) = \mathbb{L}_l(R) \cap \mathbb{L}_r(R)$. In order to work with these three sets simultaneously we use the following notation $\mathbb{L}_*(R)$ where $* \in \{l, r, \emptyset\}$ and $\emptyset$ is the empty set ($\mathbb{L}(R) = \mathbb{L}_\emptyset(R)$). Let

$$\text{ass}\, \mathbb{L}_*(R) = \{\text{ass}_R(S) \mid S \in \mathbb{L}_*(R)\}. \tag{3.3}$$

For an ideal $\mathfrak{a}$ of R, let $\mathbb{L}_*(R, \mathfrak{a}) = \{S \in \mathbb{L}_*(R) \mid \text{ass}_R(S) = \mathfrak{a}\}$. Then

$$\mathbb{L}_*(R) = \coprod_{\mathfrak{a} \in \text{ass}\, \mathbb{L}_*(R)} \mathbb{L}_*(R, \mathfrak{a}) \tag{3.4}$$

is a disjoint union of non-empty sets.

The ideals $\mathfrak{a}(S)$, $'\mathfrak{a}(S)$ **and** $\mathfrak{a}'(S)$. For each element $r \in R$, let $r\cdot : R \to R$, $x \mapsto rx$ and $\cdot r : R \to R$, $x \mapsto xr$. The sets

$'\mathcal{C}_R := \{r \in R \mid \ker(\cdot r) = 0\}$ and $\mathcal{C}'_R := \{r \in R \mid \ker(r\cdot) = 0\}$

are called the *sets of left and right regular elements* of R, respectively. Their intersection

$\mathcal{C}_R = {}'\mathcal{C}_R \cap \mathcal{C}'_R$

is the *set of regular elements* of R. The rings

$Q_{l,cl}(R) := \mathcal{C}_R^{-1}R$ and $Q_{r,cl}(R) := R\mathcal{C}_R^{-1}$

are called the *classical left and right quotient rings* of R, respectively. Goldie's Theorem states that the ring $Q_{l,cl}(R)$ is a semisimple Artinian ring iff the ring R is semiprime, $\text{udim}(R) < \infty$ and the ring R satisfies the a.c.c. on left annihilators (udim stands for the uniform dimension).

Proposition 3.1 ([3, Proposition 1.1]) *Let R be a ring and S be a non-empty subset of R.*

1. *Suppose that there exists an ideal $\mathfrak{b}$ of R such that $(S+\mathfrak{b})/\mathfrak{b} \subseteq \mathcal{C}_{R/\mathfrak{b}}$. Then there is the least ideal, say $\mathfrak{a} = \mathfrak{a}(S)$, that satisfies this property.*
2. *Suppose that there exists an ideal $\mathfrak{b}$ of R such that $(S+\mathfrak{b})/\mathfrak{b} \subseteq {}'\mathcal{C}_{R/\mathfrak{b}}$. Then there is the least ideal, say ${}'\mathfrak{a} = {}'\mathfrak{a}(S)$, that satisfies this property; and ${}'\mathfrak{a}(S) \subseteq \mathfrak{a}(S)$.*
3. *Suppose that there exists an ideal $\mathfrak{b}$ of R such that $(S+\mathfrak{b})/\mathfrak{b} \subseteq \mathcal{C}'_{R/\mathfrak{b}}$. Then there is the least ideal, say $\mathfrak{a}' = \mathfrak{a}'(S)$, that satisfies this property; and $\mathfrak{a}'(S) \subseteq \mathfrak{a}(S)$.*

We have the inclusion

$$\mathfrak{a}(S) \subseteq \mathrm{ass}_*(R) \tag{3.5}$$

where $* \in \{l, r, \emptyset\}$, [3, Lemma 1.2]. The proof of Proposition 3.1 contains an explicit description of the ideal $\mathfrak{a}(S)$. The ideal $\mathfrak{a}(S)$ is the key part in the definition of *perfect localizable sets* that are introduced in [3].

The structure of the ring $R\langle S^{-1}\rangle$ and its universal property. Recall that for a ring R, we denote by $R^\times$ its group of units. Theorem 3.2 describes the structure and the universal property of the ring $R\langle S^{-1}\rangle$.

Theorem 3.2 ([3, Theorem 1.3]) *Let $S \in \mathbb{L}_*(R, \mathfrak{a})$ where $* \in \{l, \emptyset\}$, $\overline{R} = R/\mathfrak{a}$, $\pi : R \to \overline{R}$, $r \mapsto \overline{r} = r + \mathfrak{a}$ and $\overline{S} = \pi(S)$. Then*

1. $\overline{S} \in \mathrm{Den}_*(\overline{R}, 0)$.
2. *The ring $R\langle S^{-1}\rangle$ is R-isomorphic to the ring $\overline{S}^{-1}\overline{R}$.*
3. *Let $\mathfrak{b}$ be an ideal of R and $\pi^\dagger : R \to R^\dagger = R/\mathfrak{b}$, $r \mapsto r^\dagger = r + \mathfrak{b}$. If $S^\dagger = \pi^\dagger(S) \in \mathrm{Den}_*(R^\dagger, 0)$ then $\mathfrak{a} \subseteq \mathfrak{b}$ and the map*

   $\overline{S}^{-1}\overline{R} \to S^{\dagger -1}R^\dagger, \;\; \overline{s}^{-1}\overline{r} \mapsto s^{\dagger -1}r^\dagger$

   is a ring epimorphism with kernel $\overline{S}^{-1}(\mathfrak{b}/\mathfrak{a})$. So, the ideal $\mathfrak{a}$ is the least ideal $\mathfrak{a}$ of the ring R such that $S + \mathfrak{a} \in \mathrm{Den}_*(R/\mathfrak{a}, 0)$.
4. *Let $f : R \to Q$ be a ring homomorphism such that $f(S) \subseteq Q^\times$ and the ring Q is generated by $f(R)$ and the set $\{f(s)^{-1} \mid s \in S\}$. Then*

   (a) $\mathfrak{a} \subseteq \ker(f)$ *and the map*

   $\overline{S}^{-1}\overline{R} \to Q, \;\; \overline{s}^{-1}\overline{r} \mapsto f(s)^{-1}f(r)$

   is a ring epimorphism with kernel $\overline{S}^{-1}(\ker(f)/\mathfrak{a})$, and

   $Q = \{f(s)^{-1}f(r) \mid s \in S, r \in R\}$.

   (b) Let $\widetilde{R} = R/\ker(f)$ and $\widetilde{\pi} : R \to \widetilde{R}$, $r \mapsto \tilde{r} = r + \ker(f)$. Then $\widetilde{S} := \widetilde{\pi}(S) \in \mathrm{Den}_l(\widetilde{R}, 0)$ and $\widetilde{S}^{-1}\widetilde{R} \simeq Q$, an $\widetilde{R}$-isomorphism.

In view of Theorem 3.2.(1,2), for $S \in \mathbb{L}_*(R)$ we denote by $S^{-1}R$ the ring $R\langle S^{-1}\rangle$ for $* \in \{l, \emptyset\}$ and by RS^{-1} for $* \in \{r, \emptyset\}$. In particular, for $S \in \mathbb{L}(R)$,

$$R\langle S^{-1}\rangle = S^{-1}R \simeq RS^{-1}.$$

Elements of the rings $S^{-1}R$ and RS^{-1} are denoted by $s^{-1}r$ and rs^{-1}, respectively, where $s \in S$ and $r \in R$. Sometime, in order to make arguments shorter for $S \in \mathbb{L}_r(R)$ we denote the ring RS^{-1} by $S^{-1}R$.

For the algebra $\mathbb{S}_n$ and its multiplicative set Y, Lemma 3.3 presents explicitly all the ingredients of Proposition 3.1 and Theorem 3.2.

Lemma 3.3 *1. $Y \in \mathrm{Ore}(\mathbb{S}_n)$ and $Y \notin \mathrm{Den}_r(\mathbb{S}_n)$, $Y \subseteq {}'\mathcal{C}_{\mathbb{S}_n}$, $\mathrm{ass}_l(Y) = \mathfrak{a}_n$, and $\mathrm{ass}_r(Y) = 0$.*

2. *The ideals* $\mathfrak{a}(Y) = \mathfrak{a}(Y)' = \mathfrak{a}_n$ *and* ${}'\mathfrak{a}(Y) = 0$ *(see Proposition 3.1).*
3. *We keep the notation of Theorem 3.2. Then for all* $* \in \{l, r, \emptyset\}$,
   (a) $Y \in \mathbb{L}_*(\mathbb{S}_n, \mathfrak{a})$, $\mathfrak{a} = \mathfrak{a}_n$, *and* $Y^{-1}\mathbb{S}_n \simeq \mathbb{S}_n Y^{-1} \simeq L_n$,
   (b) $\overline{\mathbb{S}}_n := \mathbb{S}_n/\mathfrak{a} = \mathbb{S}_n/\mathfrak{a}_n = L_n$,
   (c) $\overline{Y} = \tilde{Y} \in \mathrm{Den}_*(\overline{\mathbb{S}}_n, 0)$.

Proof 1. The equalities $y_i x_i = 1$, $i = 1, \ldots, n$, implies that $y_i \in {}'\mathcal{C}_{\mathbb{S}_n}$, and so $Y \subseteq {}'\mathcal{C}_{\mathbb{S}_n}$. Hence, $\mathrm{ass}_r(Y) = 0$. By Proposition 2.5.(1), $\mathrm{ass}_l(Y) = \mathfrak{a}_n$. Hence, $Y \notin \mathrm{Den}_r(\mathbb{S}_n)$ (since $0 \neq \mathfrak{a}_n = \mathrm{ass}_l(Y) \nsubseteq \mathrm{ass}_r(Y) = 0$).

By Proposition 2.5.(1), $Y \in \mathrm{Ore}_l(\mathbb{S}_n)$. To finish the proof of statement 1, it remains to show that $Y \in \mathrm{Ore}_r(\mathbb{S}_n)$. Since $\mathbb{S}_n = \mathbb{S}_1^{\otimes n}$, it suffice to prove the statement for $n = 1$, that is $Y = \{y^i \mid i \geq 0\}$, we drop the subscript '1'. The algebra $\mathbb{S}_1$ is generated by the elements x and y, and $Y = \{y^i \mid i \geq 0\}$. So, it suffices to check that the right Ore condition holds for the elements $x \in \mathbb{S}_1$ and $y \in Y$, i.e. to prove that there are elements $a \in \mathbb{S}_1$ and y^i such that $xy^i = ya$. It suffices to take $i = 2$ and $a = 1 - E_{11}$:

$$xy^2 = (1 - (1 - xy))y = (1 - E_{00})y = y - E_{01} = y(1 - E_{11}).$$

2. By statement 1, $Y \subseteq {}'\mathcal{C}_{\mathbb{S}_n}$, and so ${}'\mathfrak{a}(Y) = 0$. By Proposition 2.5.(1), $Y \in \mathrm{Den}_l(\mathbb{S}_n, \mathfrak{a}_n)$. Hence, $\mathfrak{a}(Y) = \mathfrak{a}_n$. On the one hand, $\mathfrak{a}(Y)' \subseteq \mathfrak{a}(Y) = \mathfrak{a}_n$, by Proposition 3.1.(3). On the other hand, $\mathfrak{a}_n \subseteq \mathfrak{a}(Y)'$, by (2.2). Therefore, $\mathfrak{a}(Y)' = \mathfrak{a}_n$.

3. The case $* = l$ follows from the fact that $Y \in \mathrm{Den}_l(\mathbb{S}_n, \mathfrak{a}_n)$ (Proposition 2.5.(1)). It suffices to consider the case where $* = r$. By statement 1, $\mathrm{ass}_l(Y) = \mathfrak{a}_n$. Clearly, $\mathrm{ass}_l(Y) \subseteq \mathrm{ass}_R(Y)$. Since $\mathbb{S}_n/\mathrm{ass}_l(Y) = \mathbb{S}_n/\mathfrak{a}_n = L_n$ and the elements of the set Y are units in the Laurent polynomial ring L_n, we have that $\mathrm{ass}_R(Y) = \mathfrak{a}_n$, $Y \in \mathbb{L}_r(\mathbb{S}_n, \mathfrak{a}_n)$ and $\mathbb{S}_n Y^{-1} \simeq L_n$. Now statements (b) and (c) follows. □

For the algebra $\mathbb{S}_n$ and its multiplicative set X, Lemma 3.4 presents explicitly all the ingredients of Proposition 3.1 and Theorem 3.2.

Lemma 3.4 *1.* $X \in \mathrm{Ore}(\mathbb{S}_n)$ *and* $X \notin \mathrm{Den}_l(\mathbb{S}_n)$, $X \subseteq \mathcal{C}'_{\mathbb{S}_n}$, $\mathrm{ass}_r(X) = \mathfrak{a}_n$, *and* $\mathrm{ass}_l(X) = 0$.
2. *The ideals* $\mathfrak{a}(X) = {}'\mathfrak{a}(X) = \mathfrak{a}_n$ *and* $\mathfrak{a}(X)' = 0$ *(see Proposition 3.1).*
3. *We keep the notation of Theorem 3.2. Then for all* $* \in \{l, r, \emptyset\}$,
   (a) $X \in \mathbb{L}_*(\mathbb{S}_n, \mathfrak{a})$, $\mathfrak{a} = \mathfrak{a}_n$, *and* $X^{-1}\mathbb{S}_n \simeq \mathbb{S}_n X^{-1} \simeq L_n$,
   (b) $\overline{\mathbb{S}}_n := \mathbb{S}_n/\mathfrak{a} = \mathbb{S}_n/\mathfrak{a}_n = L_n$,
   (c) $\overline{X} = \tilde{X} \in \mathrm{Den}_*(\overline{\mathbb{S}}_n, 0)$.

Proof Since $\eta(Y) = X$ and $\eta(\mathfrak{a}_n) = \mathfrak{a}_n$, the lemma follows from Lemma 3.3. □

For each element $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, let

$$\mathrm{supp}(\alpha) = \{i \mid \alpha_i \neq 0, i \in \{1, \ldots, n\}\}.$$

Recall that $\sigma : \mathbb{S}_n \to \mathbb{S}_n/\mathfrak{a}_n = L_n$, $a \mapsto a + \mathfrak{a}_n$. Notice that

$$\mathbb{S}_n = \bigoplus_{\alpha,\beta\in\mathbb{N}^n, \mathrm{supp}(\alpha)\cap\mathrm{supp}(\beta)=\emptyset} Kx^\alpha y^\beta \oplus \mathfrak{a}_n. \tag{3.6}$$

Consider a subgroup of units $\tilde{Z} := \{x^\alpha \mid \alpha \in \mathbb{Z}^n\}$ of the algebra L_n. Its pre-image

$$Z := \sigma^{-1}(\tilde{Z}) = \{x^\alpha y^\beta + \mathfrak{a}_n \mid \alpha, \beta \in \mathbb{N}^n, \mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta) = \emptyset\} \tag{3.7}$$

is a submonoid of $(\mathbb{S}_n, \cdot)$.

Lemma 3.5 $Z \in \mathrm{Den}(\mathbb{S}_n, \mathfrak{a}_n)$ *and* $Z^{-1}\mathbb{S}_n \simeq \mathbb{S}_n Y^{-1} \simeq L_n$.

Proof Clearly, $Y \subseteq Z$. Then, by Lemma 3.3.(1), $\mathfrak{a}_n = \mathrm{ass}_l(Y) \subseteq \mathrm{ass}_l(Z)$. Similarly, $X \subseteq Z$. Then, by Lemma 3.4.(1), $\mathfrak{a}_n = \mathrm{ass}_r(X) \subseteq \mathrm{ass}_r(Z)$. The algebra $\mathbb{S}_n/\mathfrak{a}_n = L_n$ is a domain. Hence,

$\mathrm{ass}_l(Z) = \mathrm{ass}_r(Z) = \mathfrak{a}_n$.

Since the set $\tilde{Z} = \sigma(Z)$ is a group of units in the algebra L_n, we must have $Z \in \mathrm{Den}(\mathbb{S}_n, \mathfrak{a}_n)$ and $Z^{-1}\mathbb{S}_n \simeq \mathbb{S}_n Y^{-1} \simeq L_n$. □

Lemma 3.6 *Let R be a ring, $S \in \mathbb{L}_*(R, \mathfrak{a})$ and $T \in \mathbb{L}_*(R, \mathfrak{b})$ such that $S \subseteq T$ where $* \in \{l, r, \emptyset\}$. Then $\mathfrak{a} \subseteq \mathfrak{b}$ and for $* \in \{l, \emptyset\}$ the map $S^{-1}R \to T^{-1}R$, $s^{-1}r \mapsto t^{-1}r$ is an R-homomorphism with kernel $S^{-1}(\mathfrak{b}/\mathfrak{a}) = \overline{S}^{-1}(\mathfrak{b}/\mathfrak{a})$ where $\overline{S} = \{s + \mathfrak{a} \mid s \in S\}$. A similar result holds for $* = r$.*

Proof Recall that $\mathfrak{a} = \mathrm{ass}_R(S)$ and $\mathfrak{b} = \mathrm{ass}_R(T)$. Let Q be a subring of $T^{-1}R$ which is generated by the images of the ring R and the set $\{s^{-1} \mid s \in S\}$ in $T^{-1}R$ (recall that $S \subseteq T$). Applying Theorem 3.2.(4a) to the ring homomorphism $R \to Q \subseteq T^{-1}R$, $r \mapsto \frac{r}{1}$ we obtain the ring R-homomorphism

$S^{-1}R \to T^{-1}R, \ \ s^{-1}r \mapsto s^{-1}r.$

Since $S^{-1}R = \overline{S}^{-1}\overline{R}$ and $T^{-1}R = \overline{T}^{-1}(R/\mathfrak{b})$ where $\overline{T} = \{t + \mathfrak{b} \mid t \in T\}$, the kernel of the R-homomorphism is $\overline{S}^{-1}(\mathfrak{b}/\mathfrak{a})$. □

The posets $(\mathbb{L}_*(R), \subseteq)$ **and** $(\mathbb{L}\mathrm{oc}_*(R), \to)$. The set $(\mathbb{L}\mathrm{oc}_*(R, \mathfrak{a}), \to)$ is a poset where $A_1 \to A_2$ if $A_1 = R\langle S_1^{-1}\rangle$ and $A_2 = R\langle S_2^{-1}\rangle$ for some localizable sets $S_1, S_2 \in \mathbb{L}_*(R, \mathfrak{a})$ such that the map $A_1 \to A_2$, $s_1^{-1}r \mapsto s_1^{-1}r$ if $* \in \{l, \emptyset\}$ (resp., $rs_1^{-1} \mapsto rs_1^{-1}$ if $* = r$) is a well-defined homomorphism. Moreover, enlarging if necessary the denominator set S_2 we can assume that $S_1 \subseteq S_2$ (for example, by taking $S_2 = \sigma_2^{-1}(A_2^\times)$ where $\sigma_2 : R \to A_2$, $r \mapsto \frac{r}{1}$, see Proposition 3.8.(2)). By Proposition 3.8.(2),

$A_1 \to A_2$ iff $\mathcal{S}_l(R, \mathfrak{a}, A_1) \subseteq \mathcal{S}_l(R, \mathfrak{a}, A_2)$.

In the same way, the poset $(\mathbb{L}\mathrm{oc}_*(R), \to)$ is defined, i.e. $A_1 \to A_2$ if there exist $S_1, S_2 \in \mathbb{L}_*(R)$ such that $S_1 \subseteq S_2$, $A_1 = R\langle S_1^{-1}\rangle$ and $A_2 = R\langle S_2^{-1}\rangle$, $A_1 \to A_2$ stands for the map $\varphi : A_1 \to A_2$, $s_1^{-1}r \mapsto s_1^{-1}r$ if $* \in \{l, \emptyset\}$ (resp., $rs_1^{-1} \mapsto rs_1^{-1}$ if $* = r$). The map

$$\mathbb{L}_*(R) \to \mathbb{L}\mathrm{oc}_*(R), \ \ S \mapsto R\langle S^{-1}\rangle, \tag{3.8}$$

is an epimorphism from the poset $(\mathbb{L}_*(R), \subseteq)$ to $(\mathbb{L}\mathrm{oc}_l(R), \to)$. For each ideal $\mathfrak{a} \in \mathrm{Ass}_*(R)$, it induces the epimorphism of the posets $(\mathbb{L}_*(R, \mathfrak{a}), \subseteq)$ and $(\mathbb{L}\mathrm{oc}_*(R, \mathfrak{a}), \to)$,

$$\mathbb{L}_*(R, \mathfrak{a}) \to \mathbb{L}\mathrm{oc}_*(R, \mathfrak{a}), \ \ S \mapsto R\langle S^{-1}\rangle. \tag{3.9}$$

The sets $\mathbb{L}_*(R)$ and $\mathbb{L}\mathrm{oc}_*(R)$ are the disjoint unions

$$\mathbb{L}_*(R) = \bigsqcup_{\mathfrak{a} \in \mathrm{Ass}_*(R)} \mathbb{L}_*(R, \mathfrak{a}), \ \ \mathbb{L}\mathrm{oc}_l(R) = \bigsqcup_{\mathfrak{a} \in \mathrm{Ass}_l(R)} \mathbb{L}\mathrm{oc}_*(R, \mathfrak{a}). \tag{3.10}$$

For each ideal $\mathfrak{a} \in \mathrm{Ass}_*(R)$, the set $\mathbb{L}_*(R, \mathfrak{a})$ is the disjoint union

$$\mathbb{L}_*(R, \mathfrak{a})) = \bigsqcup_{A \in \mathbb{L}\mathrm{oc}_*(R, \mathfrak{a}))} \mathbb{L}_*(R, \mathfrak{a}, A) \tag{3.11}$$

where $\mathbb{L}_*(R, \mathfrak{a}, A) := \{S \in \mathbb{L}_*(R, \mathfrak{a}) \mid R\langle S^{-1}\rangle \simeq A$, an R-isomorphism$\}$.

The largest element $\mathcal{S}_*(R, \mathfrak{a}, S^{-1}R)$ **in** $(\mathbb{L}_*(R, \mathfrak{a}, S^{-1}R) \subseteq)$ **and its characterizations where** $S \in \mathbb{L}_*(R, \mathfrak{a})$. Proposition 3.7.(1) is a practical criterion for a multiplicative set S of a ring R to belong to the set $\mathbb{L}_*(R, \mathfrak{a})$.

Proposition 3.7 *Let S be a multiplicative set of a ring R.*

1. *Suppose that there exists an ideal $\mathfrak{a}$ of R such that $\mathfrak{a} \subseteq \mathrm{ass}_R(S)$ and $\overline{S} := \pi(S) \in \mathrm{Den}_*(\overline{R}, 0)$ where $\pi : R \to \overline{R} := R/\mathfrak{a}$, $a \mapsto a + \mathfrak{a}$. Then $\mathfrak{a} = \mathrm{ass}_R(S)$ and $S \in \mathbb{L}_*(R, \mathfrak{a})$.*
2. *$S \in \mathbb{L}_*(R, \mathfrak{b})$ iff there is an ideal $\mathfrak{a}$ of R such that $\mathfrak{a} \subseteq \mathfrak{b}$ and $\overline{S} := \pi(S) \in \mathrm{Den}_*(\overline{R}, 0)$ where $\pi : R \to \overline{R} := R/\mathfrak{a}$, $a \mapsto a + \mathfrak{a}$.*

Proof 1. Since the elements of the set $\overline{S}$ are invertible in the localization $\mathcal{R}$ of the ring $\overline{R}$ at $\overline{S}$, there is an R-epimorphism from $R\langle S^{-1}\rangle$ to $\mathcal{R}$. In particular, $\mathrm{ass}_R(S) \subseteq \mathfrak{a}$. Hence $\mathfrak{a} = \mathrm{ass}_R(S)$, and then $S \in \mathbb{L}_*(R, \mathfrak{a})$ (by Theorem 3.2.(1)), and statement 1 follows.

2. ($\Rightarrow$) If $S \in \mathbb{L}_*(R, \mathfrak{b})$ then it suffices to take $\mathfrak{a} = \mathfrak{b}$, by Theorem 3.2.(1).

($\Leftarrow$) This implication follows from Theorem 3.2.(3). $\square$

Proposition 3.8.(2), is an explicit description of the largest element $\mathcal{S}_*(R, \mathfrak{a}, S^{-1}A)$ of the partially ordered set $(\mathbb{L}_*(R, \mathfrak{a}, S^{-1}R), \subseteq)$.

Proposition 3.8 *Let $S \in \mathbb{L}_*(R, \mathfrak{a})$, $\pi : R \to \overline{R} := R/\mathfrak{a}$, $a \mapsto a + \mathfrak{a}$, and $\sigma : R \to S^{-1}R$, $r \mapsto r/1$ where $* \in \{l, r, \emptyset\}$.*

1. *Suppose that $T \in \mathrm{Den}_*(S^{-1}R, 0)$ be such that $\pi(S), \pi(S)^{-1} \subseteq T$. Then $\overline{T} := T \cap \overline{R} \in \mathrm{Den}_l(\overline{R}, 0)$, $\overline{S} := \pi(S) \subseteq \overline{T}$, $\overline{S}^{-1}\overline{R} \subseteq \overline{T}^{-1}\overline{R} \simeq T^{-1}(S^{-1}R)$, and $T' := \sigma^{-1}(T) \in \mathbb{L}_*(R, \mathfrak{a}, T^{-1}(S^{-1}R))$.*
2. *The set $\mathcal{S}_*(R, \mathfrak{a}, S^{-1}A) := \sigma^{-1}((S^{-1}R)^\times)$ is the largest element of the partially ordered set $(\mathbb{L}_*(R, \mathfrak{a}, S^{-1}R), \subseteq)$.*

Proof 1. (i) $\overline{T} := T \cap \overline{R} \in \mathrm{Den}_l(\overline{R}, 0)$, $\overline{S} := \pi(S) \subseteq \overline{T}$, *and* $\overline{S}^{-1}\overline{R} \subseteq \overline{T}^{-1}\overline{R} \simeq T^{-1}(S^{-1}R)$: This is a particular case of [2, Lemma 3.3.(1)].

(ii) $T' \in \mathbb{L}_*(R, \mathfrak{a}, \overline{T}^{-1}\overline{R} = T^{-1}(S^{-1}R))$: The set $T' = \sigma^{-1}(T)$ is a multiplicative set in R that contains S. Since $S \subseteq T'$, we have the inclusion of ideals

$$\mathfrak{a} = \mathrm{ass}_*(S) \subseteq \mathrm{ass}_*(T).$$

Since $\pi(T') = \pi(\sigma^{-1}(T)) = \sigma(\sigma^{-1}(T)) = T \cap \overline{R} = \overline{T}$ and $\overline{T} \in \mathrm{Den}_l(\overline{R}, 0)$, we have that

$$T' \in \mathbb{L}_*(R, \mathfrak{a}),$$

by Proposition 3.7. (1). Since $\overline{T}^{-1}\overline{R} \simeq T^{-1}(S^{-1}R)$ (the statement (i)), $T' \in \mathbb{L}_*(R, \mathfrak{a}, T^{-1}(S^{-1}R))$.

2. Clearly, $(S^{-1}R)^\times \in \mathrm{Den}_*(S^{-1}R, 0) \subseteq \mathbb{L}_*(S^{-1}R, 0)$. By statement 1,

$$\sigma^{-1}((S^{-1}R)^\times) \in \mathbb{L}_*(R, \mathfrak{a}, S^{-1}R).$$

On the other hand, if $\mathcal{T} \in \mathbb{L}_*(R, \mathfrak{a}, S^{-1}R)$, then $\pi(\mathcal{T}) \subseteq (S^{-1}R)^\times$, and so

$$\mathcal{T} \subseteq \sigma^{-1}((S^{-1}R)^\times).$$

Therefore, the set $\sigma^{-1}((S^{-1}R)^\times)$ is the largest element of the poset $(\mathbb{L}_*(R, \mathfrak{a}, S^{-1}R), \subseteq)$. $\square$

By Lemma 3.5,

$$Z \in \mathrm{Den}(\mathbb{S}_n, \mathfrak{a}_n) \quad \text{and} \quad Z^{-1}\mathbb{S}_n \simeq \mathbb{S}_n Z^{-1} \simeq L_n.$$

For the algebra $\mathbb{S}_n$ and its multiplicative set $S = Z$ (see (3.7)), Lemma 3.9 gives an explicit description of the set $S_*(\mathbb{S}_n, \mathfrak{a}_n, L_n)$, see Proposition 3.8.(2).

Lemma 3.9 *We keep the notation as above. Then for all $* \in \{l, r, \emptyset\}$,*

$$S_*(\mathbb{S}_n, \mathfrak{a}_n, L_n) = \{K^\times x^\alpha y^\beta + \mathfrak{a}_n \mid \alpha, \beta \in \mathbb{N}^n\}.$$

Proof Let $\sigma : \mathbb{S}_n \to Z^{-1}\mathbb{S}_n \simeq L_n, r \mapsto \frac{r}{1}$. Now, the result follows from Proposition 3.8.(2),

$$S_*(\mathbb{S}_n, \mathfrak{a}_n, L_n) = \sigma^{-1}(L_n^\times)) = \{K^\times x^\alpha y^\beta + \mathfrak{a}_n \mid \alpha, \beta \in \mathbb{N}^n, \mathrm{supp}(\alpha) \cap \mathrm{supp}(\beta) = \emptyset\}$$

since $L_n^\times = \{K^\times x^\gamma \mid \gamma \in \mathbb{Z}^n\}$ and $\mathbb{S}_n = \bigoplus_{\alpha,\beta\in\mathbb{N}^n, \mathrm{supp}(\alpha)\cap\mathrm{supp}(\beta)=\emptyset} K x^\alpha y^\beta \oplus \mathfrak{a}_n$. □

Theorem 3.10 is another characterization of the set $\mathcal{S}_*(R, \mathfrak{a}, R\langle S^{-1}\rangle)$ in terms of the five saturations.

Theorem 3.10 *Let R be a ring, $\mathfrak{a}$ be an ideal of R, $\pi : R \to \overline{R} := R/\mathfrak{a}$, $a \mapsto a + \mathfrak{a}$, and $\overline{S} := \pi(S)$ for a subset S of R (in statements 1–6 below, saturations of $\overline{S}$ are given in the ring $\overline{R}$).*

1. *If $S \in \mathbb{L}_l(R, \mathfrak{a})$ then $\mathcal{S}_l(R, \mathfrak{a}, S^{-1}R) = \pi^{-1}(\overline{S}_l^{sat})$.*
2. *If $S \in \mathbb{L}_r(R, \mathfrak{a})$ then $\mathcal{S}_r(R, \mathfrak{a}, RS^{-1}) = \pi^{-1}(\overline{S}_r^{sat})$.*
3. *If $S \in \mathbb{L}(R, \mathfrak{a})$ then $\mathcal{S}(R, \mathfrak{a}, S^{-1}R) = \mathcal{S}_l(R, \mathfrak{a}, S^{-1}R) = \mathcal{S}_r(R, \mathfrak{a}, RS^{-1}) = \pi^{-1}(\overline{S}_l^{sat}) = \pi^{-1}(\overline{S}_r^{sat}) = \pi^{-1}(\overline{S}^{ws})$ and $\overline{S}^{ws} = \overline{S}_l^{sat} \cap \overline{S}_r^{sat}$.*
4. *If $S \in \mathbb{L}_l(R, \mathfrak{a})$ and the ring $S^{-1}R$ is finite then $\mathcal{S}_l(R, \mathfrak{a}, S^{-1}R) = \pi^{-1}(\overline{S}_l^{sat}) = \pi^{-1}(\overline{S}_l^{ws})$.*
5. *If and $S \in \mathbb{L}_r(R, \mathfrak{a})$ and the ring RS^{-1} is a finite ring then $\mathcal{S}_r(R, \mathfrak{a}, RS^{-1}) = \pi^{-1}(\overline{S}_r^{sat}) = \pi^{-1}(\overline{S}_r^{ws})$.*
6. *If $S \in \mathbb{L}(R, \mathfrak{a})$ and the ring $S^{-1}R$ is a finite ring then*
$$\begin{aligned}\mathcal{S}(R, \mathfrak{a}, S^{-1}R) &= \mathcal{S}_l(R, \mathfrak{a}, S^{-1}R) = \mathcal{S}_r(R, \mathfrak{a}, RS^{-1}) = \pi^{-1}(\overline{S}_l^{sat}) = \pi^{-1}(\overline{S}_r^{sat})\\ &= \pi^{-1}(\overline{S}^{ws}) = \pi^{-1}(\overline{S}_l^{ws}) = \pi^{-1}(\overline{S}_r^{ws}).\end{aligned}$$

Proof Given $S \in \mathbb{L}_*(R, \mathfrak{a})$. Let $A = R\langle S^{-1}\rangle$ be the localization of the ring R at the localizable set S. By Theorem 3.2.(1,2), $\overline{S} \in \mathrm{Den}_*(\overline{R}, 0)$ and the ring A is R-isomorphic to the localization of the ring $\overline{R}$ at the denominator set $\overline{S}$. Let $\overline{\sigma} : \overline{R} \to A, \overline{a} \mapsto \frac{\overline{a}}{1}$. Then the map $\sigma : R \to A, r \mapsto \frac{r}{1}$ is the composition of the composition map $\sigma = \overline{\sigma}\pi$. Therefore, by Proposition 3.8.(2),

$$\mathcal{S}_*(R, \mathfrak{a}, A) = \sigma^{-1}(A^\times) = \pi^{-1}(\overline{\sigma}^{-1}(A^\times)).$$

Now, statements 1–6 follow from statements 1–6 of Theorem 1.2, respectively. □

Corollary 3.11 *We keep the notation of Theorem 3.10.*

1. *Suppose that $S \in \mathbb{L}_l(R)$ and the ring $S^{-1}R$ is either a domain or a one-sided Noetherian ring or does not contain an infinite direct sum of one-sided ideals then $\mathcal{S}_l(R, \mathfrak{a}, S^{-1}R) = \pi^{-1}(\overline{S}_l^{sat}) = \pi^{-1}(\overline{S}_l^{ws})$.*
2. *Suppose that $S \in \mathbb{L}_r(R)$ and the ring RS^{-1} is either a domain or a one-sided Noetherian ring or does not contain an infinite direct sum of one-sided ideals then $\mathcal{S}_r(R, \mathfrak{a}, RS^{-1}) = \pi^{-1}(\overline{S}_r^{sat}) = \pi^{-1}(\overline{S}_r^{ws})$.*
3. *Suppose that $S \in \mathbb{L}(R)$ and the ring $S^{-1}R$ is either a domain or a one-sided Noetherian ring or does not contain an infinite direct sum of one-sided ideals then*
$$\begin{aligned}\mathcal{S}(R, \mathfrak{a}, S^{-1}R) &= \mathcal{S}_l(R, \mathfrak{a}, S^{-1}R) = \mathcal{S}_r(R, \mathfrak{a}, RS^{-1}) = \pi^{-1}(\overline{S}_l^{sat}) = \pi^{-1}(\overline{S}_r^{sat})\\ &= \pi^{-1}(\overline{S}^{ws}) = \pi^{-1}(\overline{S}_l^{ws}) = \pi^{-1}(\overline{S}_r^{ws}).\end{aligned}$$

Proof The ring is a finite ring provided it is either a domain or a one-sised Noetherian ring or does not contain an infinite direct sum of one-sided ideals. Now, statements 1-3 follow from Theorem 3.10.(4–6). □

By [3, Theorem 1.6.(1,2)], if $S \in \mathrm{Ore}(R)$ then $S \in \mathbb{L}(R)$ and

$$\mathrm{ass}_R(S) = \{a \in R \mid sat = 0 \ \text{ for some } \ s, t \in S\}. \tag{3.12}$$

Proof of Theorem 1.4 We keep the notation of Theorem 3.10. Given elements $\overline{a}, \overline{b} \in \overline{R}$. Then $\overline{a}\overline{b} \in \overline{S}$ iff $ba \in S + \mathfrak{a}$ iff $s_1bas_2 \in S$ for some elements $s_1, s_2 \in S$, by (3.12).

1. Now, statement 1 follows from Theorem 3.10.(3) and (3.12).
2. Similarly, statement 2 follows from Corollary 3.11.(3) and (3.12). □

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Declarations

Conflict of interest Up to my knowledge there is none.

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