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# Weyl Sets in a Non-degenerate Truncated Matricial Hausdorff Moment Problem

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## Abstract

Given a point $w$ in the upper half-plane $\Pi_+$, we describe the set of all possible values $F(w)$ of transforms $F(z) := \int_{[\alpha,\beta]}(x-z)^{-1}\sigma(\mathrm{d}x)$, $z \in \Pi_+$, corresponding to solutions $\sigma$ to a (non-degenerate) truncated matricial Hausdorff moment problem. This set turns out to be the intersection of two matrix balls the parameters of which are explicitly constructed from the given data.

**Keywords** Matricial Hausdorff moment problem · Orthogonal matrix polynomials · Weyl sets · Matrix balls

**Mathematics Subject Classification** 44A60 · 47A57 · 30E05

Dedicated to the memory of Heinz Langer.

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## 1 Introduction

This paper covers a research issue which arises from the following truncated matricial Hausdorff type moment problem: Given real numbers $\alpha$ and $\beta$ such that $\alpha < \beta$ and a finite sequence $(s_j)_{j=0}^{m}$ of complex $q \times q$ matrices, describe the set of all non-negative Hermitian $q \times q$ measures $\sigma$ which satisfy $s_j = \int_{[\alpha,\beta]} t^j \sigma(\mathrm{d}t)$ for every choice of $j \in \mathbb{Z}_{0,m}$. In fact, the solutions $\sigma$ to this matricial moment problem are in an one-to-one correspondence with certain holomorphic matrix-valued functions $F$. The core objective of our investigations is to characterize the set of all possible values $F(w)$ which these matrix functions can take at an arbitrarily fixed point $w$ of the open upper complex half-plane $\Pi_+$.

In the papers [14, 27, 36], the Weyl sets for a matricial truncated Hamburger moment problem and a matricial truncated Stieltjes moment problem were determined. The main goal of this paper is a representation of the corresponding Weyl set for a matricial truncated Hausdorff moment problem. In contrast to our considerations in the Hamburger and Stieltjes case (see [14, 36]), we want to restrict our following considerations to the so-called non-degenerate case. It turns out that similar as in the Stieltjes case, in the Hausdorff case one can represent the corresponding Weyl set as the intersection of two matrix balls whose parameters are constructed explicitly from the given data. In order to prove the main results of this work, a couple of statements of a technical nature are required. For the convenience of the reader, we also find it advantageous to formulate some results in this work that have been proven elsewhere.

This paper is organized as follows: In Sect. 2, the considered moment problems are formulated. In Sect. 3, known solvability conditions for the matricial truncated Hamburger and Hausdorff power moment problems are stated. These necessary and sufficient conditions for solvability are formulated with the help of classes of special sequences of complex matrices that are also essential for us in the following. In Sect. 4, the Hamburger and the Hausdorff moment problems are transformed into interpolation problems for special classes of holomorphic matrix functions by means of integral transformations, as is usually the case. These transformed problems are the focus of our further considerations. In particular, there we formulate our goal in detail. In the following sections, several known and new technical tools are provided. Thus, Sect. 5 is aimed to recall the parameterizations of the classes of sequences of matrices which play key roles in the matricial Hamburger and Hausdorff moment problems. In Sect. 6, the parameters introduced for the Hamburger case are combined with orthogonal matrix polynomials, while in Sect. 7 the corresponding matrix balls are then explicitly represented with the help of some family of rational matrix functions. In Sect. 8, the classes of sequences of complex matrices, which are relevant for the Hausdorff moment problem, are linked with orthogonal matrix polynomials. In Sect. 9, we repeat shortly the $\mathcal{F}_{\alpha,\beta}$-transformation for matrix sequences, which was introduced in [21] and constitutes the elementary step of a Schur algorithm in the class of $[\alpha, \beta]$-non-negative definite sequences. In Sect. 10, special classes of meromorphic matrix functions are presented. In Sects. 11 and 12, we prove descriptions of the solution set of the matricial truncated Hausdorff moment problem. In contrast to the corresponding parameterization in [23], the corresponding first main result of the present paper, proven in Theorem 11.29, contains a representation using orthogonal matrix polynomials, whereby a certain additional requirement is made. Section 13 is aimed at proving that the interesting set of all possible values of solutions of the (transformed) moment problem is a subset of two matrix balls. In Sects. 14 and 15, the main concern of the present work is realized, namely the description of the values of the solutions of the truncated matricial Hausdorff moment problem in the non-degenerate case.

## 2 Preliminaries

Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}_0$, and $\mathbb{N}$ be the set of all complex numbers, the set of all real numbers, the set of all non-negative integers, and the set of all positive integers, respectively. For every choice of $\upsilon, \omega \in \mathbb{R} \cup \{-\infty, \infty\}$, let $\mathbb{Z}_{\upsilon,\omega}$ be the set of all integers $k$ such that $\upsilon \leq k \leq \omega$. Throughout this paper, if not explicitly mentioned otherwise, let $p, q \in \mathbb{N}$ and let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$. If $\mathcal{X}$ is a non-empty set, then $\mathcal{X}^{p\times q}$ stands for the set of all $p \times q$ matrices each entry of which belongs to $\mathcal{X}$, and $\mathcal{X}^p$ is short for $\mathcal{X}^{p\times 1}$. We use $\mathbb{C}^{q\times q}_{\succcurlyeq}$ (resp. $\mathbb{C}^{q\times q}_{\succ}$) to designate the set of all non-negative Hermitian (resp. positive Hermitian) complex $q \times q$ matrices. Furthermore, let $\mathbb{C}^{q\times q}_{\mathrm{H}}$ be the set of all Hermitian complex $q \times q$ matrices. If $A$ and $B$ are complex $q \times q$ matrices, then we will write $A \preccurlyeq B$ (or $B \succcurlyeq A$) to indicate that $A$ and $B$ are Hermitian matrices such that $B - A$ is a non-negative Hermitian matrix. Furthermore, for all $A \in \mathbb{C}^{q\times q}$, we will use $\Re A$ and $\Im A$ to denote the real part of $A$ and the imaginary part of $A$, respectively: $\Re A := \frac{1}{2}(A + A^*)$ and $\Im A := \frac{1}{2\mathrm{i}}(A - A^*)$. If $A \in \mathbb{C}^{q\times q}$, then let $\det A$ be the determinant of $A$. For all $A \in \mathbb{C}^{p\times q}$, let $\|A\|$ be the operator norm of $A$.

If $(\Omega, \mathscr{A})$ is a measurable space, then each $\sigma$-additive mapping defined on $\mathscr{A}$ with values in $\mathbb{C}^{q\times q}_{\succcurlyeq}$ is called a non-negative Hermitian $q \times q$ measure on $(\Omega, \mathscr{A})$ and we will use the notation $\mathcal{M}^{\succcurlyeq}_q(\Omega, \mathscr{A})$ in order to denote the set of all non-negative Hermitian $q \times q$ measures on $(\Omega, \mathscr{A})$. If $\mu = [\mu_{jk}]_{j,k=1}^q$ is a non-negative Hermitian $q \times q$ measure on $(\Omega, \mathscr{A})$, then the notation $\mathcal{L}^1(\Omega, \mathscr{A}, \mu; \mathbb{C})$ stands for the set of all Borel measurable functions $f\colon \Omega \to \mathbb{C}$ that satisfy $\int_\Omega |f| \mathrm{d}\nu_{jk} < \infty$ for all $j, k \in \mathbb{Z}_{1,q}$, where $\nu_{jk}$ is the variation of the complex measure $\mu_{jk}$. If $f \in \mathcal{L}^1(\Omega, \mathscr{A}, \mu; \mathbb{C})$, then let $\int_A f \mathrm{d}\mu := [\int_A f \mathrm{d}\mu_{jk}]_{j,k=1}^q$ for each $A \in \mathscr{A}$ and we will also write $\int_A f(\omega)\mu(\mathrm{d}\omega)$ for this integral. Observe that there are equivalent criteria for integrability (see, e.g. [36, Lem. B.1] or [24, Rem. 7.1]). Let $\mathfrak{B}_\mathbb{R}$ (resp. $\mathfrak{B}_\mathbb{C}$) be the $\sigma$-algebra of all Borel subsets of $\mathbb{R}$ (resp. $\mathbb{C}$). For all $\Omega \in \mathfrak{B}_\mathbb{R} \setminus \{\emptyset\}$, let $\mathfrak{B}_\Omega$ be the $\sigma$-algebra of all Borel subsets of $\Omega$ and let $\mathcal{M}^{\succcurlyeq}_q(\Omega)$ be the set of all non-negative Hermitian complex $q \times q$ measures on $(\Omega, \mathfrak{B}_\Omega)$, i.e., $\mathcal{M}^{\succcurlyeq}_q(\Omega)$ is short for $\mathcal{M}^{\succcurlyeq}_q(\Omega, \mathfrak{B}_\Omega)$. For all $\Omega \in \mathfrak{B}_\mathbb{R} \setminus \{\emptyset\}$, let $\mathcal{M}^{\succcurlyeq}_{q,\kappa}(\Omega)$ be the set of all $\sigma \in \mathcal{M}^{\succcurlyeq}_q(\Omega)$ such that, for all $j \in \mathbb{Z}_{0,\kappa}$, the function $f_j\colon \Omega \to \mathbb{C}$ defined by $f_j(t) = t^j$ belongs to $\mathcal{L}^1(\Omega, \mathfrak{B}_\Omega, \sigma; \mathbb{C})$. If $\mu \in \mathcal{M}^{\succcurlyeq}_{q,\kappa}(\Omega)$, then, for all $j \in \mathbb{Z}_{0,\kappa}$, the matrix $s_j^{(\mu)} := \int_\Omega t^j \mu(\mathrm{d}t)$ is called the power moment of $\mu$ of order $j$. Obviously, we have $\mathcal{M}^{\succcurlyeq}_q(\Omega) = \mathcal{M}^{\succcurlyeq}_{q,0}(\Omega) \subseteq \mathcal{M}^{\succcurlyeq}_{q,\ell}(\Omega) \subseteq \mathcal{M}^{\succcurlyeq}_{q,\ell+1}(\Omega) \subseteq \mathcal{M}^{\succcurlyeq}_{q,\infty}(\Omega)$ for all $\ell \in \mathbb{N}_0$ and

$s_0^{(\mu)} = \mu(\Omega)$ for all $\mu \in \mathcal{M}_q^{\succcurlyeq}(\Omega)$. If $\Omega$ is bounded, then one can easily see that $\mathcal{M}_q^{\succcurlyeq}(\Omega) = \mathcal{M}_{q,\infty}^{\succcurlyeq}(\Omega)$.

We now state the general form of the moment problems lying in the background of our considerations:

**Problem** $\mathsf{MP}[\Omega; (s_j)_{j=0}^{\kappa}, =]$: Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. Parametrize the set $\mathcal{M}_q^{\succcurlyeq}[\Omega; (s_j)_{j=0}^{\kappa}, =]$ of all $\sigma \in \mathcal{M}_{q,\kappa}^{\succcurlyeq}(\Omega)$ satisfying $s_j^{(\sigma)} = s_j$ for all $j \in \mathbb{Z}_{0,\kappa}$.

**Problem** $\mathsf{MP}[\Omega; (s_j)_{j=0}^{m}, \preccurlyeq]$: Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{m}$ be a sequence of complex $q \times q$ matrices. Parametrize the set $\mathcal{M}_q^{\succcurlyeq}[\Omega; (s_j)_{j=0}^{m}, \preccurlyeq]$ of all $\sigma \in \mathcal{M}_{q,m}^{\succcurlyeq}(\Omega)$ for which the matrix $s_m - s_m^{(\sigma)}$ is non-negative Hermitian and, in the case $m \geq 1$, for which additionally $s_j^{(\sigma)} = s_j$ holds true for all $j \in \mathbb{Z}_{0,m-1}$.

Throughout this paper, let $\alpha$ and $\beta$ be two arbitrarily given real numbers satisfying $\alpha < \beta$ and let $\delta := \beta - \alpha$. In what follows, we mainly consider the Hausdorff moment problem, i.e., the case that $\Omega$ is the compact interval $[\alpha, \beta]$ of the real axis $\mathbb{R}$. As mentioned above, we have $\mathcal{M}_q^{\succcurlyeq}([\alpha, \beta]) = \mathcal{M}_{q,\infty}^{\succcurlyeq}([\alpha, \beta])$. Since each solution of $\mathsf{MP}[[\alpha, \beta]; (s_j)_{j=0}^{\kappa}, =]$ generates in a natural way solutions of $\mathsf{MP}[[\alpha, \infty); (s_j)_{j=0}^{\kappa}, =]$, $\mathsf{MP}[(-\infty, \beta]; (s_j)_{j=0}^{\kappa}, =]$, and $\mathsf{MP}[\mathbb{R}; (s_j)_{j=0}^{\kappa}, =]$, we will also use results concerning the treatment of these moment problems.

We would like to point out that we are going to apply a somewhat more complex notation for the matrix polynomials used in this work. The reason for this is that simpler notations for the corresponding polynomials in connection with the Hamburger and Stieltjes moment problems have already been used by the authors and we want to avoid any confusion.

## 3 Solvability Criteria for the Matricial Hamburger and Hausdorff Power Moment Problems

In order to state solvability criteria of the matricial Hamburger moment problem ($\Omega = \mathbb{R}$) and the matricial Hausdorff moment problem ($\Omega = [\alpha, \beta]$), we introduce certain sets of sequences of complex $q \times q$ matrices which are determined by properties of particular block Hankel matrices built of them.

Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n}$ be a sequence of complex $q \times q$ matrices. Then $(s_j)_{j=0}^{2n}$ is called $\mathbb{R}$-*non-negative definite* (resp. $\mathbb{R}$-*non-positive definite*) if the block Hankel matrix

$$H_n := [s_{j+k}]_{j,k=0}^{n} \tag{3.1}$$

is non-negative Hermitian (resp. positive Hermitian). For all $n \in \mathbb{N}_0$, we will write $\mathcal{H}_{q,2n}^{\succcurlyeq}$ (resp. $\mathcal{H}_{q,2n}^{\succ}$) for the set of all sequences $(s_j)_{j=0}^{2n}$ of complex $q \times q$ matrices which are $\mathbb{R}$-non-negative definite (resp. $\mathbb{R}$-positive definite). If $n \in \mathbb{N}_0$ and if $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\succcurlyeq}$ (resp. $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\succ}$), then, for each $m \in \mathbb{Z}_{0,n}$, and preceding

$(s_j)_{j=0}^{2m}$ obviously belongs to $\mathcal{H}_{q,2m}^{\succcurlyeq}$ (resp. $\mathcal{H}_{q,2m}^{\succ}$). Thus, let $\mathcal{H}_{q,\infty}^{\succcurlyeq}$ (resp. $\mathcal{H}_{q,\infty}^{\succ}$) be the set of all sequences $(s_j)_{j=0}^{\infty}$ of complex $q \times q$ matrices such that, for all $n \in \mathbb{N}_0$, the sequence $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{H}_{q,2n}^{\succcurlyeq}$ (resp. $\mathcal{H}_{q,2n}^{\succ}$). Note that in [12] and preceding publications $\mathbb{R}$-non-negative definite (resp. $\mathbb{R}$-positive definite) sequences of complex $q \times q$ matrices are also said to be Hankel non-negative definite (resp. Hankel positive definite).

A well-known solvability criterion for Problem MP$[\mathbb{R}; (s_j)_{j=0}^{2n}, \preccurlyeq]$ is the following:

**Theorem 3.1** *Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n}$ be a sequence of complex $q \times q$ matrices. Then $\mathcal{M}_q^{\succcurlyeq}[\mathbb{R}; (s_j)_{j=0}^{2n}, \preccurlyeq] \neq \emptyset$ if and only if $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\succcurlyeq}$.*

There are different proofs of Theorem 3.1, for example [3, Thm. 3.2], [30, Lem. 4.2], and [12, Thm. 4.16]. In particular, the proof given in [3] is based on a Schur-type algorithm as well as a matricial version of the Theorem of Hamburger–Nevanlinna.

For all $n \in \mathbb{N}_0$, let $\mathcal{H}_{q,2n}^{\succcurlyeq,\mathrm{e}}$ (resp. $\mathcal{H}_{q,2n}^{\succ,\mathrm{e}}$) be the set of all sequences $(s_j)_{j=0}^{2n}$ of complex $q \times q$ matrices for which there exist complex $q \times q$ matrices $s_{2n+1}$ and $s_{2n+2}$ such that $(s_j)_{j=0}^{2(n+1)}$ belongs to $\mathcal{H}_{q,2n+2}^{\succcurlyeq}$ (resp. $\mathcal{H}_{q,2n+2}^{\succ}$). Furthermore, for all $n \in \mathbb{N}_0$, we will use $\mathcal{H}_{q,2n+1}^{\succcurlyeq,\mathrm{e}}$ (resp. $\mathcal{H}_{q,2n+1}^{\succ,\mathrm{e}}$) to denote the set of all sequences $(s_j)_{j=0}^{2n+1}$ of complex $q \times q$ matrices for which there exists a complex $q \times q$ matrix $s_{2n+2}$ such that $(s_j)_{j=0}^{2(n+1)}$ belongs to $\mathcal{H}_{q,2n+2}^{\succcurlyeq}$ (resp. $\mathcal{H}_{q,2n+2}^{\succ}$). For each $m \in \mathbb{N}_0$, the elements of the set $\mathcal{H}_{q,m}^{\succcurlyeq,\mathrm{e}}$ are called $\mathbb{R}$-*non-negative definite extendable* (or Hankel non-negative definite extendable) sequences. Observe that $\mathcal{H}_{q,2n}^{\succ,\mathrm{e}} = \mathcal{H}_{q,2n}^{\succ}$ for all $n \in \mathbb{N}_0$. For technical reasons, we set $\mathcal{H}_{q,\infty}^{\succcurlyeq,\mathrm{e}} := \mathcal{H}_{q,\infty}^{\succcurlyeq}$ and $\mathcal{H}_{q,\infty}^{\succ,\mathrm{e}} := \mathcal{H}_{q,\infty}^{\succ}$.

Now we can formulate the announced well-known solvability criterion for Problem MP$[\mathbb{R}; (s_j)_{j=0}^{\kappa}, =]$:

**Theorem 3.2** ([12, Thm. 4.17]) *Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. Then $\mathcal{M}_q^{\succcurlyeq}[\mathbb{R}; (s_j)_{j=0}^{\kappa}, =] \neq \emptyset$ if and only if $(s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q,\kappa}^{\succcurlyeq,\mathrm{e}}$.*

The proof of Theorem 3.2 in [12] is a modification of the proof in [2, Lem. 2.10], where $\kappa$ is supposed to be an even non-negative integer. For the case $\kappa = \infty$, a proof of Theorem 3.2 is stated in [15, Thm. 6.6].

In order to formulate a solvability criterion for the matricial Hausdorff power moment problem, we introduce some further block Hankel matrices. Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Then let the block Hankel matrices $K_n$ and $G_n$ be given by $K_n := [s_{j+k+1}]_{j,k=0}^{n}$ for all $n \in \mathbb{N}_0$ such that $2n+1 \leq \kappa$, and by $G_n := [s_{j+k+2}]_{j,k=0}^{n}$ for all $n \in \mathbb{N}_0$ fulfilling $2n+2 \leq \kappa$.

**Notation 3.3** Suppose $\kappa \geq 1$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Then let the sequences $(a_j)_{j=0}^{\kappa-1}$ and $(b_j)_{j=0}^{\kappa-1}$ be given by

$$a_j := -\alpha s_j + s_{j+1} \qquad \text{and} \qquad b_j := \beta s_j - s_{j+1}, \tag{3.2}$$

respectively. Furthermore, if $\kappa \geq 2$, then let the sequence $(c_j)_{j=0}^{\kappa-2}$ be given by

$$c_j := -\alpha\beta s_j + (\alpha+\beta)s_{j+1} - s_{j+2}. \tag{3.3}$$

To emphasize that a certain (block) matrix $X$ is built from a sequence $(s_j)_{j=0}^{\kappa}$, we sometimes write $X^{\langle s\rangle}$ for $X$.

**Notation 3.4** For each matrix $X_k = X_k^{\langle s\rangle}$ built from the sequence $(s_j)_{j=0}^{\kappa}$, we denote (if possible) by $X_{\alpha,k,\bullet} := X_k^{\langle a\rangle}$, by $X_{\bullet,k,\beta} := X_k^{\langle b\rangle}$, and by $X_{\alpha,k,\beta} := X_k^{\langle c\rangle}$ the corresponding matrix built from the sequences $(a_j)_{j=0}^{\kappa-1}$, $(b_j)_{j=0}^{\kappa-1}$, and $(c_j)_{j=0}^{\kappa-2}$ (given by Notation 3.3) instead of $(s_j)_{j=0}^{\kappa}$, respectively.

In view of Notations 3.3 and 3.4, we get in particular $H_{\alpha,n,\bullet} = -\alpha H_n + K_n$ and $H_{\bullet,n,\beta} = \beta H_n - K_n$ for all $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$ and $H_{\alpha,n,\beta} = -\alpha\beta H_n + (\alpha + \beta)K_n - G_n$ for all $n \in \mathbb{N}_0$ with $2n + 2 \leq \kappa$.

**Definition 3.5** (cf. [19, Def. 4.2]) Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha < \beta$. Then let $\mathcal{F}^{\succcurlyeq}_{q,0,\alpha,\beta}$ (resp. $\mathcal{F}^{\succ}_{q,0,\alpha,\beta}$) be the set of all sequences $(s_j)_{j=0}^{0}$ of complex $q \times q$ matrices for which the block Hankel matrix $H_0$ is non-negative (resp. positive) Hermitian, i.e., for which $s_0 \in \mathbb{C}^{q\times q}_{\succcurlyeq}$ (resp. $s_0 \in \mathbb{C}^{q\times q}_{\succ}$) holds true. For each $n \in \mathbb{N}$, denote by $\mathcal{F}^{\succcurlyeq}_{q,2n,\alpha,\beta}$ (resp. $\mathcal{F}^{\succ}_{q,2n,\alpha,\beta}$) the set of all sequences $(s_j)_{j=0}^{2n}$ of complex $q \times q$ matrices, for which the block Hankel matrices $H_n$ and $H_{\alpha,n-1,\beta}$ are both non-negative (resp. positive) Hermitian. For each $n \in \mathbb{N}_0$, denote by $\mathcal{F}^{\succcurlyeq}_{q,2n+1,\alpha,\beta}$ (resp. $\mathcal{F}^{\succ}_{q,2n+1,\alpha,\beta}$) the set of all sequences $(s_j)_{j=0}^{2n+1}$ of complex $q \times q$ matrices for which the block Hankel matrices $H_{\alpha,n,\bullet}$ and $H_{\bullet,n,\beta}$ are both non-negative (resp. positive) Hermitian. Furthermore, denote by $\mathcal{F}^{\succcurlyeq}_{q,\infty,\alpha,\beta}$ (resp. $\mathcal{F}^{\succ}_{q,\infty,\alpha,\beta}$) the set of all sequences $(s_j)_{j=0}^{\infty}$ of complex $q \times q$ matrices satisfying $(s_j)_{j=0}^{m} \in \mathcal{F}^{\succcurlyeq}_{q,m,\alpha,\beta}$ (resp. $(s_j)_{j=0}^{m} \in \mathcal{F}^{\succ}_{q,m,\alpha,\beta}$) for all $m \in \mathbb{N}_0$. For each $\tau \in \mathbb{N}_0 \cup \{\infty\}$, the sequences belonging to $\mathcal{F}^{\succcurlyeq}_{q,\tau,\alpha,\beta}$ (resp. $\mathcal{F}^{\succ}_{q,\tau,\alpha,\beta}$) are said to be $[\alpha, \beta]$-*non-negative definite* (resp. $[\alpha, \beta]$-*positive definite*).

The reason for defining $\mathcal{F}^{\succcurlyeq}_{q,\infty,\alpha,\beta}$ and $\mathcal{F}^{\succ}_{q,\infty,\alpha,\beta}$ in the described way can be seen from Proposition 5.10 below. Note that in [19], the sequences belonging to $\mathcal{F}^{\succcurlyeq}_{q,\kappa,\alpha,\beta}$ were called $[\alpha, \beta]$-*Hausdorff non-negative definite*.

A necessary and sufficient condition for the solvability of the Hausdorff power moment Problem $\mathsf{MP}[[\alpha, \beta]; (s_j)_{j=0}^{\kappa}, =]$ is the following:

**Theorem 3.6** (cf. [6, Thm. 1.3] and [7, Thm. 1.3]) *Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha < \beta$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. Then $\mathcal{M}^{\succcurlyeq}_q[[\alpha, \beta]; (s_j)_{j=0}^{\kappa}, =] \neq \emptyset$ if and only if $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{\succcurlyeq}_{q,\kappa,\alpha,\beta}$.*

At the end of the section, let us observe the following remark.

***Remark 3.7*** Let $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{\succcurlyeq}_{q,\kappa,\alpha,\beta}$ and let $\sigma \in \mathcal{M}^{\succcurlyeq}_q[[\alpha, \beta]; (s_j)_{j=0}^{\kappa}, =]$. In view of [16, Prop. B.5], one can easily check the following statements: If $\kappa \geq 1$, then $\mu_{\mathrm{a}} : \mathfrak{B}_{[\alpha,\beta]} \to \mathbb{C}^{q\times q}$ given by $\mu_{\mathrm{a}}(B) := \int_B (t - \alpha)\sigma(\mathrm{d}t)$ belongs to $\mathcal{M}^{\succcurlyeq}_q[[\alpha, \beta]; (a_j)_{j=0}^{\kappa-1}, =]$ and $\mu_{\mathrm{b}} : \mathfrak{B}_{[\alpha,\beta]} \to \mathbb{C}^{q\times q}$ defined by $\mu_{\mathrm{b}}(B) := \int_B (\beta - t)\sigma(\mathrm{d}t)$ belongs to $\mathcal{M}^{\succcurlyeq}_q[[\alpha, \beta]; (b_j)_{j=0}^{\kappa-1}, =]$. Furthermore, if $\kappa \geq 2$, then $\mu_{\mathrm{c}} : \mathfrak{B}_{[\alpha,\beta]} \to \mathbb{C}^{q\times q}$ given by $\mu_{\mathrm{c}}(B) := \int_B (\beta - t)(t - \alpha)\sigma(\mathrm{d}t)$ belongs to $\mathcal{M}^{\succcurlyeq}_q[[\alpha, \beta]; (c_j)_{j=0}^{\kappa-2}, =]$.

## 4 Reformulation of the Hamburger and Hausdorff Moment Problems

In the following sections, we consider several classes of functions. For this reason, we introduce some further notation. If $\mathcal{G}$ is a non-empty subset of $\mathbb{C}$ and if $f: \mathcal{G} \to \mathbb{C}$ is a complex-valued function, then let $\mathcal{Z}(f) := \{z \in \mathcal{G}: f(z) = 0\}$. Now let $\mathcal{G}$ be a non-empty open subset of $\mathbb{C}$. If $g$ is a complex-valued function meromorphic in $\mathcal{G}$, then we use $\mathcal{H}(g)$ in order to denote the set of all points at which $g$ is holomorphic and we have $\mathcal{Z}(g) = \{z \in \mathcal{H}(g): g(z) = 0\}$. A $p \times q$ matrix-valued function $G = [g_{jk}]_{\substack{j=1,\dots,p \\ k=1,\dots,q}}$ is called meromorphic in $\mathcal{G}$ if $g_{jk}$ is meromorphic in $\mathcal{G}$ for all $j \in \mathbb{Z}_{1,p}$ and all $k \in \mathbb{Z}_{1,q}$. In this case, let $\mathcal{H}(G) := \bigcap_{j=1}^{p} \bigcap_{k=1}^{q} \mathcal{H}(g_{jk})$. Let $\Pi_+ := \{z \in \mathbb{C}: \Im(z) \in (0, \infty)\}$ and let $\Pi_- := \{z \in \mathbb{C}: \Im(z) \in (-\infty, 0)\}$. The class $\mathcal{R}_q(\Pi_+)$ of all $q \times q$ *Herglotz–Nevanlinna functions* in $\Pi_+$ consists of all matrix-valued functions $F: \Pi_+ \to \mathbb{C}^{q\times q}$ which are holomorphic in $\Pi_+$ and which satisfy $\Im F(z) \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ for all $z \in \Pi_+$. Detailed observations on matrix-valued Herglotz–Nevanlinna functions can be found in [16, 25].

For our consideration, the subclass $\mathcal{R}_{0,q}(\Pi_+)$ of $\mathcal{R}_q(\Pi_+)$ given by

$$\mathcal{R}_{0,q}(\Pi_+) := \left\{ F \in \mathcal{R}_q(\Pi_+) : \sup_{y\in[1,\infty)} y\|F(\mathrm{i}y)\| < \infty \right\}$$

plays a key role. The functions belonging to $\mathcal{R}_{0,q}(\Pi_+)$ admit a further integral representation. This is a well-known matricial generalization of a classical result due to Nevanlinna [29]:

**Theorem 4.1** *(a) For each $F \in \mathcal{R}_{0,q}(\Pi_+)$, there exists a unique $\sigma \in \mathcal{M}_q^{\succcurlyeq}(\mathbb{R})$ such that*

$$F(w) = \int_{\mathbb{R}} \frac{1}{x - w} \sigma(\mathrm{d}x) \qquad \textit{for each } w \in \Pi_+. \tag{4.1}$$

*(b) If $\sigma \in \mathcal{M}_q^{\succcurlyeq}(\mathbb{R})$, then $F: \Pi_+ \to \mathbb{C}^{q\times q}$ defined by (4.1) belongs to $\mathcal{R}_{0,q}(\Pi_+)$.*

***Remark 4.2*** If $F \in \mathcal{R}_{0,q}(\Pi_+)$, then the unique $\sigma \in \mathcal{M}_q^{\succcurlyeq}(\mathbb{R})$ for which (4.1) holds true is called the $\mathbb{R}$-*Stieltjes measure* (or $\mathbb{R}$-spectral measure or matricial spectral measure) of $F$. We also write $\sigma_F$ instead of $\sigma$ to indicate that $\sigma_F$ is the $\mathbb{R}$-Stieltjes measure of $F$. Conversely, if $\sigma \in \mathcal{M}_q^{\succcurlyeq}(\mathbb{R})$ is given, then $F: \Pi_+ \to \mathbb{C}^{q\times q}$ defined by (4.1) is said to be the $\mathbb{R}$-*Stieltjes transform of* $\sigma$.

Now one can reformulate Problems MP[$\mathbb{R}$; $(s_j)_{j=0}^{\kappa}$, $=$] and MP[$\mathbb{R}$; $(s_j)_{j=0}^{2n}$, $\preccurlyeq$] in terms of the following interpolation problems in the class $\mathcal{R}_{0,q}(\Pi_+)$:

**Problem** RP[$\Pi_+$; $(s_j)_{j=0}^{\kappa}$, $=$]: Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. Parametrize the set $\mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{\kappa}, =]$ of all matrix-valued functions $F \in \mathcal{R}_{0,q}(\Pi_+)$ the $\mathbb{R}$-Stieltjes measure of which belongs to $\mathcal{M}_q^{\succcurlyeq}[\mathbb{R}; (s_j)_{j=0}^{\kappa}, =]$.

**Problem** RP$[\Pi_+; (s_j)_{j=0}^{2n}, \preccurlyeq]$: Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n}$ be a sequence of complex $q \times q$ matrices. Parametrize the set $\mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \preccurlyeq]$ of all matrix-valued functions $F \in \mathcal{R}_{0,q}(\Pi_+)$ the $\mathbb{R}$-Stieltjes measure of which belongs to $\mathcal{M}_q^{\succcurlyeq}[\mathbb{R}; (s_j)_{j=0}^{2n}, \preccurlyeq]$.

Kovalishina [27] was the first who gave a parameterization of the solution set of Problem RP$[\Pi_+; (s_j)_{j=0}^{2n}, \preccurlyeq]$ in the so-called non-degenerate matrix case. Parameterizations of the solution sets of the above formulated problems are treated in the general matrix case in [2–4, 13, 17, 35].

In order to reformulate Problem MP$[[\alpha, \beta]; (s_j)_{j=0}^{\kappa}, =]$ in an analogous form, we consider the class $\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$ of all matrix-valued functions $F: \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}^{q\times q}$ which are holomorphic in $\mathbb{C} \setminus [\alpha, \beta]$ and which satisfy $\Im F(z) \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ for all $z \in \Pi_+$ as well as $F(x) \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ for all $x \in (-\infty, \alpha)$ and $-F(x) \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ for all $x \in (\beta, \infty)$. The functions belonging to $\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$ admit an integral representation as well:

**Theorem 4.3** ([6, Thm. 1.1]) *Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha < \beta$. If $F \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$, then there exists a unique measure $\ddot{\sigma} \in \mathcal{M}_q^{\succcurlyeq}([\alpha, \beta])$ such that*

$$F(z) = \int_{[\alpha,\beta]} \frac{1}{t-z} \ddot{\sigma}(\mathrm{d}t) \tag{4.2}$$

*holds true for all $z \in \mathbb{C} \setminus [\alpha, \beta]$. Conversely, if $\ddot{\sigma} \in \mathcal{M}_q^{\succcurlyeq}([\alpha, \beta])$, then $F: \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}^{q\times q}$ defined by (4.2) belongs to $\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$.*

If $F \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$, then the unique non-negative Hermitian measure $\ddot{\sigma}$ which belongs to $\mathcal{M}_q^{\succcurlyeq}([\alpha, \beta])$ and which fulfills (4.2) for all $z \in \mathbb{C} \setminus [\alpha, \beta]$ is called the *$\mathcal{R}[\alpha, \beta]$ -measure of $F$* and will be denoted by $\ddot{\sigma}_F$.

By virtue of Theorem 4.3, Problem MP$[[\alpha, \beta]; (s_j)_{j=0}^{\kappa}, =]$ admits a reformulation as an equivalent problem for functions belonging to the class $\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$:

**Problem** FP$[[\alpha, \beta]; (s_j)_{j=0}^{\kappa}, =]$: Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha < \beta$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. Parametrize the set $\mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^{\kappa}, =]$ of all $F \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$ with $\mathcal{R}[\alpha, \beta]$-measure $\ddot{\sigma}_F$ belonging to $\mathcal{M}_q^{\succcurlyeq}[[\alpha, \beta]; (s_j)_{j=0}^{\kappa}, =]$.

In particular, Problem FP$[[\alpha, \beta]; (s_j)_{j=0}^{\kappa}, =]$ has a solution if and only if Problem MP$[[\alpha, \beta]; (s_j)_{j=0}^{\kappa}, =]$ has a solution. From Theorems 3.6 and 4.1, one can see that the set $\mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^{\kappa}, =]$ is non-empty if and only if the sequence $(s_j)_{j=0}^{\kappa}$ belongs to $\mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq}$.

Now we can exactly formulate the main goal of this paper. For each $w \in \Pi_+$, we are going to parametrize the set

$$\left\{F(w): F \in \mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^{m}, =]\right\} \tag{4.3}$$

where we suppose that an arbitrary $m \in \mathbb{N}_0$ and an arbitrary sequence $(s_j)_{j=0}^{m} \in \mathcal{F}_{q,m,\alpha,\beta}^{\succcurlyeq}$ are given.

At the end of this section, let us observe that in [23, Notation 4.19], from a function of class $\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$ the following three additional functions are derived:

***Remark 4.4*** Let $F \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$ with $\mathcal{R}[\alpha, \beta]$-measure $\ddot{\sigma}_F$. Then the functions $F_{\mathrm{a}}, F_{\mathrm{b}}, F_{\mathrm{c}} \colon \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}^{q \times q}$ defined by

$$F_{\mathrm{a}}(z) := (z - \alpha)F(z) + \ddot{\sigma}_F([\alpha, \beta]), \qquad F_{\mathrm{b}}(z) := (\beta - z)F(z) - \ddot{\sigma}_F([\alpha, \beta])$$

and

$$F_{\mathrm{c}}(z) := (\beta - z)(z - \alpha)F(z) + (\alpha + \beta - z)\ddot{\sigma}_F([\alpha, \beta]) - \int_{[\alpha,\beta]} t\ddot{\sigma}_F(\mathrm{d}t) \quad (4.4)$$

belong to $\mathcal{R}_q(\mathbb{C} \setminus [\alpha, \beta])$ and $\mu_{\mathrm{a}}$, $\mu_{\mathrm{b}}$, and $\mu_{\mathrm{c}}$, respectively, given in Remark 3.7 are the corresponding $\mathcal{R}[\alpha, \beta]$-measures (see [23, Prop. 4.20]).

***Remark 4.5*** ([23, Rem. 5.11]) Let $(s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq}$ and let $F \in \mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^{\kappa}, =]$. If $\kappa \geq 1$, then $F_{\mathrm{a}} \in \mathcal{R}_q[[\alpha, \beta]; (a_j)_{j=0}^{\kappa-1}, =]$ and $F_{\mathrm{b}} \in \mathcal{R}_q[[\alpha, \beta]; (b_j)_{j=0}^{\kappa-1}, =]$. Moreover, if $\kappa \geq 2$, then $F_{\mathrm{c}} \in \mathcal{R}_q[[\alpha, \beta]; (c_j)_{j=0}^{\kappa-2}, =]$.

## 5 Parameterization of Sequences of Matrices Belonging to the Classes $\mathcal{H}_{q,2\kappa}^{\succcurlyeq}$ and $\mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq}$

In this section, we recall some results concerning parameterizations of sequences belonging to subclasses of $\mathcal{H}_{q,\infty}^{\succcurlyeq}$.

***Remark 5.1*** $\mathcal{H}_{q,2\tau}^{\succ,\mathrm{e}} = \mathcal{H}_{q,2\tau}^{\succ} \subseteq \mathcal{H}_{q,2\tau}^{\succcurlyeq,\mathrm{e}} \subseteq \mathcal{H}_{q,2\tau}^{\succcurlyeq}$ and $\mathcal{H}_{q,2\tau}^{\succ} \neq \mathcal{H}_{q,2\tau}^{\succcurlyeq,\mathrm{e}}$ for all $\tau \in \mathbb{N}_0 \cup \{\infty\}$.

Given $m \in \mathbb{N}$ and complex matrices $A_1, A_2, \ldots, A_m$ which have the same number of columns (resp. rows), then let

$$\operatorname{col}(A_j)_{j=0}^{m} := \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} \qquad (\text{resp. } \operatorname{row}(A_k)_{k=0}^{m} := [A_1, A_2, \ldots, A_m]).$$

We will work with Moore–Penrose inverses of complex matrices. If $A \in \mathbb{C}^{p \times q}$, then there is a unique matrix $X \in \mathbb{C}^{q \times p}$ such that the four equations $AXA = A$, $XAX = X$, $(AX)^* = AX$, and $(XA)^* = XA$ are fulfilled, namely the Moore–Penrose inverse $X$ of $A$. For each $A \in \mathbb{C}^{p \times q}$, we use $A^{\dagger}$ to denote the Moore–Penrose inverse of $A$.

**Notation 5.2** Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. For all $\ell, m \in \mathbb{N}_0$ such that $\ell \le m \le \kappa$, let $y_{\ell,m} := \operatorname{col}(s_j)_{j=\ell}^{m}$ and $z_{\ell,m} := \operatorname{row}(s_k)_{k=\ell}^{m}$. Let $\Theta_0 := O_{p\times q}$ and $\Theta_n := z_{n,2n-1} H_{n-1}^{\dagger} y_{n,2n-1}$ for each $n \in \mathbb{N}$ such that $2n-1 \le \kappa$. For all $n \in \mathbb{N}_0$ with $2n \le \kappa$, let $L_n := s_{2n} - \Theta_n$. For every choice of $n \in \mathbb{N}$ fulfilling $2n-1 \le \kappa$, we set $\Sigma_n := z_{n,2n-1} H_{n-1}^{\dagger} K_{n-1} H_{n-1}^{\dagger} y_{n,2n-1}$. For each $n \in \mathbb{N}$ fulfilling $2n \le \kappa$, let $M_n := z_{n,2n-1} H_{n-1}^{\dagger} y_{n+1,2n}$ and $N_n := z_{n+1,2n} H_{n-1}^{\dagger} y_{n,2n-1}$. Furthermore, let $\Lambda_0 := O_{p\times q}$ and $\Lambda_n := M_n + N_n - \Sigma_n$ for all $n \in \mathbb{N}$ fulfilling $2n \le \kappa$.

If $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is the block representation of a complex $(p+q)\times(r+s)$ matrix $M$ with $p \times r$ block $A$, then $M/A := D - CA^{\dagger}B$ is called the Schur complement of $A$ in $M$. We will work with this notion in particular with respect of the first block representation of $H_n$ in the following remark.

***Remark 5.3*** Suppose $\kappa \ge 2$. For all $n \in \mathbb{N}$ fulfilling $2n \le \kappa$, the block representations $H_n = \left[\begin{smallmatrix} H_{n-1} & y_{n,2n-1} \\ z_{n,2n-1} & s_{2n} \end{smallmatrix}\right]$ and $H_n = \left[\begin{smallmatrix} y_{0,n-1} & K_{n-1} \\ s_n & z_{n+1,2n} \end{smallmatrix}\right]$ as well as $H_n = \left[\begin{smallmatrix} z_{0,n-1} & s_n \\ K_{n-1} & y_{n+1,2n} \end{smallmatrix}\right]$ and $H_n = \left[\begin{smallmatrix} s_0 & z_{1,n} \\ y_{1,n} & G_{n-1} \end{smallmatrix}\right]$ hold true.

Let $\mathcal{H}_{p\times q,0}^{\mathrm{r}} := \left\{ (s_j)_{j=0}^{0} \colon s_0 \in \mathbb{C}^{p\times q} \right\}$. Furthermore, for all $\tau \in \mathbb{N} \cup \{\infty\}$, let $\mathcal{H}_{p\times q,\tau}^{\mathrm{r}}$ be the set of all sequences $(s_j)_{j=0}^{\tau}$ of complex $p \times q$ matrices such that $\mathcal{R}(y_{n,2n-1}) \subseteq \mathcal{R}(H_{n-1})$ and $\mathcal{N}(H_{n-1}) \subseteq \mathcal{N}(z_{n,2n-1})$ hold true for all $n \in \mathbb{N}$ fulfilling $2n-1 \le \tau$.

***Remark 5.4*** (cf. [12, Remarks 2.1 and A.3]) $\mathcal{H}_{q,2\tau}^{\succcurlyeq} \subseteq \mathcal{H}_{q\times q,2\tau}^{\mathrm{r}}$ for all $\tau \in \mathbb{N}_0 \cup \{\infty\}$.

Using Notation 5.2, we are able to define the announced $\mathcal{H}$-parameter sequence of a given sequence $(s_j)_{j=0}^{\kappa}$ of complex $q \times q$ matrices. This notion will play a key role in our further considerations.

**Definition 5.5** ([12, Def. 2.28], [13, Def. 5.5]). Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. For each $k \in \mathbb{N}_0$ fulfilling $2k \le \kappa$, let $\mathfrak{h}_{2k} := s_{2k} - \Theta_k$ and, for each $k \in \mathbb{N}_0$ fulfilling $2k+1 \le \kappa$, let $\mathfrak{h}_{2k+1} := s_{2k+1} - \Lambda_k$. Then $(\mathfrak{h}_j)_{j=0}^{\kappa}$ is called the *$\mathcal{H}$-parameter sequence (or sequence of canonical Hankel parameters) of* $(s_j)_{j=0}^{\kappa}$.

In particular, we have

$$\mathfrak{h}_0 = s_0, \qquad \mathfrak{h}_1 = s_1, \qquad \text{and} \qquad \mathfrak{h}_2 = s_2 - s_1 s_0^{\dagger} s_1. \tag{5.1}$$

In [12, 15], one can find a couple of results on $\mathcal{H}$-parameters. Here we touch only a few aspects.

***Remark 5.6*** Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices, and let $(\mathfrak{h}_j)_{j=0}^{\kappa}$ be the $\mathcal{H}$-parameter sequence of $(s_j)_{j=0}^{\kappa}$. For all $m \in \mathbb{Z}_{0,\kappa}$, then $(\mathfrak{h}_j)_{j=0}^{m}$ coincides with the $\mathcal{H}$-parameter sequence of $(s_j)_{j=0}^{m}$.

Furthermore, we recall a connection to the Schur complement $L_n$ given by Notation 5.2 (see also Remark 5.3):

***Remark 5.7*** Let the assumptions of Definition 5.5 be fulfilled. In view of Definition 5.5 and Notation 5.2, we have $\mathfrak{h}_{2n} = L_n$ for all $n \in \mathbb{N}_0$ fulfilling $2n \leq \kappa$.

**Proposition 5.8** ([12, Prop. 2.30(d)], [15, Prop. 2.15(c)]) *Let $(s_j)_{j=0}^{2\kappa}$ be a sequence of complex $q \times q$ matrices with $\mathcal{H}$-parameter sequence $(\mathfrak{h}_j)_{j=0}^{2\kappa}$. Then $(s_j)_{j=0}^{2\kappa}$ belongs to $\mathcal{H}_{q,2\kappa}^{\succ}$ if and only if $\mathfrak{h}_{2\kappa} \in \mathbb{C}_{\succ}^{q\times q}$ for all $k \in \mathbb{Z}_{0,\kappa}$ and $\mathfrak{h}_{2k-1}^{*} = \mathfrak{h}_{2k-1}$ for all $k \in \mathbb{Z}_{1,\kappa}$.*

Now we again turn our attention to the sets $\mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq}$ and $\mathcal{F}_{q,\kappa,\alpha,\beta}^{\succ}$ considered above. For each $\rho \in \mathbb{N}_0 \cup \{\infty\}$ and each non-empty set $\mathcal{X}$, we denote by $\mathfrak{S}_{\rho}(\mathcal{X})$ the set of all sequences $(X_j)_{j=0}^{\rho}$ of elements belonging to $\mathcal{X}$. Obviously, the class $\mathcal{F}_{q,0,\alpha,\beta}^{\succcurlyeq}$ coincides with the set of all sequences $(s_j)_{j=0}^{0}$ with $s_0 \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. Furthermore, we have

$$\mathcal{F}_{q,2n,\alpha,\beta}^{\succcurlyeq} = \left\{(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\succcurlyeq} : (c_j)_{j=0}^{2(n-1)} \in \mathcal{H}_{q,2(n-1)}^{\succcurlyeq}\right\} \tag{5.2}$$

and

$$\mathcal{F}_{q,2n,\alpha,\beta}^{\succ} = \left\{(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\succ} : (c_j)_{j=0}^{2(n-1)} \in \mathcal{H}_{q,2(n-1)}^{\succ}\right\} \tag{5.3}$$

for all $n \in \mathbb{N}$ as well as

$$\mathcal{F}_{q,2n+1,\alpha,\beta}^{\succcurlyeq} = \left\{(s_j)_{j=0}^{2n+1} \in \mathfrak{S}_{2n+1}(\mathbb{C}^{q\times q}) : \{(a_j)_{j=0}^{2n}, (b_j)_{j=0}^{2n}\} \subseteq \mathcal{H}_{q,2n}^{\succcurlyeq}\right\} \tag{5.4}$$

and

$$\mathcal{F}_{q,2n+1,\alpha,\beta}^{\succ} = \left\{(s_j)_{j=0}^{2n+1} \in \mathfrak{S}_{2n+1}(\mathbb{C}^{q\times q}) : \{(a_j)_{j=0}^{2n}, (b_j)_{j=0}^{2n}\} \subseteq \mathcal{H}_{q,2n}^{\succ}\right\} \tag{5.5}$$

for all $n \in \mathbb{N}_0$.

***Remark 5.9*** $\mathcal{F}_{q,\kappa,\alpha,\beta}^{\succ} \subseteq \mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq}$.

Note that essential parts of the following two propositions, which are proved by algebraic arguments in [19], can immediately also be obtained from Theorem 3.6 and Remark 3.7.

**Proposition 5.10** ([19, Prop. 7.7]) *If $(s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq}$, then $(s_j)_{j=0}^{m} \in \mathcal{F}_{q,m,\alpha,\beta}^{\succcurlyeq}$ for all $m \in \mathbb{Z}_{0,\kappa}$. Moreover, if $(s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,\kappa,\alpha,\beta}^{\succ}$, then $(s_j)_{j=0}^{m} \in \mathcal{F}_{q,m,\alpha,\beta}^{\succ}$ for all $m \in \mathbb{Z}_{0,\kappa}$.*

In view of Proposition 5.10, we have in particular

$$\begin{aligned}\mathcal{F}_{q,\infty,\alpha,\beta}^{\succcurlyeq} &= \left\{(s_j)_{j=0}^{\infty} \in \mathcal{H}_{q,\infty}^{\succcurlyeq} : (c_j)_{j=0}^{\infty} \in \mathcal{H}_{q,\infty}^{\succcurlyeq}\right\}\\ &= \left\{(s_j)_{j=0}^{\infty} \in \mathfrak{S}_{\infty}(\mathbb{C}^{q\times q}) : \{(a_j)_{j=0}^{\infty}, (b_j)_{j=0}^{\infty}\} \in \mathcal{H}_{q,\infty}^{\succcurlyeq}\right\}.\end{aligned} \tag{5.6}$$

**Proposition 5.11** ([19, Prop. 9.1]) *Let* $(s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq}$. *If* $\kappa \geq 1$, *then both sequences* $(a_j)_{j=0}^{\kappa-1}$ *and* $(b_j)_{j=0}^{\kappa-1}$ *belong to* $\mathcal{F}_{q,\kappa-1,\alpha,\beta}^{\succcurlyeq}$. *Furthermore, if* $\kappa \geq 2$, *then* $(c_j)_{j=0}^{\kappa-2} \in \mathcal{F}_{q,\kappa-2,\alpha,\beta}^{\succcurlyeq}$.

**Proposition 5.12** ([19, Prop. 7.10]) $\mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq} \subseteq \mathcal{H}_{q,\kappa}^{\succcurlyeq,\mathrm{e}}$ *and* $\mathcal{F}_{q,\kappa,\alpha,\beta}^{\succ} \subseteq \mathcal{H}_{q,\kappa}^{\succ,\mathrm{e}}$.

**Proposition 5.13** ([19, Propositions 4.8, 7.7 and 11.12]) *Let* $m \in \mathbb{N}_0$ *and let* $(s_j)_{j=0}^{m} \in \mathcal{F}_{q,m,\alpha,\beta}^{\succcurlyeq}$ *(resp.* $\mathcal{F}_{q,m,\alpha,\beta}^{\succ}$*). For all* $\kappa \in \mathbb{Z}_{m+1,\infty}$, *then there is a sequence* $(s_k)_{k=m+1}^{\kappa}$ *of complex* $q \times q$ *matrices such that* $(s_j)_{j=0}^{\kappa}$ *belongs to* $\mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq}$ *(resp.* $\mathcal{F}_{q,\kappa,\alpha,\beta}^{\succ}$*).*

**Definition 5.14** ([19, Definitions 10.3 and 10.11]) Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha < \beta$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. In view of Notations 3.4 and 5.2, then the sequences $(\mathsf{A}_j)_{j=0}^{\kappa}$ and $(\mathsf{B}_j)_{j=0}^{\kappa}$ given by $\mathsf{A}_{2k} := \alpha s_{2k} + \Theta_{\alpha,k,\bullet}$ and $\mathsf{B}_{2k} := \beta s_{2k} - \Theta_{\bullet,k,\beta}$ for all $k \in \mathbb{N}_0$ with $2k \leq \kappa$ and by $\mathsf{A}_{2k+1} := \Theta_{k+1}$ and $\mathsf{B}_{2k+1} := -\alpha\beta s_{2k} + (\alpha+\beta)s_{2k+1} - \Theta_{\alpha,k,\beta}$ for all $k \in \mathbb{N}_0$ with $2k+1 \leq \kappa$ are called *the sequence of left matricial interval endpoints associated with* $(s_j)_{j=0}^{\kappa}$ *and* $[\alpha, \beta]$ and *the sequence of right matricial interval endpoints associated with* $(s_j)_{j=0}^{\kappa}$ *and* $[\alpha, \beta]$, respectively. Furthermore, the sequence $(\mathsf{D}_j)_{j=0}^{\kappa}$ given by $\mathsf{D}_j := \mathsf{B}_j - \mathsf{A}_j$ is said to be the *sequence of* $[\alpha, \beta]$*-interval lengths associated with* $(s_j)_{j=0}^{\kappa}$.

By virtue of Notation 5.2 and (5.7), we have in particular

$$\mathsf{A}_0 = \alpha s_0, \qquad \mathsf{B}_0 = \beta s_0, \qquad \mathsf{A}_1 = s_1 s_0^{\dagger} s_1, \qquad \mathsf{B}_1 = -\alpha\beta s_0 + (\alpha+\beta)s_1 \tag{5.7}$$

as well as

$$\mathsf{D}_0 = \delta s_0 \qquad \text{and} \qquad \mathsf{D}_1 = -\alpha\beta s_0 + (\alpha+\beta)s_1 - s_1 s_0^{\dagger} s_1. \tag{5.8}$$

***Remark 5.15*** (cf. [23, Rem. 3.23]) Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices with sequence of left matricial interval endpoints $(\mathsf{A}_j)_{j=0}^{\kappa}$, sequence of right matricial interval endpoints $(\mathsf{B}_j)_{j=0}^{\kappa}$, and sequence of $[\alpha, \beta]$-interval lengths $(\mathsf{D}_j)_{j=0}^{\kappa}$. For each $k \in \mathbb{Z}_{0,\kappa}$, the matrices $\mathsf{A}_k$, $\mathsf{B}_k$, and $\mathsf{D}_k$ are built from the matrices $s_0, s_1, \ldots, s_k$. In particular, for each $m \in \mathbb{Z}_{0,\kappa}$, the sequence of left matricial interval endpoints, the sequence of right matricial interval endpoints and the sequence of $[\alpha, \beta]$-interval lengths associated with $(s_j)_{j=0}^{m}$ coincide with $(\mathsf{A}_j)_{j=0}^{m}$, $(\mathsf{B}_j)_{j=0}^{m}$, and $(\mathsf{D}_j)_{j=0}^{m}$, respectively.

**Proposition 5.16** ([19, Prop. 10.15]) *If* $(s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq}$, *then* $\mathsf{D}_j \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ *for each* $j \in \mathbb{Z}_{0,\kappa}$. *Moreover, if* $(s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,\kappa,\alpha,\beta}^{\succ}$, *then* $\mathsf{D}_j \in \mathbb{C}_{\succ}^{q\times q}$ *for each* $j \in \mathbb{Z}_{0,\kappa}$.

**Definition 5.17** ([19, Def. 10.6]) Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha < \beta$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Then the sequence $(\mathfrak{A}_j)_{j=0}^{\kappa}$ given by $\mathfrak{A}_0 := s_0$ and by $\mathfrak{A}_j := s_j - \mathsf{A}_{j-1}$ for $j \in \mathbb{Z}_{1,\kappa}$ is called the *sequence of lower Schur complements associated with* $(s_j)_{j=0}^{\kappa}$ *and* $[\alpha, \beta]$. Furthermore, if $\kappa \geq 1$, then the sequence $(\mathfrak{B}_j)_{j=1}^{\kappa}$ given by $\mathfrak{B}_j := \mathsf{B}_{j-1} - s_j$ is called the *sequence of upper Schur complements associated with* $(s_j)_{j=0}^{\kappa}$ *and* $[\alpha, \beta]$.

In view of (5.7), we have in particular $\mathfrak{A}_1 = a_0$, $\mathfrak{B}_1 = b_0$, and $\mathfrak{B}_2 = c_0$.

Taking into account Notations 3.4 and 5.2, the following remark shows the reason for choosing the notions introduced in Definition 5.17:

***Remark 5.18*** ([23, Rem. 3.26]) Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Then $\mathfrak{A}_{2n} = L_n$ for all $n \in \mathbb{N}_0$ with $2n \leq \kappa$ and $\mathfrak{A}_{2n+1} = L_{\alpha,n,\bullet}$ for all $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$. In particular, if $n \in \mathbb{N}$ fulfills $2n \leq \kappa$, then $\mathfrak{A}_{2n}$ is the Schur complement of $H_{n-1}$ in $H_n$ and, in the case that $n \in \mathbb{N}$ is such that $2n+1 \leq \kappa$, moreover $\mathfrak{A}_{2n+1}$ is the Schur complement of $H_{\alpha,n-1,\bullet}$ in $H_{\alpha,n,\bullet}$. Furthermore, $\mathfrak{B}_{2n+1} = L_{\bullet,n,\beta}$ for all $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$ and $\mathfrak{B}_{2n+2} = L_{\alpha,n,\beta}$ for all $n \in \mathbb{N}_0$ with $2n+2 \leq \kappa$. In particular, if $n \in \mathbb{N}$ fulfills $2n+1 \leq \kappa$, then $\mathfrak{B}_{2n+1}$ is the Schur complement of $H_{\bullet,n-1,\beta}$ in $H_{\bullet,n,\beta}$ and, if $n \in \mathbb{N}$ is such that $2n+2 \leq \kappa$, then $\mathfrak{B}_{2n+2}$ is the Schur complement of $H_{\alpha,n-1,\beta}$ in $H_{\alpha,n,\beta}$.

Now we turn our attention to the $\mathcal{F}_{\alpha,\beta}$-parameter sequence, which was studied in detail in [20].

**Definition 5.19** ([20, Def. 6.1]) Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha < \beta$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Let the sequence $(\mathfrak{f}_j)_{j=0}^{2\kappa}$ be given by $\mathfrak{f}_0 := \mathfrak{A}_0$, by $\mathfrak{f}_{4k+1} := \mathfrak{A}_{2k+1}$ and $\mathfrak{f}_{4k+2} := \mathfrak{B}_{2k+1}$ for all $k \in \mathbb{N}_0$ with $2k+1 \leq \kappa$, and by $\mathfrak{f}_{4k+3} := \mathfrak{B}_{2k+2}$ and $\mathfrak{f}_{4k+4} := \mathfrak{A}_{2k+2}$ for all $k \in \mathbb{N}_0$ with $2k+2 \leq \kappa$. Then we call $(\mathfrak{f}_j)_{j=0}^{2\kappa}$ the *$\mathcal{F}_{\alpha,\beta}$-parameter sequence of* $(s_j)_{j=0}^{\kappa}$.

In particular, we have

$$\mathfrak{f}_0 = s_0, \quad \mathfrak{f}_1 = a_0 = s_1 - \alpha s_0, \quad \mathfrak{f}_2 = b_0 = \beta s_0 - s_1, \tag{5.9}$$

$$\mathfrak{f}_3 = c_0 = -\alpha\beta s_0 + (\alpha+\beta)s_1 - s_2, \quad \text{and} \quad \mathfrak{f}_4 = s_2 - s_1 s_0^{\dagger} s_1. \tag{5.10}$$

The $\mathcal{F}_{\alpha,\beta}$-parameter sequence plays in the context of the Hausdorff moment problem a similar role as the $\mathcal{H}$-parameter sequence in the context of the Hamburger moment problem.

**Proposition 5.20** ([20, Propositions 6.14 and 6.15]) *Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. Then $(s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq}$ if and only if $\mathfrak{f}_j \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ for all $j \in \mathbb{Z}_{0,2\kappa}$. Moreover $(s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,\kappa,\alpha,\beta}^{\succ}$ if and only if $\mathfrak{f}_j \in \mathbb{C}_{\succ}^{q\times q}$ for all $j \in \mathbb{Z}_{0,2\kappa}$.*

***Remark 5.21*** In view of Definition 5.19, we have $\{\mathfrak{f}_{2m-1}, \mathfrak{f}_{2m}\} = \{\mathfrak{A}_m, \mathfrak{B}_m\}$ for all $m \in \mathbb{Z}_{1,\kappa}$.

***Remark 5.22*** Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices with $\mathcal{F}_{\alpha,\beta}$-parameter sequence $(\mathfrak{f}_j)_{j=0}^{2\kappa}$. From Definition 5.19 and Remark 5.18 one can easily see that $\mathfrak{f}_{4n} = L_n$ for all $n \in \mathbb{N}_0$ fulfilling $2n \leq \kappa$, that $\mathfrak{f}_{4n+1} = L_{\alpha,n,\bullet}$ and $\mathfrak{f}_{4n+2} = L_{\bullet,n,\beta}$ for all $n \in \mathbb{N}_0$ fulfilling $2n+1 \leq \kappa$, and that $\mathfrak{f}_{4n+3} = L_{\alpha,n,\beta}$ for all $n \in \mathbb{N}_0$ fulfilling $2n+2 \leq \kappa$ hold true, where $L_n$, $L_{\alpha,n,\bullet}$, $L_{\bullet,n,\beta}$, and $L_{\alpha,n,\beta}$ are given by Notations 5.2 and 3.4.

***Remark 5.23*** ([23, Rem. 3.38]) Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices with $\mathcal{F}_{\alpha,\beta}$-parameter sequence $(\mathfrak{f}_j)_{j=0}^{2\kappa}$. Then $\mathfrak{f}_0 = s_0$ and, for each $k \in \mathbb{Z}_{1,\kappa}$, the matrices $\mathfrak{f}_{2k-1}$ and $\mathfrak{f}_{2k}$ are built only from the matrices $s_0, s_1, \ldots, s_k$. In particular, for each $m \in \mathbb{Z}_{0,\kappa}$, the $\mathcal{F}_{\alpha,\beta}$-parameter sequence of $(s_j)_{j=0}^{m}$ coincides with $(\mathfrak{f}_j)_{j=0}^{2m}$.

***Remark 5.24*** ([20, Rem. 6.16]) Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. For all $k \in \mathbb{Z}_{1,\kappa}$, then $\mathfrak{f}_{2k-1} = \mathsf{D}_{k-1} - \mathfrak{f}_{2k}$.

***Remark 5.25*** ([20, Rem. 6.18]) Let $(s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq}$. Then $\mathcal{R}(\mathsf{D}_0) = \mathcal{R}(\mathfrak{f}_0)$ and $\mathcal{N}(\mathsf{D}_0) = \mathcal{N}(\mathfrak{f}_0)$. Furthermore, if $\kappa \geq 1$, then $\mathcal{R}(\mathsf{D}_j) = \mathcal{R}(\mathfrak{f}_{2j-1}) \cap \mathcal{R}(\mathfrak{f}_{2j})$ and $\mathcal{N}(\mathsf{D}_j) = \mathcal{N}(\mathfrak{f}_{2j-1}) + \mathcal{N}(\mathfrak{f}_{2j})$ for all $j \in \mathbb{Z}_{1,\kappa}$ and $\mathcal{R}(\mathsf{D}_j) = \mathcal{R}(\mathfrak{f}_{2j+1}) + \mathcal{R}(\mathfrak{f}_{2j+2})$ and $\mathcal{N}(\mathsf{D}_j) = \mathcal{N}(\mathfrak{f}_{2j+1}) \cap \mathcal{N}(\mathfrak{f}_{2j+2})$ for all $j \in \mathbb{Z}_{0,\kappa-1}$.

## 6 Orthogonal Matrix Polynomials and $\mathcal{H}$-parameters

Let $P$ be a complex $p \times q$ matrix polynomial. Then, for each $n \in \mathbb{N}_0$, let

$$Y_n(P) := \operatorname{col}(A_j)_{j=0}^{n} \tag{6.1}$$

where $(A_j)_{j=0}^{\infty}$ is the uniquely determined sequence of complex $p \times q$ matrices such that $P(w) = \sum_{j=0}^{\infty} w^j A_j$ holds true for all $w \in \mathbb{C}$. Denote by $\deg P := \sup\left\{j \in \mathbb{N}_0 \colon A_j \neq O_{p\times q}\right\}$ the *degree of* $P$. If $k := \deg P$ fulfills $k \geq 0$, then the matrix $A_k$ is called the *leading coefficient matrix of* $P$.

***Remark 6.1*** Let $P$ be a complex $q \times q$ matrix polynomial. Then $P = E_n Y_n(P)$ for all $n \in \mathbb{N}_0$ fulfilling $n \geq \deg P$, where $E_n \colon \mathbb{C} \to \mathbb{C}^{q\times(n+1)q}$ is defined by $E_n(z) := [z^0 I_q, z^1 I_q, z^2 I_q, \ldots, z^n I_q]$.

We recall a notion which has been proved to be useful already, e. g., in [15, Sec. 5] and [14, Sec. 6].

**Definition 6.2** Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{2\kappa}$ be a sequence of complex $q \times q$ matrices. A sequence $(P_k)_{k=0}^{\kappa}$ of complex $q \times q$ matrix polynomials is called *monic right orthogonal system* (short: *MROS*) *of matrix polynomials with respect to* $(s_j)_{j=0}^{2\kappa}$, if the following two conditions hold true:

(I) For each $k \in \mathbb{Z}_{0,\kappa}$, the polynomial $P_k$ has degree $k$ with leading coefficient matrix $I_q$.
(II) $[Y_n(P_j)]^* H_n [Y_n(P_k)] = O_{q\times q}$ for all $j, k \in \mathbb{Z}_{0,\kappa}$ with $j \neq k$, where $n := \max\{j, k\}$ and the block Hankel matrix $H_n$ is given by (3.1).

***Remark 6.3*** Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $(s_j)_{j=0}^{2\kappa}$ be a sequence of complex $q \times q$ matrices, and let $(P_k)_{k=0}^{\kappa}$ be an MROS of matrix polynomials with respect to $(s_j)_{j=0}^{2\kappa}$. For all $m \in \mathbb{Z}_{0,\kappa}$, then $(P_k)_{k=0}^{m}$ is an MROS with respect to $(s_j)_{j=0}^{2m}$.

We are now going to consider sequences $(s_j)_{j=0}^{2\kappa}$ of complex $q \times q$ matrices for which the set $\mathcal{M}_q^{\succcurlyeq}[\mathbb{R}; (s_j)_{j=0}^{2\kappa}, =]$ is non-empty. Then the orthogonality condition of

Definition 6.2 can be rewritten in terms of the corresponding integral with respect to an arbitrary measure $\sigma$ belonging to $\mathcal{M}_q^{\succcurlyeq}[\mathbb{R}; (s_j)_{j=0}^{2\kappa}, =]$. In order to do this, we will use some standard results of the integration theory of non-negative Hermitian measures (see also Appendix B).

***Remark 6.4*** ([15, Rem. 6.1]) Let $(s_j)_{j=0}^{2\kappa} \in \mathcal{H}_{q,2\kappa}^{\succcurlyeq,\mathrm{e}}$. In view of Theorem 3.2, let $\sigma \in \mathcal{M}_q^{\succcurlyeq}[\mathbb{R}; (s_j)_{j=0}^{2\kappa}, =]$.

(a) Let $k \in \mathbb{Z}_{0,\kappa}$. In view of Remark 6.1, then $\int_{\mathbb{R}}[E_k(t)]^*\sigma(\mathrm{d}t)E_k(t) = H_k$. In particular, the matrix $H_k$ is non-negative Hermitian.

(b) Let $P$ and $Q$ be $q \times q$ matrix polynomials of degrees $j$ and $k$, respectively. Suppose that $n := \max\{j, k\}$ fulfills $n \le \kappa$. In view of Remark 6.1 and part (a), then it is readily checked that $\int_{\mathbb{R}}[P(t)]^*\sigma(\mathrm{d}t)Q(t) = [Y_n(P)]^* H_n Y_n(Q)$.

Remark 6.4 leads to the notion of monic right orthogonal systems of matrix polynomials with respect to non-negative Hermitian measures in a natural way (see, e. g., also [15, Def. 6.7]). Supplementary to the definition, an MROS of matrix polynomials can be characterized as follows:

***Remark 6.5*** Let $(s_j)_{j=0}^{2\kappa} \in \mathcal{H}_{q,2\kappa}^{\succ}$ and let $(P_k)_{k=0}^{\kappa}$ be a sequence of $q \times q$ matrix polynomials. In view of Definition 6.2 we can easily see from [15, Propositions 5.8 and 5.9] that $(P_k)_{k=0}^{\kappa}$ is an MROS of matrix polynomials with respect to $(s_j)_{j=0}^{2\kappa}$ if and only if $Y_0(P_0) = I_q$ and the following condition holds true:

(I) If $\kappa \ge 1$, then, for each $k \in \mathbb{Z}_{1,\kappa}$, the equation $[O_{q\times kq}, I_q]Y_k(P_k) = I_q$ holds true and the matrix $X_k := -[I_{kq}, O_{kq\times q}]Y_k(P_k)$ fulfills $H_{k-1}X_k = y_{k,2k-1}$.

***Remark 6.6*** Suppose $\kappa \ge 1$. Let $(s_j)_{j=0}^{2\kappa} \in \mathcal{H}_{q,2\kappa}^{\succ}$. From [12, Rem. 2.1] one can easily see then that, for all $k \in \mathbb{Z}_{1,\kappa}$, the set $\mathcal{L}_k := \left\{\xi_k \in \mathbb{C}^{kq\times q} \colon H_{k-1}\xi_k = y_{k,2k-1}\right\}$ is non-empty.

For each $\tau \in \mathbb{N}_0 \cup \{\infty\}$, let

$$\langle\!\langle \tau ] := \sup\{k \in \mathbb{N}_0 \colon 2k - 1 \le \tau\}. \tag{6.2}$$

With the help of the $\mathcal{H}$-parameter sequence of a sequence $(s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q,2\kappa}^{\succ}$, it is, in view of [3, Formula (4.14)], possible, to construct recursively an MROS with respect to $(s_j)_{j=0}^{\kappa}$. This system of matrix polynomials plays an essential role in [13–15, 36].

**Definition 6.7** Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices, and let $(\mathfrak{h}_j)_{j=0}^{\kappa}$ be the $\mathcal{H}$-parameter sequence of $(s_j)_{j=0}^{\kappa}$. Let $\mathfrak{a}_0, \mathfrak{b}_0, \mathfrak{c}_0, \mathfrak{d}_0 \colon \mathbb{C} \to \mathbb{C}^{q\times q}$ be defined by

$$\mathfrak{a}_0(z) := O_{q\times q}, \qquad \mathfrak{b}_0(z) := I_q, \qquad \mathfrak{c}_0(z) := O_{q\times q}, \qquad \text{and} \qquad \mathfrak{d}_0(z) := I_q.$$

If $\kappa \ge 1$, then let $\mathfrak{a}_1, \mathfrak{b}_1, \mathfrak{c}_1, \mathfrak{d}_1 \colon \mathbb{C} \to \mathbb{C}^{q\times q}$ be given via

$$\mathfrak{a}_1(z) := \mathfrak{h}_0, \qquad \mathfrak{b}_1(z) := zI_q - \mathfrak{h}_0^{\dagger}\mathfrak{h}_1, \qquad \mathfrak{c}_1(z) := \mathfrak{h}_0, \qquad \mathfrak{d}_1(z) := zI_q - \mathfrak{h}_1\mathfrak{h}_0^{\dagger}.$$

If $\kappa \geq 2$, then, for all $k \in \mathbb{Z}_{2,\infty}$ fulfilling $2k-1 \leq \kappa$, let $\mathfrak{a}_k, \mathfrak{b}_k, \mathfrak{c}_k, \mathfrak{d}_k \colon \mathbb{C} \to \mathbb{C}^{q\times q}$ be defined recursively by

$$\mathfrak{a}_k(z) := \mathfrak{a}_{k-1}(z)(zI_q - \mathfrak{h}_{2k-2}^{\dagger}\mathfrak{h}_{2k-1}) - \mathfrak{a}_{k-2}(z)\mathfrak{h}_{2k-4}^{\dagger}\mathfrak{h}_{2k-2},$$
$$\mathfrak{b}_k(z) := \mathfrak{b}_{k-1}(z)(zI_q - \mathfrak{h}_{2k-2}^{\dagger}\mathfrak{h}_{2k-1}) - \mathfrak{b}_{k-2}(z)\mathfrak{h}_{2k-4}^{\dagger}\mathfrak{h}_{2k-2},$$
$$\mathfrak{c}_k(z) := (zI_q - \mathfrak{h}_{2k-1}\mathfrak{h}_{2k-2}^{\dagger})\mathfrak{c}_{k-1}(z) - \mathfrak{h}_{2k-2}\mathfrak{h}_{2k-4}^{\dagger}\mathfrak{c}_{k-2}(z),$$

and

$$\mathfrak{d}_k(z) := (zI_q - \mathfrak{h}_{2k-1}\mathfrak{h}_{2k-2}^{\dagger})\mathfrak{d}_{k-1}(z) - \mathfrak{h}_{2k-2}\mathfrak{h}_{2k-4}^{\dagger}\mathfrak{d}_{k-2}(z).$$

Regarding (6.2), we call the quadruple $[(\mathfrak{a}_k)_{k=0}^{\langle\!\langle\kappa]}, (\mathfrak{b}_k)_{k=0}^{\langle\!\langle\kappa]}, (\mathfrak{c}_k)_{k=0}^{\langle\!\langle\kappa]}, (\mathfrak{d}_k)_{k=0}^{\langle\!\langle\kappa]}]$ the $\mathbb{R}$-*quadruple* (or *canonical quadruple*) *of matrix polynomials*, abbreviating $\mathbb{R}$-QMP, *associated with* $(s_j)_{j=0}^{\kappa}$.

***Remark 6.8*** Under the assumptions of Definition 6.7, for all $k \in \mathbb{N}_0$ such that $2k-1 \leq \kappa$, the matrix-valued functions $\mathfrak{a}_k$, $\mathfrak{b}_k$, $\mathfrak{c}_k$, and $\mathfrak{d}_k$ indeed are matrix polynomials, where the matrix polynomials $\mathfrak{b}_k$ and $\mathfrak{d}_k$ both have degree $k$ and the same leading coefficient matrix $I_q$ (see also [15, Thm. 5.5]).

***Remark 6.9*** Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices, and let $[(\mathfrak{a}_k)_{k=0}^{\langle\!\langle\kappa]}, (\mathfrak{b}_k)_{k=0}^{\langle\!\langle\kappa]}, (\mathfrak{c}_k)_{k=0}^{\langle\!\langle\kappa]}, (\mathfrak{d}_k)_{k=0}^{\langle\!\langle\kappa]}]$ be the $\mathbb{R}$-QMP associated with $(s_j)_{j=0}^{\kappa}$. In view of Remark 5.6 and Definition 6.7, for all $m \in \mathbb{Z}_{0,\kappa}$, then $[(\mathfrak{a}_k)_{k=0}^{\langle\!\langle m]}, (\mathfrak{b}_k)_{k=0}^{\langle\!\langle m]}, (\mathfrak{c}_k)_{k=0}^{\langle\!\langle m]}, (\mathfrak{d}_k)_{k=0}^{\langle\!\langle m]}]$ is exactly the $\mathbb{R}$-QMP associated with $(s_j)_{j=0}^{m}$.

**Proposition 6.10** (cf. [15, Thm. 5.5(a)]) *Let* $(s_j)_{j=0}^{2\kappa} \in \mathcal{H}_{q,2\kappa}^{\succcurlyeq}$ *with* $\mathbb{R}$-*QMP* $[(\mathfrak{a}_k)_{k=0}^{\kappa}, (\mathfrak{b}_k)_{k=0}^{\kappa}, (\mathfrak{c}_k)_{k=0}^{\kappa}, (\mathfrak{d}_k)_{k=0}^{\kappa}]$. *Then* $(\mathfrak{b}_k)_{k=0}^{\kappa}$ *is an MROS of matrix polynomials with respect to* $(s_j)_{j=0}^{2\kappa}$.

For each sequence $(s_j)_{j=0}^{\kappa}$ of complex $p \times q$ matrices and each $m \in \mathbb{Z}_{0,\kappa}$, let the block Toeplitz matrices $\mathbf{S}_m$ and $\mathbb{S}_m$ be given by

$$\mathbf{S}_m := \begin{bmatrix} s_0 & O & O & \dots & O \\ s_1 & s_0 & O & \dots & O \\ s_2 & s_1 & s_0 & \dots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_m & s_{m-1} & s_{m-2} & \dots & s_0 \end{bmatrix} \quad \text{and} \quad \mathbb{S}_m := \begin{bmatrix} s_0 & s_1 & s_2 & \dots & s_m \\ O & s_0 & s_1 & \dots & s_{m-1} \\ O & O & s_0 & \dots & s_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \dots & s_0 \end{bmatrix}.$$

In view of [8, Sec. 4] and [22, Notation E.2], we introduce the following notation:

**Notation 6.11** Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices and let $P$ be a complex $q \times q$ matrix polynomial with degree $k := \deg P$ satisfying $k \leq \kappa + 1$. Then let $P^{[\![s]\!]} \colon \mathbb{C} \to \mathbb{C}^{q\times q}$ be defined by $P^{[\![s]\!]}(z) = O_{q\times q}$ if $k \leq 0$ and by $P^{[\![s]\!]}(z) = E_{k-1}(z)[O_{kq\times q}, \mathbb{S}_{k-1}]Y_k(P)$ if $k \geq 1$.

***Remark 6.12*** Under the assumptions of Notation 6.11 we see that $P^{[\![s]\!]}$ is a matrix polynomial with $\deg P^{[\![s]\!]} \leq k-1$.

The transformation described in Notation 6.11 fulfills specific linearity properties (see Appendix C).

**Proposition 6.13** ([14, Prop. 6.13]) *Let* $(s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q,\kappa}^{\succcurlyeq,\mathrm{e}}$ *with associated* $\mathbb{R}$*-QMP* $[(\mathfrak{a}_k)_{k=0}^{\langle\!\langle\kappa]}, (\mathfrak{b}_k)_{k=0}^{\langle\!\langle\kappa]}, (\mathfrak{c}_k)_{k=0}^{\langle\!\langle\kappa]}, (\mathfrak{d}_k)_{k=0}^{\langle\!\langle\kappa]}]$. *For all* $k \in \mathbb{N}_0$ *with* $2k-1 \leq \kappa$, *then* $\mathfrak{a}_k = \mathfrak{b}_k^{[\![s]\!]}$.

***Remark 6.14*** ([14, Rem. 6.15]) Let $(s_j)_{j=0}^{\kappa}$ be a sequence of Hermitian complex $q \times q$ matrices and let $[(\mathfrak{a}_k)_{k=0}^{\langle\!\langle\kappa]}, (\mathfrak{b}_k)_{k=0}^{\langle\!\langle\kappa]}, (\mathfrak{c}_k)_{k=0}^{\langle\!\langle\kappa]}, (\mathfrak{d}_k)_{k=0}^{\langle\!\langle\kappa]}]$ be the $\mathbb{R}$-QMP associated with $(s_j)_{j=0}^{\kappa}$. Then $\mathfrak{c}_k(z) = [\mathfrak{a}_k(\overline{z})]^*$ and $\mathfrak{d}_k(z) = [\mathfrak{b}_k(\overline{z})]^*$ hold true for every choice of $z \in \mathbb{C}$ and $k \in \mathbb{N}_0$ fulfilling $2k-1 \leq \kappa$.

In [12, 14, 15], one can find further results concerning the $\mathbb{R}$-QMP. Moreover, now we turn our attention to a further system of matrix polynomials which was already used in [14, Sec. 6]. We are going to consider a quadruple of $q \times q$ matrix polynomials which give a connection between the $\mathbb{R}$-QMP associated with $(s_j)_{j=0}^{2n}$ introduced in Definition 6.7 and the particular sequence $(\hat{s}_j)_{j=0}^{2n+1}$ introduced in [14, Def. 5.5]:

**Notation 6.15** Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices with $\mathcal{H}$-parameter sequence $(\mathfrak{h}_j)_{j=0}^{\kappa}$ and $\mathbb{R}$-QMP $[(\mathfrak{a}_k)_{k=0}^{\langle\!\langle\kappa]}, (\mathfrak{b}_k)_{k=0}^{\langle\!\langle\kappa]}, (\mathfrak{c}_k)_{k=0}^{\langle\!\langle\kappa]}, (\mathfrak{d}_k)_{k=0}^{\langle\!\langle\kappa]}]$, where $\langle\!\langle\kappa]$ is given in (6.2). Let $\mathring{\mathfrak{a}}_1, \mathring{\mathfrak{b}}_1, \mathring{\mathfrak{c}}_1, \mathring{\mathfrak{d}}_1 \colon \mathbb{C} \to \mathbb{C}^{q\times q}$ be defined by

$$\mathring{\mathfrak{a}}_1(z) := \mathfrak{h}_0, \qquad \mathring{\mathfrak{b}}_1(z) := zI_q, \qquad \mathring{\mathfrak{c}}_1(z) := \mathfrak{h}_0, \qquad \text{and} \qquad \mathring{\mathfrak{d}}_1(z) := zI_q. \tag{6.3}$$

For all $k \in \mathbb{Z}_{2,\infty}$ fulfilling $2k-2 \leq \kappa$, let $\mathring{\mathfrak{a}}_k, \mathring{\mathfrak{b}}_k, \mathring{\mathfrak{c}}_k, \mathring{\mathfrak{d}}_k \colon \mathbb{C} \to \mathbb{C}^{q\times q}$ be given by

$$\begin{aligned}
\mathring{\mathfrak{a}}_k(z) &:= z\mathfrak{a}_{k-1}(z) - \mathfrak{a}_{k-2}(z)\mathfrak{h}_{2k-4}^{\dagger}\mathfrak{h}_{2k-2}, \\
\mathring{\mathfrak{b}}_k(z) &:= z\mathfrak{b}_{k-1}(z) - \mathfrak{b}_{k-2}(z)\mathfrak{h}_{2k-4}^{\dagger}\mathfrak{h}_{2k-2}, \\
\mathring{\mathfrak{c}}_k(z) &:= z\mathfrak{c}_{k-1}(z) - \mathfrak{h}_{2k-2}\mathfrak{h}_{2k-4}^{\dagger}\mathfrak{c}_{2k-2}(z),
\end{aligned}$$

and

$$\mathring{\mathfrak{d}}_k(z) := z\mathfrak{d}_{k-1}(z) - \mathfrak{h}_{2k-2}\mathfrak{h}_{2k-4}^{\dagger}\mathfrak{d}_{2k-2}(z).$$

***Remark 6.16*** Let the assumptions of Notation 6.15 be fulfilled. From [14, Def. 5.5 and Lem. 6.18] then one can easily see that, for all $n \in \mathbb{N}_0$ fulfilling $2n \leq \kappa$, the quadruple $[\mathring{\mathfrak{a}}_{n+1}, \mathring{\mathfrak{b}}_{n+1}, \mathring{\mathfrak{c}}_{n+1}, \mathring{\mathfrak{d}}_{n+1}]$ is completely determined by the sequence $(s_j)_{j=0}^{2n}$.

***Remark 6.17*** Under the assumptions of Notation 6.15, for all $k \in \mathbb{N}_0$ such that $2k-2 \leq \kappa$, the matrix-valued functions $\mathring{\mathfrak{a}}_k$, $\mathring{\mathfrak{b}}_k$, $\mathring{\mathfrak{c}}_k$, and $\mathring{\mathfrak{d}}_k$ are matrix polynomials.

**Lemma 6.18** ([14, Lemmata 6.19 and 6.20]) *Let* $(s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q,\kappa}^{\succcurlyeq,\mathrm{e}}$ *with* $\mathbb{R}$*-QMP* $[(\mathfrak{a}_k)_{k=0}^{\langle\langle\kappa]}, (\mathfrak{b}_k)_{k=0}^{\langle\langle\kappa]}, (\mathfrak{c}_k)_{k=0}^{\langle\langle\kappa]}, (\mathfrak{d}_k)_{k=0}^{\langle\langle\kappa]}]$. *Then* $\det \mathfrak{b}_k(z) \neq 0$ *and* $\det \mathfrak{d}_k(z) \neq 0$ *hold true for every choice of* $k \in \mathbb{N}_0$ *fulfilling* $2k - 1 \leq \kappa$ *and for each* $z \in \mathbb{C} \setminus \mathbb{R}$. *Moreover, for all* $n \in \mathbb{N}_0$ *fulfilling* $2n \leq \kappa$ *and all* $z \in \mathbb{C} \setminus \mathbb{R}$, *then* $\det \mathring{\mathfrak{b}}_{n+1}(z) \neq 0$ *and* $\det \mathring{\mathfrak{d}}_{n+1}(z) \neq 0$.

***Remark 6.19*** Let $(s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q,\kappa}^{\succcurlyeq,\mathrm{e}}$ with $\mathbb{R}$-QMP $[(\mathfrak{a}_k)_{k=0}^{\langle\langle\kappa]}, (\mathfrak{b}_k)_{k=0}^{\langle\langle\kappa]}, (\mathfrak{c}_k)_{k=0}^{\langle\langle\kappa]}, (\mathfrak{d}_k)_{k=0}^{\langle\langle\kappa]}]$ and let $n \in \mathbb{N}_0$ be such that $2n-1 \leq \kappa$. According to Remark 6.8, the functions $\det \mathfrak{b}_n$ and $\det \mathfrak{d}_n$ are polynomials for which, in view of Lemma 6.18, the sets $\mathcal{Z}(\det \mathfrak{b}_n)$ and $\mathcal{Z}(\det \mathfrak{d}_n)$ are finite and, in particular, discrete sets fulfilling $\mathcal{Z}(\det \mathfrak{b}_n) \cup \mathcal{Z}(\det \mathfrak{d}_n) \subseteq \mathbb{R}$. Consequently, $\mathfrak{b}_n^{-1}$ and $\mathfrak{d}_n^{-1}$ are matrix-valued functions meromorphic in $\mathbb{C}$, which fulfill $\mathbb{C} \setminus \mathcal{Z}(\det \mathfrak{b}_n) \subseteq \mathcal{H}(\mathfrak{b}_n^{-1})$ and $\mathbb{C} \setminus \mathcal{Z}(\det \mathfrak{d}_n) \subseteq \mathcal{H}(\mathfrak{d}_n^{-1})$. Especially, $\mathbb{C} \setminus \mathbb{R} \subseteq \mathcal{H}(\mathfrak{b}_n^{-1}) \cap \mathcal{H}(\mathfrak{d}_n^{-1})$.

***Remark 6.20*** Let $(s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q,\kappa}^{\succcurlyeq,\mathrm{e}}$ and let $n \in \mathbb{N}_0$ be such that $2n \leq \kappa$. Analogous to Remark 6.19 and in view of Remark 6.17, the functions $\det \mathring{\mathfrak{b}}_n$ and $\det \mathring{\mathfrak{d}}_n$ are polynomials for which, in view of Lemma 6.18, the respective sets $\mathcal{Z}(\det \mathring{\mathfrak{b}}_n)$ and $\mathcal{Z}(\det \mathring{\mathfrak{d}}_n)$ are finite and, in particular, discrete sets fulfilling $\mathcal{Z}(\det \mathring{\mathfrak{b}}_n) \cup \mathcal{Z}(\det \mathring{\mathfrak{d}}_n) \subseteq \mathbb{R}$. Consequently, $\mathring{\mathfrak{b}}_n^{-1}$ and $\mathring{\mathfrak{d}}_n^{-1}$ are also matrix-valued functions meromorphic in $\mathbb{C}$ which fulfill $\mathbb{C} \setminus \mathcal{Z}(\det \mathring{\mathfrak{b}}_n) \subseteq \mathcal{H}(\mathring{\mathfrak{b}}_n^{-1})$ and $\mathbb{C} \setminus \mathcal{Z}(\det \mathring{\mathfrak{d}}_n) \subseteq \mathcal{H}(\mathring{\mathfrak{d}}_n^{-1})$. In particular, $\mathbb{C} \setminus \mathbb{R} \subseteq \mathcal{H}(\mathring{\mathfrak{b}}_n^{-1}) \cap \mathcal{H}(\mathring{\mathfrak{d}}_n^{-1})$.

## 7 Weyl Matrix Balls of the Truncated Hamburger Moment Problem

In the context of the matricial Hamburger moment problem, for $w \in \Pi_+$ the representation of the set

$$\left\{F(w) \colon F \in \mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \preccurlyeq]\right\} \tag{7.1}$$

as matrix ball was proved in [14] and is stated in Theorem 7.11 below. This representation plays also a key role in our following considerations for constructing a parameterization of the set (4.3). Let $\mathbb{K}_{p\times q} := \left\{K \in \mathbb{C}^{p\times q} \colon \|K\| \leq 1\right\}$. If matrices $M \in \mathbb{C}^{p\times q}$, $A \in \mathbb{C}^{p\times p}$, and $B \in \mathbb{C}^{q\times q}$ are given, then

$$\mathfrak{K}(M; A, B) := \left\{M + AKB \colon K \in \mathbb{K}_{p\times q}\right\} \tag{7.2}$$

is called the matrix ball with center $M$, left semi-radius $A$, and right semi-radius $B$. The theory of matrix balls goes back to Yu. L. Shmul'yan [34], who, moreover, worked out the operator case in the context of Hilbert spaces. Observe that the particular case of matrices is elaborated in [11, Sec. 1.5]. An essential tool to check that the Weyl set (7.1) can be represented as matrix ball was the following system of rational matrix-valued functions. This system was studied in [14, Sec. 7] in detail.

**Definition 7.1** (see [14, Def. 7.2]) Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q,\kappa}^{\succcurlyeq,\mathrm{e}}$ with $\mathcal{H}$-parameter sequence $(\mathfrak{h}_j)_{j=0}^{\kappa}$ and $\mathbb{R}$-QMP $[(\mathfrak{a}_k)_{k=0}^{\langle\langle\kappa]}, (\mathfrak{b}_k)_{k=0}^{\langle\langle\kappa]}, (\mathfrak{c}_k)_{k=0}^{\langle\langle\kappa]}, (\mathfrak{d}_k)_{k=0}^{\langle\langle\kappa]}]$. Let

$\chi_{-1}\colon \mathbb{C} \to \mathbb{C}^{q\times q}$ be defined by $\chi_{-1}(z) := O_{q\times q}$. For all $n \in \mathbb{N}_0$ such that $2n \leq \kappa$, let $\chi_{2n} := \mathfrak{h}_{2n}\mathfrak{b}_n^{-1}\mathring{\mathfrak{b}}_{n+1}$. Furthermore, for all $n \in \mathbb{N}_0$ fulfilling $2n+1 \leq \kappa$, let $\chi_{2n+1} := \mathfrak{h}_{2n}\mathfrak{b}_n^{-1}\mathfrak{b}_{n+1}$. Then $(\chi_j)_{j=-1}^{\kappa}$ is called the *sequence of* $\chi$*-functions associated with* $(s_j)_{j=0}^{\kappa}$.

Note that, according to Remark 6.19, the $\chi$-functions defined in Definition 7.1 are well-defined rational matrix-valued functions meromorphic in $\mathbb{C}$. In view of Definition 6.7, (6.3), (5.1), and the validity of $\mathfrak{h}_{2k-2}\mathfrak{h}_{2k-2}^{\dagger}\mathfrak{h}_{2k-1} = \mathfrak{h}_{2k-1}$ and $\mathfrak{h}_{2k-1}\mathfrak{h}_{2k-2}^{\dagger}\mathfrak{h}_{2k-2} = \mathfrak{h}_{2k-1}$ for all $k \in \mathbb{N}$ fulfilling $2k-1 \leq \kappa$ (see [14, Rem. 6.21, (6.14)]), for all $z \in \mathbb{C}$, we obtain

$$\chi_{-1}(z) = O_{q\times q}, \quad \chi_0(z) = z\mathfrak{h}_0 = zs_0, \quad \text{and} \quad \chi_1(z) = z\mathfrak{h}_0 - \mathfrak{h}_1 = zs_0 - s_1. \tag{7.3}$$

***Remark 7.2*** ([14, Rem. 7.3]) Let $(s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q,\kappa}^{\succcurlyeq,\mathrm{e}}$ with sequence of $\chi$-functions $(\chi_j)_{j=-1}^{\kappa}$ and let $m \in \mathbb{Z}_{0,\kappa}$. Then $(s_j)_{j=0}^{m} \in \mathcal{H}_{q,m}^{\succcurlyeq,\mathrm{e}}$ and $(\chi_j)_{j=-1}^{m}$ is exactly the sequence of $\chi$-functions associated with $(s_j)_{j=0}^{m}$.

For each $\tau \in \mathbb{N}_0 \cup \{\infty\}$, let

$$\langle\tau] := \sup\{k \in \mathbb{N}_0 \colon 2k \leq \tau\}. \tag{7.4}$$

Therefore, if $m = 2n$ or $m = 2n+1$ for some $n \in \mathbb{N}_0$, we have $\langle m] = n$.

***Remark 7.3*** ([14, Rem. 7.4]) Let $(s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q,\kappa}^{\succcurlyeq,\mathrm{e}}$ with sequence of $\chi$-functions $(\chi_j)_{j=-1}^{\kappa}$. In view of Definition 7.1 and Remark 6.19, then $\mathbb{C}\setminus\mathbb{R} \subseteq \mathcal{Z}(\det \mathfrak{b}_{\langle m]}) \subseteq \mathcal{H}(\chi_m)$ for all $m \in \mathbb{Z}_{0,\kappa}$.

***Remark 7.4*** ([14, Rem. 7.5]) Let the assumptions of Definition 7.1 be fulfilled. In view of Remark 6.19, then $\det \mathfrak{b}_n(z) \neq 0$ and $\chi_{2n}(z) = \mathfrak{h}_{2n}[\mathfrak{b}_n(z)]^{-1}\mathring{\mathfrak{b}}_{n+1}(z)$ for all $n \in \mathbb{N}_0$ such that $2n \leq \kappa$ and all $z \in \mathbb{C}\setminus\mathbb{R}$ as well as $\det \mathfrak{b}_n(z) \neq 0$ and $\chi_{2n+1}(z) = \mathfrak{h}_{2n}[\mathfrak{b}_n(z)]^{-1}\mathfrak{b}_{n+1}(z)$ for all $n \in \mathbb{N}_0$ such that $2n+1 \leq \kappa$ and all $z \in \mathbb{C}\setminus\mathbb{R}$.

***Remark 7.5*** ([14, Cor. 7.18]) Let $(s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q,\kappa}^{\succcurlyeq,\mathrm{e}}$ with sequence of $\chi$-functions $(\chi_j)_{j=-1}^{\kappa}$. Then $(\Im z)^{-1}\Im\chi_m(z) \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ for every choice of $m \in \mathbb{Z}_{-1,\kappa}$ and $z \in \mathbb{C}\setminus\mathbb{R}$.

**Lemma 7.6** ([14, Lemmata 8.3 and 8.5]) *Let* $(s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q,\kappa}^{\succcurlyeq,\mathrm{e}}$ *with* $\mathcal{H}$*-parameter sequence* $(\mathfrak{h}_j)_{j=0}^{\kappa}$, $\mathbb{R}$*-QMP* $[(\mathfrak{a}_k)_{k=0}^{\langle\langle\kappa]}, (\mathfrak{b}_k)_{k=0}^{\langle\langle\kappa]}, (\mathfrak{c}_k)_{k=0}^{\langle\langle\kappa]}, (\mathfrak{d}_k)_{k=0}^{\langle\langle\kappa]}]$, *and sequence of* $\chi$*-functions* $(\chi_j)_{j=-1}^{\kappa}$. *Furthermore, let* $n \in \mathbb{N}_0$ *be such that* $2n \leq \kappa$ *and let* $\mathring{\mathfrak{d}}_{n+1}$ *be given by Notation 6.15. For each* $z \in \mathbb{C}\setminus\mathbb{R}$, *then* $\det([\chi_{2n+1}(z)]^*\mathfrak{h}_{2n}^{\dagger}\mathfrak{d}_n(z) - \mathfrak{d}_{n+1}(z)) \neq 0$ *and* $\det([\chi_{2n}(z)]^*\mathfrak{h}_{2n}^{\dagger}\mathfrak{d}_n(z) - \mathring{\mathfrak{d}}_{n+1}(z)) \neq 0$.

Now we introduce the central object of this section, which we need to describe the set (4.3) as an intersection of matrix balls. Observe that the following constructions are well defined due to Remarks 6.16 and 7.5 as well as Lemmata 6.18 and 7.6:

**Notation 7.7** Let $(s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q,\kappa}^{\succcurlyeq,\mathrm{e}}$ with $\mathcal{H}$-parameter sequence $(\mathfrak{h}_j)_{j=0}^{\kappa}$, $\mathbb{R}$-QMP $[(\mathfrak{a}_k)_{k=0}^{\langle\langle\kappa]}, (\mathfrak{b}_k)_{k=0}^{\langle\langle\kappa]}, (\mathfrak{c}_k)_{k=0}^{\langle\langle\kappa]}, (\mathfrak{d}_k)_{k=0}^{\langle\langle\kappa]}]$, and sequence of $\chi$-functions $(\chi_j)_{j=-1}^{\kappa}$.

(a) For all $n \in \mathbb{N}_0$ such that $2n \le \kappa$, let $\mathscr{A}_{2n}, \mathscr{B}_{2n}, \mathscr{C}_{2n} : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{q\times q}$ be defined by

$$\mathscr{A}_{2n}(z) := [\mathfrak{d}_n(z)]^{-1}\mathfrak{h}_{2n}\sqrt{(\Im z)^{-1}\Im\chi_{2n}(z)}^{\dagger},$$
$$\mathscr{B}_{2n}(z) := \sqrt{(\Im z)^{-1}\Im\chi_{2n}(z)}^{\dagger}\mathfrak{h}_{2n}[\mathfrak{b}_n(z)]^{-1},$$

and

$$\mathscr{C}_{2n}(z) := -\left([\chi_{2n}(z)]^*\mathfrak{h}_{2n}^{\dagger}\mathfrak{d}_n(z) - \mathring{\mathfrak{d}}_{n+1}(z)\right)^{-1} \times \left([\chi_{2n}(z)]^*\mathfrak{h}_{2n}^{\dagger}\mathfrak{c}_n(z) - \mathring{\mathfrak{c}}_{n+1}(z)\right).$$

(b) Suppose $\kappa \ge 1$. For all $n \in \mathbb{N}_0$ such that $\mathscr{B}_{2n+1} \le \kappa$, let $\mathscr{A}_{2n+1}, \mathscr{B}2n+1, \mathscr{C}_{2n+1} : \mathbb{C}\setminus \mathbb{R} \to \mathbb{C}^{q\times q}$ be given by

$$\mathscr{A}_{2n+1}(z) := [\mathfrak{d}_n(z)]^{-1}\mathfrak{h}_{2n}\sqrt{(\Im z)^{-1}\Im\chi_{2n+1}(z)}^{\dagger},$$
$$\mathscr{B}_{2n+1}(z) := \sqrt{(\Im z)^{-1}\Im\chi_{2n+1}(z)}^{\dagger}\mathfrak{h}_{2n}[\mathfrak{b}_n(z)]^{-1},$$

and

$$\mathscr{C}_{2n+1}(z) := -\left([\chi_{2n+1}(z)]^*\mathfrak{h}_{2n}^{\dagger}\mathfrak{d}_n(z) - \mathfrak{d}_{n+1}(z)\right)^{-1} \times \left([\chi_{2n+1}(z)]^*\mathfrak{h}_{2n}^{\dagger}\mathfrak{c}_n(z) - \mathfrak{c}_{n+1}(z)\right)$$

At first view it looks a bit surprising why the Notation 7.7 is introduced for two different indices $2n$ and $2n+1$, because Notation 7.7 will be applied to describe the Weyl matrix ball associated with a sequence $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\succcurlyeq,\mathrm{e}}$. The reason is a technical one. It turns out soon (see Lemma 7.9 below) that the corresponding matrices introduced in parts (a) and (b) of Notation 7.7 coincide. We will express them in terms of Notation 7.7(a).

***Remark 7.8*** ([14, Rem. 8.8]) Let $(s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q,\kappa}^{\succcurlyeq,\mathrm{e}}$. Regarding Notations 7.7 and 6.15, then one can see from Remarks 5.6, 6.9 and 7.2, that for each $m \in \mathbb{Z}_{0,\kappa}$, the functions $\mathscr{A}_m$, $\mathscr{B}_m$, and $\mathscr{C}_m$ are only built from the matrices $s_0, s_1, \dots, s_m$ and therefore do not depend on the matrices $s_j$ with $j \ge m+1$.

Now we are going to formulate the announced statement containing the coincidence of the corresponding matrices introduced in Notation 7.7:

**Lemma 7.9** ([14, Lemmata 8.10 and 8.12]) *Suppose* $\kappa \geq 1$. *Let* $(s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q,\kappa}^{\succcurlyeq,\mathrm{e}}$ *and let* $n \in \mathbb{N}_0$ *be such that* $2n+1 \leq \kappa$. *Then* $\mathscr{A}_{2n}(z) = \mathscr{A}_{2n+1}(z)$, $\mathscr{B}_{2n}(z) = \mathscr{B}_{2n+1}(z)$, *and* $\mathscr{C}_{2n}(z) = \mathscr{C}_{2n+1}(z)$ *for all* $z \in \mathbb{C} \setminus \mathbb{R}$.

***Remark 7.10*** More precisely as stated in Remark 7.8, Lemma 7.9 even shows that $\mathscr{A}_m$, $\mathscr{B}_m$, and $\mathscr{C}_m$ are independent of the matrices $s_j$ with $j \geq \langle m] + 1$, where $\langle m]$ is given by (7.4).

Using Notation 7.7(a), now we are able to formulate the announced result, which is a generalization of Kovalishina's result [27, §2], who studied the (non-degenerate) case that the given sequence $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{H}_{q,2n}^{\succ}$:

**Theorem 7.11** ([14, Thm. 8.7]) *Let* $n \in \mathbb{N}_0$ *and let* $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\succcurlyeq,\mathrm{e}}$. *For each* $w \in \Pi_+$, *then*

$$\left\{F(w)\colon F \in \mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \preccurlyeq]\right\} = \mathfrak{K}(\mathscr{C}_{2n}(w); (w-\overline{w})^{-1}\mathscr{A}_{2n}(w), \mathscr{B}_{2n}(w)),$$

*where* $\mathscr{A}_{2n}$, $\mathscr{B}_{2n}$, *and* $\mathscr{C}_{2n}$ *are given by Notation 7.7(a).*

Note that a further representation of the matrix ball stated in Theorem 7.11 is given in [14, Thm. 8.26, Prop. 8.27].

## 8 Orthogonal Matrix Polynomials and $\mathcal{F}_{\alpha,\beta}$-parameters

Given a sequence $(s_j)_{j=0}^{2\kappa} \in \mathcal{H}_{q,2\kappa}^{\succcurlyeq,\mathrm{e}}$, the sequence $(\mathfrak{h}_j)_{j=0}^{2\kappa}$ of its $\mathcal{H}$-parameters can be recovered from a system of monic orthogonal polynomials with respect to some measure $\sigma$ belonging to $\mathcal{M}_q^{\succcurlyeq}[\mathbb{R}; (s_j)_{j=0}^{2\kappa}, =]$:

**Theorem 8.1** ([15, Thm. 6.9(b1)]) *Let* $\kappa \in \mathbb{N} \cup \{\infty\}$, *let* $(s_j)_{j=0}^{2\kappa} \in \mathcal{H}_{q,2\kappa}^{\succcurlyeq,\mathrm{e}}$ *with* $\mathcal{H}$*-parameter sequence* $(\mathfrak{h}_j)_{j=0}^{2\kappa}$, *let* $\sigma \in \mathcal{M}_q^{\succcurlyeq}[\mathbb{R}; (s_j)_{j=0}^{2\kappa}, =]$, *and let* $(P_k)_{k=0}^{\kappa}$ *be an MROS of matrix polynomials with respect to* $(s_j)_{j=0}^{2\kappa}$. *Then* $\int_{\mathbb{R}} [P_k(t)]^* \sigma(\mathrm{d}t)[P_k(t)] = \mathfrak{h}_{2k}$ *for all* $k \in \mathbb{Z}_{0,\kappa}$ *and* $\int_{\mathbb{R}} t[P_k(t)]^* \sigma(\mathrm{d}t)[P_k(t)] = \mathfrak{h}_{2k-1}$ *for all* $k \in \mathbb{Z}_{1,\kappa}$.

From Theorem 8.1 we obtain the following result which shows a first connection between monic right orthogonal systems of matrix polynomials introduced in Definition 6.2 and the $\mathcal{F}_{\alpha,\beta}$-parameter sequence given in Definition 5.19:

**Corollary 8.2** *Let* $(s_j)_{j=0}^{2\kappa+2} \in \mathcal{F}_{q,2\kappa+2,\alpha,\beta}^{\succcurlyeq}$ *with* $\mathcal{F}_{\alpha,\beta}$*-parameter sequence* $(\mathfrak{f}_j)_{j=0}^{4\kappa+4}$ *and let* $\sigma \in \mathcal{M}_q^{\succcurlyeq}[[\alpha,\beta]; (s_j)_{j=0}^{2\kappa+2}, =]$. *Furthermore, let* $(P_k)_{k=0}^{\kappa+1}$ *be an MROS of matrix polynomials with respect to* $(s_j)_{j=0}^{2\kappa+2}$ *and let* $(A_k)_{k=0}^{\kappa}$, $(B_k)_{k=0}^{\kappa}$, *and* $(C_k)_{k=0}^{\kappa}$ *be an MROS of matrix polynomials with respect to* $(a_j)_{j=0}^{2\kappa}$, *to* $(b_j)_{j=0}^{2\kappa}$, *and to* $(c_j)_{j=0}^{2\kappa}$, *respectively. For all* $k \in \mathbb{Z}_{0,\kappa}$, *then*

$$\int_{[\alpha,\beta]} (t-\alpha)[A_k(t)]^* \sigma(\mathrm{d}t)[A_k(t)] = \mathfrak{f}_{4k+1},$$
$$\int_{[\alpha,\beta]} (\beta-t)[B_k(t)]^* \sigma(\mathrm{d}t)[B_k(t)] = \mathfrak{f}_{4k+2},$$
$$\int_{[\alpha,\beta]} (\beta-t)(t-\alpha)[C_k(t)]^* \sigma(\mathrm{d}t)[C_k(t)] = \mathfrak{f}_{4k+3}, \tag{8.1}$$

*and*

$$\int_{[\alpha,\beta]} [P_{k+1}(t)]^* \sigma(\mathrm{d}t)[P_{k+1}(t)] = \mathfrak{f}_{4k+4}. \tag{8.2}$$

***Proof*** From Proposition 5.12 we get $(s_j)_{j=0}^{2\kappa+2} \in \mathcal{H}_{q,2\kappa+2}^{\succcurlyeq,\mathrm{e}}$. Moreover, using additionally Propositions 5.10 and 5.11, we have $\{(a_j)_{j=0}^{2\kappa}, (b_j)_{j=0}^{2\kappa}, (c_j)_{j=0}^{2\kappa}\} \subseteq \mathcal{H}_{q,2\kappa}^{\succcurlyeq,\mathrm{e}}$. Let the measures $\mu$, $\mu_\mathrm{a}$, $\mu_\mathrm{b}$, and $\mu_\mathrm{c}$ be given by Remark 3.7. According to Remark 3.7, then $\mu$ and $\mu_\mathrm{A}, \mu_\mathrm{B}, \mu_\mathrm{C} \colon \mathfrak{B}_\mathbb{R} \to \mathbb{C}^{q\times q}$ defined by $\mu_\mathrm{A}(M) := \mu_\mathrm{a}(M \cap [\alpha,\beta])$, $\mu_\mathrm{B}(M) := \mu_\mathrm{b}(M \cap [\alpha,\beta])$, and $\mu_\mathrm{C}(M) := \mu_\mathrm{c}(M \cap [\alpha,\beta])$, respectively, fulfill $\mu \in \mathcal{M}_q^{\succcurlyeq}[\mathbb{R}; (s_j)_{j=0}^{2\kappa+2}, =]$, $\mu_\mathrm{A} \in \mathcal{M}_q^{\succcurlyeq}[\mathbb{R}; (a_j)_{j=0}^{2\kappa}, =]$, $\mu_\mathrm{B} \in \mathcal{M}_q^{\succcurlyeq}[\mathbb{R}; (b_j)_{j=0}^{2\kappa}, =]$, and $\mu_\mathrm{C} \in \mathcal{M}_q^{\succcurlyeq}[\mathbb{R}; (c_j)_{j=0}^{2\kappa}, =]$. Now we consider the case $\kappa \geq 1$. Using Theorem 8.1, Remarks 5.7 and 5.18 as well as the notations given in Definitions 5.5, 5.14, 5.17 and 5.19, for each $k \in \mathbb{Z}_{0,\kappa}$, we get

$$\int_{[\alpha,\beta]} [P_{k+1}(t)]^* \sigma(\mathrm{d}t)[P_{k+1}(t)] = \int_\mathbb{R} [P_{k+1}(t)]^* \mu(\mathrm{d}t)[P_{k+1}(t)]$$
$$= \mathfrak{h}_{2k+2} = L_{k+1} = \mathfrak{A}_{2k+2} = \mathfrak{f}_{4k+4},$$

where $(\mathfrak{h}_j)_{j=0}^{2\kappa+2}$ denotes the $\mathcal{H}$-parameter sequence of $(s_j)_{j=0}^{2\kappa+2}$. Thus, equation (8.2) is proved. Analogously, if $\kappa \geq 1$, then, using the same arguments and Proposition B.1 additionally, one can check the other three asserted identities by applying Theorem 8.1 to $(a_j)_{j=0}^{2\kappa}$ and $\mu_\mathrm{A}$, to $(b_j)_{j=0}^{2\kappa}$ and $\mu_\mathrm{B}$ as well as to $(c_j)_{j=0}^{2\kappa}$ and $\mu_\mathrm{C}$, respectively. If $\kappa = 0$, then, combining Theorem 8.1, Definitions 5.5, 5.14, 5.17 and 5.19 as well as (5.1), (5.9), (5.10), and [24, Lem. 7.2], the asserted equations follow by straightforward calculations. □

Conversely, outgoing from a sequence $(s_j)_{j=0}^{2\kappa+2} \in \mathcal{F}_{q,2\kappa+2,\alpha,\beta}^{\succcurlyeq}$, one can explicitly construct MROS of matrix polynomials with respect to $(s_j)_{j=0}^{2\kappa+2}$, $(a_j)_{j=0}^{2\kappa}$, $(b_j)_{j=0}^{2\kappa}$, and $(c_j)_{j=0}^{2\kappa}$, respectively, with the help of their respective $\mathcal{F}_{\alpha,\beta}$-parameter sequence. For this reason, we introduce some further matrix polynomials. For each subspace $\mathcal{U}$ of $\mathbb{C}^q$, let $\mathbb{P}_\mathcal{U}$ be the orthogonal projection matrix onto $\mathcal{U}$ (see also Remarks A.4 and A.5).

**Notation 8.3** Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q\times q$ matrices with $\mathcal{F}_{\alpha,\beta}$-parameter sequence $(\mathfrak{f}_j)_{j=0}^{2\kappa}$ and sequence of $[\alpha,\beta]$-interval lengths $(\mathsf{D}_j)_{j=0}^{\kappa}$.

Then let $\ddot{\mathbf{p}}_0, \ddot{\mathbf{q}}_0, \ddot{\mathbf{p}}_1, \ddot{\mathbf{q}}_1, \ddot{\mathbf{p}}_2, \ddot{\mathbf{q}}_2 : \mathbb{C} \to \mathbb{C}^{q\times q}$ be defined by

$$\ddot{\mathbf{p}}_0(z) := I_q, \qquad \ddot{\mathbf{q}}_0(z) := O_{q\times q}, \quad \ddot{\mathbf{p}}_1(z) := (z-\alpha)I_q, \quad \ddot{\mathbf{q}}_1(z) := \mathfrak{f}_0, \tag{8.3}$$

$$\ddot{\mathbf{p}}_2(z) := -(\beta - z)I_q, \qquad \text{and} \qquad \ddot{\mathbf{q}}_2(z) := \mathfrak{f}_0. \tag{8.4}$$

Now suppose $\kappa \geq 1$ and let $(\Gamma_j)_{j=1}^{\kappa}$ be a sequence of complex $q \times q$ matrices. For each $j \in \mathbb{Z}_{1,\kappa}$, let

$$\mathbf{A}_j := \mathsf{D}_{j-1}^{\dagger}\mathfrak{f}_{2j} + \mathbb{P}_{\mathcal{N}(\mathsf{D}_{j-1})}\Gamma_j \quad \text{and} \quad \mathbf{B}_j := \mathsf{D}_{j-1}^{\dagger}\mathfrak{f}_{2j-1} + \mathbb{P}_{\mathcal{N}(\mathsf{D}_{j-1})}(I_q - \Gamma_j). \tag{8.5}$$

For all $k \in \mathbb{Z}_{2,\kappa+1}$, then let $\ddot{\mathbf{p}}_{2k-1}, \ddot{\mathbf{q}}_{2k-1}, \ddot{\mathbf{p}}_{2k}, \ddot{\mathbf{q}}_{2k} : \mathbb{C} \to \mathbb{C}^{q\times q}$ be recursively defined as follows: If $k = 2\ell$ with some $\ell \in \mathbb{N}$, then let

$$\ddot{\mathbf{p}}_{4\ell-1}(z) := -(\beta - z)\ddot{\mathbf{p}}_{4\ell-3}(z)\mathbf{A}_{2\ell-1} + (z-\alpha)\ddot{\mathbf{p}}_{4\ell-2}(z)\mathbf{B}_{2\ell-1}, \tag{8.6}$$

$$\ddot{\mathbf{q}}_{4\ell-1}(z) := -(\beta - z)\ddot{\mathbf{q}}_{4\ell-3}(z)\mathbf{A}_{2\ell-1} + (z-\alpha)\ddot{\mathbf{q}}_{4\ell-2}(z)\mathbf{B}_{2\ell-1} \tag{8.7}$$

and

$$\ddot{\mathbf{p}}_{4\ell}(z) := \ddot{\mathbf{p}}_{4\ell-3}(z)\mathbf{A}_{2\ell-1} + \ddot{\mathbf{p}}_{4\ell-2}(z)\mathbf{B}_{2\ell-1}, \tag{8.8}$$

$$\ddot{\mathbf{q}}_{4\ell}(z) := \ddot{\mathbf{q}}_{4\ell-3}(z)\mathbf{A}_{2\ell-1} + \ddot{\mathbf{q}}_{4\ell-2}(z)\mathbf{B}_{2\ell-1}. \tag{8.9}$$

If $k = 2\ell + 1$ with some $\ell \in \mathbb{N}$, then let

$$\ddot{\mathbf{p}}_{4\ell+1}(z) := \ddot{\mathbf{p}}_{4\ell-1}(z)\mathbf{A}_{2\ell} + (z-\alpha)\ddot{\mathbf{p}}_{4\ell}(z)\mathbf{B}_{2\ell}, \tag{8.10}$$

$$\ddot{\mathbf{q}}_{4\ell+1}(z) := \ddot{\mathbf{q}}_{4\ell-1}(z)\mathbf{A}_{2\ell} + (z-\alpha)\ddot{\mathbf{q}}_{4\ell}(z)\mathbf{B}_{2\ell} \tag{8.11}$$

and

$$\ddot{\mathbf{p}}_{4\ell+2}(z) := \ddot{\mathbf{p}}_{4\ell-1}(z)\mathbf{A}_{2\ell} - (\beta - z)\ddot{\mathbf{p}}_{4\ell}(z)\mathbf{B}_{2\ell}, \tag{8.12}$$

$$\ddot{\mathbf{q}}_{4\ell+2}(z) := \ddot{\mathbf{q}}_{4\ell-1}(z)\mathbf{A}_{2\ell} - (\beta - z)\ddot{\mathbf{q}}_{4\ell}(z)\mathbf{B}_{2\ell}. \tag{8.13}$$

***Remark 8.4*** Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices with $\mathcal{F}_{\alpha,\beta}$-parameter sequence $(\mathfrak{f}_j)_{j=0}^{2\kappa}$. In view of Notation 8.3 and Remarks 5.15 and 5.23, for each $k \in \mathbb{Z}_{1,\kappa+1}$, then $\ddot{\mathbf{p}}_{2k-1}, \ddot{\mathbf{q}}_{2k-1}, \ddot{\mathbf{p}}_{2k}$, and $\ddot{\mathbf{q}}_{2k}$ are matrix polynomials which are built only from the matrices $\mathfrak{f}_0, \mathfrak{f}_1, \ldots, \mathfrak{f}_{2k-2}$ and, therefore, which depend only on the matrices $s_0, s_1, \ldots, s_{k-1}$.

***Remark 8.5*** Suppose $\kappa \geq 1$. Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. For every choice of $\ell \in \mathbb{N}$ such that $2\ell - 1 \leq \kappa$ and for all $z \in \mathbb{C}$, then, adding suitable multiples of (8.6)–(8.13), it is readily checked that

$$(z-\alpha)\ddot{\mathbf{p}}_{4\ell}(z) - \ddot{\mathbf{p}}_{4\ell-1}(z) = \delta\ddot{\mathbf{p}}_{4\ell-3}(z)\mathbf{A}_{2\ell-1},$$

$$(z-\alpha)\ddot{\mathbf{q}}_{4\ell}(z) - \ddot{\mathbf{q}}_{4\ell-1}(z) = \delta\ddot{\mathbf{q}}_{4\ell-3}(z)\mathbf{A}_{2\ell-1}$$

and

$$(\beta - z)\ddot{\mathbf{p}}_{4\ell}(z) + \ddot{\mathbf{p}}_{4\ell-1}(z) = \delta\ddot{\mathbf{p}}_{4\ell-2}(z)\mathbf{B}_{2\ell-1},$$
$$(\beta - z)\ddot{\mathbf{q}}_{4\ell}(z) + \ddot{\mathbf{q}}_{4\ell-1}(z) = \delta\ddot{\mathbf{q}}_{4\ell-2}(z)\mathbf{B}_{2\ell-1}$$

hold true. Furthermore, for all $\ell \in \mathbb{N}$ such that $2\ell \leq \kappa$ and for all $z \in \mathbb{C}$, straightforward calculations yield the identities

$$\ddot{\mathbf{p}}_{4\ell+1}(z) - \ddot{\mathbf{p}}_{4\ell+2}(z) = \delta\ddot{\mathbf{p}}_{4\ell}(z)\mathbf{B}_{2\ell}, \quad \ddot{\mathbf{q}}_{4\ell+1}(z) - \ddot{\mathbf{q}}_{4\ell+2}(z) = \delta\ddot{\mathbf{q}}_{4\ell}(z)\mathbf{B}_{2\ell},$$
$$(\beta - z)\ddot{\mathbf{p}}_{4\ell+1}(z) + (z - \alpha)\ddot{\mathbf{p}}_{4\ell+2}(z) = \delta\ddot{\mathbf{p}}_{4\ell-1}(z)\mathbf{A}_{2\ell},$$

and

$$(\beta - z)\ddot{\mathbf{q}}_{4\ell+1}(z) + (z - \alpha)\ddot{\mathbf{q}}_{4\ell+2}(z) = \delta\ddot{\mathbf{q}}_{4\ell-1}(z)\mathbf{A}_{2\ell}.$$

**Lemma 8.6** *Suppose $\kappa \geq 1$. Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. Then $\mathbf{A}_j + \mathbf{B}_j = I_q$ and, in particular, $\mathbf{A}_j\mathbf{B}_j = \mathbf{B}_j\mathbf{A}_j$ hold true for all $j \in \mathbb{Z}_{1,\kappa}$.*

***Proof*** In view of Notation 8.3, one can see from Remarks 5.24 and A.8, that $\mathbf{A}_j + \mathbf{B}_j = \mathsf{D}_{j-1}^{\dagger}(\mathfrak{f}_{2j} + \mathfrak{f}_{2j-1}) + \mathbb{P}_{\mathcal{N}(\mathsf{D}_{j-1})} = \mathsf{D}_{j-1}^{\dagger}\mathsf{D}_{j-1} + (I_q - \mathsf{D}_{j-1}^{\dagger}\mathsf{D}_{j-1}) = I$ and, therefore, $\mathbf{A}_j\mathbf{B}_j = \mathbf{A}_j(I_q - \mathbf{A}_j) = \mathbf{A}_j - \mathbf{A}_j^2 = (I_q - \mathbf{A}_j)\mathbf{A}_j = \mathbf{B}_j\mathbf{A}_j$ for all $j \in \mathbb{Z}_{1,\kappa}$. □

**Lemma 8.7** *Suppose $\kappa \geq 1$. Let $(s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq}$. For all $j \in \mathbb{Z}_{1,\kappa}$, then $\mathsf{D}_{j-1}\mathbf{A}_j = \mathfrak{f}_{2j}$, $\delta\mathfrak{f}_{2j-1}\mathbf{A}_j = \mathsf{D}_j$, $\mathsf{D}_{j-1}\mathbf{B}_j = \mathfrak{f}_{2j-1}$, and $\delta\mathfrak{f}_{2j}\mathbf{B}_j = \mathsf{D}_j$ as well as, in particular, $\mathfrak{f}_{2j-1}\mathbf{A}_j = \mathfrak{f}_{2j}\mathbf{B}_j$.*

***Proof*** Let $j \in \mathbb{Z}_{1,\kappa}$. According to Remark 5.25, we have $\mathcal{R}(\mathsf{D}_{j-1}) = \mathcal{R}(\mathfrak{f}_{2j}) + \mathcal{R}(\mathfrak{f}_{2j-1}) \supseteq \mathcal{R}(\mathfrak{f}_{2j}) \cup \mathcal{R}(\mathfrak{f}_{2j-1})$. Thus, Remark A.1 yields $\mathsf{D}_{j-1}\mathsf{D}_{j-1}^{\dagger}\mathfrak{f}_{2j} = \mathfrak{f}_{2j}$ and $\mathsf{D}_{j-1}\mathsf{D}_{j-1}^{\dagger}\mathfrak{f}_{2j-1} = \mathfrak{f}_{2j-1}$. From (8.5) we can conclude $\mathsf{D}_{j-1}\mathbf{A}_j = \mathfrak{f}_{2j}$ and $\mathsf{D}_{j-1}\mathbf{B}_j = \mathfrak{f}_{2j-1}$. Remark 5.25 moreover shows $\mathcal{N}(\mathsf{D}_{j-1}) = \mathcal{N}(\mathfrak{f}_{2j-1}) \cap \mathcal{N}(\mathfrak{f}_{2j})$. Consequently, $\mathfrak{f}_{2j-1}\mathbb{P}_{\mathcal{N}(\mathsf{D}_{j-1})} = O$ and $\mathfrak{f}_{2j}\mathbb{P}_{\mathcal{N}(\mathsf{D}_{j-1})} = O$. Furthermore, combining [19, Cor. 10.21] and Remark 5.21 provides $\delta\mathfrak{f}_{2j-1}\mathsf{D}_{j-1}^{\dagger}\mathfrak{f}_{2j} = \mathsf{D}_j$ and $\delta\mathfrak{f}_{2j}\mathsf{D}_{j-1}^{\dagger}\mathfrak{f}_{2j-1} = \mathsf{D}_j$. In view of (8.5), this implies $\delta\mathfrak{f}_{2j-1}\mathbf{A}_j = \delta\mathfrak{f}_{2j-1}\mathsf{D}_{j-1}^{\dagger}\mathfrak{f}_{2j} + \delta\mathfrak{f}_{2j-1}\mathbb{P}_{\mathcal{N}(\mathsf{D}_{j-1})}\Gamma_j = \mathsf{D}_j$ and, analogously, $\delta\mathfrak{f}_{2j}\mathbf{B}_j = \mathsf{D}_j$. Because of $\delta > 0$, then $\mathfrak{f}_{2j-1}\mathbf{A}_j = \mathfrak{f}_{2j}\mathbf{B}_j$ follows as well. □

***Remark 8.8*** Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. In view of Notation 8.3 and Lemma 8.6, one can inductively see that the following statements hold true:

(a) For each $\ell \in \mathbb{N}_0$ such that $2\ell - 1 \leq \kappa$, the function $\ddot{\mathbf{p}}_{4\ell}$ is a complex $q \times q$ matrix polynomial of degree $\ell$ with leading coefficient matrix $I_q$.
(b) For each $\ell \in \mathbb{N}_0$ such that $2\ell \leq \kappa$, the functions $\ddot{\mathbf{p}}_{4\ell+1}$ and $\ddot{\mathbf{p}}_{4\ell+2}$ are complex $q \times q$ matrix polynomials of degree $\ell+1$, where both leading coefficient matrices coincide with $I_q$, fulfilling $\ddot{\mathbf{p}}_{4\ell+1}(\alpha) = O_{q\times q}$ and $\ddot{\mathbf{p}}_{4\ell+2}(\beta) = O_{q\times q}$.

(c) If $\kappa \geq 1$, then, for each $\ell \in \mathbb{N}_0$ such that $2\ell + 1 \leq \kappa$, the function $\ddot{\mathbf{p}}_{4\ell+3}$ is a complex $q \times q$ matrix polynomial of degree $\ell + 2$ with leading coefficient matrix $I_q$, fulfilling $\ddot{\mathbf{p}}_{4\ell+3}(\alpha) = O_{q\times q}$ and $\ddot{\mathbf{p}}_{4\ell+4}(\beta) = O_{q\times q}$.

***Remark 8.9*** Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. In view of Notation 8.3, Remark 8.8, and $\alpha \neq \beta$, the following matrix polynomials can be defined:

(a) For each $\ell \in \mathbb{N}_0$ such that $2\ell - 1 \leq \kappa$, let $\ddot{\mathbf{r}}_\ell := \ddot{\mathbf{p}}_{4\ell}$.
(b) For all $\ell \in \mathbb{N}_0$ fulfilling $2\ell \leq \kappa$, let $\ddot{\mathbf{t}}_\ell$ and $\ddot{\mathbf{v}}_\ell$ be the uniquely determined $q \times q$ matrix polynomials fulfilling $\ddot{\mathbf{p}}_{4\ell+1}(z) = (z-\alpha)\ddot{\mathbf{t}}_\ell(z)$ and $\ddot{\mathbf{p}}_{4\ell+2}(z) = -(\beta - z)\ddot{\mathbf{v}}_\ell(z)$ for all $z \in \mathbb{C}$.
(c) Remark 8.8 moreover shows that, for each $\ell \in \mathbb{N}_0$ such that $2\ell + 1 \leq \kappa$, there is a unique complex $q \times q$ matrix polynomial $\ddot{\mathbf{x}}_\ell$ for which $\ddot{\mathbf{p}}_{4\ell+3}(z) = -(\beta - z)(z-\alpha)\ddot{\mathbf{x}}_\ell$ is fulfilled for every choice of $z \in \mathbb{C}$.

The matrix polynomials $\ddot{\mathbf{r}}_l$, $\ddot{\mathbf{t}}_l$, $\ddot{\mathbf{v}}_l$, and $\ddot{\mathbf{x}}_l$, which we introduced in Remark 8.9 and whose notation we will continue to use, stand in an recursive interrelationship:

**Lemma 8.10** *Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. For all $z \in \mathbb{C}$, then $\ddot{\mathbf{r}}_0(z) = \ddot{\mathbf{t}}_0(z) = \ddot{\mathbf{v}}_0(z) = I_q$ as well as, in the case $\kappa \geq 1$, moreover*

$$\ddot{\mathbf{x}}_{\ell-1}(z) = \ddot{\mathbf{t}}_{\ell-1}(z)\mathbf{A}_{2\ell-1} + \ddot{\mathbf{v}}_{\ell-1}(z)\mathbf{B}_{2\ell-1} \tag{8.14}$$

*and*

$$\ddot{\mathbf{r}}_\ell(z) = (z-\alpha)\ddot{\mathbf{t}}_{\ell-1}(z)\mathbf{A}_{2\ell-1} - (\beta - z)\ddot{\mathbf{v}}_{\ell-1}(z)\mathbf{B}_{2\ell-1} \tag{8.15}$$

*for every choice of $\ell \in \mathbb{N}$ such that $2\ell - 1 \leq \kappa$. Furthermore, if $\kappa \geq 2$, then*

$$\ddot{\mathbf{t}}_\ell(z) = -(\beta - z)\ddot{\mathbf{x}}_{\ell-1}(z)\mathbf{A}_{2\ell} + \ddot{\mathbf{r}}_\ell(z)\mathbf{B}_{2\ell} \tag{8.16}$$

*and*

$$\ddot{\mathbf{v}}_\ell(z) = (z-\alpha)\ddot{\mathbf{x}}_{\ell-1}(z)\mathbf{A}_{2\ell} + \ddot{\mathbf{r}}_\ell(z)\mathbf{B}_{2\ell} \tag{8.17}$$

*hold true for all $\ell \in \mathbb{N}$ such that $2\ell \leq \kappa$ and all $z \in \mathbb{C}$.*

***Proof*** In view of Remark 8.9, (8.3), and (8.4), we have $\ddot{\mathbf{r}}_0(z) = \ddot{\mathbf{p}}_0(z) = I_q$, $(z-\alpha)\ddot{\mathbf{t}}_0(z) = \ddot{\mathbf{p}}_1(z) = (z-\alpha)I_q$, and $-(\beta - z)\ddot{\mathbf{v}}_0(z) = \ddot{\mathbf{p}}_2(z) = -(\beta - z)I_q$ for all $z \in \mathbb{C}$. Now assume $\kappa \geq 1$ and let $\ell \in \mathbb{N}$ be such that $2\ell - 1 \leq \kappa$. Taking into account Remark 8.9 as well as (8.6) and (8.8), we conclude

$$\begin{aligned} &-(\beta - z)(z-\alpha)\ddot{\mathbf{x}}_{\ell-1}(z) = \ddot{\mathbf{p}}_{4\ell-1}(z) \\ &\quad = -(\beta - z)\ddot{\mathbf{p}}_{4\ell-3}(z)\mathbf{A}_{2\ell-1} + (z-\alpha)\ddot{\mathbf{p}}_{4\ell-2}(z)\mathbf{B}_{2\ell-1} \\ &\quad = -(\beta - z)(z-\alpha)[\ddot{\mathbf{t}}_{\ell-1}(z)\mathbf{A}_{2\ell-1} + \ddot{\mathbf{v}}_{\ell-1}(z)\mathbf{B}_{2\ell-1}]. \end{aligned} \tag{8.18}$$

Comparing the left-hand side of (8.18) with the right-hand side of (8.18) delivers (8.14). Equation (8.15) can be seen by $\ddot{\mathbf{r}}_\ell(z) = \ddot{\mathbf{p}}_{4\ell}(z) = \ddot{\mathbf{p}}_{4\ell-3}(z)\mathbf{A}_{2\ell-1} + \ddot{\mathbf{p}}_{4\ell-2}(z)\mathbf{B}_{2\ell-1} =$

$(z-\alpha)\ddot{\mathbf{t}}_{\ell-1}(z)\mathbf{A}_{2\ell-1} - (\beta - z)\ddot{\mathbf{v}}_{\ell-1}(z)\mathbf{B}_{2\ell-1}$. If $\kappa \geq 2$ and if $\ell \in \mathbb{N}$ fulfills $2\ell \leq \kappa$, then using Remark 8.9 as well as (8.10) and (8.12), we obtain analogously (8.16) and (8.17). □

***Remark 8.11*** Suppose $\kappa \geq 1$. Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. In view of Lemma 8.10, then it is readily checked that, for all $\ell \in \mathbb{N}$ such that $2\ell - 1 \leq \kappa$ and for all $z \in \mathbb{C}$, the equations

$$(\beta - z)\ddot{\mathbf{x}}_{\ell-1}(z) + \ddot{\mathbf{r}}_\ell(z) = \delta\ddot{\mathbf{t}}_{\ell-1}(z)\mathbf{A}_{2\ell-1} \tag{8.19}$$

and

$$(z-\alpha)\ddot{\mathbf{x}}_{\ell-1}(z) - \ddot{\mathbf{r}}_\ell(z) = \delta\ddot{\mathbf{v}}_{\ell-1}(z)\mathbf{B}_{2\ell-1} \tag{8.20}$$

hold true. Moreover, if $\kappa \geq 2$, then Lemma 8.10 immediately implies the identities $\ddot{\mathbf{v}}_\ell(z) - \ddot{\mathbf{t}}_\ell(z) = \delta\ddot{\mathbf{x}}_{\ell-1}(z)\mathbf{A}_{2\ell}$ and $(\beta - z)\ddot{\mathbf{v}}_\ell(z) + (z-\alpha)\ddot{\mathbf{t}}_\ell(z) = \delta\ddot{\mathbf{r}}_\ell(z)\mathbf{B}_{2\ell}$ for all $\ell \in \mathbb{N}$ such that $2\ell \leq \kappa$ and all $z \in \mathbb{C}$.

**Lemma 8.12** *Suppose $\kappa \geq 2$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. For all $\ell \in \mathbb{N}$ such that $2\ell \leq \kappa$ and all $z \in \mathbb{C}$, then*

$$\begin{aligned}\ddot{\mathbf{t}}_\ell(z) &= \ddot{\mathbf{r}}_\ell(z) - \ddot{\mathbf{t}}_{\ell-1}(z)(\delta\mathbf{A}_{2\ell-1}\mathbf{A}_{2\ell})\\ &= -(\beta - z)\ddot{\mathbf{x}}_{\ell-1}(z) + \ddot{\mathbf{t}}_{\ell-1}(z)(\delta\mathbf{A}_{2\ell-1}\mathbf{B}_{2\ell})\end{aligned} \tag{8.21}$$

*and*

$$\begin{aligned}\ddot{\mathbf{v}}_\ell(z) &= \ddot{\mathbf{r}}_\ell(z) + \ddot{\mathbf{v}}_{\ell-1}(z)(\delta\mathbf{B}_{2\ell-1}\mathbf{A}_{2\ell})\\ &= (z-\alpha)\ddot{\mathbf{x}}_{\ell-1}(z) - \ddot{\mathbf{v}}_{\ell-1}(z)(\delta\mathbf{B}_{2\ell-1}\mathbf{B}_{2\ell})\end{aligned} \tag{8.22}$$

*are valid. For all $\ell \in \mathbb{Z}_{2,\infty}$ such that $2\ell - 1 \leq \kappa$ and all $z \in \mathbb{C}$, moreover*

$$\begin{aligned}\ddot{\mathbf{x}}_{\ell-1}(z) &= \ddot{\mathbf{v}}_{\ell-1}(z) - \ddot{\mathbf{x}}_{\ell-2}(z)(\delta\mathbf{A}_{2\ell-2}\mathbf{A}_{2\ell-1})\\ &= \ddot{\mathbf{t}}_{\ell-1}(z) + \ddot{\mathbf{x}}_{\ell-2}(z)(\delta\mathbf{A}_{2\ell-2}\mathbf{B}_{2\ell-1})\end{aligned} \tag{8.23}$$

$$\ddot{\mathbf{r}}_\ell(z) = -(\beta - z)\ddot{\mathbf{v}}_{\ell-1}(z) + \ddot{\mathbf{r}}_{\ell-1}(z)(\delta\mathbf{B}_{2\ell-2}\mathbf{A}_{2\ell-1}), \tag{8.24}$$

*and*

$$\ddot{\mathbf{r}}_\ell(z) = (z-\alpha)\ddot{\mathbf{t}}_{\ell-1}(z) - \ddot{\mathbf{r}}_{\ell-1}(z)(\delta\mathbf{B}_{2\ell-2}\mathbf{B}_{2\ell-1}). \tag{8.25}$$

***Proof*** We consider an arbitrary $z \in \mathbb{C}$. Let $\ell \in \mathbb{N}$ be such that $2\ell \leq \kappa$. One can easily see that (8.19) is equivalent to $-(\beta - z)\ddot{\mathbf{x}}_{\ell-1}(z) = \ddot{\mathbf{r}}_\ell(z) - \delta\ddot{\mathbf{t}}_{\ell-1}(z)\mathbf{A}_{2\ell-1}$ which, by multiplying with $\mathbf{A}_{2\ell}$ from the right, leads to

$$-(\beta - z)\ddot{\mathbf{x}}_{\ell-1}(z)\mathbf{A}_{2\ell} = \ddot{\mathbf{r}}_\ell(z)\mathbf{A}_{2\ell} - \delta\ddot{\mathbf{t}}_{\ell-1}(z)\mathbf{A}_{2\ell-1}\mathbf{A}_{2\ell}. \tag{8.26}$$

Multiplying (8.19) from the right by $\mathbf{B}_{2\ell}$ delivers

$$\ddot{\mathbf{r}}_{\ell}(z)\mathbf{B}_{2\ell} = \delta\ddot{\mathbf{t}}_{\ell-1}(z)\mathbf{A}_{2\ell-1}\mathbf{B}_{2\ell} - (\beta - z)\ddot{\mathbf{x}}_{\ell-1}(z)\mathbf{B}_{2\ell}. \tag{8.27}$$

In the same manner, by multiplying (8.20) from the right by $\mathbf{A}_{2\ell}$, we get

$$(z-\alpha)\ddot{\mathbf{x}}_{\ell-1}(z)\mathbf{A}_{2\ell} = \delta\ddot{\mathbf{v}}_{\ell-1}(z)\mathbf{B}_{2\ell-1}\mathbf{A}_{2\ell} + \ddot{\mathbf{r}}_{\ell}(z)\mathbf{A}_{2\ell} \tag{8.28}$$

and, by multiplying (8.20) from the right by $\mathbf{B}_{2\ell}$, we obtain

$$\ddot{\mathbf{r}}_{\ell}(z)\mathbf{B}_{2\ell} = (z-\alpha)\ddot{\mathbf{x}}_{\ell-1}(z)\mathbf{B}_{2\ell} - \delta\ddot{\mathbf{v}}_{\ell-1}(z)\mathbf{B}_{2\ell-1}\mathbf{B}_{2\ell}. \tag{8.29}$$

Lemma 8.10 yields (8.14), (8.15), (8.16), and (8.17). Inserting (8.26) into (8.16), from Lemma 8.6 we conclude

$$\ddot{\mathbf{t}}_{\ell}(z) = \ddot{\mathbf{r}}_{\ell}(z)(\mathbf{A}_{2\ell} + \mathbf{B}_{2\ell}) - \delta\ddot{\mathbf{t}}_{\ell-1}(z)\mathbf{A}_{2\ell-1}\mathbf{A}_{2\ell} = \ddot{\mathbf{r}}_{\ell}(z) - \ddot{\mathbf{t}}_{\ell-1}(z)(\delta\mathbf{A}_{2\ell-1}\mathbf{A}_{2\ell}).$$

Analogously, using (8.27) instead of (8.26), we get

$$\begin{aligned}\ddot{\mathbf{t}}_{\ell}(z) &= -(\beta - z)\ddot{\mathbf{x}}_{\ell-1}(z)\mathbf{A}_{2\ell} + \delta\ddot{\mathbf{t}}_{\ell-1}(z)\mathbf{A}_{2\ell-1}\mathbf{B}_{2\ell} - (\beta - z)\ddot{\mathbf{x}}_{\ell-1}(z)\mathbf{B}_{2\ell} \\ &= -(\beta - z)\ddot{\mathbf{x}}_{\ell-1}(z) + \ddot{\mathbf{t}}_{\ell-1}(z)(\delta\mathbf{A}_{2\ell-1}\mathbf{B}_{2\ell}).\end{aligned}$$

Hence, (8.21) is proved. The application of (8.28), (8.29), (8.17), and Lemma 8.10 yields analogously (8.22). Now we are going to show the remaining equations in (8.23) and (8.25). For this reason, we consider an arbitrary $\ell \in \mathbb{Z}_{2,\infty}$ fulfilling $2\ell - 1 \leq \kappa$. Lemma 8.10 shows (8.14) and (8.15). Remark 8.11 yields $\ddot{\mathbf{v}}_{\ell-1}(z) - \ddot{\mathbf{t}}_{\ell-1}(z) = \delta\ddot{\mathbf{x}}_{\ell-2}(z)\mathbf{A}_{2\ell-2}$ and $(\beta - z)\ddot{\mathbf{v}}_{\ell-1}(z) + (z-\alpha)\ddot{\mathbf{t}}_{\ell-1}(z) = \delta\ddot{\mathbf{r}}_{\ell-1}(z)\mathbf{B}_{2\ell-2}$, which implies

$$\begin{aligned}\ddot{\mathbf{t}}_{\ell-1}(z)\mathbf{A}_{2\ell-1} &= \ddot{\mathbf{v}}_{\ell-1}(z)\mathbf{A}_{2\ell-1} - \delta\ddot{\mathbf{x}}_{\ell-2}(z)\mathbf{A}_{2\ell-2}\mathbf{A}_{2\ell-1}, && (8.30)\\ \ddot{\mathbf{v}}_{\ell-1}(z)\mathbf{B}_{2\ell-1} &= \delta\ddot{\mathbf{x}}_{\ell-2}(z)\mathbf{A}_{2\ell-2}\mathbf{B}_{2\ell-1} + \ddot{\mathbf{t}}_{\ell-1}(z)\mathbf{B}_{2\ell-1}, && (8.31)\\ (z-\alpha)\ddot{\mathbf{t}}_{\ell-1}(z)\mathbf{A}_{2\ell-1} &= \delta\ddot{\mathbf{r}}_{\ell-1}(z)\mathbf{B}_{2\ell-2}\mathbf{A}_{2\ell-1} - (\beta - z)\ddot{\mathbf{v}}_{\ell-1}(z)\mathbf{A}_{2\ell-1}, && (8.32)\end{aligned}$$

and

$$-(\beta - z)\ddot{\mathbf{v}}_{\ell-1}(z)\mathbf{B}_{2\ell-1} = (z-\alpha)\ddot{\mathbf{t}}_{\ell-1}(z)\mathbf{B}_{2\ell-1} - \delta\ddot{\mathbf{r}}_{\ell-1}(z)\mathbf{B}_{2\ell-2}\mathbf{B}_{2\ell-1}. \tag{8.33}$$

Keeping Lemma 8.6 in mind, inserting (8.30) directly into (8.14) delivers the first equation in (8.23), and inserting instead (8.31) directly into (8.14) shows analogously the second one. Combining (8.15) and (8.32) provides (8.24). Inserting (8.33) into (8.15) implies similarly (8.25). □

From Lemma 8.12 we get for every of the four systems of matrix polynomials 3-term recurrence relations:

**Lemma 8.13** *Suppose $\kappa \geq 4$. Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices.*

(a) *If* $\kappa \geq 5$*, for every choice of* $\ell \in \mathbb{Z}_{3,\infty}$ *such that* $2\ell - 1 \leq \kappa$ *and all* $z \in \mathbb{C}$*, then*

$$\begin{aligned}\ddot{\mathbf{r}}_{\ell}(z) &= \ddot{\mathbf{r}}_{\ell-1}(z)(zI_q - [\alpha I_q + \delta(\mathbf{A}_{2\ell-3}\mathbf{A}_{2\ell-2} + \mathbf{B}_{2\ell-2}\mathbf{B}_{2\ell-1})]) \\ &\quad - \ddot{\mathbf{r}}_{\ell-2}(z)(\delta^2\mathbf{B}_{2\ell-4}\mathbf{B}_{2\ell-3}\mathbf{A}_{2\ell-3}\mathbf{A}_{2\ell-2}) \\ &= \ddot{\mathbf{r}}_{\ell-1}(z)(zI_q - [\beta I_q - \delta(\mathbf{B}_{2\ell-3}\mathbf{A}_{2\ell-2} + \mathbf{B}_{2\ell-2}\mathbf{A}_{2\ell-1})]) \\ &\quad - \ddot{\mathbf{r}}_{\ell-2}(z)(\delta^2\mathbf{B}_{2\ell-4}\mathbf{A}_{2\ell-3}\mathbf{B}_{2\ell-3}\mathbf{A}_{2\ell-2}).\end{aligned}$$

(b) *For all* $\ell \in \mathbb{Z}_{2,\infty}$ *such that* $2\ell \leq \kappa$ *and all* $z \in \mathbb{C}$*, moreover*

$$\begin{aligned}\ddot{\mathbf{t}}_{\ell}(z) &= \ddot{\mathbf{t}}_{\ell-1}(z)(zI_q - [\alpha I_q + \delta(\mathbf{B}_{2\ell-2}\mathbf{B}_{2\ell-1} + \mathbf{A}_{2\ell-1}\mathbf{A}_{2\ell})]) \\ &\quad -\ddot{\mathbf{t}}_{\ell-2}(z)(\delta^2\mathbf{A}_{2\ell-3}\mathbf{A}_{2\ell-2}\mathbf{B}_{2\ell-2}\mathbf{B}_{2\ell-1})\end{aligned}$$

*and*

$$\begin{aligned}\ddot{\mathbf{v}}_{\ell}(z) &= \ddot{\mathbf{v}}_{\ell-1}(z)(zI_q - [\beta I_q - \delta(\mathbf{B}_{2\ell-2}\mathbf{A}_{2\ell-1} + \mathbf{B}_{2\ell-1}\mathbf{A}_{2\ell})]) \\ &\quad -\ddot{\mathbf{v}}_{\ell-2}(z)(\delta^2\mathbf{B}_{2\ell-3}\mathbf{A}_{2\ell-2}\mathbf{B}_{2\ell-2}\mathbf{A}_{2\ell-1}).\end{aligned}$$

(c) *If* $\kappa \geq 5$*, then, for all* $\ell \in \mathbb{Z}_{2,\infty}$ *such that* $2\ell + 1 \leq \kappa$ *and all* $z \in \mathbb{C}$*, moreover*

$$\begin{aligned}\ddot{\mathbf{x}}_{\ell}(z) &= \ddot{\mathbf{x}}_{\ell-1}(z)(zI_q - [\alpha I_q + \delta(\mathbf{B}_{2\ell-1}\mathbf{B}_{2\ell} + \mathbf{A}_{2\ell}\mathbf{A}_{2\ell+1})]) \\ &\quad -\ddot{\mathbf{x}}_{\ell-2}(z)(\delta^2\mathbf{A}_{2\ell-2}\mathbf{A}_{2\ell-1}\mathbf{B}_{2\ell-1}\mathbf{B}_{2\ell})\end{aligned}$$

*and*

$$\begin{aligned}\ddot{\mathbf{x}}_{\ell}(z) &= \ddot{\mathbf{x}}_{\ell-1}(z)(zI_q - [\beta I_q - \delta(\mathbf{A}_{2\ell-1}\mathbf{B}_{2\ell} + \mathbf{A}_{2\ell}\mathbf{B}_{2\ell+1})]) \\ &\quad -\ddot{\mathbf{x}}_{\ell-2}(z)(\delta^2\mathbf{A}_{2\ell-2}\mathbf{B}_{2\ell-1}\mathbf{A}_{2\ell-1}\mathbf{B}_{2\ell}).\end{aligned}$$

***Proof*** Let $z \in \mathbb{C}$. In order to prove part (a), we suppose $\kappa \geq 5$. Assume that $\ell \in \mathbb{Z}_{3,\infty}$ is such that $2\ell - 1 \leq \kappa$. According to Lemma 8.12, we know that (8.24) and (8.25) as well as

$$\ddot{\mathbf{v}}_{\ell-1}(z) = \ddot{\mathbf{r}}_{\ell-1}(z) + \ddot{\mathbf{v}}_{\ell-2}(z)(\delta\mathbf{B}_{2\ell-3}\mathbf{A}_{2\ell-2}), \tag{8.34}$$

$$\ddot{\mathbf{t}}_{\ell-1}(z) = \ddot{\mathbf{r}}_{\ell-1}(z) - \ddot{\mathbf{t}}_{\ell-2}(z)(\delta\mathbf{A}_{2\ell-3}\mathbf{A}_{2\ell-2}), \tag{8.35}$$

and

$$\begin{aligned}\ddot{\mathbf{r}}_{\ell-1}(z) &= -(\beta - z)\ddot{\mathbf{v}}_{\ell-2}(z) + \ddot{\mathbf{r}}_{\ell-2}(z)(\delta\mathbf{B}_{2\ell-4}\mathbf{A}_{2\ell-3}) \\ &= (z - \alpha)\ddot{\mathbf{t}}_{\ell-2}(z) - \ddot{\mathbf{r}}_{\ell-2}(z)(\delta\mathbf{B}_{2\ell-4}\mathbf{B}_{2\ell-3})\end{aligned} \tag{8.36}$$

hold true. From (8.36) we get

$$\begin{aligned}-(\beta - z)\ddot{\mathbf{v}}_{\ell-2}(z)(\delta\mathbf{B}_{2\ell-3}\mathbf{A}_{2\ell-2}) &= \ddot{\mathbf{r}}_{\ell-1}(z)(\delta\mathbf{B}_{2\ell-3}\mathbf{A}_{2\ell-2}) \\ &\quad -\ddot{\mathbf{r}}_{\ell-2}(z)(\delta\mathbf{B}_{2\ell-4}\mathbf{A}_{2\ell-3})(\delta\mathbf{B}_{2\ell-3}\mathbf{A}_{2\ell-2}),\end{aligned} \tag{8.37}$$

and

$$(z-\alpha)\ddot{\mathbf{t}}_{\ell-2}(z)(\delta\mathbf{A}_{2\ell-3}\mathbf{A}_{2\ell-2})$$
$$=\ddot{\mathbf{r}}_{\ell-1}(z)(\delta\mathbf{A}_{2\ell-3}\mathbf{A}_{2\ell-2})+\ddot{\mathbf{r}}_{\ell-2}(z)(\delta\mathbf{B}_{2\ell-4}\mathbf{B}_{2\ell-3})(\delta\mathbf{A}_{2\ell-3}\mathbf{A}_{2\ell-2}). \quad (8.38)$$

In view of (8.24), (8.34), and (8.37), consequently

$$\begin{aligned}\ddot{\mathbf{r}}_\ell(z) &= -(\beta-z)[\ddot{\mathbf{r}}_{\ell-1}(z)+\ddot{\mathbf{v}}_{\ell-2}(z)(\delta\mathbf{B}_{2\ell-3}\mathbf{A}_{2\ell-2})]+\ddot{\mathbf{r}}_{\ell-1}(z)(\delta\mathbf{B}_{2\ell-2}\mathbf{A}_{2\ell-1})\\ &= \ddot{\mathbf{r}}_{\ell-1}(z)(zI_q-[\beta I_q-\delta(\mathbf{B}_{2\ell-3}\mathbf{A}_{2\ell-2}+\mathbf{B}_{2\ell-2}\mathbf{A}_{2\ell-1})])\\ &\quad -\ddot{\mathbf{r}}_{\ell-2}(z)(\delta^2\mathbf{B}_{2\ell-4}\mathbf{A}_{2\ell-3}\mathbf{B}_{2\ell-3}\mathbf{A}_{2\ell-2}),\end{aligned}$$

whereas (8.25), (8.35), and (8.38) yield

$$\begin{aligned}\ddot{\mathbf{r}}_\ell(z) &= (z-\alpha)[\ddot{\mathbf{r}}_{\ell-1}(z)-\ddot{\mathbf{t}}_{\ell-2}(z)(\delta\mathbf{A}_{2\ell-3}\mathbf{A}_{2\ell-2})]-\ddot{\mathbf{r}}_{\ell-1}(z)(\delta\mathbf{B}_{2\ell-2}\mathbf{B}_{2\ell-1})\\ &= \ddot{\mathbf{r}}_{\ell-1}(z)(zI_q-[\alpha I_q+\delta(\mathbf{A}_{2\ell-3}\mathbf{A}_{2\ell-2}+\mathbf{B}_{2\ell-2}\mathbf{B}_{2\ell-1})])\\ &\quad -\ddot{\mathbf{r}}_{\ell-2}(z)(\delta^2\mathbf{B}_{2\ell-4}\mathbf{B}_{2\ell-3}\mathbf{A}_{2\ell-3}\mathbf{A}_{2\ell-2}).\end{aligned}$$

Thus part (a) is proved. Using Lemma 8.12, parts (b) and (c) can be checked analogously. □

For the special case that $\alpha = 0$ and $\beta = 1$ hold true, in [9, Thm. 4.1], it is shown that under certain regularity assumptions and in view of (8.5), the specific structure of the recursion coefficients from Lemma 8.13 is both necessary and sufficient for the existence of a matrix-valued measure concentrated on the interval [0, 1] for which the recursive constructed matrix polynomials are orthogonal. In accordance with that, it is now evident that the corresponding matrix polynomials given in Remark 8.9 are monic right orthogonal systems, whereby the sequences introduced in Notation 3.3 occur again. We continue to use the notation introduced in (7.4), in order to formulate the main result of this section:

**Theorem 8.14** *Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha < \beta$, let $\tau \in \mathbb{Z}_{2,\infty}$, and let $(s_j)_{j=0}^{\tau} \in \mathcal{F}^{\succcurlyeq}_{q,\tau,\alpha,\beta}$. Then the sequence $(\ddot{\mathbf{r}}_\ell)_{\ell=0}^{\langle\tau]}$ forms an MROS of matrix polynomials with respect to $(s_j)_{j=0}^{2\langle\tau]}$, the sequence $(\ddot{\mathbf{t}}_\ell)_{\ell=0}^{\langle\tau-1]}$ forms an MROS of matrix polynomials with respect to $(a_j)_{j=0}^{2\langle\tau-1]}$, the sequence $(\ddot{\mathbf{v}}_\ell)_{\ell=0}^{\langle\tau-1]}$ forms an MROS of matrix polynomials with respect to $(b_j)_{j=0}^{2\langle\tau-1]}$, and the sequence $(\ddot{\mathbf{x}}_\ell)_{\ell=0}^{\langle\tau-2]}$ forms an MROS of matrix polynomials with respect to $(c_j)_{j=0}^{2\langle\tau-2]}$.*

***Proof*** Let $(\mathfrak{f}_j)_{j=0}^{2\tau}$ be the $\mathcal{F}_{\alpha,\beta}$-parameter sequence of $(s_j)_{j=0}^{\tau}$ and let $(\mathsf{D}_j)_{j=0}^{\tau}$ be the sequence of $[\alpha, \beta]$-interval lengths associated with $(s_j)_{j=0}^{\tau}$. Now our proof is divided into two parts.

**Part 1:** First we discuss the case that $\tau = 2\rho + 2$ with some $\rho \in \mathbb{N}_0 \cup \{\infty\}$. Then we first observe that (7.4) yields $\langle\tau] = \rho + 1$ and $\langle\tau - 1] = \rho$ as well as $\langle\tau - 2] = \rho$. In view of Remark 8.9, from Remark 8.8 we easily see that, for all $\ell \in \mathbb{Z}_{0,\rho+1}$, the function $\ddot{\mathbf{r}}_\ell$ is a matrix polynomial of degree $\ell$ with leading coefficient matrix $I_q$ and, for all $\ell \in \mathbb{Z}_{0,\rho}$, the functions $\ddot{\mathbf{t}}_\ell$, $\ddot{\mathbf{v}}_\ell$, and $\ddot{\mathbf{x}}_\ell$ are all complex matrix polynomials of degree $\ell$ with leading coefficient matrix $I_q$. In particular, for each $\ell \in \mathbb{Z}_{1,\rho+1}$, there is an $r_\ell \in \mathbb{C}^{\ell q \times q}$ such that the block representation

$$Y_\ell(\ddot{\mathbf{r}}_\ell) = \begin{bmatrix} -r_\ell \\ I_q \end{bmatrix} \tag{8.39}$$

is valid, where we used the notation given in (6.1). Because of Lemmata 8.10 and 8.6, we obtain $\ddot{\mathbf{r}}_1(z) = z(\mathbf{A}_1 + \mathbf{B}_1) - (\alpha\mathbf{A}_1 + \beta\mathbf{B}_1) = zI_q - (\alpha\mathbf{A}_1 + \beta\mathbf{B}_1)$ for all $z \in \mathbb{C}$. Hence, $r_1 = \alpha\mathbf{A}_1 + \beta\mathbf{B}_1$. From Notation 5.2, (5.9), and (3.1) we get $-\mathfrak{f}_1 + y_{1,1} = -\mathfrak{f}_1 + s_1 = \alpha s_0 = \alpha H_0$ and $\mathfrak{f}_2 + y_{1,1} = \mathfrak{f}_2 + s_1 = \beta s_0 = \beta H_0$. Thus, in view of Lemma 8.7 and $\delta \neq 0$ as well as Remark 8.6, we can conclude

$$\begin{aligned} H_0 r_1 &= \alpha H_0 \mathbf{A}_1 + \beta H_0 \mathbf{B}_1 \\ &= (-\mathfrak{f}_1\mathbf{A}_1 + \mathfrak{f}_2\mathbf{B}_1) + y_{1,1}(\mathbf{A}_1 + \mathbf{B}_1) = \delta^{-1}(-\mathsf{D}_1 + \mathsf{D}_1) + y_{1,1}I_q = y_{1,1}. \end{aligned} \tag{8.40}$$

Now we consider the case $\rho \geq 1$. For all $\ell \in \mathbb{Z}_{1,\rho}$, then there are matrices $t_\ell$, $v_\ell$, $x_\ell \in \mathbb{C}^{\ell q \times q}$ such that block representations

$$Y_\ell(\ddot{\mathbf{t}}_\ell) = \begin{bmatrix} -t_\ell \\ I_q \end{bmatrix}, \qquad Y_\ell(\ddot{\mathbf{v}}_\ell) = \begin{bmatrix} -v_\ell \\ I_q \end{bmatrix}, \qquad \text{and} \qquad Y_\ell(\ddot{\mathbf{x}}_\ell) = \begin{bmatrix} -x_\ell \\ I_q \end{bmatrix} \tag{8.41}$$

are valid. We are going to prove inductively that

(I$_\ell$) $H^{\langle a\rangle}_{\ell-1} t_\ell = y^{\langle a\rangle}_{\ell,2\ell-1}$, $H^{\langle b\rangle}_{\ell-1} v_\ell = y^{\langle b\rangle}_{\ell,2\ell-1}$, $H^{\langle c\rangle}_{\ell-1} x_\ell = y^{\langle c\rangle}_{\ell,2\ell-1}$, and $H_\ell r_{\ell+1} = y_{\ell+1,2\ell+1}$ hold true for all $\ell \in \mathbb{Z}_{1,\rho}$.

According to Lemma 8.10, the equations in (8.16) and (8.17) as well as

$$\ddot{\mathbf{x}}_\ell(z) = \ddot{\mathbf{t}}_\ell(z)\mathbf{A}_{2\ell+1} + \ddot{\mathbf{v}}_\ell(z)\mathbf{B}_{2\ell+1} \tag{8.42}$$

and

$$\ddot{\mathbf{r}}_{\ell+1}(z) = (z-\alpha)\ddot{\mathbf{t}}_\ell(z)\mathbf{A}_{2\ell+1} - (\beta - z)\ddot{\mathbf{v}}_\ell(z)\mathbf{B}_{2\ell+1} \tag{8.43}$$

are valid for all $\ell \in \mathbb{Z}_{1,\rho}$ and all $z \in \mathbb{C}$. In particular, from (8.16), (8.17), (8.39), (8.41), (8.42), and (8.43), we get

$$-t_1 = -\beta\mathbf{A}_2 - r_1\mathbf{B}_2, \quad -v_1 = -\alpha\mathbf{A}_2 - r_1\mathbf{B}_2, \quad -x_1 = -t_1\mathbf{A}_3 - v_1\mathbf{B}_3 \tag{8.44}$$

and

$$-r_2 = \left(\begin{bmatrix} O_{q\times q} \\ -t_1 \end{bmatrix} - \alpha\begin{bmatrix} -t_1 \\ I_q \end{bmatrix}\right)\mathbf{A}_3 - \left(\beta\begin{bmatrix} -v_1 \\ I_q \end{bmatrix} - \begin{bmatrix} O_{q\times q} \\ -v_1 \end{bmatrix}\right)\mathbf{B}_3. \tag{8.45}$$

In view of (5.10) as well as Notations 5.2, 3.3 and 3.4, we obtain

$$\mathfrak{f}_3 + y_{1,1}^{\langle a\rangle} = c_0 + a_1 = \beta a_0 = \beta H_0^{\langle a\rangle}, \quad -\mathfrak{f}_3 + y_{1,1}^{\langle b\rangle} = -c_0 + b_1 = \alpha b_0 = \alpha H_0^{\langle b\rangle}. \tag{8.46}$$

According to Remark 5.3, for all $\ell \in \mathbb{Z}_{1,\rho}$, we have $\left[\begin{smallmatrix} * & * \\ K_{\ell-1} & y_{\ell+1,2\ell}\end{smallmatrix}\right] = H_\ell = \left[\begin{smallmatrix} H_{\ell-1} & y_{\ell,2\ell-1} \\ * & * \end{smallmatrix}\right]$, consequently, $([O_{\ell q\times q}, I_{\ell q}] - \alpha[I_{\ell q}, O_{\ell q\times q}])H_\ell = [K_{\ell-1}, y_{\ell+1,2\ell}] - \alpha[H_{\ell-1}, y_{\ell,2\ell-1}] = [H_{\ell-1}^{\langle a\rangle}, y_{\ell,2\ell-1}^{\langle a\rangle}]$ and, analogously, $(\beta[I_{\ell q}, O_{\ell q\times q}] - [O_{\ell q\times q}, I_{\ell q}])H_\ell = [H_{\ell-1}^{\langle b\rangle}, y_{\ell,2\ell-1}^{\langle b\rangle}]$. For all $\ell \in \mathbb{Z}_{1,\rho}$, therefore,

$$\begin{aligned} H_{\ell-1}^{\langle a\rangle} r_\ell &= -[H_{\ell-1}^{\langle a\rangle}, y_{\ell,2\ell-1}^{\langle a\rangle}] \begin{bmatrix} -r_\ell \\ I_q \end{bmatrix} + y_{\ell,2\ell-1}^{\langle a\rangle} \\ &= (\alpha[I_{\ell q}, O_{\ell q\times q}] - [O_{\ell q\times q}, I_{\ell q}])H_\ell \begin{bmatrix} -r_\ell \\ I_q \end{bmatrix} + y_{\ell,2\ell-1}^{\langle a\rangle} \end{aligned} \tag{8.47}$$

and, analogously,

$$H_{\ell-1}^{\langle b\rangle} r_\ell = -(\beta[I_{\ell q}, O_{\ell q\times q}] - [O_{\ell q\times q}, I_{\ell q}])H_\ell \begin{bmatrix} -r_\ell \\ I_q \end{bmatrix} + y_{\ell,2\ell-1}^{\langle b\rangle}. \tag{8.48}$$

Since $(s_j)_{j=0}^{2\rho+2}$ belongs to $\mathcal{F}_{q,2\rho+2,\alpha,\beta}^{\succcurlyeq}$, from (5.2) and (5.6) we see that $(s_j)_{j=0}^{2\rho+2}$ belongs to $\mathcal{H}_{q,2\rho+2}^{\succcurlyeq}$. Thus, because of Remark 5.4, furthermore $(s_j)_{j=0}^{2\rho+2} \in \mathcal{H}_{q\times q,2\rho+2}^{\mathrm{r}}$. Consequently, $\mathcal{R}(y_{1,1}) \subseteq \mathcal{R}(H_0)$ and $\mathcal{N}(H_0) \subseteq \mathcal{N}(z_{1,1})$. In view of Definition 5.19, we have $\mathfrak{f}_4 = \mathfrak{A}_2$. Hence, from (8.40), Lemma A.10, and Remark 5.18 we get $H_1\left[\begin{smallmatrix} -r_1 \\ I_q \end{smallmatrix}\right] = \left[\begin{smallmatrix} O_{q\times q} \\ \mathfrak{f}_4 \end{smallmatrix}\right]$. With (8.47) it follows then

$$H_0^{\langle a\rangle} r_1 = (\alpha[I_q, O_{q\times q}] - [O_{q\times q}, I_q]) \begin{bmatrix} O_{q\times q} \\ \mathfrak{f}_4 \end{bmatrix} + y_{1,1}^{\langle a\rangle} = -\mathfrak{f}_4 + y_{1,1}^{\langle a\rangle}, \tag{8.49}$$

whereas (8.48) yields analogously $H_0^{\langle b\rangle} r_1 = \mathfrak{f}_4 + y_{1,1}^{\langle b\rangle}$. Inserting (8.46) and (8.49) into (8.44), by using Lemmata 8.7 and 8.6, we obtain that

$$\begin{aligned} H_0^{\langle a\rangle} t_1 &= \beta H_0^{\langle a\rangle} \mathbf{A}_2 + H_0^{\langle a\rangle} r_1 \mathbf{B}_2 = (\mathfrak{f}_3 + y_{1,1}^{\langle a\rangle})\mathbf{A}_2 + (-\mathfrak{f}_4 + y_{1,1}^{\langle a\rangle})\mathbf{B}_2 \\ &= (\mathfrak{f}_3\mathbf{A}_2 - \mathfrak{f}_4\mathbf{B}_2) + y_{1,1}^{\langle a\rangle}(\mathbf{A}_2 + \mathbf{B}_2) = y_{1,1}^{\langle a\rangle}. \end{aligned} \tag{8.50}$$

From the second equation in (8.44), and Lemmata 8.7 and 8.6, we conclude analogously that

$$H_0^{\langle b\rangle} v_1 = y_{1,1}^{\langle b\rangle}. \tag{8.51}$$

Since, because of Notation 3.3, for all $\ell \in \mathbb{Z}_{1,\rho+1}$, obviously

$$H_{\ell-1}^{\langle c\rangle} = -\alpha H_{\ell-1}^{\langle b\rangle} + K_{\ell-1}^{\langle b\rangle} \qquad \text{and} \qquad H_{\ell-1}^{\langle c\rangle} = \beta H_{\ell-1}^{\langle a\rangle} - K_{\ell-1}^{\langle a\rangle} \tag{8.52}$$

are valid, one can verify analogous to (8.47) that the equations

$$H_{\ell-1}^{\langle c\rangle} v_\ell = (\alpha[I_{\ell q}, O_{\ell q\times q}] - [O_{\ell q\times q}, I_{\ell q}]) H_\ell^{\langle b\rangle} \begin{bmatrix} -v_\ell \\ I_q \end{bmatrix} + y_{\ell,2\ell-1}^{\langle c\rangle} \tag{8.53}$$

and

$$H_{\ell-1}^{\langle c\rangle} t_\ell = ([O_{\ell q\times q}, I_{\ell q}] - \beta[I_{\ell q}, O_{\ell q\times q}]) H_\ell^{\langle a\rangle} \begin{bmatrix} -t_\ell \\ I_q \end{bmatrix} + y_{\ell,2\ell-1}^{\langle c\rangle} \tag{8.54}$$

hold true for all $\ell \in \mathbb{Z}_{1,\rho}$. Because of Proposition 5.10 as well as (5.4) and (5.6), the sequences $(a_j)_{j=0}^{2\rho}$ and $(b_j)_{j=0}^{2\rho}$ both belong to $\mathcal{H}_{q,2\rho}^{\succcurlyeq}$, and, in view of Remark 5.4, to $\mathcal{H}_{q\times q,2\rho}^{\mathrm{r}}$ as well. In particular, we have $\mathcal{R}(y_{1,1}^{\langle a\rangle}) \subseteq \mathcal{R}(H_0^{\langle a\rangle})$ and $\mathcal{N}(H_0^{\langle a\rangle}) \subseteq \mathcal{N}(z_{1,1}^{\langle a\rangle})$ as well as $\mathcal{R}(y_{1,1}^{\langle b\rangle}) \subseteq \mathcal{R}(H_0^{\langle b\rangle})$ and $\mathcal{N}(H_0^{\langle b\rangle}) \subseteq \mathcal{N}(z_{1,1}^{\langle b\rangle})$. Since Definition 5.19 yields $\mathfrak{f}_5 = \mathfrak{A}_3$ and $\mathfrak{f}_6 = \mathfrak{B}_3$, from (8.50) and (8.51) we can see then by applying Lemma A.10 and Remark 5.18 that

$$H_1^{\langle a\rangle} \begin{bmatrix} -t_1 \\ I_q \end{bmatrix} = \begin{bmatrix} O_{q\times q} \\ \mathfrak{f}_5 \end{bmatrix} \qquad \text{and} \qquad H_1^{\langle b\rangle} \begin{bmatrix} -v_1 \\ I_q \end{bmatrix} = \begin{bmatrix} O_{q\times q} \\ \mathfrak{f}_6 \end{bmatrix}. \tag{8.55}$$

Inserting the first equation of (8.55) in (8.54) for $\ell = 1$ yields to

$$H_0^{\langle c\rangle} t_1 = ([O_{q\times q}, I_q] - \beta[I_q, O_{q\times q}]) \begin{bmatrix} O_{q\times q} \\ \mathfrak{f}_5 \end{bmatrix} + y_{1,1}^{\langle c\rangle} = \mathfrak{f}_5 + y_{1,1}^{\langle c\rangle}.$$

Analogously, inserting the second equation of (8.55) into (8.53) for $\ell = 1$ shows $H_0^{\langle c\rangle} v_1 = -\mathfrak{f}_6 + y_{1,1}^{\langle c\rangle}$. By multiplying the third equation in (8.44) from the left by $-H_0^{\langle c\rangle}$ and using the recently shown identities, we get by additional application of Lemmata 8.7 and 8.6 moreover

$$H_0^{\langle c\rangle} x_1 = H_0^{\langle c\rangle} t_1 \mathbf{A}_3 + H_0^{\langle c\rangle} v_1 \mathbf{B}_3 = (\mathfrak{f}_5 + y_{1,1}^{\langle c\rangle})\mathbf{A}_3 + (-\mathfrak{f}_6 + y_{1,1}^{\langle c\rangle})\mathbf{B}_3 = y_{1,1}^{\langle c\rangle}. \tag{8.56}$$

For all $\ell \in \mathbb{Z}_{0,\rho}$, since $[H_\ell, y_{\ell+1,2\ell+1}] = [y_{0,\ell}, K_\ell]$ is valid, we know from Remark 5.3 that

$$\begin{aligned} H_\ell \begin{bmatrix} O_{q\times q} \\ -t_\ell \end{bmatrix} &= [H_\ell, y_{\ell+1,2\ell+1}] \begin{bmatrix} O_{q\times q} \\ -t_\ell \\ I_q \end{bmatrix} - y_{\ell+1,2\ell+1} \\ &= K_\ell \begin{bmatrix} -t_\ell \\ I_q \end{bmatrix} - y_{\ell+1,2\ell+1} \end{aligned} \tag{8.57}$$

is fulfilled and, analogously, that

$$H_\ell \begin{bmatrix} O_{q\times q} \\ -v_\ell \end{bmatrix} = K_\ell \begin{bmatrix} -v_\ell \\ I_q \end{bmatrix} - y_{\ell+1,2\ell+1} \tag{8.58}$$

holds true. Since, for all $\ell \in \mathbb{Z}_{0,\rho}$, according to Notation 3.3, moreover $H_\ell^{\langle a\rangle} = -\alpha H_\ell + K_\ell$ and $H_\ell^{\langle b\rangle} = \beta H_\ell - K_\ell$ are valid, we get by applying (8.57) that

$$H_\ell \begin{bmatrix} O_{q\times q} \\ -t_\ell \end{bmatrix} - \alpha H_\ell \begin{bmatrix} -t_\ell \\ I_q \end{bmatrix} = (K_\ell - \alpha H_\ell) \begin{bmatrix} -t_\ell \\ I_q \end{bmatrix} - y_{\ell+1,2\ell+1}$$
$$= H_\ell^{\langle a\rangle} \begin{bmatrix} -t_\ell \\ I_q \end{bmatrix} - y_{\ell+1,2\ell+1} \tag{8.59}$$

and, by applying (8.58), similarly that

$$\beta H_\ell \begin{bmatrix} -v_\ell \\ I_q \end{bmatrix} - H_\ell \begin{bmatrix} O_{q\times q} \\ -v_\ell \end{bmatrix} = (\beta H_\ell - K_\ell) \begin{bmatrix} -v_\ell \\ I_q \end{bmatrix} + y_{\ell+1,2\ell+1}$$
$$= H_\ell^{\langle b\rangle} \begin{bmatrix} -v_\ell \\ I_q \end{bmatrix} + y_{\ell+1,2\ell+1}. \tag{8.60}$$

Looking at the case $\ell = 1$ and using (8.55) yields $H_1\left[\begin{smallmatrix} O_{q\times q} \\ -t_1 \end{smallmatrix}\right] - \alpha H_1\left[\begin{smallmatrix} -t_1 \\ I_q \end{smallmatrix}\right] = \left[\begin{smallmatrix} O_{q\times q} \\ \mathfrak{f}_5 \end{smallmatrix}\right] - y_{2,3}$ and $\beta H_1\left[\begin{smallmatrix} -v_1 \\ I_q \end{smallmatrix}\right] - H_1\left[\begin{smallmatrix} O_{q\times q} \\ -v_1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} O_{q\times q} \\ \mathfrak{f}_6 \end{smallmatrix}\right] + y_{2,3}$. Consequently, multiplying (8.45) from the left by $-H_1$ and using additionally Lemmata 8.7 and 8.6 moreover

$$\begin{aligned} H_1 r_2 &= -\left(H_1 \begin{bmatrix} O_{q\times q} \\ -t_1 \end{bmatrix} - \alpha H_1 \begin{bmatrix} -t_1 \\ I_q \end{bmatrix}\right) \mathbf{A}_3 + \left(\beta H_1 \begin{bmatrix} -v_1 \\ I_q \end{bmatrix} - H_1 \begin{bmatrix} O_{q\times q} \\ -v_1 \end{bmatrix}\right) \mathbf{B}_3 \\ &= -\left(\begin{bmatrix} O_{q\times q} \\ \mathfrak{f}_5 \end{bmatrix} - y_{2,3}\right) \mathbf{A}_3 + \left(\begin{bmatrix} O_{q\times q} \\ \mathfrak{f}_6 \end{bmatrix} + y_{2,3}\right) \mathbf{B}_3 \\ &= \begin{bmatrix} O_{q\times q} \\ -\mathfrak{f}_5 \mathbf{A}_3 + \mathfrak{f}_6 \mathbf{B}_3 \end{bmatrix} + y_{2,3}(\mathbf{A}_3 + \mathbf{B}_3) = y_{2,3}. \end{aligned} \tag{8.61}$$

Finally, using (8.50), (8.51), (8.56), and (8.61) shows ($\mathrm{I}_1$).

Now let $\rho \geq 2$ and we assume that ($\mathrm{I}_{\ell-1}$) is true for some $\ell \in \mathbb{Z}_{2,\rho}$. We are going to verify ($\mathrm{I}_\ell$). In view of (8.16), (8.17), (8.39), (8.41), (8.42), and (8.43), we have

$$-t_\ell = -\left(\beta \begin{bmatrix} -x_{\ell-1} \\ I_q \end{bmatrix} - \begin{bmatrix} O_{q\times q} \\ -x_{\ell-1} \end{bmatrix}\right) \mathbf{A}_{2\ell} - r_\ell \mathbf{B}_{2\ell}, \tag{8.62}$$

$$-v_\ell = \left(\begin{bmatrix} O_{q\times q} \\ -x_{\ell-1} \end{bmatrix} - \alpha \begin{bmatrix} -x_{\ell-1} \\ I_q \end{bmatrix}\right) \mathbf{A}_{2\ell} - r_\ell \mathbf{B}_{2\ell},$$

$$-x_\ell = -t_\ell \mathbf{A}_{2\ell+1} - v_\ell \mathbf{B}_{2\ell+1}, \tag{8.63}$$

and

$$-r_{\ell+1} = \left(\begin{bmatrix} O_{q\times q} \\ -t_\ell \end{bmatrix} - \alpha \begin{bmatrix} -t_\ell \\ I_q \end{bmatrix}\right) \mathbf{A}_{2\ell+1} - \left(\beta \begin{bmatrix} -v_\ell \\ I_q \end{bmatrix} - \begin{bmatrix} O_{q\times q} \\ -v_\ell \end{bmatrix}\right) \mathbf{B}_{2\ell+1}. \tag{8.64}$$

Since $[H^{\langle a\rangle}_{\ell-1}, y^{\langle a\rangle}_{\ell,2\ell-1}] = [y^{\langle a\rangle}_{0,\ell-1}, K^{\langle a\rangle}_{\ell-1}]$ as well as $[H^{\langle b\rangle}_{\ell-1}, y^{\langle b\rangle}_{\ell,2\ell-1}] = [y^{\langle b\rangle}_{0,\ell-1}, K^{\langle b\rangle}_{\ell-1}]$ are valid, we get, analogous to (8.57) and (8.58), the identities

$$H^{\langle a\rangle}_{\ell-1}\begin{bmatrix} O_{q\times q} \\ -x_{\ell-1}\end{bmatrix} = K^{\langle a\rangle}_{\ell-1}\begin{bmatrix} -x_{\ell-1} \\ I_q\end{bmatrix} - y^{\langle a\rangle}_{\ell,2\ell-1} \tag{8.65}$$

and

$$H^{\langle b\rangle}_{\ell-1}\begin{bmatrix} O_{q\times q} \\ -x_{\ell-1}\end{bmatrix} = K^{\langle b\rangle}_{\ell-1}\begin{bmatrix} -x_{\ell-1} \\ I_q\end{bmatrix} - y^{\langle b\rangle}_{\ell,2\ell-1}. \tag{8.66}$$

Because of Proposition 5.10 and the formulas (5.2) and (5.6), the sequence $(c_j)_{j=0}^{2\rho}$ belongs to $\mathcal{H}^{\succcurlyeq}_{q,2\rho}$, and, in view of Remark 5.4, in particular to $\mathcal{H}^{\mathrm{r}}_{q\times q,2\rho}$ as well. Thus, $\mathcal{R}(y^{\langle c\rangle}_{\ell-1,2\ell-3}) \subseteq \mathcal{R}(H^{\langle c\rangle}_{\ell-2})$ and $\mathcal{N}(H^{\langle c\rangle}_{\ell-2}) \subseteq \mathcal{N}(z^{\langle c\rangle}_{\ell-1,2\ell-3})$. In view of Definition 5.19, we have $\mathfrak{f}_{4\ell-1} = \mathfrak{B}_{2\ell}$. Hence, using additionally the induction assumption ($\mathrm{I}_{\ell-1}$), we can see then, by applying Lemma A.10 and Remark 5.18, that $H^{\langle c\rangle}_{\ell-1}\left[\begin{smallmatrix} -x_{\ell-1} \\ I_q\end{smallmatrix}\right] = \left[\begin{smallmatrix} O_{(\ell-1)q\times q} \\ \mathfrak{f}_{4\ell-1}\end{smallmatrix}\right]$ holds true. Using additionally (8.65) and (8.52), then

$$\begin{aligned}\beta H^{\langle a\rangle}_{\ell-1}\begin{bmatrix} -x_{\ell-1} \\ I_q\end{bmatrix} - H^{\langle a\rangle}_{\ell-1}\begin{bmatrix} O_{q\times q} \\ -x_{\ell-1}\end{bmatrix} &= (\beta H^{\langle a\rangle}_{\ell-1} - K^{\langle a\rangle}_{\ell-1})\begin{bmatrix} -x_{\ell-1} \\ I_q\end{bmatrix} + y^{\langle a\rangle}_{\ell,2\ell-1} \\ = H^{\langle c\rangle}_{\ell-1}\begin{bmatrix} -x_{\ell-1} \\ I_q\end{bmatrix} + y_{\ell,2\ell-1} &= \begin{bmatrix} O_{(\ell-1)q\times q} \\ \mathfrak{f}_{4\ell-1}\end{bmatrix} + y^{\langle a\rangle}_{\ell,2\ell-1}\end{aligned} \tag{8.67}$$

and, in view of (8.66), moreover

$$\begin{aligned}H^{\langle b\rangle}_{\ell-1}\begin{bmatrix} O_{q\times q} \\ -x_{\ell-1}\end{bmatrix} - \alpha H^{\langle b\rangle}_{\ell-1}\begin{bmatrix} -x_{\ell-1} \\ I_q\end{bmatrix} &= (K^{\langle b\rangle}_{\ell-1} - \alpha H^{\langle b\rangle}_{\ell-1})\begin{bmatrix} -x_{\ell-1} \\ I_q\end{bmatrix} - y^{\langle b\rangle}_{\ell,2\ell-1} \\ = H^{\langle c\rangle}_{\ell-1}\begin{bmatrix} -x_{\ell-1} \\ I_q\end{bmatrix} - y_{\ell,2\ell-1} &= \begin{bmatrix} O_{(\ell-1)q\times q} \\ \mathfrak{f}_{4\ell-1}\end{bmatrix} - y^{\langle b\rangle}_{\ell,2\ell-1}\end{aligned} \tag{8.68}$$

follow. Because $(s_j)_{j=0}^{2\rho+2}$ belongs to $\mathcal{H}^{\mathrm{r}}_{q\times q,2\rho+2}$, we have $\mathcal{R}(y_{\ell,2\ell-1}) \subseteq \mathcal{R}(H_{\ell-1})$ and $\mathcal{N}(H_{\ell-1}) \subseteq \mathcal{N}(z_{\ell,2\ell-1})$. In view of $\mathfrak{f}_{4\ell} = \mathfrak{A}_{2\ell}$, Remark 5.3, ($\mathrm{I}_{\ell-1}$), Lemma A.10 and Remark 5.18 yield $H_\ell\left[\begin{smallmatrix} -r_\ell \\ I_q\end{smallmatrix}\right] = \left[\begin{smallmatrix} O_{\ell q\times q} \\ \mathfrak{f}_{4\ell}\end{smallmatrix}\right]$. Inserting the last equation directly into (8.47) delivers

$$\begin{aligned}H^{\langle a\rangle}_{\ell-1} r_\ell &= (\alpha[I_{\ell q}, O_{\ell q\times q}] - [O_{\ell q\times q}, I_{\ell q}])\begin{bmatrix} O_{\ell q\times q} \\ \mathfrak{f}_{4\ell}\end{bmatrix} + y^{\langle a\rangle}_{\ell,2\ell-1} \\ &= -\begin{bmatrix} O_{(\ell-1)q\times q} \\ \mathfrak{f}_{4\ell}\end{bmatrix} + y^{\langle a\rangle}_{\ell,2\ell-1}.\end{aligned} \tag{8.69}$$

In the same manner, taking into account (8.48), one gets

$$H^{\langle b\rangle}_{\ell-1} r_\ell = ([O_{\ell q\times q}, I_{\ell q}] - \beta [I_{\ell q}, O_{\ell q\times q}]) \begin{bmatrix} O_{\ell q\times q} \\ \mathfrak{f}_{4\ell} \end{bmatrix} + y^{\langle b\rangle}_{\ell,2\ell-1}$$
$$= \begin{bmatrix} O_{(\ell-1)q\times q} \\ \mathfrak{f}_{4\ell} \end{bmatrix} + y^{\langle b\rangle}_{\ell,2\ell-1}. \tag{8.70}$$

In view of (8.62), (8.67), and (8.69), Lemmata 8.7 and 8.6 provide

$$H^{\langle a\rangle}_{\ell-1} t_\ell = \left(\beta H^{\langle a\rangle}_{\ell-1} \begin{bmatrix} -x_{\ell-1} \\ I_q \end{bmatrix} - H^{\langle a\rangle}_{\ell-1} \begin{bmatrix} O_{q\times q} \\ -x_{\ell-1} \end{bmatrix}\right) \mathbf{A}_{2\ell} + H^{\langle a\rangle}_{\ell-1} r_\ell \mathbf{B}_{2\ell}$$
$$= \left(\begin{bmatrix} O_{(\ell-1)q\times q} \\ \mathfrak{f}_{4\ell-1} \end{bmatrix} + y^{\langle a\rangle}_{\ell,2\ell-1}\right) \mathbf{A}_{2\ell} + \left(-\begin{bmatrix} O_{(\ell-1)q\times q} \\ \mathfrak{f}_{4\ell} \end{bmatrix} + y^{\langle a\rangle}_{\ell,2\ell-1}\right) \mathbf{B}_{2\ell}$$
$$= \begin{bmatrix} O_{(\ell-1)q\times q} \\ \mathfrak{f}_{4\ell-1}\mathbf{A}_{2\ell} - \mathfrak{f}_{4\ell}\mathbf{B}_{2\ell} \end{bmatrix} + y^{\langle a\rangle}_{\ell,2\ell-1}(\mathbf{A}_{2\ell} + \mathbf{B}_{2\ell}) = y^{\langle a\rangle}_{\ell,2\ell-1}. \tag{8.71}$$

In view of (8.62), (8.68), and (8.70), Lemmata 8.7 and 8.6 yield analogously $H^{\langle b\rangle}_{\ell-1} v_\ell = y^{\langle b\rangle}_{\ell,2\ell-1}$. Since $(a_j)_{j=0}^{2\rho}$ belongs to $\mathcal{H}^{\mathrm{r}}_{q\times q,2\rho}$, we have $\mathcal{R}(y^{\langle a\rangle}_{\ell,2\ell-1}) \subseteq \mathcal{R}(H^{\langle a\rangle}_{\ell-1})$ and $\mathcal{N}(H^{\langle a\rangle}_{\ell-1}) \subseteq \mathcal{N}(z^{\langle a\rangle}_{\ell,2\ell-1})$. In view of Remark 5.3, $\mathfrak{f}_{4\ell+1} = \mathfrak{A}_{2\ell+1}$, (8.71), Lemma A.10, and Remark 5.18, we obtain then

$$H^{\langle a\rangle}_{\ell} \begin{bmatrix} -t_\ell \\ I_q \end{bmatrix} = \begin{bmatrix} O_{\ell q\times q} \\ \mathfrak{f}_{4\ell+1} \end{bmatrix}. \tag{8.72}$$

Inserting the last equation into (8.54) delivers

$$H^{\langle c\rangle}_{\ell-1} t_\ell = ([O_{\ell q\times q}, I_{\ell q}] - \beta [I_{\ell q}, O_{\ell q\times q}]) \begin{bmatrix} O_{\ell q\times q} \\ \mathfrak{f}_{4\ell+1} \end{bmatrix} + y^{\langle c\rangle}_{\ell,2\ell-1}$$
$$= \begin{bmatrix} O_{(\ell-1)q\times q} \\ \mathfrak{f}_{4\ell+1} \end{bmatrix} + y^{\langle c\rangle}_{\ell,2\ell-1}. \tag{8.73}$$

Since $(b_j)_{j=0}^{2\rho} \in \mathcal{H}^{\mathrm{r}}_{q\times q,2\rho}$ and $\mathfrak{f}_{4\ell+2} = \mathfrak{B}_{2\ell+1}$ are valid, from Remark 5.3, $H^{\langle b\rangle}_{\ell-1} v_\ell = y^{\langle b\rangle}_{\ell,2\ell-1}$, Lemma A.10, and Remark 5.18, we analogously get

$$H^{\langle b\rangle}_{\ell} \begin{bmatrix} -v_\ell \\ I_q \end{bmatrix} = \begin{bmatrix} O_{\ell q\times q} \\ \mathfrak{f}_{4\ell+2} \end{bmatrix}. \tag{8.74}$$

Inserting (8.74) into (8.53) yields

$$H^{\langle c\rangle}_{\ell-1} v_\ell = -\begin{bmatrix} O_{(\ell-1)q\times q} \\ \mathfrak{f}_{4\ell+2} \end{bmatrix} + y^{\langle c\rangle}_{\ell,2\ell-1}. \tag{8.75}$$

Now, in view of (8.63), (8.73), and (8.75), applying Lemmata 8.7 and 8.6, we conclude

$$\begin{aligned} H_{\ell-1}^{\langle c\rangle} x_\ell &= H_{\ell-1}^{\langle c\rangle} t_\ell \mathbf{A}_{2\ell+1} + H_{\ell-1}^{\langle c\rangle} v_\ell \mathbf{B}_{2\ell+1} \\ &= \begin{bmatrix} O_{(\ell-1)q\times q} \\ \mathfrak{f}_{4\ell+1}\mathbf{A}_{2\ell+1} - \mathfrak{f}_{4\ell+2}\mathbf{B}_{2\ell+1} \end{bmatrix} + y_{\ell,2\ell-1}^{\langle c\rangle}(\mathbf{A}_{2\ell+1} + \mathbf{B}_{2\ell+1}) = y_{\ell,2\ell-1}^{\langle c\rangle}. \end{aligned} \tag{8.76}$$

From (8.72) and (8.59) we infer

$$H_\ell \begin{bmatrix} O_{q\times q} \\ -t_\ell \end{bmatrix} - \alpha H_\ell \begin{bmatrix} -t_\ell \\ I_q \end{bmatrix} = \begin{bmatrix} O_{\ell q\times q} \\ \mathfrak{f}_{4\ell+1} \end{bmatrix} - y_{\ell+1,2\ell+1}. \tag{8.77}$$

Combining (8.74) and (8.60), we obtain

$$\beta H_\ell \begin{bmatrix} -v_\ell \\ I_q \end{bmatrix} - H_\ell \begin{bmatrix} O_{q\times q} \\ -v_\ell \end{bmatrix} = \begin{bmatrix} O_{\ell q\times q} \\ \mathfrak{f}_{4\ell+2} \end{bmatrix} + y_{\ell+1,2\ell+1}. \tag{8.78}$$

Now, by virtue of (8.64), (8.77), and (8.78), Lemmata 8.7 and 8.6 provide

$$\begin{aligned} &H_\ell r_{\ell+1} \\ &= -\left( H_\ell \begin{bmatrix} O_{q\times q} \\ -t_\ell \end{bmatrix} - \alpha H_\ell \begin{bmatrix} -t_\ell \\ I_q \end{bmatrix} \right) \mathbf{A}_{2\ell+1} + \left( \beta H_\ell \begin{bmatrix} -v_\ell \\ I_q \end{bmatrix} - H_\ell \begin{bmatrix} O_{q\times q} \\ -v_\ell \end{bmatrix} \right) \mathbf{B}_{2\ell+1} \\ &= -\left( \begin{bmatrix} O_{\ell q\times q} \\ \mathfrak{f}_{4\ell+1} \end{bmatrix} - y_{\ell+1,2\ell+1} \right) \mathbf{A}_{2\ell+1} + \left( \begin{bmatrix} O_{\ell q\times q} \\ \mathfrak{f}_{4\ell+2} \end{bmatrix} + y_{\ell+1,2\ell+1} \right) \mathbf{B}_{2\ell+1} \\ &= \begin{bmatrix} O_{\ell q\times q} \\ -\mathfrak{f}_{4\ell+1}\mathbf{A}_{2\ell+1} + \mathfrak{f}_{4\ell+2}\mathbf{B}_{2\ell+1} \end{bmatrix} + y_{\ell+1,2\ell+1}(\mathbf{A}_{2\ell+1} + \mathbf{B}_{2\ell+1}) = y_{\ell+1,2\ell+1}. \end{aligned} \tag{8.79}$$

From (8.71), $H_{\ell-1}^{\langle b\rangle} v_\ell = y_{\ell,2\ell-1}^{\langle b\rangle}$, (8.76), and (8.79) finally ($\mathrm{I}_\ell$) follows. Thus, ($\mathrm{I}_\ell$) is fulfilled for all $\ell \in \mathbb{Z}_{1,\rho}$. Under additional consideration of (8.40), the application of Remark 6.5 to $(s_j)_{j=0}^{2\rho+2}$ and $(\ddot{\mathbf{r}}_\ell)_{\ell=0}^{\rho+1}$, to $(a_j)_{j=0}^{2\rho}$ and $(\ddot{\mathbf{t}}_\ell)_{\ell=0}^{\rho}$, to $(b_j)_{j=0}^{2\rho}$ and $(\ddot{\mathbf{v}}_\ell)_{\ell=0}^{\rho}$, and to $(c_j)_{j=0}^{2\rho}$ and $(\ddot{\mathbf{x}}_\ell)_{\ell=0}^{\rho}$, respectively, finishes the proof in the case that $\tau$ is a positive even integer or $\tau = \infty$.

**Part 2:** Now assume $\tau = 2\rho + 1$ with some $\rho \in \mathbb{N}$. Because of (7.4), we have then $\langle\tau] = \rho$ and $\langle\tau - 1] = \rho$ as well as $\langle\tau - 2] = \rho - 1$. According to Remark 8.4, for each $k \in \mathbb{Z}_{1,2\rho+1}$, then the matrix polynomials $\ddot{\mathbf{p}}_{2k-1}$, $\ddot{\mathbf{q}}_{2k-1}$, $\ddot{\mathbf{p}}_{2k}$, and $\ddot{\mathbf{q}}_{2k}$ are built only by use of the matrices $\mathfrak{f}_0, \mathfrak{f}_1, \dots, \mathfrak{f}_{4k-2}$ and, in view of Remark 5.23, consequently only depend on the matrices $s_0, s_1, \dots, s_{2k-1}$. Thus, in view Remark 8.9, this is also true for the polynomials $\ddot{\mathbf{r}}_k$, $\ddot{\mathbf{t}}_{k-1}$, $\ddot{\mathbf{v}}_{k-1}$, and $\ddot{\mathbf{x}}_{k-1}$ constructed from it. Moreover, we have $\ddot{\mathbf{r}}_0 = \ddot{\mathbf{p}}_0 = I_q$. Since $(s_j)_{j=0}^{2\rho+1}$ belongs to $\mathcal{F}_{q,2\rho+1,\alpha,\beta}^{\succcurlyeq}$, we know from Proposition 5.13 that there is a sequence $(s_j)_{j=2\rho+2}^{\infty}$ of complex $q \times q$ matrices such that $(s_j)_{j=0}^{\infty}$ belongs to $\mathcal{F}_{q,\infty,\alpha,\beta}^{\succcurlyeq}$. Since the case $\tau = \infty$ is already checked in Part 1 of the proof, the sequence $(\ddot{\mathbf{r}}_\ell)_{\ell=0}^{\infty}$ forms an MROS of matrix polynomials with respect to $(s_j)_{j=0}^{\infty}$

and likewise, the sequences $(\ddot{\mathbf{t}}_\ell)_{\ell=0}^{\infty}$, $(\ddot{\mathbf{v}}_\ell)_{\ell=0}^{\infty}$, and $(\ddot{\mathbf{x}}_\ell)_{\ell=0}^{\infty}$ form MROS of matrix polynomials with respect to $(a_j)_{j=0}^{\infty}$, $(b_j)_{j=0}^{\infty}$, and $(c_j)_{j=0}^{\infty}$, respectively. Remark 6.3 provides the corresponding result for the subsystems and, in particular, for the case $\tau = 2\rho + 1$. Hence, the proof of Part 2 is complete as well. □

We again turn our attention to the matrix polynomials $\ddot{\mathbf{p}}_j$ and $\ddot{\mathbf{q}}_j$. They are connected to each other by the transformation described in Notation 6.11.

**Lemma 8.15** *Let $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{\succcurlyeq}_{q,\kappa,\alpha,\beta}$ with $\mathcal{F}_{\alpha,\beta}$-parameter sequence $(\mathfrak{f}_j)_{j=0}^{2\kappa}$. For all $j \in \mathbb{Z}_{0,2\kappa+2}$, let the matrix polynomial $P_j : \mathbb{C} \to \mathbb{C}^{q\times q}$ by given by $P_j(w) := w\ddot{\mathbf{p}}_j(w)$. For all $z \in \mathbb{C}$, then $P_0^{[\![s]\!]}(z) = z\ddot{\mathbf{p}}_0^{[\![s]\!]}(z) + \mathfrak{f}_0$ as well as $P_1^{[\![s]\!]}(z) = z\ddot{\mathbf{p}}_1^{[\![s]\!]}(z) + \mathfrak{f}_1$ and $P_2^{[\![s]\!]}(z) = z\ddot{\mathbf{p}}_2^{[\![s]\!]}(z) - \mathfrak{f}_2$ hold true. Moreover, if $\kappa \geq 1$, then $P_3^{[\![s]\!]}(z) = z\ddot{\mathbf{p}}_3^{[\![s]\!]}(z) - \mathfrak{f}_3$ for each $z \in \mathbb{C}$ and $P_j^{[\![s]\!]}(z) = z\ddot{\mathbf{p}}_j^{[\![s]\!]}(z)$ for every choice of $j \in \mathbb{Z}_{4,2\kappa+2}$ and $z \in \mathbb{C}$.*

***Proof*** Throughout this proof, we consider an arbitrary $z \in \mathbb{C}$. In view of Proposition 5.13 and Remarks 5.23 and 8.4, it is sufficient to consider the case $\kappa = \infty$. Let $Q : \mathbb{C} \to \mathbb{C}^{q\times q}$ be given by $Q(w) := wI_q$. Because of Notation 6.11, (5.9), and (5.10), we conclude $Q^{[\![s]\!]}(z) = s_0 = \mathfrak{f}_0$, $Q^{[\![a]\!]}(z) = a_0 = \mathfrak{f}_1$, $Q^{[\![b]\!]}(z) = b_0 = \mathfrak{f}_2$, and $Q^{[\![c]\!]}(z) = c_0 = \mathfrak{f}_3$. Taking into account (8.3), we see that $P_0 = Q$. By virtue of Notation 6.11, we get $\ddot{\mathbf{p}}_0^{[\![s]\!]}(z) = O_{q\times q}$ and $P_0^{[\![s]\!]}(z) = Q^{[\![s]\!]}(z) = \mathfrak{f}_0 = z\ddot{\mathbf{p}}_0^{[\![s]\!]}(z) + \mathfrak{f}_0$. Since, according to (8.3) and (8.4), we have $\ddot{\mathbf{p}}_1(w) = (w - \alpha)I_q$ and $\ddot{\mathbf{p}}_2(w) = (w - \beta)I_q$, then $P_1(w) = (w - \alpha)Q(w)$ and $P_2(w) = -[(\beta - w)Q(w)]$ hold true for all $w \in \mathbb{C}$. Lemma C.3 yields $\ddot{\mathbf{p}}_1^{[\![s]\!]}(z) = s_0$ and $\ddot{\mathbf{p}}_2^{[\![s]\!]}(z) = s_0$ as well as, using additionally Remark C.1, moreover $P_1^{[\![s]\!]}(z) = Q^{[\![a]\!]}(z) + s_0Q(z)$ and $P_2^{[\![s]\!]}(z) = -[Q^{[\![b]\!]}(z) - s_0Q(z)]$. Therefore, $P_1^{[\![s]\!]}(z) = \mathfrak{f}_1 + zs_0 = \mathfrak{f}_1 + z\ddot{\mathbf{p}}_1^{[\![s]\!]}(z)$ and $P_2^{[\![s]\!]}(z) = -\mathfrak{f}_2 + zs_0 = -\mathfrak{f}_2 + z\ddot{\mathbf{p}}_2^{[\![s]\!]}(z)$. Equation (8.6) implies with (8.3), (8.4), and Lemma 8.6 that $\ddot{\mathbf{p}}_3(z) = -(\beta - z)(z - \alpha)(\mathbf{A}_1 + \mathbf{B}_1) = -[(\beta - z)(z - \alpha)I_q]$. Consequently, $P_3(z) = -[(\beta - z)(z - \alpha)Q(z)]$. As a result of Remark C.1 and Lemma C.4, we have $\ddot{\mathbf{p}}_3^{[\![s]\!]}(z) = -[(\alpha+\beta-z)s_0 - s_1]$ and $P_3^{[\![s]\!]}(z) = -(Q^{[\![c]\!]}(z) + [(\alpha+\beta-z)s_0 - s_1]Q(z))$ and, hence, $P_3^{[\![s]\!]}(z) = -\mathfrak{f}_3 - z[(\alpha + \beta - z)s_0 - s_1] = -\mathfrak{f}_3 + z\ddot{\mathbf{p}}_3^{[\![s]\!]}(z)$. Now we consider an arbitrary $\ell \in \mathbb{N}$. In view of Notation 8.9, we first of all have $\ddot{\mathbf{p}}_{4\ell} = \ddot{\mathbf{r}}_\ell$ and $P_{4\ell} = w\ddot{\mathbf{r}}_\ell(w)$ for all $w \in \mathbb{C}$. By virtue of Theorem 8.14, the sequence $(\ddot{\mathbf{r}}_k)_{k=0}^{\infty}$ forms an MROS of matrix polynomials with respect to $(s_j)_{j=0}^{\infty}$. In particular, we know that the matrix polynomial $\ddot{\mathbf{r}}_\ell$ is of degree $\ell$ with leading coefficient matrix $I_q$. Because of (5.6), the sequence $(s_j)_{j=0}^{\infty}$ belongs to $\mathcal{H}^{\succcurlyeq}_{q,\infty}$. Then one can now see from Lemma C.2 in connection with Remark 6.5 that $P_{4\ell}^{[\![s]\!]}(z) = z\ddot{\mathbf{r}}_\ell^{[\![s]\!]}(z) = z\ddot{\mathbf{p}}_{4\ell}^{[\![s]\!]}(z)$. In view of Notation 8.9, we have furthermore

$$\ddot{\mathbf{p}}_{4\ell+1}(w) = (w - \alpha)\ddot{\mathbf{t}}_\ell(w) \quad \text{and} \quad \ddot{\mathbf{p}}_{4\ell+2}(w) = -[(\beta - w)\ddot{\mathbf{v}}_\ell(w)] \tag{8.80}$$

for all $w \in \mathbb{C}$. Let $T_\ell, V_\ell : \mathbb{C} \to \mathbb{C}^{q\times q}$ be given by $T_\ell(w) := w\ddot{\mathbf{t}}_\ell(w)$ and $V_\ell(w) := w\ddot{\mathbf{v}}_\ell(w)$. Thus,

$$P_{4\ell+1}(w) = (w-\alpha)T_\ell(w) \quad \text{and} \quad P_{4\ell+2}(w) = -[(\beta - w)V_\ell(w)] \tag{8.81}$$

for all $w \in \mathbb{C}$. According to Theorem 8.14, the sequence $(\ddot{\mathbf{t}}_k)_{k=0}^\infty$ forms an MROS of matrix polynomials with respect to $(a_j)_{j=0}^\infty$ and the sequence $(\ddot{\mathbf{v}}_k)_{k=0}^\infty$ forms an MROS of matrix polynomials with respect to $(b_j)_{j=0}^\infty$. In particular, the matrix polynomials $\ddot{\mathbf{t}}_\ell$ and $\ddot{\mathbf{v}}_\ell$ are of degree $\ell$ and have both the leading coefficient matrix $I_q$. According to (5.6), both sequences $(a_j)_{j=0}^\infty$ and $(b_j)_{j=0}^\infty$ belong to $\mathcal{H}_{q,\infty}^{\succcurlyeq}$. One can see from Lemma C.2 in connection with Remark 6.5 that $T_\ell^{[\![a]\!]}(w) = w\ddot{\mathbf{t}}_\ell^{[\![a]\!]}(w)$ and $V_\ell^{[\![b]\!]}(w) = w\ddot{\mathbf{v}}_\ell^{[\![b]\!]}(w)$ hold true for all $w \in \mathbb{C}$. Therefore, because of (8.81), (8.80), and Lemma C.3, we obtain

$$P_{4\ell+1}^{[\![s]\!]}(z) = T_\ell^{[\![a]\!]}(z) + s_0 T_\ell(z) = z\left[\ddot{\mathbf{t}}_\ell^{[\![a]\!]}(z) + s_0\ddot{\mathbf{t}}_\ell(z)\right] = z\ddot{\mathbf{p}}_{4\ell+1}^{[\![s]\!]}(z)$$

and, using additionally Remark C.1, analogously

$$P_{4\ell+2}^{[\![s]\!]}(z) = -\left[V_\ell^{[\![b]\!]}(z) - s_0 V_\ell(z)\right] = z\left(-\left[\ddot{\mathbf{v}}_\ell^{[\![b]\!]}(z) - s_0\ddot{\mathbf{v}}_\ell(z)\right]\right) = z\ddot{\mathbf{p}}_{4\ell+2}^{[\![s]\!]}(z).$$

In view of Notation 8.9, we have furthermore $\ddot{\mathbf{p}}_{4\ell+3}(w) = -[(\beta - w)(w-\alpha)\ddot{\mathbf{x}}_\ell(w)]$ for all $w \in \mathbb{C}$ and with $X_\ell(w) := w\ddot{\mathbf{x}}_\ell(w)$ then

$$P_{4\ell+3}(w) = -[(\beta - w)(w-\alpha)X_\ell(w)] \tag{8.82}$$

for all $w \in \mathbb{C}$. According to Theorem 8.14, the sequence $(\ddot{\mathbf{x}}_k)_{k=0}^\infty$ forms an MROS of matrix polynomials with respect to $(c_j)_{j=0}^\infty$. In particular, the matrix polynomial $\ddot{\mathbf{x}}_\ell$ is of degree $\ell$ and has the leading coefficient matrix $I_q$. Because of (5.6), the sequence $(c_j)_{j=0}^\infty$ belongs to $\mathcal{H}_{q,\infty}^{\succcurlyeq}$. One can see from Lemma C.2 in connection with Remark 6.5 that $X_\ell^{[\![c]\!]}(z) = z\ddot{\mathbf{x}}_\ell^{[\![c]\!]}(z)$ holds true. Therefore, because of (8.82), Remark C.1, and Lemma C.4, we obtain $P_{4\ell+3}^{[\![s]\!]}(z) = -(z\ddot{\mathbf{x}}_\ell^{[\![c]\!]}(z) + [(\alpha+\beta-z)s_0 - s_1][z\ddot{\mathbf{x}}_\ell(z)]) = z\ddot{\mathbf{p}}_{4\ell+3}^{[\![s]\!]}(z)$. Hence, $P_j^{[\![s]\!]}(z) = z\ddot{\mathbf{p}}_j^{[\![s]\!]}(z)$ for all $j \in \mathbb{Z}_{4,\infty}$. □

**Lemma 8.16** *Let* $(s_j)_{j=0}^\kappa \in \mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq}$. *Then* $\ddot{\mathbf{q}}_j = \ddot{\mathbf{p}}_j^{[\![s]\!]}$ *for all* $j \in \mathbb{Z}_{0,2\kappa+2}$.

***Proof*** We use the notation given in Lemma 8.15. Our proof works inductively. We consider an arbitrary $z \in \mathbb{C}$. In view of (8.3) and Notation 6.11, we have $\deg \ddot{\mathbf{p}}_0 = 0$ and $\ddot{\mathbf{p}}_0^{[\![s]\!]}(z) = O_{q\times q} = \ddot{\mathbf{q}}_0(z)$. Because of (8.3) and (8.4), moreover $\ddot{\mathbf{p}}_1(z) = (z-\alpha)\ddot{\mathbf{p}}_0(z)$ and $\ddot{\mathbf{p}}_2(z) = (z-\beta)\ddot{\mathbf{p}}_0(z)$ are fulfilled. Applying Remark C.1, (8.3) as well as Lemma 8.15 and the notation given there, leads with the use of Notation 6.11 to $\ddot{\mathbf{p}}_1^{[\![s]\!]}(z) = P_0^{[\![s]\!]}(z) - \alpha\ddot{\mathbf{p}}_0^{[\![s]\!]}(z) = (z-\alpha)\ddot{\mathbf{p}}_0^{[\![s]\!]}(z) + \mathfrak{f}_0 = \mathfrak{f}_0 = \ddot{\mathbf{q}}_1(z)$ and, analogously, to $\ddot{\mathbf{p}}_2^{[\![s]\!]}(z) = \mathfrak{f}_0 = \ddot{\mathbf{q}}_2(z)$.

Now let $\kappa \geq 1$. From (8.6) we get $\ddot{\mathbf{p}}_3(z) = -\beta\ddot{\mathbf{p}}_1(z)\mathbf{A}_1 + P_1(z)\mathbf{A}_1 + P_2(z)\mathbf{B}_1 - \alpha\ddot{\mathbf{p}}_2(z)\mathbf{B}_1$. Thus, applying Remark C.1 and Lemma 8.15 as well as Lemma 8.7, from (8.7) and the already shown equations, then we conclude

$$\begin{aligned}
&\ddot{\mathbf{p}}_3^{[\![s]\!]}(z)\\
&= -\left(\beta\ddot{\mathbf{p}}_1^{[\![s]\!]}(z) - \left[z\ddot{\mathbf{p}}_1^{[\![s]\!]}(z) + \mathfrak{f}_1\right]\right)\mathbf{A}_1 + \left(\left[z\ddot{\mathbf{p}}_2^{[\![s]\!]}(z) - \mathfrak{f}_2\right] - \alpha\ddot{\mathbf{p}}_2^{[\![s]\!]}(z)\right)\mathbf{B}_1\\
&= -[\beta\ddot{\mathbf{q}}_1(z) - z\ddot{\mathbf{q}}_1(z)]\mathbf{A}_1 + [z\ddot{\mathbf{q}}_2(z) - \alpha\ddot{\mathbf{q}}_2(z)]\mathbf{B}_1 + (\mathfrak{f}_1\mathbf{A}_1 - \mathfrak{f}_2\mathbf{B}_1)\\
&= -[(\beta - z)\ddot{\mathbf{q}}_1(z)]\mathbf{A}_1 + [(z-\alpha)\ddot{\mathbf{q}}_2(z)]\mathbf{B}_1 = \ddot{\mathbf{q}}_3(z).
\end{aligned}$$

Keeping in mind (8.8) and (8.9) and using Remark C.1, then $\ddot{\mathbf{p}}_4^{[\![s]\!]}(z) = \ddot{\mathbf{p}}_1^{[\![s]\!]}(z)\mathbf{A}_1 + \ddot{\mathbf{p}}_2^{[\![s]\!]}(z)\mathbf{B}_1 = \ddot{\mathbf{q}}_1(z)\mathbf{A}_1 + \ddot{\mathbf{q}}_2(z)\mathbf{B}_1 = \ddot{\mathbf{q}}_4(z)$ follows as well. Now we suppose that $\kappa \geq 2$ and that we already know that there is an $\ell \in \mathbb{N}$ fulfilling $2\ell \leq \kappa$ such that $\ddot{\mathbf{p}}_{4m-1}^{[\![s]\!]} = \ddot{\mathbf{q}}_{4m-1}$ and $\ddot{\mathbf{p}}_{4m}^{[\![s]\!]} = \ddot{\mathbf{q}}_{4m}$ are valid for all $m \in \mathbb{Z}_{1,\ell}$. From (8.10) and (8.11) we get $\ddot{\mathbf{p}}_{4\ell+1}(z) = \ddot{\mathbf{p}}_{4\ell-1}(z)\mathbf{A}_{2\ell} + P_{4\ell}(z)\mathbf{B}_{2\ell} - \alpha\ddot{\mathbf{p}}_{4\ell}(z)\mathbf{B}_{2\ell}$ and $\ddot{\mathbf{p}}_{4\ell+2}(z) = \ddot{\mathbf{p}}_{4\ell-1}(z)\mathbf{A}_{2\ell} + P_{4\ell}(z)\mathbf{B}_{2\ell} - \beta\ddot{\mathbf{p}}_{4\ell}(z)\mathbf{B}_{2\ell}$. Thus, because of (8.10)–(8.13), Remark C.1, and Lemma 8.15, we obtain

$$\begin{aligned}
\ddot{\mathbf{p}}_{4\ell+1}^{[\![s]\!]}(z) &= \ddot{\mathbf{p}}_{4\ell-1}^{[\![s]\!]}(z)\mathbf{A}_{2\ell} + z\ddot{\mathbf{p}}_{4\ell}^{[\![s]\!]}(z)\mathbf{B}_{2\ell} - \alpha\ddot{\mathbf{p}}_{4\ell}^{[\![s]\!]}(z)\mathbf{B}_{2\ell}\\
&= \ddot{\mathbf{q}}_{4\ell-1}(z)\mathbf{A}_{2\ell} + (z-\alpha)\ddot{\mathbf{q}}_{4\ell}(z)\mathbf{B}_{2\ell} = \ddot{\mathbf{q}}_{4\ell+1}(z)
\end{aligned}$$

and, analogously, $\ddot{\mathbf{p}}_{4\ell+2}^{[\![s]\!]}(z) = \ddot{\mathbf{q}}_{4\ell+2}(z)$. If $2\ell + 1 \leq \kappa$, then (8.6) yields

$$\begin{aligned}
\ddot{\mathbf{p}}_{4\ell+3}(z) &= -(\beta - z)\ddot{\mathbf{p}}_{4\ell+1}(z)\mathbf{A}_{2\ell+1} + (z-\alpha)\ddot{\mathbf{p}}_{4\ell+2}(z)\mathbf{B}_{2\ell+1}\\
&= -\beta\ddot{\mathbf{p}}_{4\ell+1}(z)\mathbf{A}_{2\ell+1} + P_{4\ell+1}(z)\mathbf{A}_{2\ell+1} + P_{4\ell+2}(z)\mathbf{B}_{2\ell+1} - \alpha\ddot{\mathbf{p}}_{4\ell+2}(z)\mathbf{B}_{2\ell+1}.
\end{aligned}$$

From (8.8) we conclude $\ddot{\mathbf{p}}_{4\ell+4}(z) = \ddot{\mathbf{p}}_{4\ell+1}(z)\mathbf{A}_{2\ell+1} + \ddot{\mathbf{p}}_{4\ell+2}(z)\mathbf{B}_{2\ell+1}$, which, in view of Remark C.1, Lemma 8.15, and (8.13), implies

$$\begin{aligned}
\ddot{\mathbf{p}}_{4\ell+3}^{[\![s]\!]}(z) &= -(\beta - z)\ddot{\mathbf{p}}_{4\ell+1}^{[\![s]\!]}(z)\mathbf{A}_{2\ell+1} + (z-\alpha)\ddot{\mathbf{p}}_{4\ell+2}^{[\![s]\!]}(z)\mathbf{B}_{2\ell+1}\\
&= -(\beta - z)\ddot{\mathbf{q}}_{4\ell+1}(z)\mathbf{A}_{2\ell+1} + (z-\alpha)\ddot{\mathbf{q}}_{4\ell+2}(z)\mathbf{B}_{2\ell+1} = \ddot{\mathbf{q}}_{4\ell+3}(z).
\end{aligned}$$

Analogously, in view of (8.9), we get $\ddot{\mathbf{p}}_{4\ell+4}^{[\![s]\!]}(z) = \ddot{\mathbf{q}}_{4\ell+4}(z)$. Therefore, the lemma is proved inductively. □

## 9 The $\mathcal{F}_{\alpha,\beta}$-transformation for Matricial Sequences

We continue by stating the construction of a certain transformation for sequences of matrices. This transformation was introduced in [21] and constitutes the elementary step of a Schur type algorithm in the class of $[\alpha, \beta]$-non-negative definite sequences.

**Definition 9.1** ([21, Def. 8.8]) Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices, and let $\beta \in \mathbb{R}$. Further, let $b_{-1} := -s_0$ and, in the case $\kappa \geq 1$, let $(b_j)_{j=0}^{\kappa-1}$ be given by (3.2). Then we call the sequence $(\mathbf{b}_j)_{j=0}^{\kappa}$ defined by $\mathbf{b}_j := b_{j-1}$ the $(-\infty, \beta]$ *-modification of* $(s_j)_{j=0}^{\kappa}$.

If $(s_j)_{j=0}^{\kappa}$ and $(t_j)_{j=0}^{\kappa}$ are sequences of complex $p \times q$ and $q \times r$ matrices, then we use the Cauchy product $(x_j)_{j=0}^{\kappa}$ of $(s_j)_{j=0}^{\kappa}$ and $(t_j)_{j=0}^{\kappa}$ which is given by $x_j := \sum_{\ell=0}^{j} s_\ell t_{j-\ell}$.

**Definition 9.2** ([21, Def. 8.14]) Let $\kappa \in \mathbb{N} \cup \{\infty\}$, let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha < \beta$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. In view of (3.2), denote by $(g_j)_{j=0}^{\kappa-1}$ the $(-\infty, \beta]$-modification of $(a_j)_{j=0}^{\kappa-1}$ and by $(x_j)_{j=0}^{\kappa-1}$ the Cauchy product of $(b_j)_{j=0}^{\kappa-1}$ and $(g_j)_{j=0}^{\kappa-1}$. Then we call the sequence $(t_j)_{j=0}^{\kappa-1}$ given by $t_j := -a_0 s_0^{\dagger} x_j a_0$ the $\mathcal{F}_{\alpha,\beta}$ *-transform of* $(s_j)_{j=0}^{\kappa}$.

***Remark 9.3*** ([21, Rem. 8.15]) Suppose $\kappa \geq 1$. Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices with $\mathcal{F}_{\alpha,\beta}$-transform $(t_j)_{j=0}^{\kappa-1}$. Then, for each $k \in \mathbb{Z}_{0,\kappa-1}$, the matrix $t_k$ is built from the matrices $s_0, s_1, \ldots, s_{k+1}$. In particular, for all $m \in \mathbb{Z}_{1,\kappa}$, the $\mathcal{F}_{\alpha,\beta}$-transform of $(s_j)_{j=0}^{m}$ coincides with $(t_j)_{j=0}^{m-1}$.

We are now going to iterate the $\mathcal{F}_{\alpha,\beta}$-transform introduced in Definition 9.2:

**Definition 9.4** ([21, Def. 9.1]) Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha < \beta$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Let the sequence $(s_j^{\{0\}})_{j=0}^{\kappa}$ be given by $s_j^{\{0\}} := s_j$. If $\kappa \geq 1$, then, for all $k \in \mathbb{Z}_{1,\kappa}$, let the sequence $(s_j^{\{k\}})_{j=0}^{\kappa-k}$ be recursively defined to be the $\mathcal{F}_{\alpha,\beta}$-transform of the sequence $(s_j^{\{k-1\}})_{j=0}^{\kappa-(k-1)}$. For all $k \in \mathbb{Z}_{0,\kappa}$, then we call the sequence $(s_j^{\{k\}})_{j=0}^{\kappa-k}$ the *k-th* $\mathcal{F}_{\alpha,\beta}$ *-transform of* $(s_j)_{j=0}^{\kappa}$.

***Remark 9.5*** ([21, Rem. 9.3]) Let $\kappa \in \mathbb{Z}_{0,\kappa}$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices with $k$-th $\mathcal{F}_{\alpha,\beta}$-transform $(u_j)_{j=0}^{\kappa-k}$. In view of Remark 9.3, we see that, for each $\ell \in \mathbb{Z}_{0,\kappa-k}$, the matrix $u_\ell$ is built only from the matrices $s_0, s_1, \ldots, s_{\ell+k}$. In particular, for each $m \in \mathbb{Z}_{k,\kappa}$, the $k$-th $\mathcal{F}_{\alpha,\beta}$-transform of $(s_j)_{j=0}^{m}$ coincides with $(u_j)_{j=0}^{m-k}$.

***Remark 9.6*** ([21, Cor. 9.9]) If $(s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq}$, then $\mathsf{D}_j = \delta^{-(j-1)} s_0^{\{j\}}$ for all $j \in \mathbb{Z}_{0,\kappa}$ where, as introduced above, $\delta = \beta - \alpha$.

## 10 The Classes $\mathcal{PR}_q(\Pi_+)$ and $\mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta])$

In this section, we consider particular pairs of meromorphic matrix functions, which are used as parameters in the context of matricial moment problems.

***Remark 10.1*** The matrix

$$\tilde{J}_q := \begin{bmatrix} O_{q\times q} & -\mathrm{i}I_q \\ \mathrm{i}I_q & O_{q\times q} \end{bmatrix} \tag{10.1}$$

is a $2q \times 2q$ signature matrix, i.e., $\tilde{J}_q^* = \tilde{J}_q$ and $\tilde{J}_q^2 = I_{2q}$ hold true. Moreover for every choice of $A, B \in \mathbb{C}^{q\times q}$, we have $\begin{bmatrix} A \\ B \end{bmatrix}^* (-\tilde{J}_q) \begin{bmatrix} A \\ B \end{bmatrix} = 2\Im(B^* A)$. In particular, $\begin{bmatrix} A \\ I_q \end{bmatrix}^* (-\tilde{J}_q) \begin{bmatrix} A \\ I_q \end{bmatrix} = 2\Im(A)$ is valid for each $A \in \mathbb{C}^{q\times q}$.

***Remark 10.2*** Let $M \in \mathbb{C}_{\mathrm{H}}^{q\times q}$. Then it is readily checked that $\begin{bmatrix} I_q & O_{q\times q} \\ M & I_q \end{bmatrix}^* \tilde{J}_q \begin{bmatrix} I_q & O_{q\times q} \\ M & I_q \end{bmatrix} = \tilde{J}_q$ and $\begin{bmatrix} I_q & M \\ O_{q\times q} & I_q \end{bmatrix}^* \tilde{J}_q \begin{bmatrix} I_q & M \\ O_{q\times q} & I_q \end{bmatrix} = \tilde{J}_q$.

For our following considerations, we observe that, for each meromorphic matrix-valued function $F$, the set $\mathcal{P}(F)$ of all poles of $F$ is discrete.

The following class can be used as set of parameters in the context of the matricial Hamburger moment problem (see, e. g. [13]). Denote by $\mathcal{PR}_q(\Pi_+)$ the set of all ordered pairs $[P; Q]$ consisting of $\mathbb{C}^{q\times q}$-valued functions $P$ and $Q$ which are meromorphic in $\Pi_+$ and for which a discrete subset $\mathcal{D}$ of $\Pi_+$ exists such that the following three conditions are fulfilled:

(I) $\mathcal{P}(P) \cup \mathcal{P}(Q) \subseteq \mathcal{D}$.
(II) $\operatorname{rank}\begin{bmatrix} P(z) \\ Q(z) \end{bmatrix} = q$ for all $z \in \Pi_+ \setminus \mathcal{D}$.
(III) $\Im([Q(z)]^* P(z)) \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ for all $z \in \Pi_+ \setminus \mathcal{D}$.

Observe that the class $\mathcal{PR}_q(\Pi_+)$ can be used as set of parameters to describe the solution set of the matricial Hamburger moment problem (see, e. g. [13]).

***Remark 10.3*** Let $[P; Q] \in \mathcal{PR}_q(\Pi_+)$ be such that $\det Q$ does not vanish identically. Then one can easily check the well-known fact that the function $PQ^{-1}$ belongs to $\mathcal{R}_q(\Pi_+)$.

If $[P; Q] \in \mathcal{PR}_q(\Pi_+)$, then, for each $q \times q$ matrix-valued function $R$ meromorphic in $\Pi_+$ such that the function $\det R$ does not vanish identically, one can see that the pair $[PR; QR]$ belongs to $\mathcal{PR}_q(\Pi_+)$ as well. Two pairs $[P_1; Q_1], [P_2; Q_2] \in \mathcal{PR}_q(\Pi_+)$ are said to be equivalent, if there exist a $q \times q$ matrix-valued function $R$ meromorphic in $\Pi_+$ and a discrete subset $\mathcal{D}$ of $\Pi_+$ such that $P_1, Q_1, P_2, Q_2$, and $R$ are holomorphic in $\Pi_+ \setminus \mathcal{D}$ and that $\det R(w) \neq 0$ as well as $P_2(w) = P_1(w)R(w)$ and $Q_2(w) = Q_1(w)R(w)$ hold true for each $w \in \Pi_+ \setminus \mathcal{D}$. Indeed, this relation defines an equivalence relation on $\mathcal{PR}_q(\Pi_+)$. For each $[P; Q] \in \mathcal{PR}_q(\Pi_+)$, we denote by $\langle [P; Q] \rangle$ the equivalence class generated by $[P; Q]$. Furthermore, if $\mathcal{M}$ is a subset of $\mathcal{PR}_q(\Pi_+)$, then let $\langle \mathcal{M} \rangle := \{\langle [P; Q] \rangle : [P; Q] \in \mathcal{M}\}$.

Now we want to study special subclasses of the class $\mathcal{PR}_q(\Pi_+)$. For each linear subspace $\mathcal{U}$ of $\mathbb{C}^q$, we again use $\mathbb{P}_{\mathcal{U}}$ to denote the orthogonal projection matrix onto $\mathcal{U}$ (see also Remarks A.4 and A.5).

**Notation 10.4** Let $M \in \mathbb{C}^{q\times p}$. We denote by $\mathcal{P}[M]$ the set of all pairs $[P; Q] \in \mathcal{PR}_q(\Pi_+)$ such that $\mathbb{P}_{\mathcal{R}(M)} P = P$ is fulfilled.

Observe, if $M \in \mathbb{C}^{p\times q}$ is such that $\operatorname{rank} M = q$, then $\mathcal{P}[M] = \mathcal{PR}_q(\Pi_+)$.

The construction in Notation 10.4 will be used later to treat Problem $\mathsf{MP}[\mathbb{R}; (s_j)_{j=0}^{2n}, \preccurlyeq]$ by choosing $M = \mathfrak{h}_{2n}$, where $\mathfrak{h}_{2n}$ is given by Definition 5.5.

Now we consider a further class of meromorphic matrix-valued functions, which is connected with the matricial Hausdorff moment problem (see [23]).

**Notation 10.5** (cf. [6, Def. 5.2]) Denote by $\mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta])$ the set of all ordered pairs $[P; Q]$ consisting of $q \times q$ matrix-valued functions $P$ and $Q$ which are meromorphic in $\mathbb{C} \setminus [\alpha, \beta]$ and for which a discrete subset $\mathcal{D}$ of $\mathbb{C} \setminus [\alpha, \beta]$ exists such that $P$ and $Q$ are both holomorphic in $\mathbb{C} \setminus ([\alpha, \beta] \cup \mathcal{D})$ and that the following two conditions are fulfilled:

(I) $\operatorname{rank}\left[\begin{smallmatrix} P(z) \\ Q(z) \end{smallmatrix}\right] = q$ for each $z \in \mathbb{C}\setminus([\alpha, \beta] \cup \mathcal{D})$.

(II) $(\Im z)^{-1}\Im((z-\alpha)[Q(z)]^*[P(z)]) \in \mathbb{C}^{q\times q}_{\succcurlyeq}$ and $(\Im z)^{-1}\Im((\beta-z)[Q(z)]^*[P(z)]) \in \mathbb{C}^{q\times q}_{\succcurlyeq}$ for all $z \in \mathbb{C}\setminus(\mathbb{R} \cup \mathcal{D})$.

For each $[P; Q] \in \mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta])$, we denote by $\ddot{\mathcal{E}}([P; Q])$ the set of all $z \in \mathbb{C} \setminus ([\alpha, \beta] \cup \mathcal{P}(P) \cup \mathcal{P}(Q))$ satisfying $\operatorname{rank}\left[\begin{smallmatrix} P(z) \\ Q(z) \end{smallmatrix}\right] \neq q$.

Two ordered pairs $[P; Q]$ and $[S; T]$ belonging to $\mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta])$ are said to be equivalent, if there is a $\mathbb{C}^{q\times q}$ matrix-valued function $R$ meromorphic in $\mathbb{C}\setminus[\alpha, \beta]$ such that $\det R$ does not vanish identically in $\mathbb{C}\setminus[\alpha, \beta]$ and such that $S = PR$ and $T = QR$ hold true. Indeed, this relations defines an equivalence relation on $\mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta])$. Moreover, let $\langle[P; Q]\rangle$ denote the equivalence class of the pair $[P; Q] \in \mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta])$. For each subset $\mathfrak{Q} \subseteq \mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta])$, let $\langle\mathfrak{Q}\rangle := \{\langle[S; T]\rangle : [S; T] \in \mathfrak{Q}\}$ be the set of all equivalence classes of pairs belonging to $\mathfrak{Q}$.

The pairs belonging to a certain subclass of the set $\mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta])$ introduced below generate the equivalence classes, which will be used later as parameters in the description of the set of all solutions to Problem $\mathsf{FP}[[\alpha, \beta]; (s_j)_{j=0}^m, =]$:

**Notation 10.6** ([23, Notation 8.1]) For each $M \in \mathbb{C}^{q\times q}$, let $\ddot{\mathcal{P}}[M]$ be the set of all pairs $[F; G] \in \mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta])$ for which there exists a $z_0 \in \mathbb{C}\setminus([\alpha, \beta] \cup \mathcal{P}(F) \cup \mathcal{P}(G) \cup \ddot{\mathcal{E}}([F; G]))$ such that $\mathcal{R}(F(z_0) \subseteq \mathcal{R}(M)$.

***Remark 10.7*** ([23, Rem. 8.2]) If $M \in \mathbb{C}^{q\times q}$ fulfills $\operatorname{rank} M = q$, then $\ddot{\mathcal{P}}[M] = \mathcal{PR}_q(\mathbb{C} \setminus [\alpha, \beta])$.

The class $\ddot{\mathcal{P}}[M]$ can be characterized as follows:

**Lemma 10.8** ([23, Lem. 8.3]) *Let $M \in \mathbb{C}^{q\times q}$ and let $[F; G] \in \mathcal{PR}_q(\mathbb{C}\setminus[\alpha, \beta])$. Then $[F; G] \in \ddot{\mathcal{P}}[M]$ if and only if $\mathbb{P}_{\mathcal{R}(M)}F = F$. In this case, $\mathcal{R}(F(z)) \subseteq \mathcal{R}(M)$ is valid for $z \in \mathbb{C} \setminus ([\alpha, \beta] \cup \mathcal{P}(F))$.*

## 11 Description of the Set of Solutions of the Matricial Truncated Hausdorff Moment Problem via Linear Fractional Transformations

In this section, we draw the attention of the reader to certain transformations for matrix functions, which are closely interrelated to the $\mathcal{F}_{\alpha,\beta}$-transformation for sequences of complex matrices.

**Definition 11.1** ([23, Def. 9.1]) Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha < \beta$, let $\mathcal{D}$ be a non-empty subset of $\mathbb{C}$, let $F : \mathcal{D} \to \mathbb{C}^{p\times q}$ be a matrix-valued function, and let $M \in \mathbb{C}^{p\times q}$. Then the pair $[G_1; G_2]$ given by the matrix-valued functions $G_1 : \mathcal{D} \to \mathbb{C}^{p\times q}$ and $G_2 : \mathcal{D} \to \mathbb{C}^{q\times q}$ defined by $G_1(z) := (\beta - z)F(z) - M$ and $G_2(z) := (\beta - z)[(z - \alpha)M^\dagger F(z) + \mathbb{P}_{\mathcal{R}(M^*)}] + \delta\mathbb{P}_{\mathcal{N}(M)}$ is called the $\mathcal{F}_{\alpha,\beta}(M)$ *-transformed pair of* $F$.

**Definition 11.2** ([23, Def. 9.5]) Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha < \beta$, let $\mathcal{G}$ be a domain of $\mathbb{C}$, let $G_1$ be a $\mathbb{C}^{p\times q}$-valued function meromorphic in $\mathcal{G}$ and let $G_2$ be a $\mathbb{C}^{q\times q}$-valued function meromorphic in $\mathcal{G}$. Moreover, let $M \in \mathbb{C}^{p\times q}$ and let the functions $g, h\colon \mathcal{G} \to \mathbb{C}$ be given by $g(z) := z - \alpha$ and $h(z) := \beta - z$. Furthermore, let $F_1 := h\mathbb{P}_{\mathcal{R}(M)}G_1 + MG_2$ and $F_2 := -hgM^\dagger G_1 + hG_2$. If $\det F_2$ does not vanish identically in $\mathcal{G}$, then we call the (in $\mathcal{G}$ meromorphic and $\mathbb{C}^{p\times q}$-valued) function $F := F_1F_2^{-1}$ the *inverse* $\mathcal{F}_{\alpha,\beta}(M)$ *-transform of* $[G_1; G_2]$.

The following complex $(p+q)\times(p+q)$ matrix polynomial (see, e. g. [23, Notation 9.6]) is connected to the inverse $\mathcal{F}_{\alpha,\beta}(M)$-transform:

**Notation 11.3** Let $M \in \mathbb{C}^{p\times q}$. Then let $\ddot{V}_M\colon \mathbb{C} \to \mathbb{C}^{(p+q)\times(p+q)}$ be given by

$$\ddot{V}_M(z) := \begin{bmatrix} (\beta - z)\mathbb{P}_{\mathcal{R}(M)} & M \\ -(\beta - z)(z-\alpha)M^\dagger & (\beta - z)I_q \end{bmatrix}.$$

We extend the concept used in Definitions 11.1 and 11.2 and Notation 11.3. Now we want to introduce a transformation depending on two complex matrices $A$ and $M$ of the same size, which will take in the role of the Hermitian matrices $a_0$ and $s_0$ determined by a given sequence $(s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq}$ (see (3.2)). Then $B := \delta M - A$ and $N := A + \alpha M$ correspond to the matrices $b_0$ and $s_1$, and we have $A = -\alpha M + N$ and $B = \beta M - N$.

**Definition 11.4** ([23, Def. 10.1]) Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha < \beta$, let $\mathcal{D}$ be a non-empty subset of $\mathbb{C}$, let $G\colon \mathcal{D} \to \mathbb{C}^{p\times q}$ be a matrix function and let $A$ and $M$ be two complex $p\times q$ matrices. Then we call the matrix function $G\colon \mathcal{D} \to \mathbb{C}^{p\times q}$ given by $G(z) := AM^\dagger[(\beta - z)F(z) - M]((\beta - z)[(z-\alpha)F(z) + M])^\dagger A$ the $\mathcal{F}_{\alpha,\beta}(A, M)$ *-transform of* $F$.

**Definition 11.5** ([23, Def. 10.4]) Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha < \beta$ and let $\delta := \beta - \alpha$. Let $\mathcal{D}$ be a non-empty subset of $\mathbb{C}$, let $G\colon \mathcal{D} \to \mathbb{C}^{p\times q}$ be a matrix function, and let $A$ and $M$ be two complex $p\times q$ matrices. Let $B := \delta M - A$. Then we call the function $F\colon \mathcal{D} \to \mathbb{C}^{p\times q}$ given by

$$\begin{aligned} F(z) \;:=\; & -[(\beta - z)MA^\dagger G(z) + A + M\mathbb{P}_{\mathcal{N}(A)}M^\dagger B] \\ & \times((\beta - z)[(z-\alpha)A^\dagger G(z) - M^\dagger A] + (z-\alpha)\mathbb{P}_{\mathcal{N}(A)}M^\dagger B)^\dagger \end{aligned}$$

the *inverse* $\mathcal{F}_{\alpha,\beta}(A, M)$ *-transform of* $G$.

**Notation 11.6** Let $A, M \in \mathbb{C}^{p\times q}$ and let $B := \delta M - A$. Then let $\ddot{V}_{A,M}\colon \mathbb{C} \to \mathbb{C}^{(p+q)\times(p+q)}$ be defined by

$$\ddot{V}_{A,M}(z) := \left[\begin{array}{c:c} (\beta - z)MA^\dagger & A + M\mathbb{P}_{\mathcal{N}(A)}M^\dagger B \\ \hdashline -(\beta - z)(z-\alpha)A^\dagger & V_{22}(z) \end{array}\right],$$

where $V_{22}(z) := (\beta - z)(\delta\mathbb{P}_{\mathcal{N}(M)} + M^\dagger A) - (z-\alpha)\mathbb{P}_{\mathcal{N}(A)}M^\dagger B$.

**Definition 11.7** ([23, Def. 13.1]) Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha < \beta$, let $\mathcal{D}$ be a non-empty subset of $\mathbb{C}$, let $F\colon \mathcal{D} \to \mathbb{C}^{p\times q}$ be a matrix-valued function, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Let $\ddot{\mathbf{G}}_0(F; (s_j)_{j=0}^{\kappa}) := F$. Recursively, in view of (3.2), for all $k \in \mathbb{Z}_{1,\kappa}$, we denote by $\ddot{\mathbf{G}}_k(F; (s_j)_{j=0}^{\kappa})$ the $\mathcal{F}_{\alpha,\beta}(a_0^{\{k-1\}}, s_0^{\{k-1\}})$-transform of $\ddot{\mathbf{G}}_{k-1}(F; (s_j)_{j=0}^{\kappa})$. Moreover, for all $k \in \mathbb{Z}_{0,\kappa}$, let $\mathbf{P}\ddot{\mathbf{G}}_k(F; (s_j)_{j=0}^{\kappa})$ denote the $\mathcal{F}_{\alpha,\beta}(s_0^{\{k\}})$-transformed pair of $\ddot{\mathbf{G}}_k(F; (s_j)_{j=0}^{\kappa})$. For all $m \in \mathbb{Z}_{0,\kappa}$, then we call $\mathbf{P}\ddot{\mathbf{G}}_m(F; (s_j)_{j=0}^{\kappa})$ the $m$-th $\mathcal{F}_{\alpha,\beta}$-transformed pair of $F$ with respect to $(s_j)_{j=0}^{\kappa}$ and $\ddot{\mathbf{G}}_m(F; (s_j)_{j=0}^{\kappa})$ the $m$-th $\mathcal{F}_{\alpha,\beta}$-transform of $F$ with respect to $(s_j)_{j=0}^{\kappa}$.

***Remark 11.8*** Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. In view of Notation 11.3, for each $m \in \mathbb{Z}_{0,\kappa}$, then $\ddot{\mathfrak{V}}_m := V_0 V_1 \cdots V_m$ is a complex matrix polynomial with $\deg \ddot{\mathfrak{V}}_m \leq 2(m+1)$, where $V_m := \ddot{V}_{s_0^{\{m\}}}$ and where, in the case $m \geq 1$, moreover $V_k := \ddot{V}_{a_0^{\{k\}}, s_0^{\{k\}}}$ for all $k \in \mathbb{Z}_{0,m-1}$.

Now we obtain the following description of the set of all solutions to Problem $\mathsf{FP}[[\alpha, \beta]; (s_j)_{j=0}^{m}, =]$ via linear fractional transformation:

**Theorem 11.9** ([23, Thm. 14.2]) *Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha < \beta$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathcal{F}^{\succcurlyeq}_{q,m,\alpha,\beta}$. Let $\ddot{\mathfrak{V}}_m\colon \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ be given by Remark 11.8 and Notation 11.3, and let $\begin{bmatrix} \tilde{\mathfrak{w}}_m & \tilde{\mathfrak{x}}_m \\ \tilde{\mathfrak{y}}_m & \tilde{\mathfrak{z}}_m \end{bmatrix}$ be the $q \times q$ block representation of the restriction of $\ddot{\mathfrak{V}}_m$ onto $\mathbb{C} \setminus [\alpha, \beta]$. If $\Gamma \in \langle \ddot{\mathcal{P}}[s_0^{\{m\}}] \rangle$ and if $[G_1; G_2] \in \Gamma$, then $\det(\tilde{\mathfrak{y}}_m G_1 + \tilde{\mathfrak{z}}_m G_2)$ does not vanish identically in $\mathbb{C} \setminus [\alpha, \beta]$ and the matrix-valued function $F := (\tilde{\mathfrak{w}}_m G_1 + \tilde{\mathfrak{x}}_m G_2)(\tilde{\mathfrak{y}}_m G_1 + \tilde{\mathfrak{z}}_m G_2)^{-1}$ belongs to $\mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^{m}, =]$. Conversely, for each $F \in \mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^{m}, =]$, there exists a unique equivalence class $\Gamma \in \langle \ddot{\mathcal{P}}[s_0^{\{m\}}] \rangle$ such that $F = (\tilde{\mathfrak{w}}_m G_1 + \tilde{\mathfrak{x}}_m G_2)(\tilde{\mathfrak{y}}_m G_1 + \tilde{\mathfrak{z}}_m G_2)^{-1}$ holds true for all $[G_1; G_2] \in \Gamma$, namely the equivalence class $\langle \mathbf{P}\ddot{\mathbf{G}}_m(F; (s_j)_{j=0}^{m}) \rangle$ of the $m$-th $\mathcal{F}_{\alpha,\beta}$-transformed pair $\mathbf{P}\ddot{\mathbf{G}}_m(F; (s_j)_{j=0}^{m})$ of $F$ with respect to $(s_j)_{j=0}^{m}$.*

Proposition 11.27 below shows that the blocks of the $q \times q$ block representation $\ddot{\mathfrak{V}}_m = \begin{bmatrix} \mathfrak{w}_m & \mathfrak{x}_m \\ \mathfrak{y}_m & \mathfrak{z}_m \end{bmatrix}$ of the matrix polynomial $\ddot{\mathfrak{V}}_m$ defined in Remark 11.8 and used in Theorem 11.9, are related to the matrix polynomials $\ddot{\mathbf{p}}_j$ and $\ddot{\mathbf{q}}_j$ given in Notation 8.3, which moreover are transferable to each other (see Lemma 8.16). For now, we want to factorize $\ddot{\mathfrak{V}}_m$ in an alternative way in comparison with Remark 11.8. Therefore, we deduce a connection between $\ddot{\mathfrak{V}}_m$ and $\ddot{\mathfrak{V}}_{m-1}$, which (considering Theorem 11.9) will relate the solution sets fulfilling $\mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^{m}, =] \subseteq \mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^{m-1}, =]$.

**Notation 11.10** Let $A, M \in \mathbb{C}^{p\times q}$. Let $B := \delta M - A$ and let $D := AM^{\dagger}B$. Then let $\ddot{U}_{A,M}\colon \mathbb{C} \to \mathbb{C}^{(p+q)\times(p+q)}$ be defined by

$$\ddot{U}_{A,M}(z) := \left[\begin{array}{c:c} U_{11}(z) & B \\ \hdashline -(\beta - z)(z - \alpha)M^{\dagger}AD^{\dagger} & (\beta - z)(\delta \mathbb{P}_{\mathcal{N}(M)} + M^{\dagger}A) \end{array}\right],$$

where $U_{11}(z) := M[(\beta - z)\mathbb{P}_{\mathcal{R}(A^*)} M^\dagger B + (z-\alpha)\mathbb{P}_{\mathcal{N}(A)} M^\dagger A] D^\dagger$.

**Lemma 11.11** ([23, Lem. 14.10]) *Suppose $\kappa \geq 1$. Let $(s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq}$. For all $m \in \mathbb{Z}_{1,\kappa}$, then*

$$\ddot{\mathfrak{V}}_m = \ddot{\mathfrak{V}}_{m-1} \ddot{U}_{a_0^{\{m-1\}}, s_0^{\{m-1\}}}. \tag{11.1}$$

**Lemma 11.12** ([23, Lem. 14.12]) *Suppose $\kappa \geq 1$. Let $(s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq}$ with $\mathcal{F}_{\alpha,\beta}$-parameter sequence $(\mathfrak{f}_j)_{j=0}^{2\kappa}$ and sequence of $[\alpha,\beta]$-interval lengths $(\mathsf{D}_j)_{j=0}^{\kappa}$. For all $k \in \mathbb{Z}_{0,\kappa-1}$ and all $z \in \mathbb{C}$, then*

$$\ddot{U}_{a_0^{\{k\}}, s_0^{\{k\}}}(z) = \begin{bmatrix} U_{11}(z) & U_{12}(z) \\ U_{21}(z) & U_{22}(z) \end{bmatrix}, \tag{11.2}$$

*where*

$$\begin{aligned} U_{11}(z) &= \mathsf{D}_k \Big\{ (\beta - z)\mathfrak{f}_{2k+1}^\dagger \mathfrak{f}_{2k+1} \mathsf{D}_k^\dagger \mathfrak{f}_{2k+2} \\ &\qquad + (z-\alpha)(I_q - \mathfrak{f}_{2k+1}^\dagger \mathfrak{f}_{2k+1}) \mathsf{D}_k^\dagger \mathfrak{f}_{2k+1} \Big\} \mathsf{D}_{k+1}^\dagger, \\ U_{12}(z) &= \delta^k \mathfrak{f}_{2k+2}, \quad U_{21}(z) = -(\beta - z)(z-\alpha)\delta^{-k+1} \mathsf{D}_k^\dagger \mathfrak{f}_{2k+1} \mathsf{D}_{k+1}^\dagger, \end{aligned}$$

*and*

$$U_{22}(z) = (\beta - z)\delta[(I_q - \mathsf{D}_k^\dagger \mathsf{D}_k + \mathsf{D}_k^\dagger \mathfrak{f}_{2k+1}].$$

In some situations, the recursion coefficients $\mathbf{A}_j$ and $\mathbf{B}_j$ defined in (8.5) occur when applying the specific choice $\Gamma_j = O_{q\times q}$. In the following, we will again use the notations introduced in Notation 8.3. Hence, for a given sequence $(s_j)_{j=0}^{\kappa}$ of complex $q \times q$ matrices with $\mathcal{F}_{\alpha,\beta}$-parameter sequence $(\mathfrak{f}_j)_{j=0}^{2\kappa}$ and sequence of $[\alpha,\beta]$-interval lengths $(\mathsf{D}_j)_{j=0}^{\kappa}$, let in the following the systems $(\ddot{\mathbf{p}}_j)_{j=0}^{2\kappa+2}$ and $(\ddot{\mathbf{q}}_j)_{j=0}^{2\kappa+2}$ of complex matrix polynomials be defined recursively in accordance to Notation 8.3, where in the case $\kappa \geq 1$, we use the special given sequence $(\Gamma_j)_{j=1}^{\kappa}$ defined by $\Gamma_j := O_{q\times q}$ for all $j \in \mathbb{Z}_{1,\kappa}$. In particular, the sequences $(\mathbf{A}_j)_{j=1}^{\kappa}$ and $(\mathbf{B}_j)_{j=1}^{\kappa}$ introduced in (8.5) then admit, for every choice of $j \in \mathbb{Z}_{1,\kappa}$, the representations

$$\mathbf{A}_j = \mathsf{D}_{j-1}^\dagger \mathfrak{f}_{2j} \qquad \text{and} \qquad \mathbf{B}_j = \mathsf{D}_{j-1}^\dagger \mathfrak{f}_{2j-1} + \mathbb{P}_{\mathcal{N}(\mathsf{D}_{j-1})}. \tag{11.3}$$

**Lemma 11.13** *Suppose $\kappa \geq 1$. Let $(s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq}$ with $\mathcal{F}_{\alpha,\beta}$-parameter sequence $(\mathfrak{f}_j)_{j=0}^{2\kappa}$ and sequence of $[\alpha,\beta]$-interval lengths $(\mathsf{D}_j)_{j=0}^{\kappa}$. Furthermore, suppose that $k \in \mathbb{Z}_{0,\kappa-1}$ is such that*

$$\mathsf{D}_k(I_q - \mathfrak{f}_{2k+1}^\dagger \mathfrak{f}_{2k+1}) \mathsf{D}_k^\dagger \mathfrak{f}_{2k+1} \mathsf{D}_{k+1}^\dagger = O. \tag{11.4}$$

*For all* $z \in \mathbb{C}$, *then*

$$\ddot{U}_{a_0^{\{k\}}, s_0^{\{k\}}}(z) = \left[\begin{array}{c|c} (\beta - z)\mathsf{D}_k \mathbf{A}_{k+1} \mathsf{D}_{k+1}^{\dagger} & \delta^k \mathsf{D}_k \mathbf{A}_{k+1} \\ \hline -(\beta - z)(z - \alpha)\delta^{-k+1}\mathbf{B}_{k+1}\mathsf{D}_{k+1}^{\dagger} & (\beta - z)\delta \mathbf{B}_{k+1} \end{array}\right].$$

***Proof.*** Let $k \in \mathbb{Z}_{0,\kappa-1}$ and let $z \in \mathbb{C}$. Furthermore, let (11.2) be the $q \times q$ block representation of $\ddot{U}_{a_0^{\{k\}}, s_0^{\{k\}}}(z)$. Lemma 8.7 yields $\mathsf{D}_k \mathbf{A}_{k+1} = \mathfrak{f}_{2k+2}$. In view of Lemma 11.12, then $U_{12}(z) = \delta^k \mathsf{D}_k \mathbf{A}_{k+1}$ follows. Using (11.3), Remark A.8, and Lemma 11.12, we also get $U_{22}(z) = (\beta - z)\delta \mathbf{B}_{k+1}$. From [19, Cor. 10.20] we obtain $\mathcal{N}(\mathsf{D}_k) \subseteq \mathcal{N}(\mathsf{D}_{k+1})$. Because of Remarks A.6 and A.3, we have $\mathcal{N}(\mathsf{D}_{k+1})^{\perp} = \mathcal{R}(\mathsf{D}_{k+1}^{\dagger})$. Since $\mathcal{N}(\mathbb{P}_{\mathcal{N}(\mathsf{D}_k)}) = \mathcal{N}(\mathsf{D}_k)^{\perp}$ holds true, Remark A.5 implies $\mathcal{R}(\mathsf{D}_{k+1}^{\dagger}) = \mathcal{N}(\mathsf{D}_{k+1})^{\perp} \subseteq \mathcal{N}(\mathsf{D}_k)^{\perp} = \mathcal{N}(\mathbb{P}_{\mathcal{N}(\mathsf{D}_k)})$. Consequently, $\mathbb{P}_{\mathcal{N}(\mathsf{D}_k)}\mathsf{D}_{k+1}^{\dagger} = O$. In view of (11.3), thus we obtain $\mathbf{B}_{k+1}\mathsf{D}_{k+1}^{\dagger} = \mathsf{D}_k^{\dagger}\mathfrak{f}_{2k+1}\mathsf{D}_{k+1}^{\dagger}$. Hence, Lemma 11.12 gives $U_{21}(z) = -(\beta - z)(z - \alpha)\delta^{-k+1}\mathbf{B}_{k+1}\mathsf{D}_{k+1}^{\dagger}$. Taking into account Remark A.8 and $\mathbb{P}_{\mathcal{N}(\mathsf{D}_k)}\mathsf{D}_{k+1}^{\dagger} = O$, we infer $\mathsf{D}_k^{\dagger}\mathsf{D}_k\mathsf{D}_{k+1}^{\dagger} = (I_q - \mathbb{P}_{\mathcal{N}(\mathsf{D}_k)})\mathsf{D}_{k+1}^{\dagger} = \mathsf{D}_{k+1}^{\dagger}$. Proposition 5.16 shows us that $\mathsf{D}_j \in \mathbb{C}_{\succcurlyeq}^{q \times q} \subseteq \mathbb{C}_{\mathrm{H}}^{q \times q}$ is valid for all $j \in \mathbb{Z}_{0,\kappa}$, whereas Proposition 5.20 yields $\mathfrak{f}_l \in \mathbb{C}_{\succcurlyeq}^{q \times q} \subseteq \mathbb{C}_{\mathrm{H}}^{q \times q}$ for all $l \in \mathbb{Z}_{0,2\kappa}$. Thus, Remark A.6 provides us $\mathcal{R}(\mathsf{D}_{k+1}^{\dagger}) = \mathcal{R}(\mathsf{D}_{k+1}^{*}) = \mathcal{R}(\mathsf{D}_{k+1})$ and $\mathcal{R}(\mathfrak{f}_{2k+1}^{\dagger}) = \mathcal{R}(\mathfrak{f}_{2k+1}^{*}) = \mathcal{R}(\mathfrak{f}_{2k+1})$. Therefore, since we know $\mathcal{R}(\mathsf{D}_{k+1}) \subseteq \mathcal{R}(\mathfrak{f}_{2k+1})$ from Remark 5.25, we have then $\mathcal{R}(\mathsf{D}_{k+1}^{\dagger}) \subseteq \mathcal{R}(\mathfrak{f}_{2k+1}^{\dagger})$, which, in view of Remarks A.6 and A.1(a), implies $\mathfrak{f}_{2k+1}^{\dagger}\mathfrak{f}_{2k+1}\mathsf{D}_{k+1}^{\dagger} = \mathfrak{f}_{2k+1}^{\dagger}(\mathfrak{f}_{2k+1}^{\dagger})^{\dagger}\mathsf{D}_{k+1}^{\dagger} = \mathsf{D}_{k+1}^{\dagger}$. Hence, using additionally $\mathsf{D}_k^{\dagger}\mathsf{D}_k\mathsf{D}_{k+1}^{\dagger} = \mathsf{D}_{k+1}^{\dagger}$ we conclude $\mathfrak{f}_{2k+1}^{\dagger}\mathfrak{f}_{2k+1}\mathsf{D}_k^{\dagger}\mathsf{D}_k\mathsf{D}_{k+1}^{\dagger} = \mathfrak{f}_{2k+1}^{\dagger}\mathfrak{f}_{2k+1}\mathsf{D}_{k+1}^{\dagger} = \mathsf{D}_{k+1}^{\dagger} = \mathsf{D}_k^{\dagger}\mathsf{D}_k\mathsf{D}_{k+1}^{\dagger}$. Since Remark 5.24 yields that $\mathfrak{f}_{2k+2} = \mathsf{D}_k - \mathfrak{f}_{2k+1}$ is valid, consequently, the assumption (11.4) shows that

$$\begin{aligned}
\mathsf{D}_k\mathfrak{f}_{2k+1}^{\dagger}\mathfrak{f}_{2k+1}\mathsf{D}_k^{\dagger}\mathfrak{f}_{2k+2}\mathsf{D}_{k+1}^{\dagger} &= \mathsf{D}_k\mathfrak{f}_{2k+1}^{\dagger}\mathfrak{f}_{2k+1}\mathsf{D}_k^{\dagger}(\mathsf{D}_k - \mathfrak{f}_{2k+1})\mathsf{D}_{k+1}^{\dagger} \\
&= \mathsf{D}_k\mathfrak{f}_{2k+1}^{\dagger}\mathfrak{f}_{2k+1}\mathsf{D}_k^{\dagger}\mathsf{D}_k\mathsf{D}_{k+1}^{\dagger} - \mathsf{D}_k\mathfrak{f}_{2k+1}^{\dagger}\mathfrak{f}_{2k+1}\mathsf{D}_k^{\dagger}\mathfrak{f}_{2k+1}\mathsf{D}_{k+1}^{\dagger} \\
&= \mathsf{D}_k\mathsf{D}_k^{\dagger}\mathsf{D}_k\mathsf{D}_{k+1}^{\dagger} - \mathsf{D}_k\mathsf{D}_k^{\dagger}\mathfrak{f}_{2k+1}\mathsf{D}_{k+1}^{\dagger} \\
&= \mathsf{D}_k\mathsf{D}_k^{\dagger}(\mathsf{D}_k - \mathfrak{f}_{2k+1})\mathsf{D}_{k+1}^{\dagger} = \mathsf{D}_k\mathsf{D}_k^{\dagger}\mathfrak{f}_{2k+2}\mathsf{D}_{k+1}^{\dagger}.
\end{aligned} \tag{11.5}$$

Using Lemma 11.12 as well as (11.5), (11.4), and (11.3), we get finally

$$\begin{aligned}
&U_{11}(z) \\
&= \mathsf{D}_k\left[(\beta - z)\mathfrak{f}_{2k+1}^{\dagger}\mathfrak{f}_{2k+1}\mathsf{D}_k^{\dagger}\mathfrak{f}_{2k+2} + (z - \alpha)(I_q - \mathfrak{f}_{2k+1}^{\dagger}\mathfrak{f}_{2k+1})\mathsf{D}_k^{\dagger}\mathfrak{f}_{2k+1}\right]\mathsf{D}_{k+1}^{\dagger} \\
&= (\beta - z)\mathsf{D}_k\mathfrak{f}_{2k+1}^{\dagger}\mathfrak{f}_{2k+1}\mathsf{D}_k^{\dagger}\mathfrak{f}_{2k+2}\mathsf{D}_{k+1}^{\dagger} \\
&= (\beta - z)\mathsf{D}_k\mathsf{D}_k^{\dagger}\mathfrak{f}_{2k+2}\mathsf{D}_{k+1}^{\dagger} = (\beta - z)\mathsf{D}_k\mathbf{A}_{k+1}\mathsf{D}_{k+1}^{\dagger}.
\end{aligned}$$

□

If condition (11.4) is satisfied for all $k \in \mathbb{Z}_{0,\kappa-1}$, then we obtain recursively constructed systems of complex $q \times q$ matrix polynomials given by Notation 8.3 utilizing the specific choice $\Gamma_j := O_{q\times q}$ for all $j \in \mathbb{Z}_{0,\kappa}$. First, we consider a corresponding subclass:

**Notation 11.14** If $\kappa \geq 1$, then let $\mathcal{F}^{\succcurlyeq,c}_{q,\kappa,\alpha,\beta}$ be the set of all sequences $(s_j)_{j=0}^{\kappa}$ belonging to $\mathcal{F}^{\succcurlyeq}_{q,\kappa,\alpha,\beta}$ such that $\mathcal{R}(\mathfrak{f}_{2m-1}\mathsf{D}_m^\dagger) \subseteq \mathcal{R}(\mathsf{D}_{m-1}\mathfrak{f}_{2m-1}^\dagger)$ is fulfilled for all $m \in \mathbb{Z}_{1,\kappa}$, where $(\mathfrak{f}_j)_{j=0}^{2\kappa}$ is the $\mathcal{F}_{\alpha,\beta}$-parameter sequence and where $(\mathsf{D}_j)_{j=0}^{\kappa}$ is the sequence of $[\alpha, \beta]$-interval lengths of $(s_j)_{j=0}^{\kappa}$. Furthermore, let $\mathcal{F}^{\succcurlyeq,c}_{q,0,\alpha,\beta} := \mathcal{F}^{\succcurlyeq}_{q,0,\alpha,\beta}$.

The next result shows that the class introduced in Notation 11.14 satisfies the conditions (11.4) in Lemma 11.13.

**Lemma 11.15** *Suppose $\kappa \geq 1$. Let $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{\succcurlyeq}_{q,\kappa,\alpha,\beta}$ and let $(\mathfrak{f}_j)_{j=0}^{2\kappa}$ be the $\mathcal{F}_{\alpha,\beta}$-parameter sequence as well as $(\mathsf{D}_j)_{j=0}^{\kappa}$ be the sequence of $[\alpha, \beta]$-interval lengths of $(s_j)_{j=0}^{\kappa}$. Furthermore, let $m \in \mathbb{Z}_{1,\kappa}$. Then the following conditions are equivalent:*

(i) $\mathcal{R}(\mathfrak{f}_{2m-1}\mathsf{D}_m^\dagger) \subseteq \mathcal{R}(\mathsf{D}_{m-1}\mathfrak{f}_{2m-1}^\dagger)$.
(ii) $\mathcal{R}(\mathsf{D}_{m-1}^\dagger\mathfrak{f}_{2m-1}\mathsf{D}_m^\dagger) \subseteq \mathcal{R}(\mathfrak{f}_{2m-1}^\dagger)$.
(iii) $\mathbb{P}_{\mathcal{R}(\mathsf{D}_m)}\mathfrak{f}_{2m-1}\mathsf{D}_{m-1}^\dagger\mathbb{P}_{\mathcal{N}(\mathfrak{f}_{2m-1})} = O$.
(iv) $\mathsf{D}_{m-1}(I_q - \mathfrak{f}_{2m-1}^\dagger\mathfrak{f}_{2m-1})\mathsf{D}_{m-1}^\dagger\mathfrak{f}_{2m-1}\mathsf{D}_m^\dagger = O$.

***Proof*** Because of Propositions 5.16 and 5.20, we see that the matrices $M := \mathsf{D}_{m-1}$, $D := \mathsf{D}_m$, and $A := \mathfrak{f}_{2m-1}$ are all non-negative Hermitian. In particular, Remarks A.7 and A.6 yield then

$$(M^\dagger)^* = M^\dagger, \quad A^\dagger(A^*)^\dagger = AA^+ = A^\dagger A, \quad \mathcal{R}(A^\dagger) = \mathcal{R}(A^*) = \mathcal{R}(A). \tag{11.6}$$

Furthermore, Remark 5.25 provides $\mathcal{R}(A) \subseteq \mathcal{R}(M)$ and $\mathcal{R}(A^*) \subseteq \mathcal{R}(M^*)$, whereas Remark A.6 shows that $\mathcal{R}(A^\dagger) \subseteq \mathcal{R}(M^\dagger)$. Applying Remark A.1(a), we get then

$$MM^\dagger A = A \qquad \text{and} \qquad M^\dagger MA^\dagger = M^\dagger(M^\dagger)^\dagger A^\dagger = A^\dagger. \tag{11.7}$$

(i) ⇒ (ii) Suppose (i), i. e., $\mathcal{R}(AD^\dagger) \subseteq \mathcal{R}(MA^\dagger)$. Using (11.7), we get

$$A\mathcal{R}(D^\dagger) = \mathcal{R}(AD^\dagger) \subseteq \mathcal{R}(MA^\dagger) = \mathcal{R}(MA^\dagger AA^\dagger) = (MA^\dagger A)\mathcal{R}(A^\dagger)$$

and, in view of (11.7) and (11.6), consequently $\mathcal{R}(M^\dagger AD^\dagger) = M^\dagger A\mathcal{R}(D^\dagger) \subseteq M^\dagger MA^\dagger A\mathcal{R}(A^\dagger) = A^\dagger A\mathcal{R}(A^\dagger) = \mathcal{R}(A^\dagger)$. Hence, (ii) holds true.

(ii) ⇒ (i) Assume (ii), i. e., $\mathcal{R}(M^\dagger AD^\dagger) \subseteq \mathcal{R}(A^\dagger)$. Therefore, $M^\dagger A\mathcal{R}(D^\dagger) \subseteq \mathcal{R}(A^\dagger)$. Consequently, (11.7) yields $\mathcal{R}(AD^\dagger) = A\mathcal{R}(D^\dagger) = MM^\dagger A\mathcal{R}(D^\dagger) \subseteq M\mathcal{R}(A^\dagger) = \mathcal{R}(MA^\dagger)$. Thus, (i) is fulfilled.

(ii) ⇔ (iv) First assume (ii), i. e., $\mathcal{R}(M^\dagger AD^\dagger) \subseteq \mathcal{R}(A)$. By virtue of Remark A.1(a) and (11.6), this is equivalent to $AA^\dagger M^\dagger AD^\dagger = M^\dagger AD^\dagger$, which, in view of (11.6) is equivalent to

$$A^\dagger AM^\dagger AD^\dagger = M^\dagger AD^\dagger. \tag{11.8}$$

Obviously, (11.8) implies

$$MA^\dagger AM^\dagger AD^\dagger = MM^\dagger AD^\dagger. \tag{11.9}$$

Conversely, because of (11.7), equation (11.9) yields

$$M^\dagger AD^\dagger = M^\dagger MM^\dagger AD^\dagger = M^\dagger MA^\dagger AM^\dagger AD^\dagger = A^\dagger AM^\dagger AD^\dagger,$$

i.e., (11.8). Clearly, (11.9) is equivalent to $M(I_q - A^\dagger A)M^\dagger AD^\dagger = O$, i.e., to (iv). Hence, (ii) holds if and only if (iv) is valid.

(ii) ⇔ (iii) We already know that (ii) is equivalent to (11.8). Hence, (ii) is valid if and only if $(I_q - A^\dagger A)M^\dagger AD^\dagger = O$ holds true, which is equivalent to $(I_q - A^\dagger A)M^\dagger AD^\dagger D = O$. According to Remark A.8, the last equation is equivalent to

$$\mathbb{P}_{\mathcal{N}(A)}M^\dagger A\mathbb{P}_{\mathcal{R}(D^*)} = O_{q\times q}. \tag{11.10}$$

Since all the matrices $A$, $D$, and $M$ are Hermitian, one can see by adjoining both sides of equation (11.10), under consideration of Remarks A.5 and A.7, that (11.10) is equivalent to $\mathbb{P}_{\mathcal{R}(D)}AM^\dagger\mathbb{P}_{\mathcal{N}(A)} = O_{q\times q}$, i.e., equivalent to (iii). Consequently, (iii) is necessary and sufficient for (ii). □

***Remark 11.16*** Let the assumptions of Notation 11.14 be fulfilled. In the scalar case $q = 1$, we have the validity of $\mathfrak{f}_{2m-1}\mathsf{D}_{m-1}^\dagger = \mathsf{D}_{m-1}^\dagger\mathfrak{f}_{2m-1}$. Consequently, $\mathbb{P}_{\mathcal{R}(\mathsf{D}_m)}\mathfrak{f}_{2m-1}\mathsf{D}_{m-1}^\dagger\mathbb{P}_{\mathcal{N}(\mathfrak{f}_{2m-1})} = \mathbb{P}_{\mathcal{R}(\mathsf{D}_m)}\mathsf{D}_{m-1}^\dagger\mathfrak{f}_{2m-1}\mathbb{P}_{\mathcal{N}(\mathfrak{f}_{2m-1})} = O$. Hence, Lemma 11.15 yields $\mathcal{F}_{1,\kappa,\alpha,\beta}^{\succcurlyeq,c} = \mathcal{F}_{1,\kappa,\alpha,\beta}^{\succcurlyeq}$.

**Proposition 11.17** $\mathcal{F}_{q,\kappa,\alpha,\beta}^{\succ} \subseteq \mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq,\mathrm{c}}$.

***Proof*** The case $\kappa = 0$ is trivial because of Remark 5.9 and Notation 11.14. Now assume $\kappa \geq 1$ and that $(s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,\kappa,\alpha,\beta}^{\succ}$. According to Proposition 5.20, we have $\mathfrak{f}_j \in \mathbb{C}_{\succ}^{q\times q}$ for all $j \in \mathbb{Z}_{0,2\kappa}$. In particular, from Remark A.6 we get then $\mathcal{R}(\mathfrak{f}_j^\dagger) = \mathcal{R}(\mathfrak{f}_j^*) = \mathcal{R}(\mathfrak{f}_j) = \mathbb{C}^q$ for all $j \in \mathbb{Z}_{0,2\kappa}$. Applying Lemma 11.15 yields $(s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq,\mathrm{c}}$. □

Now we introduce a further subclass of $\mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq}$ which turns out to be included in $\mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq,\mathrm{c}}$ and contains on the other side $\mathcal{F}_{q,\kappa,\alpha,\beta}^{\succ}$.

**Notation 11.18** For all $\tau \in \mathbb{N} \cup \{\infty\}$, let $\mathcal{F}_{q,\tau,\alpha,\beta}^{\succcurlyeq,\mathrm{ld}}$ be the set of all sequences $(s_j)_{j=0}^{\tau}$ belonging to $\mathcal{F}_{q,\tau,\alpha,\beta}^{\succcurlyeq}$ which fulfill $\mathcal{R}(\mathfrak{f}_{2m}) \subseteq \mathcal{R}(\mathfrak{f}_{2m-1})$ for all $m \in \mathbb{Z}_{1,\tau}$, where $(\mathfrak{f}_j)_{j=0}^{2\tau}$ denotes the $\mathcal{F}_{\alpha,\beta}$-parameter sequence of $(s_j)_{j=0}^{\tau}$. Furthermore, let $\mathcal{F}_{q,0,\alpha,\beta}^{\succcurlyeq,\mathrm{ld}} := \mathcal{F}_{q,0,\alpha,\beta}^{\succcurlyeq}$.

***Remark 11.19*** Analogous to the proof of Proposition 11.17 one gets $\mathcal{F}_{q,\kappa,\alpha,\beta}^{\succ} \subseteq \mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq,\mathrm{ld}}$.

Let $\mathcal{U}$ and $\mathcal{W}$ be subspaces of the vector space $\mathbb{C}^q$. Then we use the notation $\mathcal{U}+\mathcal{W}$ to denote the Minkowski sum of the subspaces $\mathcal{U}$ and $\mathcal{W}$. We will write $\langle \cdot, \cdot \rangle_E$ for the (left) Euclidean inner product in $\mathbb{C}^q$, i. e., for all $x, y \in \mathbb{C}^q$, let $\langle x, y\rangle_E := y^* x$. If $\mathcal{M}$ is a non-empty subspace of $\mathbb{C}^q$, we use $\mathcal{M}^\perp$ to denote the orthogonal complement of $\mathcal{M}$.

The connections described in Remark 5.25 simplify for the subclass $\mathcal{F}^{\succcurlyeq,\mathrm{ld}}_{q,\kappa,\alpha,\beta}$ as follows:

**Lemma 11.20** *Let $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{\succcurlyeq,\mathrm{ld}}_{q,\kappa,\alpha,\beta}$ with $\mathcal{F}_{\alpha,\beta}$-parameter sequence $(\mathfrak{f}_j)_{j=0}^{2\kappa}$ and sequence of $[\alpha,\beta]$-interval lengths $(\mathsf{D}_j)_{j=0}^{\kappa}$. Then $\mathcal{R}(\mathsf{D}_j) = \mathcal{R}(\mathfrak{f}_{2j})$ and $\mathcal{N}(\mathsf{D}_j) = \mathcal{N}(\mathfrak{f}_{2j})$ for all $j \in \mathbb{Z}_{0,\kappa}$ and, in the case $\kappa \geq 1$, moreover $\mathcal{R}(\mathfrak{f}_{2j+2}) \subseteq \mathcal{R}(\mathfrak{f}_{2j+1}) = \mathcal{R}(\mathsf{D}_j)$ and $\mathcal{N}(\mathsf{D}_j) = \mathcal{N}(\mathfrak{f}_{2j+1}) \subseteq \mathcal{N}(\mathfrak{f}_{2j+2})$ for all $j \in \mathbb{Z}_{0,\kappa-1}$.*

***Proof*** First we observe that Remark 5.25 yields $\mathcal{R}(\mathsf{D}_0) = \mathcal{R}(\mathfrak{f}_0)$ and $\mathcal{N}(\mathsf{D}_0) = \mathcal{N}(\mathfrak{f}_0)$. If $\kappa = 0$, then the proof is complete. Now suppose $\kappa \geq 1$. From Proposition 5.20 we know that $\mathfrak{f}_j \in \mathbb{C}^{q\times q}_{\succcurlyeq} \subseteq \mathbb{C}^{q\times q}_{\mathrm{H}}$ for all $j \in \mathbb{Z}_{0,2\kappa}$. Since we have $\mathcal{R}(\mathfrak{f}_{2m}) \subseteq \mathcal{R}(\mathfrak{f}_{2m-1})$ for all $m \in \mathbb{Z}_{1,\kappa}$ by assumption, from Remark A.3 we know then that $\mathcal{N}(\mathfrak{f}_{2m-1}) = \mathcal{N}(\mathfrak{f}^*_{2m-1}) = \mathcal{R}(\mathfrak{f}_{2m-1})^\perp \subseteq \mathcal{R}(\mathfrak{f}_{2m})^\perp = \mathcal{N}(\mathfrak{f}^*_{2m}) = \mathcal{N}(\mathfrak{f}_{2m})$ is fulfilled for all $m \in \mathbb{Z}_{1,\kappa}$ as well. Taking into account this, we get from Remark 5.25 the validity of $\mathcal{R}(\mathsf{D}_j) = \mathcal{R}(\mathfrak{f}_{2j-1}) \cap \mathcal{R}(\mathfrak{f}_{2j}) = \mathcal{R}(\mathfrak{f}_{2j})$ and $\mathcal{N}(\mathsf{D}_j) = \mathcal{N}(\mathfrak{f}_{2j-1}) + \mathcal{N}(\mathfrak{f}_{2j}) = \mathcal{N}(\mathfrak{f}_{2j})$ for all $j \in \mathbb{Z}_{1,\kappa}$ as well as $\mathcal{R}(\mathsf{D}_j) = \mathcal{R}(\mathfrak{f}_{2j+1}) + \mathcal{R}(\mathfrak{f}_{2j+2}) = \mathcal{R}(\mathfrak{f}_{2j+1}) \supseteq \mathcal{R}(\mathfrak{f}_{2j+2})$ and $\mathcal{N}(\mathsf{D}_j) = \mathcal{N}(\mathfrak{f}_{2j+1}) \cap \mathcal{N}(\mathfrak{f}_{2j+2}) = \mathcal{N}(\mathfrak{f}_{2j+1}) \subseteq \mathcal{N}(\mathfrak{f}_{2j+2})$ for all $j \in \mathbb{Z}_{0,\kappa-1}$. □

**Lemma 11.21** $\mathcal{F}^{\succcurlyeq,\mathrm{ld}}_{q,\kappa,\alpha,\beta} \subseteq \mathcal{F}^{\succcurlyeq,\mathrm{c}}_{q,\kappa,\alpha,\beta}$.

***Proof*** In view of Notations 11.14 and 11.18, we have $\mathcal{F}^{\succcurlyeq,\mathrm{ld}}_{q,0,\alpha,\beta} = \mathcal{F}^{\succcurlyeq}_{q,0,\alpha,\beta} = \mathcal{F}^{\succcurlyeq,\mathrm{c}}_{q,0,\alpha,\beta}$. Now assume $\kappa \geq 1$. Consider an arbitrary sequence $(s_j)_{j=0}^{\kappa}$ belonging to $\mathcal{F}^{\succcurlyeq,\mathrm{ld}}_{q,\kappa,\alpha,\beta}$. Let $m \in \mathbb{Z}_{1,\kappa}$. According to Lemma 11.20 and the notation given there, we have $\mathcal{N}(\mathsf{D}_{m-1}) = \mathcal{N}(\mathfrak{f}_{2m-1})$ and, because of Remark A.8, consequently $\mathbb{P}_{\mathcal{N}(\mathsf{D}_{m-1})} = \mathbb{P}_{\mathcal{N}(\mathfrak{f}_{2m-1})} = I_q - \mathfrak{f}^\dagger_{2m-1}\mathfrak{f}_{2m-1}$. Therefore, $\mathsf{D}_{m-1}(I_q - \mathfrak{f}^\dagger_{2m-1}\mathfrak{f}_{2m-1}) = \mathsf{D}_{m-1}\mathbb{P}_{\mathcal{N}(\mathsf{D}_{m-1})} = O$. Consequently, Lemma 11.15 provides $\mathcal{R}(\mathfrak{f}_{2m-1}\mathsf{D}^\dagger_m) \subseteq \mathcal{R}(\mathsf{D}_{m-1}\mathfrak{f}^\dagger_{2m-1})$. Since $m \in \mathbb{Z}_{1,\kappa}$ was chosen arbitrarily, Notation 11.14 shows then that $(s_j)_{j=0}^{\kappa}$ belongs to $\mathcal{F}^{\succcurlyeq,\mathrm{c}}_{q,\kappa,\alpha,\beta}$. □

Now we continue to use the notations given in Notation 3.4.

***Remark 11.22*** Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{m} \in \mathcal{F}^{\succcurlyeq}_{q,m,\alpha,\beta}$. From Notation 11.18 and [20, Prop. 6.13] then one can easily see that the following statements hold true:

(a) If $m = 2n$ with some $n \in \mathbb{N}_0$ and if moreover, in the case $n \geq 1$, the inequalities $\det H_{\alpha,n-1,\bullet} \neq 0$ and $\det H_{\alpha,n-1,\beta} \neq 0$ hold true, then $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{F}^{\succcurlyeq,\mathrm{ld}}_{q,2n,\alpha,\beta}$.

(b) Let $m = 2n+1$ with some $n \in \mathbb{N}_0$. Suppose $\det H_{\alpha,n,\bullet} \neq 0$. If $n \geq 1$, then additionally suppose that $\det H_{\alpha,n-1,\beta} \neq 0$. Then $(s_j)_{j=0}^{2n+1}$ belongs to $\mathcal{F}^{\succcurlyeq,\mathrm{ld}}_{q,2n+1,\alpha,\beta}$.

From Proposition 5.16 and [23, Lem. 3.35] we can see that especially $[\alpha, \beta]$-positive definite sequences $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\succ}$ fulfill the conditions stated in Remark 11.22:

***Remark 11.23*** Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\succcurlyeq}$. In view of Remark 11.19, one can easily check that the following statements hold true:

(a) Let $m = 2n$ with some $n \in \mathbb{N}_0$. Suppose $\det H_n \neq 0$. If $n \geq 1$, then additionally suppose that $\det H_{\alpha,n-1,\beta} \neq 0$. Then $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{F}_{q,2n,\alpha,\beta}^{\succ}$ and, in particular, to $\mathcal{F}_{q,2n,\alpha,\beta}^{\succcurlyeq,\mathrm{ld}}$.
(b) Let $m = 2n + 1$ with some $n \in \mathbb{N}_0$. If $n \geq 1$. Suppose $\det H_{\alpha,n,\bullet} \neq 0$ and $\det H_{\bullet,n,\beta} \neq 0$. Then $(s_j)_{j=0}^{2n+1}$ belongs to $\mathcal{F}_{q,2n+1,\alpha,\beta}^{\succ}$ and, in particular, to $\mathcal{F}_{q,2n+1,\alpha,\beta}^{\succcurlyeq,\mathrm{ld}}$.

The condition $\mathcal{R}(\mathfrak{f}_{2m}) \subseteq \mathcal{R}(\mathfrak{f}_{2m-1})$ stated in Notation 11.18 is equivalent to certain rank inequalities:

**Lemma 11.24** *Suppose $\kappa \geq 1$. Let $(s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq}$ and let $m \in \mathbb{Z}_{1,\kappa}$.*

(a) *The following statements are equivalent:*
   (i) $\mathcal{R}(\mathfrak{f}_{2m}) \subseteq \mathcal{R}(\mathfrak{f}_{2m-1})$.
   (ii) $\operatorname{rank} \mathfrak{f}_{2m} \leq \operatorname{rank} \mathsf{D}_m$.
   (iii) $\operatorname{rank} \mathsf{D}_{m-1} \leq \operatorname{rank} \mathfrak{f}_{2m-1}$.
(b) *Condition (i) is valid if and only if the following three conditions are fulfilled:*
   (iv) *If $m = 1$, then* $\operatorname{rank} H_0 \leq \operatorname{rank} H_{\alpha,0,\bullet}$.
   (v) *If $m = 2n+1$ for some $n \in \mathbb{N}$, then* $\operatorname{rank} H_n + \operatorname{rank} H_{\alpha,n-1,\beta} \leq \operatorname{rank} H_{\alpha,n,\bullet} + \operatorname{rank} H_{\bullet,n-1,\beta}$.
   (vi) *If $m = 2n$ for some $n \in \mathbb{N}$, then* $\operatorname{rank} H_{\alpha,n-1,\bullet} + \operatorname{rank} H_{\bullet,n-1,\beta} \leq \operatorname{rank} H_{n-1} + \operatorname{rank} H_{\alpha,n-1,\beta}$.
(c) *Suppose (i). Then $\mathcal{R}(\mathfrak{f}_{2m}) = \mathcal{R}(\mathsf{D}_m)$ and $\mathcal{R}(\mathfrak{f}_{2m-1}) = \mathcal{R}(\mathsf{D}_{m-1})$ and all the inequalities stated in part (b) are equalities.*

***Proof*** Taking into account Remark 5.25, [23, Lem. 3.32], and [20, Prop. 6.13], the proof is straight forward. We omit the details. □

The following example shows that in general $\mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq,\mathrm{ld}} \neq \mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq,\mathrm{c}}$:

***Example 11.25*** Let $\kappa = 1$ and let the sequence $(s_j)_{j=0}^1$ be given by $s_0 := I_q$ and $s_1 := \alpha I_q$. In view of (5.9) and $\delta > 0$, we then have the $\mathcal{F}_{\alpha,\beta}$-parameters $\mathfrak{f}_0 = s_0 = I_q$, $\mathfrak{f}_1 = s_1 - \alpha s_0 = O_{q\times q}$, and $\mathfrak{f}_2 = \beta s_0 - s_1 = \delta I_q$ as well as, in view of (5.8), the $[\alpha, \beta]$-interval lengths $\mathsf{D}_0 = \delta s_0 = \delta I_q$ and $\mathsf{D}_1 = -\alpha\beta s_0 + (\alpha + \beta)s_1 - s_1 s_0^\dagger s_1 = -\alpha\beta I_q + (\alpha + \beta)\alpha I_q - \alpha^2 I_q = O_{q\times q}$. Since all $\mathcal{F}_{\alpha,\beta}$-parameters of the sequence $(s_j)_{j=0}^1$ are non-negative Hermitian, Proposition 5.20 shows that $(s_j)_{j=0}^1$ belongs to $\mathcal{F}_{q,1,\alpha,\beta}^{\succcurlyeq}$. Moreover, since $\mathfrak{f}_1 \mathsf{D}_1^\dagger = O_{q\times q}$, the sequence $(s_j)_{j=0}^1$ fulfills the condition $\mathcal{R}(\mathfrak{f}_1 \mathsf{D}_1^\dagger) \subseteq \mathcal{R}(\mathsf{D}_0 \mathfrak{f}_1^\dagger)$ and, hence, in view of Notation 11.14, belongs to $\mathcal{F}_{q,1,\alpha,\beta}^{\succcurlyeq,\mathrm{c}}$. But since $\delta > 0$, we have $\mathcal{R}(\mathfrak{f}_2) \nsubseteq \mathcal{R}(\mathfrak{f}_1)$. Hence, in view of Notation 11.18, then $(s_j)_{j=0}^1 \notin \mathcal{F}_{q,1,\alpha,\beta}^{\succcurlyeq,\mathrm{ld}}$.

Now we give a counterexample which shows, that in general $\mathcal{F}^{\succcurlyeq,\mathrm{c}}_{q,\kappa,\alpha,\beta} \neq \mathcal{F}^{\succcurlyeq}_{q,\kappa,\alpha,\beta}$:

***Example 11.26*** Let the sequence $(s_j)_{j=0}^{1}$ be given by $s_0 := \left[\begin{smallmatrix} 1 & 1 \\ 1 & 3 \end{smallmatrix}\right]$ and $s_1 := \left[\begin{smallmatrix} \alpha & \alpha \\ \alpha & 2\alpha+\beta \end{smallmatrix}\right]$. In view of (5.9) and $\delta > 0$, we then have the $\mathcal{F}_{\alpha,\beta}$-parameters $\mathfrak{f}_0 = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 3 \end{smallmatrix}\right]$, $\mathfrak{f}_1 = \delta\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, and $\mathfrak{f}_2 = \delta\left[\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right]$ as well as with the use of (5.8), the $[\alpha, \beta]$-interval lengths $\mathsf{D}_0 = \delta\left[\begin{smallmatrix} 1 & 1 \\ 1 & 3 \end{smallmatrix}\right]$ and $\mathsf{D}_1 = \frac{\delta^2}{2}\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Since all $\mathcal{F}_{\alpha,\beta}$-parameters of the sequence $(s_j)_{j=0}^{1}$ are non-negative Hermitian, Proposition 5.20 shows that $(s_j)_{j=0}^{1}$ belongs to $\mathcal{F}^{\succcurlyeq}_{2,1,\alpha,\beta}$. On the other hand, because of $\mathfrak{f}_1\mathsf{D}_1^\dagger = \frac{2}{\delta}\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $\mathsf{D}_0\mathfrak{f}_1^\dagger = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 3 \end{smallmatrix}\right]$ neither $\mathcal{R}(\mathfrak{f}_1\mathsf{D}_1^\dagger) \subseteq \mathcal{R}(\mathsf{D}_0\mathfrak{f}_1^\dagger)$ nor $\mathcal{R}(\mathfrak{f}_1\mathsf{D}_1^\dagger) \supseteq \mathcal{R}(\mathsf{D}_0\mathfrak{f}_1^\dagger)$ are fulfilled. Hence, from Notation 11.14 we conclude $(s_j)_{j=0}^{1} \notin \mathcal{F}^{\succcurlyeq,c}_{2,1,\alpha,\beta}$.

For sequences belonging to the subclass $\mathcal{F}^{\succcurlyeq,\mathrm{c}}_{q,\kappa,\alpha,\beta}$, we get the following $q \times q$ block structure of the $2q \times 2q$ matrix polynomial $\ddot{\mathfrak{V}}_m$ introduced in Remark 11.8:

**Proposition 11.27** *Let $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{\succcurlyeq,\mathrm{c}}_{q,\kappa,\alpha,\beta}$ with sequence of $[\alpha, \beta]$-interval lengths $(\mathsf{D}_j)_{j=0}^{\kappa}$. For each $m \in \mathbb{Z}_{0,\kappa}$, let $\ddot{\mathfrak{V}}_m : \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ be given by Remark 11.8 and Notation 11.3. Then:*

*(a) For each $n \in \mathbb{N}_0$ such that $2n \leq \kappa$ and each $z \in \mathbb{C}$, then*

$$\ddot{\mathfrak{V}}_{2n}(z) = -(z-\beta)^n \delta \begin{bmatrix} -(\beta - z)\ddot{\mathbf{q}}_{4n+1}(z)\mathsf{D}_{2n}^\dagger & -\delta^{2n-1}\ddot{\mathbf{q}}_{4n+2}(z) \\ (\beta - z)\ddot{\mathbf{p}}_{4n+1}(z)\mathsf{D}_{2n}^\dagger & \delta^{2n-1}\ddot{\mathbf{p}}_{4n+2}(z) \end{bmatrix}. \tag{11.11}$$

*(b) If $\kappa \geq 1$, for each $n \in \mathbb{N}_0$ such that $2n+1 \leq \kappa$ and each $z \in \mathbb{C}$, then*

$$\ddot{\mathfrak{V}}_{2n+1}(z) = (z-\beta)^{n+1} \delta \begin{bmatrix} \ddot{\mathbf{q}}_{4n+3}(z)\mathsf{D}_{2n+1}^\dagger & -\delta^{2n}\ddot{\mathbf{q}}_{4n+4}(z) \\ -\ddot{\mathbf{p}}_{4n+3}(z)\mathsf{D}_{2n+1}^\dagger & \delta^{2n}\ddot{\mathbf{p}}_{4n+4}(z) \end{bmatrix}. \tag{11.12}$$

***Proof*** We consider an arbitrary $z \in \mathbb{C}$ and set $x := z - \alpha$ as well as $y := \beta - z$. We have $(\delta^{-1}\mathsf{D}_0)^\dagger = \delta\mathsf{D}_0^\dagger$ and, because of Remark A.8, consequently $\mathbb{P}_{\mathcal{R}(\delta^{-1}\mathsf{D}_0)} = \mathsf{D}_0\mathsf{D}_0^\dagger$. Since we know from (5.8) and (5.9) that $\mathsf{D}_0 = \delta\mathfrak{f}_0$ is valid, we get $\mathbb{P}_{\mathcal{R}(\delta^{-1}\mathsf{D}_0)} = \delta\mathfrak{f}_0\mathsf{D}_0^\dagger$. Remark 9.6 yields $s_0^{\{0\}} = \delta^{-1}\mathsf{D}_0$. Using additionally Remark 11.8, Notation 11.3 as well as (8.3) and (8.4), then

$$\begin{aligned}\ddot{\mathfrak{V}}_0(z) = V_0 = V_{s_0^{\{0\}}} = \ddot{V}_{\delta^{-1}\mathsf{D}_0}(z) &= \begin{bmatrix} y\mathbb{P}_{\mathcal{R}(\delta^{-1}\mathsf{D}_0)} & \delta^{-1}\mathsf{D}_0 \\ -yx(\delta^{-1}\mathsf{D}_0)^\dagger & yI_q \end{bmatrix} \\ &= \begin{bmatrix} y\delta\mathfrak{f}_0\mathsf{D}_0^\dagger & \mathfrak{f}_0 \\ -yx\delta\mathsf{D}_0^\dagger & yI_q \end{bmatrix} = -(-y)^0\delta \begin{bmatrix} -y\ddot{\mathbf{q}}_1(z)\mathsf{D}_0^\dagger & -\delta^{-1}\ddot{\mathbf{q}}_2(z) \\ y\ddot{\mathbf{p}}_1(z)\mathsf{D}_0^\dagger & \delta^{-1}\ddot{\mathbf{p}}_2(z) \end{bmatrix}\end{aligned} \tag{11.13}$$

follows. Hence, (11.11) is checked for $n = 0$. If $\kappa = 0$, then the proof is complete.

Now we consider the case $\kappa \geq 1$. In accordance to Notation 8.3, we use particularly the choice $\Gamma_j := O_{q\times q}$ for all $j \in \mathbb{Z}_{1,\kappa}$. Thus, the sequences $(\mathbf{A}_j)_{j=1}^{\kappa}$

and $(\mathbf{B}_j)_{j=1}^{\kappa}$ defined in (8.5) admit then the representations $\mathbf{A}_m = \mathsf{D}_{m-1}^{\dagger}\mathfrak{f}_{2m}$ and $\mathbf{B}_m = \mathsf{D}_{m-1}^{\dagger}\mathfrak{f}_{2m-1} + \mathbb{P}_{\mathcal{N}(\mathsf{D}_{m-1})}$ for every choice of $m \in \mathbb{Z}_{1,\kappa}$. This implies

$$\mathsf{D}_{m-1}^{\dagger}\mathsf{D}_{m-1}\mathbf{A}_m = \mathsf{D}_{m-1}^{\dagger}\mathfrak{f}_{2m} = \mathbf{A}_m \tag{11.14}$$

for all $m \in \mathbb{Z}_{1,\kappa}$. For all $m \in \mathbb{Z}_{1,\kappa}$, from Lemma 11.11 we know that (11.1) is valid. Since $(s_j)_{j=0}^{\kappa}$ belongs to $\mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq,c}$, according to Notation 11.14 and Lemma 11.15, moreover $\mathsf{D}_{m-1}(I_q - \mathfrak{f}_{2m-1}^{\dagger}\mathfrak{f}_{2m-1})\mathsf{D}_{m-1}^{\dagger}\mathfrak{f}_{2m-1}\mathsf{D}_m^{\dagger} = O_{q\times q}$ holds true. Thus, applying Lemma 11.13 yields

$$\ddot{U}_{a_0^{\{m-1\}},s_0^{\{m-1\}}}(z) = \begin{bmatrix} y\mathsf{D}_{m-1}\mathbf{A}_m\mathsf{D}_m^{\dagger} & \delta^{m-1}\mathsf{D}_{m-1}\mathbf{A}_m \\ -yx\delta^{-m+2}\mathbf{B}_m\mathsf{D}_m^{\dagger} & y\delta\mathbf{B}_m \end{bmatrix} \tag{11.15}$$

for all $m \in \mathbb{Z}_{1,\kappa}$. Using (11.1), (11.13), and (11.15) for $m = 1$, we get

$$\begin{aligned} \ddot{\mathfrak{V}}_1(z) &= -(-y)^0\delta \begin{bmatrix} -y\ddot{\mathbf{q}}_1(z)\mathsf{D}_0^{\dagger} & -\delta^{-1}\ddot{\mathbf{q}}_2(z) \\ y\ddot{\mathbf{p}}_1(z)\mathsf{D}_0^{\dagger} & \delta^{-1}\ddot{\mathbf{p}}_2(z) \end{bmatrix} \begin{bmatrix} y\mathsf{D}_0\mathbf{A}_1\mathsf{D}_1^{\dagger} & \delta^0\mathsf{D}_0\mathbf{A}_1 \\ -yx\delta^1\mathbf{B}_1\mathsf{D}_1^{\dagger} & y\delta\mathbf{B}_1 \end{bmatrix} \\ &= -\delta \begin{bmatrix} W_1 & X_1 \\ Y_1 & Z_1 \end{bmatrix}, \end{aligned}$$

where $W_1 := [-y\ddot{\mathbf{q}}_1(z)\mathsf{D}_0^{\dagger}](y\mathsf{D}_0\mathbf{A}_1\mathsf{D}_1^{\dagger}) + [-\delta^{-1}\ddot{\mathbf{q}}_2(z)](-yx\delta\mathbf{B}_1\mathsf{D}_1^{\dagger})$ and $X_1 := [-y\ddot{\mathbf{q}}_1(z)\mathsf{D}_0^{\dagger}](\mathsf{D}_0\mathbf{A}_1) + [-\delta^{-1}\ddot{\mathbf{q}}_2(z)](y\delta\mathbf{B}_1)$ as well as $Y_1 := [y\ddot{\mathbf{p}}_1(z)\mathsf{D}_0^{\dagger}](y\mathsf{D}_0\mathbf{A}_1\mathsf{D}_1^{\dagger}) + [\delta^{-1}\ddot{\mathbf{p}}_2(z)](-yx\delta\mathbf{B}_1\mathsf{D}_1^{\dagger})$ and $Z_1 := [y\ddot{\mathbf{p}}_1(z)\mathsf{D}_0^{\dagger}](\mathsf{D}_0\mathbf{A}_1) + [\delta^{-1}\ddot{\mathbf{p}}_2(z)](y\delta\mathbf{B}_1)$. Applying (11.14) for $m = 1$ as well as (8.6)–(8.9), it follows $W_1 = y\ddot{\mathbf{q}}_{4-1}(z)\mathsf{D}_1^{\dagger}$ and $X_1 = -y\ddot{\mathbf{q}}_4(z)$ as well as $Y_1 = -y\ddot{\mathbf{p}}_{4-1}(z)\mathsf{D}_1^{\dagger}$ and $Z_1 = y\ddot{\mathbf{p}}_4(z)$. Consequently, (11.12) is proved for $n = 0$. If $\kappa \leq 1$, then the proof is finished.

Now assume $\kappa \geq 2$. Then there exists a positive integer $l$ with $2l \leq \kappa$ such that (11.11) and (11.12) are valid for each $n \in \mathbb{Z}_{0,l-1}$. We are going to check (11.11) for $n = l$. Using (11.1) and (11.15) for $m = 2l$, we obtain

$$\begin{aligned} \ddot{\mathfrak{V}}_{2l}(z) &= \ddot{\mathfrak{V}}_{2l-1}(z)\ddot{U}_{a_0^{\{2l-1\}},s_0^{\{2l-1\}}}(z) \\ &= (-y)^l\delta \begin{bmatrix} \ddot{\mathbf{q}}_{4l-1}(z)\mathsf{D}_{2l-1}^{\dagger} & -\delta^{2l-2}\ddot{\mathbf{q}}_{4l}(z) \\ -\ddot{\mathbf{p}}_{4l-1}(z)\mathsf{D}_{2l-1}^{\dagger} & \delta^{2l-2}\ddot{\mathbf{p}}_{4l}(z) \end{bmatrix} \\ &\quad\times \begin{bmatrix} y\mathsf{D}_{2l-1}\mathbf{A}_{2l}\mathsf{D}_{2l}^{\dagger} & \delta^{2l-1}\mathsf{D}_{2l-1}\mathbf{A}_{2l} \\ -yx\delta^{-2l+2}\mathbf{B}_{2l}\mathsf{D}_{2l}^{\dagger} & y\delta\mathbf{B}_{2l} \end{bmatrix} \\ &= (-y)^l\delta \begin{bmatrix} W_{2l} & X_{2l} \\ Y_{2l} & Z_{2l} \end{bmatrix}, \end{aligned} \tag{11.16}$$

where

$$W_{2l} := [\ddot{\mathbf{q}}_{4l-1}(z)\mathsf{D}_{2l-1}^{\dagger}](y\mathsf{D}_{2l-1}\mathbf{A}_{2l}\mathsf{D}_{2l}^{\dagger}) + [-\delta^{2l-2}\ddot{\mathbf{q}}_{4l}(z)](-yx\delta^{-2l+2}\mathbf{B}_{2l}\mathsf{D}_{2l}^{\dagger}),$$
$$X_{2l} := [\ddot{\mathbf{q}}_{4l-1}(z)\mathsf{D}_{2l-1}^{\dagger}](\delta^{2l-1}\mathsf{D}_{2l-1}\mathbf{A}_{2l}) + [-\delta^{2l-2}\ddot{\mathbf{q}}_{4l}(z)](y\delta\mathbf{B}_{2l}),$$
$$Y_{2l} := [-\ddot{\mathbf{p}}_{4l-1}(z)\mathsf{D}_{2l-1}^{\dagger}](y\mathsf{D}_{2l-1}\mathbf{A}_{2l}\mathsf{D}_{2l}^{\dagger}) + [\delta^{2l-2}\ddot{\mathbf{p}}_{4l}(z)](-yx\delta^{-2l+2}\mathbf{B}_{2l}\mathsf{D}_{2l}^{\dagger}),$$

and

$$Z_{2l} := \left[-\ddot{\mathbf{p}}_{4l-1}(z)\mathsf{D}_{2l-1}^{\dagger}\right](\delta^{2l-1}\mathsf{D}_{2l-1}\mathbf{A}_{2l}) + [\delta^{2l-2}\ddot{\mathbf{p}}_{4l}(z)](y\delta\mathbf{B}_{2l}).$$

In view of $2 \le 2l+1 \le \kappa+1$, (11.14) for $m = 2l$, (8.10)–(8.13), and (11.16), we conclude

$$\ddot{\mathfrak{V}}_{2l}(z) = -(-y)^l\delta\begin{bmatrix} -y\ddot{\mathbf{q}}_{4l+1}(z)\mathsf{D}_{2l}^{\dagger} & -\delta^{2l-1}\ddot{\mathbf{q}}_{4l+2}(z) \\ y\ddot{\mathbf{p}}_{4l+1}(z)\mathsf{D}_{2l}^{\dagger} & \delta^{2l-1}\ddot{\mathbf{p}}_{4l+2}(z)\end{bmatrix}. \tag{11.17}$$

Hence, (11.11) is proved for $n = l$. If $\kappa = 2$, then the proof is complete.

We consider the case $2l+1 \le \kappa$. Using (11.1), (11.17), and (11.15) for $m = 2l+1$, we get

$$\begin{aligned}\ddot{\mathfrak{V}}_{2l+1}(z) &= \ddot{\mathfrak{V}}_{2l}(z)\ddot{U}_{a_0^{\{2l\}},s_0^{\{2l\}}}(z)\\ &= -(-y)^l\delta\begin{bmatrix} -y\ddot{\mathbf{q}}_{4l+1}(z)\mathsf{D}_{2l}^{\dagger} & -\delta^{2l-1}\ddot{\mathbf{q}}_{4l+2}(z) \\ y\ddot{\mathbf{p}}_{4l+1}(z)\mathsf{D}_{2l}^{\dagger} & \delta^{2l-1}\ddot{\mathbf{p}}_{4l+2}(z)\end{bmatrix}\\ &\quad\times\begin{bmatrix} y\mathsf{D}_{2l}\mathbf{A}_{2l+1}\mathsf{D}_{2l+1}^{\dagger} & \delta^{2l}\mathsf{D}_{2l}\mathbf{A}_{2l+1} \\ -yx\delta^{-2l+1}\mathbf{B}_{2l+1}\mathsf{D}_{2l+1}^{\dagger} & y\delta\mathbf{B}_{2l+1}\end{bmatrix}\\ &= -(-y)^l\delta\begin{bmatrix} W_{2l+1} & X_{2l+1} \\ Y_{2l+1} & Z_{2l+1}\end{bmatrix},\end{aligned}$$

where

$$\begin{aligned}W_{2l+1} &:= \left[-y\ddot{\mathbf{q}}_{4l+1}(z)\mathsf{D}_{2l}^{\dagger}\right]\left(y\mathsf{D}_{2l}\mathbf{A}_{2l+1}\mathsf{D}_{2l+1}^{\dagger}\right)\\ &\quad+\left[-\delta^{2l-1}\ddot{\mathbf{q}}_{4l+2}(z)\right]\left(-yx\delta^{-2l+1}\mathbf{B}_{2l+1}\mathsf{D}_{2l+1}^{\dagger}\right),\\ X_{2l+1} &:= \left[-y\ddot{\mathbf{q}}_{4l+1}(z)\mathsf{D}_{2l}^{\dagger}\right](\delta^{2l}\mathsf{D}_{2l}\mathbf{A}_{2l+1}) + \left[-\delta^{2l-1}\ddot{\mathbf{q}}_{4l+2}(z)\right]\left(y\delta\mathbf{B}_{2l+1}\right),\\ Y_{2l+1} &:= \left[y\ddot{\mathbf{p}}_{4l+1}(z)\mathsf{D}_{2l}^{\dagger}\right]\left(y\mathsf{D}_{2l}\mathbf{A}_{2l+1}\mathsf{D}_{2l+1}^{\dagger}\right)\\ &\quad+\left[\delta^{2l-1}\ddot{\mathbf{p}}_{4l+2}(z)\right]\left(-yx\delta^{-2l+1}\mathbf{B}_{2l+1}\mathsf{D}_{2l+1}^{\dagger}\right),\end{aligned}$$

and

$$Z_{2l+1} := \left[y\ddot{\mathbf{p}}_{4l+1}(z)\mathsf{D}_{2l}^{\dagger}\right](\delta^{2l}\mathsf{D}_{2l}\mathbf{A}_{2l+1}) + \left[\delta^{2l-1}\ddot{\mathbf{p}}_{4l+2}(z)\right](y\delta\mathbf{B}_{2l+1}).$$

Taking into account $2 \leq 2(l+1) \leq \kappa + 1$ as well as (11.14) for $m = 2l+1$ and (8.6)–(8.9), we infer

$$\ddot{\mathfrak{V}}_{2l+1}(z) = (-y)^{l+1}\delta \begin{bmatrix} \ddot{\mathbf{q}}_{4l+3}(z)\mathsf{D}_{2l+1}^{\dagger} & -\delta^{2l}\ddot{\mathbf{q}}_{4l+4}(z) \\ -\ddot{\mathbf{p}}_{4l+3}(z)\mathsf{D}_{2l+1}^{\dagger} & \delta^{2l}\ddot{\mathbf{p}}_{4l+4}(z) \end{bmatrix}.$$

Therefore, the assertion is proved inductively. □

We have already seen in Remark 8.9 that the matrix polynomials $\ddot{\mathbf{p}}_j$ are closely related to the MROS $(\ddot{\mathbf{r}}_l)_{l=0}^{\kappa}$, $(\ddot{\mathbf{t}}_l)_{l=0}^{\tau}$, $(\ddot{\mathbf{t}}_{vl})_{l=0}^{\tau}$ and $(\ddot{\mathbf{t}}_{xl})_{l=0}^{\tau}$ with respect to the sequences $(s_j)_{j=0}^{2\kappa}$, $(a_j)_{j=0}^{2\tau}$, $(b_j)_{j=0}^{2\tau}$, and $(c_j)_{j=0}^{2\rho}$, respectively, given by Notation 3.3.

**Corollary 11.28** *Assume* $\kappa \geq 2$. *Let* $(s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq,c}$ *with sequence of* $[\alpha,\beta]$*-interval lengths* $(\mathsf{D}_j)_{j=0}^{\kappa}$. *Then*

$$\begin{aligned}\ddot{\mathfrak{V}}_{2n}(z) &\\ &= -(z-\beta)^n\delta \begin{bmatrix} -(\beta - z)[\ddot{\mathbf{t}}_n^{[\![a]\!]}(z) + s_0\ddot{\mathbf{t}}_n(z)]\mathsf{D}_{2n}^{\dagger} & \delta^{2n-1}[\ddot{\mathbf{v}}_n^{[\![b]\!]}(z) - s_0\ddot{\mathbf{v}}_n(z)] \\ (\beta - z)(z-\alpha)\ddot{\mathbf{t}}_n(z)\mathsf{D}_{2n}^{\dagger} & -(\beta - z)\delta^{2n-1}\ddot{\mathbf{v}}_n(z) \end{bmatrix}\end{aligned}$$

*for every choice of* $n \in \mathbb{N}$ *such that* $2n \leq \kappa$ *and for all* $z \in \mathbb{C}$ *as well as*

$$\begin{aligned}\ddot{\mathfrak{V}}_{2n+1}(z) &\\ &= (z-\beta)^{n+1}\delta \begin{bmatrix} -(\ddot{\mathbf{x}}_n^{[\![c]\!]}(z) + [(\alpha+\beta-z)s_0 - s_1]\ddot{\mathbf{x}}_n(z))\mathsf{D}_{2n+1}^{\dagger} & -\delta^{2n}\ddot{\mathbf{r}}_{n+1}^{[\![s]\!]}(z) \\ (\beta - z)(z-\alpha)\ddot{\mathbf{x}}_n(z)\mathsf{D}_{2n+1}^{\dagger} & \delta^{2n}\ddot{\mathbf{r}}_{n+1}(z) \end{bmatrix}\end{aligned}$$

*for every choice of* $n \in \mathbb{N}$ *such that* $2n+1 \leq \kappa$ *and for all* $z \in \mathbb{C}$ *hold true.*

***Proof*** Considering an arbitrary $z \in \mathbb{C}$, according to Remark 8.9, we have $\ddot{\mathbf{p}}_{4l}(z) = \ddot{\mathbf{r}}_l(z)$ for all $l \in \mathbb{N}_0$ with $2l-1 \leq \kappa$ and, moreover, $\ddot{\mathbf{p}}_{4l+1}(z) = (z-\alpha)\ddot{\mathbf{t}}_l(z)$ and $\ddot{\mathbf{p}}_{4l+2}(z) = -(\beta - z)\ddot{\mathbf{v}}_l(z)$ for all $l \in \mathbb{N}_0$ with $2l \leq \kappa$ as well as $\ddot{\mathbf{p}}_{4l+3}(z) = -(\beta - z)(z-\alpha)\ddot{\mathbf{x}}_l(z)$ for all $l \in \mathbb{N}_0$ fulfilling $2l+1 \leq \kappa$. Taking into account Notation 11.14, we see that the sequence $(s_j)_{j=0}^{\kappa}$ belongs to $\mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq}$. Thus, using Proposition 8.16, we get $\ddot{\mathbf{q}}_{4l}(z) = \ddot{\mathbf{p}}_{4l}^{[\![s]\!]}(z) = \ddot{\mathbf{r}}_l^{[\![s]\!]}(z)$ for all $l \in \mathbb{N}_0$ with $2l-1 \leq \kappa$ and all $z \in \mathbb{C}$. Remarks 8.8 and 8.9 yield $\deg \ddot{\mathbf{t}}_l = \deg \ddot{\mathbf{v}}_l = l \geq 1$ for all $l \in \mathbb{N}$ fulfilling $2l \leq \kappa$. Because of Proposition 8.16, Lemma C.3, and Remark C.1, we conclude then $\ddot{\mathbf{q}}_{4l+1}(z) = \ddot{\mathbf{p}}_{4l+1}^{[\![s]\!]}(z) = \ddot{\mathbf{t}}_l^{[\![a]\!]}(z) + s_0\ddot{\mathbf{t}}_l(z)$ and $\ddot{\mathbf{q}}_{4l+2}(z) = \ddot{\mathbf{p}}_{4l+2}^{[\![s]\!]}(z) = -[\ddot{\mathbf{v}}_l^{[\![b]\!]}(z) - s_0\ddot{\mathbf{v}}_l(z)]$ for all $l \in \mathbb{N}$ such that $2l \leq \kappa$ and all $z \in \mathbb{C}$. Moreover, Remark 8.8 provides $\deg \ddot{\mathbf{x}}_l = l \geq 1$ for all $l \in \mathbb{N}$ fulfilling $2l+1 \leq \kappa$. Thus, using Lemmata 8.16 and C.4 and Remark C.1, we recognize that $\ddot{\mathbf{q}}_{4l+3}(z) = \ddot{\mathbf{p}}_{4l+3}^{[\![s]\!]}(z) = -(\ddot{\mathbf{x}}_l^{[\![c]\!]}(z) + [(\alpha+\beta-z)s_0 - s_1]\ddot{\mathbf{x}}_l(z))$ holds true for all $l \in \mathbb{N}$ fulfilling $2l+1 \leq \kappa$ and all $z \in \mathbb{C}$. Applying Proposition 11.27 completes the proof. □

Given arbitrary $m \in \mathbb{N}_0$ and $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\succcurlyeq}$, in [23, Thm. 14.2] a parameterization of the set $\mathcal{R}_q[[\alpha,\beta]; (s_j)_{j=0}^m, =]$ is proved. Under the assumption that the given sequence $(s_j)_{j=0}^m$ belongs to the subclass $\mathcal{F}_{q,m,\alpha,\beta}^{\succcurlyeq,\mathrm{c}}$ of $\mathcal{F}_{q,m,\alpha,\beta}^{\succcurlyeq}$, now we are going to check a parameterization of the set $\mathcal{R}_q[[\alpha,\beta]; (s_j)_{j=0}^m, =]$ by the use of orthogonal matrix polynomials.

In the scalar case $q = 1$, a parameterization of the solution set of Problem $\mathsf{FP}[[\alpha,\beta]; (s_j)_{j=0}^m, =]$ in form of a linear fractional transformation using orthogonal polynomials can be found in [28, Theorems 7.1 and 7.2] for $[\alpha,\beta]$-positive definite sequences $(s_j)_{j=0}^m$. For the non-degenerate matrix case $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\succ}$, in [6, Thm. 6.12] and [7, Thm. 6.14] parameterizations of the set $\mathcal{R}_q[[\alpha,\beta]; (s_j)_{j=0}^m, =]$ are derived, using the corresponding systems of Potapov's fundamental matrix inequalities. Choque Rivero showed in [5, Theorems 3.8 and 3.5] how these parameterizations can be represented applying orthogonal matrix polynomials. As already mentioned above, our next goal is to prove parameterizations of the set $\mathcal{R}_q[[\alpha,\beta]; (s_j)_{j=0}^m, =]$ in terms of orthogonal matrix polynomials in a more general case, which contains the non-degenerate case as a special situation.

Let $m \in \mathbb{N}_0$. Given a sequence $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\succcurlyeq}$ with sequence of $[\alpha,\beta]$-interval lengths $(\mathsf{D}_j)_{j=0}^m$, Remark 9.6 provides $s_0^{\{m\}} = \delta^{m-1}\mathsf{D}_m$. Since $\delta > 0$ is valid, we observe that $\mathcal{R}(s_0^{\{m\}}) = \mathcal{R}(\mathsf{D}_m)$ and, taking into account Notation 10.6, moreover that $\ddot{\mathcal{P}}[s_0^{\{m\}}] = \ddot{\mathcal{P}}[\mathsf{D}_m]$. Taking additionally into account Proposition 11.27, then, using Theorem 11.9, we are able to formulate the main result of this subsection. Precisely, Theorem 11.29 below points out a parameterization of the solution set of Problem $\mathsf{FP}[[\alpha,\beta]; (s_j)_{j=0}^m, =]$ for sequences belonging to the subclass $\mathcal{F}_{q,m,\alpha,\beta}^{\succcurlyeq,\mathrm{c}}$ utilizing equivalence classes of pairs of $\ddot{\mathcal{P}}[\mathsf{D}_m]$:

**Theorem 11.29** *Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha < \beta$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\succcurlyeq,\mathrm{c}}$ with sequence of $[\alpha,\beta]$-interval lengths $(\mathsf{D}_j)_{j=0}^m$. Let $\varepsilon_m \colon \mathbb{C} \to \mathbb{C}$ be defined by $\varepsilon_m(z) := z - \beta$ if the integer $m$ is even and by $\varepsilon_m(z) := 1$ if $m$ is odd. Let $\mathbf{p}_{2m+1}^{\diamond} := \varepsilon_m \ddot{\mathbf{p}}_{2m+1}$ and $\mathbf{q}_{2m+1}^{\diamond} := \varepsilon_m \ddot{\mathbf{q}}_{2m+1}$. Furthermore, let $\tilde{\mathbf{p}}_{2m+1}$, $\tilde{\mathbf{q}}_{2m+1}$, $\tilde{\mathbf{p}}_{2m+2}$, and $\tilde{\mathbf{q}}_{2m+2}$ be the restrictions of $\mathbf{p}_{2m+1}^{\diamond}$, $\mathbf{q}_{2m+1}^{\diamond}$, $\ddot{\mathbf{p}}_{2m+2}$, and $\ddot{\mathbf{q}}_{2m+2}$ onto $\mathbb{C} \setminus [\alpha,\beta]$, respectively.*

(a) *For all $\Gamma \in \langle \ddot{\mathcal{P}}[\mathsf{D}_m] \rangle$ and all $[G_1; G_2] \in \Gamma$, the function $\det(\tilde{\mathbf{p}}_{2m+1}\mathsf{D}_m^{\dagger}G_1 - \delta^{m-1}\tilde{\mathbf{p}}_{2m+2}G_2)$ does not vanish identically and $F$ defined by*

$$F = -(\tilde{\mathbf{q}}_{2m+1}\mathsf{D}_m^{\dagger}G_1 - \delta^{m-1}\tilde{\mathbf{q}}_{2m+2}G_2)(\tilde{\mathbf{p}}_{2m+1}\mathsf{D}_m^{\dagger}G_1 - \delta^{m-1}\tilde{\mathbf{p}}_{2m+2}G_2)^{-1} \tag{11.18}$$

*belongs to $\mathcal{R}_q[[\alpha,\beta]; (s_j)_{j=0}^m, =]$.*

(b) *For each $F \in \mathcal{R}_q[[\alpha,\beta]; (s_j)_{j=0}^m, =]$, there exists a unique $\Gamma \in \langle \ddot{\mathcal{P}}[\mathsf{D}_m] \rangle$ such that equation (11.18) holds true for all $[G_1; G_2] \in \Gamma$, namely the equivalence class $\Gamma$ which is generated by the $m$-th $\mathcal{F}_{\alpha,\beta}$-transformed pair $\mathbf{P}\ddot{\mathbf{G}}_m(F; (s_j)_{j=0}^m)$ of $F$ with respect to $(s_j)_{j=0}^m$.*

***Proof*** Let us consider the case that $m = 2n$ with some non-negative integer $n$. Because of the assumption, we have $(s_j)_{j=0}^{2n} \in \mathcal{F}_{q,2n,\alpha,\beta}^{\succcurlyeq,c}$. Thus, by virtue of Notation 11.14, the sequence $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{F}_{q,2n,\alpha,\beta}^{\succcurlyeq}$ as well. Consequently, Remark 9.6 shows that $s_0^{\{2n\}} = \delta^{2n-1}\mathsf{D}_{2n}$ holds true, which, because of $\delta > 0$, implies $\mathcal{R}(s_0^{\{2n\}}) = \mathcal{R}(\mathsf{D}_{2n})$ and, in view of Notation 10.6, moreover $\ddot{\mathcal{P}}[s_0^{\{2n\}}] = \ddot{\mathcal{P}}[\mathsf{D}_{2n}]$. Comparing the notations given in Theorem 11.9 and Proposition 11.27, we get

$$\tilde{\mathfrak{w}}_{2n}(z) = -(z-\beta)^n\delta\tilde{\mathbf{q}}_{4n+1}(z)\mathsf{D}_{2n}^\dagger, \tag{11.19}$$

$$\tilde{\mathfrak{y}}_{2n}(z) = -(z-\beta)^n\delta\big[-\tilde{\mathbf{p}}_{4n+1}(z)\big]\mathsf{D}_{2n}^\dagger, \tag{11.20}$$

$$\tilde{\mathfrak{x}}_{2n}(z) = -(z-\beta)^n\delta\big[-\delta^{2n-1}\tilde{\mathbf{q}}_{4n+2}(z)\big], \tag{11.21}$$

and

$$\tilde{\mathfrak{z}}_{2n}(z) = -(z-\beta)^n\delta\big[\delta^{2n-1}\tilde{\mathbf{p}}_{4n+2}(z)\big] \tag{11.22}$$

for all $z \in \mathbb{C} \setminus [\alpha, \beta]$. Taking into account $\ddot{\mathcal{P}}[s_0^{\{2n\}}] = \ddot{\mathcal{P}}[\mathsf{D}_{2n}]$, (11.19)–(11.22), and the fact that $-(z-\beta)^n\delta \neq 0$ is valid for all $z \in \mathbb{C} \setminus [\alpha, \beta]$, then the application of Theorem 11.9 completes the proof in the case of an even non-negative integer $m$. If $m = 2n + 1$ with some non-negative integer $n$, then we get with the notations given in Theorem 11.9 and Proposition 11.27 that

$$\tilde{\mathfrak{w}}_{2n+1}(z) = (z-\beta)^{n+1}\delta\tilde{\mathbf{q}}_{4n+3}(z)\mathsf{D}_{2n+1}^\dagger, \tag{11.23}$$

$$\tilde{\mathfrak{y}}_{2n+1}(z) = -(z-\beta)^{n+1}\delta\tilde{\mathbf{p}}_{4n+3}(z)\mathsf{D}_{2n+1}^\dagger, \tag{11.24}$$

$$\tilde{\mathfrak{x}}_{2n+1}(z) = -(z-\beta)^{n+1}\delta^{2n+1}\tilde{\mathbf{q}}_{4n+4}(z), \tag{11.25}$$

and

$$\tilde{\mathfrak{z}}_{2n+1}(z) = (z-\beta)^{n+1}\delta^{2n+1}\tilde{\mathbf{p}}_{4n+4}(z) \tag{11.26}$$

are valid for all $z \in \mathbb{C} \setminus [\alpha, \beta]$, so that the assertion can be checked analogously. □

Observe that, according to Remark 11.16, in the scalar case $q = 1$ the statements of Theorem 11.29 hold true if the given sequence $(s_j)_{j=0}^m$ belongs to $\mathcal{F}_{1,m,\alpha,\beta}^{\succcurlyeq}$ instead $\mathcal{F}_{1,m,\alpha,\beta}^{\succcurlyeq,c}$.

In [23, Thm. 14.2], we already obtained a result analogous to Theorem 11.29 for an arbitrarily given sequence $(s_j)_{j=0}^m$ belonging to $\mathcal{F}_{q,m,\alpha,\beta}^{\succcurlyeq}$. (With regard to Theorem 3.6, this concerns the general matrix case.) The new aspect in Theorem 11.29 consists in a parameterization where orthogonal matrix polynomials are used.

## 12 Description of the Values of all Solutions using Independent Parameters

Let us consider an arbitrary non-negative integer $m$ and an arbitrary sequence $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\succcurlyeq}$ with sequence of $[\alpha,\beta]$-interval lengths $(\mathsf{D}_j)_{j=0}^m$. Obviously, $0 \leq \operatorname{rank} \mathsf{D}_m \leq q$ holds true. We distinguish the following three cases:

(I) $\operatorname{rank} \mathsf{D}_m = q$.
(II) $1 \leq \operatorname{rank} \mathsf{D}_m \leq q-1$.
(III) $\operatorname{rank} \mathsf{D}_m = 0$.

It will turn out that the characteristics of the solutions differ. Therefore, in the following, we consider the three cases separately. Our observations are mainly guided by Theorem 11.29. We start with the non-degenerate case (I). This case can be equivalently characterized using the following:

***Remark 12.1*** ([23, Rem. 3.67]) Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\succcurlyeq}$. Then $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\succ}$ if and only if $\operatorname{rank} \mathsf{D}_m = q$.

**Theorem 12.2** *Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha < \beta$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\succ}$. Then all statements of Theorem 11.29 are valid for the class $\mathcal{PR}_q(\mathbb{C}\setminus[\alpha,\beta])$ instead of the class $\ddot{\mathcal{P}}[\mathsf{D}_m]$.*

***Proof*** From Proposition 11.17 we know that $(s_j)_{j=0}^m$ belongs to $\mathcal{F}_{q,m,\alpha,\beta}^{\succcurlyeq,\mathrm{c}}$ as well. According to Remark 12.1, we have $\operatorname{rank} \mathsf{D}_m = q$. Consequently, Remark 10.7 yields $\ddot{\mathcal{P}}[\mathsf{D}_m] = \mathcal{PR}_q(\mathbb{C}\setminus[\alpha,\beta])$. Applying Theorem 11.29 completes the proof. □

Now we turn our attention to the degenerate, but not completely degenerate case (II). If the rank $r$ of the matrix $\mathsf{D}_m = \delta^{-(m-1)} s_0^{\{m\}}$ is positive, we are able to specify parameterizations of the solution set of Problem $\mathsf{FP}[[\alpha,\beta]; (s_j)_{j=0}^m, =]$ for a sequence $(s_j)_{j=0}^m$ belonging to $\mathcal{F}_{q,m,\alpha,\beta}^{\succcurlyeq}$ (resp. to the subclass $\mathcal{F}_{q,m,\alpha,\beta}^{\succcurlyeq,\mathrm{c}}$) by utilizing equivalence classes of pairs belonging to $\mathcal{PR}_r(\mathbb{C}\setminus[\alpha,\beta])$. To do this, we are mainly guided by Theorem 11.9 or Theorem 11.29. In order to discuss the situation $1 \leq r \leq q-1$, we use, for two arbitrarily given complex matrices $A$ and $B$, the notation $A \oplus B := \begin{bmatrix} A & O \\ O & B \end{bmatrix}$.

**Theorem 12.3** *Suppose $q \geq 2$. Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha < \beta$ and let $m \in \mathbb{N}_0$. Furthermore, let $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\succcurlyeq,\mathrm{c}}$ be such that $r := \operatorname{rank} \mathsf{D}_m$ fulfills $1 \leq r \leq q-1$, where $(\mathsf{D}_j)_{j=0}^m$ is the sequence of $[\alpha,\beta]$-interval lengths associated with $(s_j)_{j=0}^m$. Let $\tilde{\mathbf{p}}_{2m+1}$, $\tilde{\mathbf{q}}_{2m+1}$, $\tilde{\mathbf{p}}_{2m+2}$, and $\tilde{\mathbf{q}}_{2m+2}$ be defined as in Theorem 11.29. Let $u_1, u_2, \ldots, u_q$ be an orthonormal basis of $\mathbb{C}^q$ such that $\{u_1, u_2, \ldots, u_r\} \subseteq \mathcal{R}(\mathsf{D}_m)$ and let $W := [u_1, u_2, \ldots, u_q]$. Then:*

*(a) For each pair $[g_1; g_2] \in \mathcal{PR}_r(\mathbb{C}\setminus[\alpha,\beta])$, the function*

$$det[\tilde{\mathbf{p}}_{2m+1}\mathsf{D}_m^\dagger W(g_1 \oplus O_{(q-r)\times(q-r)}) - \delta^{m-1}\tilde{\mathbf{p}}_{2m+2}W(g_2 \oplus I_{q-r})]$$

*does not vanish identically.*

*(b) For each pair* $[g_1; g_2] \in \mathcal{PR}_r(\mathbb{C} \setminus [\alpha, \beta])$*, let* $S_{m,W,[g_1;g_2]} \colon \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}^{q\times q}$ *be defined by*

$$\begin{aligned} &S_{m,W,[g_1;g_2]} \\ &\quad := -\left(\tilde{\mathbf{q}}_{2m+1}\mathsf{D}_m^\dagger W(g_1 \oplus O_{(q-r)\times(q-r)}) - \delta^{m-1}\tilde{\mathbf{q}}_{2m+2}W(g_2 \oplus I_{q-r})\right) \\ &\quad\quad \times\left(\tilde{\mathbf{p}}_{2m+1}\mathsf{D}_m^\dagger W(g_1 \oplus O_{(q-r)\times(q-r)}) - \delta^{m-1}\tilde{\mathbf{p}}_{2m+2}W(g_2 \oplus I_{q-r})\right)^{-1}. \end{aligned}$$

*Then* $\Sigma_{m,W} \colon \langle\mathcal{PR}_r(\mathbb{C} \setminus [\alpha, \beta])\rangle \to \mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^m, =]$ *given by*

$$\Sigma_{m,W}(\langle[g_1; g_2]\rangle) := S_{m,W,[g_1;g_2]}$$

*is well defined and bijective.*

***Proof*** Since $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\succcurlyeq,\mathrm{c}}$ is supposed, Notation 11.14 provides $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\succcurlyeq}$ as well. Thus, Remark 9.6 yields $s_0^{\{m\}} = \delta^{m-1}\mathsf{D}_m$, and, consequently, $\mathcal{R}(s_0^{\{m\}}) = \mathcal{R}(\mathsf{D}_m)$. Comparing the notations given in Theorem 11.9 and Proposition 11.27, and in the formulation of Theorem 12.3 above, we get (11.19), (11.20), (11.21), and (11.22) for all $z \in \mathbb{C} \setminus [\alpha, \beta]$ in the case that $m = 2n$ with some non-negative integer $n$. If $m = 2n + 1$ with some non-negative integer $n$, then (11.23), (11.24), (11.25), and (11.26) hold true for all $z \in \mathbb{C} \setminus [\alpha, \beta]$. Thus, applying [23, Thm. 14.5] completes the proof. □

Note that the so-called completely degenerate case (III), i. e., if $\mathsf{D}_m = O_{q\times q}$ holds true, is considered in [19, Def. 10.24] and leads to an interesting subclass of $[\alpha, \beta]$-non-negative definite sequences: Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\succcurlyeq}$. Then the sequence $(s_j)_{j=0}^m$ is called $[\alpha, \beta]$ *-completely degenerate* if $\mathsf{D}_m = O_{q\times q}$. Denote by $\mathcal{F}_{q,m,\alpha,\beta}^{\succcurlyeq,\mathrm{cd}}$ the set of all sequences $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\succcurlyeq}$ which are $[\alpha, \beta]$-completely degenerate.

If $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\succcurlyeq,\mathrm{cd}}$, then Problem $\mathsf{FP}[[\alpha, \beta]; (s_j)_{j=0}^m, =]$ admits a unique solution:

**Theorem 12.4** *Let* $\alpha, \beta \in \mathbb{R}$ *be such that* $\alpha < \beta$*, let* $m \in \mathbb{N}_0$*, and let* $(s_j)_{j=0}^m \in \mathcal{F}_{q,m,\alpha,\beta}^{\succcurlyeq,\mathrm{cd}}$*. Let* $\tilde{\mathbf{p}}_{2m+2}$ *and* $\tilde{\mathbf{q}}_{2m+2}$ *be the restrictions of* $\ddot{\mathbf{p}}_{2m+2}$ *and* $\ddot{\mathbf{q}}_{2m+2}$ *onto* $\mathbb{C}\setminus[\alpha, \beta]$*, respectively. Then* $\det \tilde{\mathbf{p}}_{2m+2}$ *does not vanish identically in* $\mathbb{C} \setminus [\alpha, \beta]$ *and*

$$\mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^m, =] = \{-\tilde{\mathbf{q}}_{2m+2}\tilde{\mathbf{p}}_{2m+2}^{-1}\}. \tag{12.1}$$

***Proof*** From [23, Thm. 14.6] and the notation given in Theorem 11.9 we know that $\det \tilde{\mathfrak{z}}_m$ does not vanish identically in $\mathbb{C} \setminus [\alpha, \beta]$ and that $\mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^m, =] = \{\tilde{\mathfrak{x}}_m\tilde{\mathfrak{z}}_m^{-1}\}$. If $m = 2n$ with some non-negative integer $n$, then the comparison of Theorem 11.9 and Proposition 11.27, and Theorem 11.29 yields (11.21) and (11.22) for all

$z \in \mathbb{C} \setminus [\alpha, \beta]$, which consequently shows that

$$\begin{aligned}
&(\tilde{\mathfrak{x}}_m \tilde{\mathfrak{z}}_m^{-1})(z) \\
&\quad = \big( - (z-\beta)^n \delta\big[ - \delta^{2n-1} \ddot{\mathbf{q}}_{4n+2}(z)\big]\big)\big( - (z-\beta)^n \delta\big[\delta^{2n-1} \ddot{\mathbf{p}}_{4n+2}(z)\big]\big)^{-1} \\
&\quad = -(\tilde{\mathbf{q}}_{2m+2} \tilde{\mathbf{p}}_{2m+2}^{-1})(z)
\end{aligned}$$

for all $z \in \mathbb{C} \setminus ([\alpha, \beta] \cup \mathcal{Z}(\det \tilde{\mathfrak{z}}_m))$. In view of $\mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^m, =] = \{\tilde{\mathfrak{x}}_m \tilde{\mathfrak{z}}_m^{-1}\}$, then (12.1) follows, because $\mathcal{Z}(\det \tilde{\mathfrak{z}}_m)$ is a discrete subset of $\mathbb{C} \setminus [\alpha, \beta]$. The case that $m = 2n + 1$ with some non-negative integer $n$ can be treated analogously, using (11.25) and (11.26). □

## 13 Description of the Values of the Solutions of the Truncated Matricial Hausdorff Moment Problem in the Non-degenerate Case and Corresponding Matrix Balls

This section and the following two sections are aimed at proving that, for every choice of $\alpha, \beta \in \mathbb{R}$ such that $\alpha < \beta$, a positive integer $m$, a sequence $(s_j)_{j=0}^m$ belonging to $\mathcal{F}_{q,m,\alpha,\beta}^{\succ}$, and $w \in \Pi_+$, the set $\big\{F(w) \colon F \in \mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^m, =]\big\}$ can be represented as intersection of two matrix balls, where the respective parameters are constructed explicitly by the given data. Initiated by investigations due to Weyl in [37], where concepts of nested families of certain sets were studied in the context of differential equations of second order, Krein and Nudelman in [28] as well as Akhiezer in [1] served contributions to this theory in the scalar case.

In the following, we continue to assume that $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, that $\alpha$ and $\beta$ are real numbers such that $\alpha < \beta$, and that $\delta := \beta - \alpha$. Furthermore, we are going to use Notation 3.3 again. In view of Remarks 8.8 and 8.9, we introduce the following notation.

**Notation 13.1** Let $(s_j)_{j=0}^\kappa$ be a sequence of complex $q \times q$ matrices. For all $l \in \mathbb{N}_0$ such that $2l - 1 \leq \kappa$, let $\ddot{\mathbf{o}}_l := \ddot{\mathbf{r}}_l^{[\![s]\!]}$. If $\kappa \geq 1$, for all $l \in \mathbb{N}_0$ such that $2l \leq \kappa$, let $\ddot{\mathbf{u}}_l := \ddot{\mathbf{t}}_l^{[\![a]\!]}$ and $\ddot{\mathbf{w}}_l := \ddot{\mathbf{v}}_l^{[\![b]\!]}$. Furthermore, if $\kappa \geq 2$, then, for all $l \in \mathbb{N}_0$ such that $2l + 1 \leq \kappa$, let $\ddot{\mathbf{y}}_l := \ddot{\mathbf{x}}_l^{[\![c]\!]}$.

The matrix polynomials defined in Notation 13.1 can be linked with the matrix polynomials $\ddot{\mathbf{p}}_j$ and $\ddot{\mathbf{q}}_j$ defined in Notation 8.3 as well:

**Lemma 13.2** *Let* $(s_j)_{j=0}^\kappa \in \mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq}$. *Then:*

(a) $\ddot{\mathbf{p}}_{4l}(z) = \ddot{\mathbf{r}}_l(z)$ *and* $\ddot{\mathbf{q}}_{4l}(z) = \ddot{\mathbf{o}}_l(z)$ *for all* $l \in \mathbb{N}_0$ *such that* $2l - 1 \leq \kappa$ *and all* $z \in \mathbb{C}$.
(b) $\ddot{\mathbf{p}}_{4l+1}(z) = (z - \alpha)\ddot{\mathbf{t}}_l(z)$ *and* $\ddot{\mathbf{p}}_{4l+2}(z) = -(\beta - z)\ddot{\mathbf{v}}_l(z)$ *for all* $l \in \mathbb{N}_0$ *such that* $2l \leq \kappa$ *and all* $z \in \mathbb{C}$.
(c) *If* $\kappa \geq 1$, *then* $\ddot{\mathbf{p}}_{4l+3}(z) = -(\beta - z)(z - \alpha)\ddot{\mathbf{x}}_l(z)$ *for all* $l \in \mathbb{N}_0$ *such that* $2l + 1 \leq \kappa$ *and all* $z \in \mathbb{C}$.

(d) *If* $\kappa \geq 2$, *then* $\ddot{\mathbf{q}}_{4l+1}(z) = \ddot{\mathbf{u}}_l(z) + s_0\ddot{\mathbf{t}}_l(z)$ *and* $\ddot{\mathbf{q}}_{4l+2}(z) = -[\ddot{\mathbf{w}}_l(z) - s_0\ddot{\mathbf{v}}_l(z)]$ *for all* $l \in \mathbb{N}$ *such that* $2l \leq \kappa$ *and all* $z \in \mathbb{C}$.

(e) *If* $\kappa \geq 3$, *then* $\ddot{\mathbf{q}}_{4l+3}(z) = -(\ddot{\mathbf{y}}_l(z) + [(\alpha + \beta - z)s_0 - s_1]\ddot{\mathbf{x}}_l(z))$ *for all* $l \in \mathbb{N}$ *such that* $2l + 1 \leq \kappa$ *and all* $z \in \mathbb{C}$.

***Proof*** Using Remarks 8.9, C.1 and 8.8, Lemmata C.3 and C.4 as well as Proposition 8.16, the assertions can be proved by straightforward calculations. We omit the details. □

In the first part of this section, we derive for the non-degenerate situation in Lemmata 13.5, 13.6 and 13.9 multiplicative relationships between the $2q \times 2q$ matrix polynomials defined in Notation 13.3 below. Here, $\ddot{\mathbf{V}}_m$ is a matrix polynomial, which is related to a parameterization of the set $\mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^m, =]$ (cf. Theorem 11.29), whereas the matrix polynomials $\mathbf{V}_{2n+1}$, $\mathbf{V}_{\mathrm{a},2t+1}$, $\mathbf{V}_{\mathrm{b},2u+1}$, and $\mathbf{V}_{\mathrm{c},2v+1}$ are related to parameterizations of the sets $\mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n+1}, =]$, $\mathcal{R}_{0,q}[\Pi_+; (a_j)_{j=0}^{2t+1}, =]$, $\mathcal{R}_{0,q}[\Pi_+; (b_j)_{j=0}^{2u+1}, =]$, and $\mathcal{R}_{0,q}[\Pi_+; (c_j)_{j=0}^{2v+1}, =]$, respectively (see also Theorem 8.14).

**Notation 13.3** Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices with $\mathcal{F}_{\alpha,\beta}$-parameter sequence $(\mathfrak{f}_j)_{j=0}^{2\kappa}$ and sequence of $[\alpha, \beta]$-interval lengths $(\mathsf{D}_j)_{j=0}^{\kappa}$. For each $l \in \mathbb{N}_0$ such that $2l + 1 \leq \kappa$, let the matrix polynomials $\ddot{\mathbf{r}}_l$, $\ddot{\mathbf{t}}_l$, $\ddot{\mathbf{v}}_l$, and $\ddot{\mathbf{x}}_l$ be given by Remark 8.9 and let $\ddot{\mathbf{u}}_l$, $\ddot{\mathbf{w}}_l$, $\ddot{\mathbf{y}}_l$, and $\ddot{\mathbf{o}}_l$ be given by Notation 13.1. In the case $\kappa \geq 1$, let the sequences $(\mathbf{A}_j)_{j=1}^{\kappa}$ and $(\mathbf{B}_j)_{j=1}^{\kappa}$ be defined by (8.5).

(a) For all $n \in \mathbb{N}_0$ such that $2n \leq \kappa$, let $\ddot{\mathbf{V}}_{2n} : \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ be given by

$$\ddot{\mathbf{V}}_{2n}(z) := \begin{bmatrix} -(\beta - z)\ddot{\mathbf{q}}_{4n+1}(z)\mathsf{D}_{2n}^{\dagger} & -\delta^{2n-1}\ddot{\mathbf{q}}_{4n+2}(z) \\ (\beta - z)\ddot{\mathbf{p}}_{4n+1}(z)\mathsf{D}_{2n}^{\dagger} & \delta^{2n-1}\ddot{\mathbf{p}}_{4n+2}(z) \end{bmatrix}.$$

(b) If $\kappa \geq 1$, then, for all $n \in \mathbb{N}_0$ such that $2n+1 \leq \kappa$, let $\mathbf{V}_{2n+1}$, $\ddot{\mathbf{V}}_{2n+1}$, $\mathbf{U}_{\mathrm{c},2n}$, $\mathbf{U}_{2n}$ : $\mathbb{C} \to \mathbb{C}^{2q\times 2q}$ be given by

$$\mathbf{V}_{2n+1}(z) := \begin{bmatrix} -\ddot{\mathbf{o}}_n(z)\mathfrak{f}_{4n}^{\dagger} & -\ddot{\mathbf{o}}_{n+1}(z) \\ \ddot{\mathbf{r}}_n(z)\mathfrak{f}_{4n}^{\dagger} & \ddot{\mathbf{r}}_{n+1}(z) \end{bmatrix},$$

$$\ddot{\mathbf{V}}_{2n+1}(z) := \begin{bmatrix} \ddot{\mathbf{q}}_{4n+3}(z)\mathsf{D}_{2n+1}^{\dagger} & -\delta^{2n}\ddot{\mathbf{q}}_{4n+4}(z) \\ -\ddot{\mathbf{p}}_{4n+3}(z)\mathsf{D}_{2n+1}^{\dagger} & \delta^{2n}\ddot{\mathbf{p}}_{4n+4}(z) \end{bmatrix},$$

$$\mathbf{U}_{\mathrm{c},2n}(z) := \begin{bmatrix} \delta^{2n-1}I_q & -\delta^{2n-1}\mathsf{D}_{2n}\mathbf{A}_{2n+1} \\ (z - \alpha)\mathsf{D}_{2n}^{\dagger} & (z - \alpha)\mathbf{B}_{2n+1} \end{bmatrix},$$

and

$$\mathbf{U}_{2n}(z) := \begin{bmatrix} \delta^{2n-1}I_q & \delta^{2n-1}\mathsf{D}_{2n}\mathbf{A}_{2n+1} \\ -(\beta - z)\mathsf{D}_{2n}^{\dagger} & (\beta - z)\mathbf{B}_{2n+1} \end{bmatrix}.$$

(c) If $\kappa \geq 2$, then, for all $n \in \mathbb{N}_0$ such that $2n+2 \leq \kappa$, let $\mathbf{V}_{a,2n+1}$, $\mathbf{V}_{b,2n+1}$, $\mathbf{U}_{a,2n+1}$, $\mathbf{U}_{b,2n+1} : \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ be given by

$$\mathbf{V}_{a,2n+1}(z) := \begin{bmatrix} -\ddot{\mathbf{u}}_n(z)\mathfrak{f}_{4n+1}^{\dagger} & -\ddot{\mathbf{u}}_{n+1}(z) \\ \ddot{\mathbf{t}}_n(z)\mathfrak{f}_{4n+1}^{\dagger} & \ddot{\mathbf{t}}_{n+1}(z) \end{bmatrix},$$

$$\mathbf{V}_{b,2n+1}(z) := \begin{bmatrix} -\ddot{\mathbf{w}}_n(z)\mathfrak{f}_{4n+2}^{\dagger} & -\ddot{\mathbf{w}}_{n+1}(z) \\ \ddot{\mathbf{v}}_n(z)\mathfrak{f}_{4n+2}^{\dagger} & \ddot{\mathbf{v}}_{n+1}(z) \end{bmatrix},$$

$$\mathbf{U}_{a,2n+1}(z) := \begin{bmatrix} \delta^{2n} I_q & -\delta^{2n}\mathsf{D}_{2n+1}\mathbf{A}_{2n+2} \\ (z-\alpha)\mathsf{D}_{2n+1}^{\dagger} & (z-\alpha)\mathbf{B}_{2n+2} \end{bmatrix},$$

and

$$\mathbf{U}_{b,2n+1}(z) := \begin{bmatrix} \delta^{2n} I_q & \delta^{2n}\mathsf{D}_{2n+1}\mathbf{A}_{2n+2} \\ -(\beta-z)\mathsf{D}_{2n+1}^{\dagger} & (\beta-z)\mathbf{B}_{2n+2} \end{bmatrix}.$$

(d) If $\kappa \geq 3$, then, for all $n \in \mathbb{N}_0$ such that $2n+3 \leq \kappa$, let $\mathbf{V}_{c,2n+1} : \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ be given by

$$\mathbf{V}_{c,2n+1}(z) := \begin{bmatrix} -\ddot{\mathbf{y}}_n(z)\mathfrak{f}_{4n+3}^{\dagger} & -\ddot{\mathbf{y}}_{n+1}(z) \\ \ddot{\mathbf{x}}_n(z)\mathfrak{f}_{4n+3}^{\dagger} & \ddot{\mathbf{x}}_{n+1}(z) \end{bmatrix}.$$

**Lemma 13.4** *Suppose $\kappa \geq 1$ and let $(s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,\kappa,\alpha,\beta}^{\succ}$. Furthermore, let $(\Gamma_j)_{j=1}^{\kappa}$ be a sequence of complex $q\times q$ matrices and let the sequences $(\mathbf{A}_j)_{j=1}^{\kappa}$ and $(\mathbf{B}_j)_{j=1}^{\kappa}$ be defined by (8.5). Then* $\det \mathsf{D}_j \neq 0$ *is fulfilled for all $j \in \mathbb{Z}_{0,\kappa}$. Moreover, $\mathbf{A}_j = \mathsf{D}_{j-1}^{-1}\mathfrak{f}_{2j}$ and $\mathbf{B}_j = \mathsf{D}_{j-1}^{-1}\mathfrak{f}_{2j-1}$ hold true for all $j \in \mathbb{Z}_{1,\kappa}$.*

***Proof*** Proposition 5.16 shows that $\mathsf{D}_j \in \mathbb{C}_{\succ}^{q\times q}$ is valid for each $j \in \mathbb{Z}_{0,\kappa}$. Therefore, we have $\det \mathsf{D}_j \neq 0$ for all $j \in \mathbb{Z}_{0,\kappa}$. Thus, $\mathbb{P}_{\mathcal{N}(\mathsf{D}_{j-1})} = O_{q\times q}$ for all $j \in \mathbb{Z}_{1,\kappa}$. Consequently, using (8.5), we get $\mathbf{A}_j = \mathsf{D}_{j-1}^{-1}\mathfrak{f}_{2j}$ and $\mathbf{B}_j = \mathsf{D}_{j-1}^{-1}\mathfrak{f}_{2j-1}$ for all $j \in \mathbb{Z}_{1,\kappa}$. □

**Lemma 13.5** *Let $n \in \mathbb{N}$ and let $(s_j)_{j=0}^{2n+1} \in \mathcal{F}_{q,2n+1,\alpha,\beta}^{\succ}$. For all $z \in \mathbb{C}$, then*

$$-\delta^{2n-1}(\beta-z)(z-\alpha)\mathbf{V}_{c,2n-1}(z) = \Delta(z)\ddot{\mathbf{V}}_{2n}(z)\mathbf{U}_{c,2n}(z) \tag{13.1}$$

*and*

$$\delta^{2n-1}(\beta-z)\mathbf{V}_{2n+1}(z) = \ddot{\mathbf{V}}_{2n}(z)\mathbf{U}_{2n}(z), \tag{13.2}$$

*where*

$$\Delta(z) := \left[\begin{array}{c|c} (\beta-z)(z-\alpha)I_q & (\alpha+\beta-z)s_0 - s_1 \\ \hline O & I_q \end{array}\right]. \tag{13.3}$$

***Proof*** Our proof is divided into four parts. Let $z \in \mathbb{C}$.
**Part 1:** Lemma 13.4 yields $\det \mathsf{D}_{2n} \neq 0$. Notation 8.3 provides

$$\ddot{\mathbf{p}}_{4n+3}(z) = -(\beta - z)\ddot{\mathbf{p}}_{4n+1}(z)\mathbf{A}_{2n+1} + (z-\alpha)\ddot{\mathbf{p}}_{4n+2}(z)\mathbf{B}_{2n+1}, \tag{13.4}$$

$$\ddot{\mathbf{q}}_{4n+3}(z) = -(\beta - z)\ddot{\mathbf{q}}_{4n+1}(z)\mathbf{A}_{2n+1} + (z-\alpha)\ddot{\mathbf{q}}_{4n+2}(z)\mathbf{B}_{2n+1}, \tag{13.5}$$

$$\ddot{\mathbf{p}}_{4n+4}(z) = \ddot{\mathbf{p}}_{4n+1}(z)\mathbf{A}_{2n+1} + \ddot{\mathbf{p}}_{4n+2}(z)\mathbf{B}_{2n+1}, \tag{13.6}$$

and

$$\ddot{\mathbf{q}}_{4n+4}(z) = \ddot{\mathbf{q}}_{4n+1}(z)\mathbf{A}_{2n+1} + \ddot{\mathbf{q}}_{4n+2}(z)\mathbf{B}_{2n+1}, \tag{13.7}$$

whereas Remark 8.5 delivers

$$(\beta - z)\ddot{\mathbf{p}}_{4n+1}(z) + (z-\alpha)\ddot{\mathbf{p}}_{4n+2}(z) = \delta\ddot{\mathbf{p}}_{4n-1}(z)\mathbf{A}_{2n}, \tag{13.8}$$

$$(\beta - z)\ddot{\mathbf{q}}_{4n+1}(z) + (z-\alpha)\ddot{\mathbf{q}}_{4n+2}(z) = \delta\ddot{\mathbf{q}}_{4n-1}(z)\mathbf{A}_{2n}, \tag{13.9}$$

and

$$\ddot{\mathbf{p}}_{4n+1}(z) - \ddot{\mathbf{p}}_{4n+2}(z) = \delta\ddot{\mathbf{p}}_{4n}(z)\mathbf{B}_{2n}, \qquad \ddot{\mathbf{q}}_{4n+1}(z) - \ddot{\mathbf{q}}_{4n+2}(z) = \delta\ddot{\mathbf{q}}_{4n}(z)\mathbf{B}_{2n}. \tag{13.10}$$

Remark 5.22 shows $L_{\alpha,n-1,\beta} = \mathfrak{f}_{4n-1}$ and $L_n = \mathfrak{f}_{4n}$ and, moreover, according to Lemma 8.7, the equations $\delta\mathfrak{f}_{4n-1}\mathbf{A}_{2n} = \mathsf{D}_{2n}$ and $\delta\mathfrak{f}_{4n}\mathbf{B}_{2n} = \mathsf{D}_{2n}$ hold true. In view of $\det \mathsf{D}_{2n} \neq 0$, then $\det L_{\alpha,n-1,\beta} \neq 0$ and $\det L_n \neq 0$ as well as

$$\delta\mathbf{A}_{2n}\mathsf{D}_{2n}^{\dagger} = L_{\alpha,n-1,\beta}^{\dagger} \qquad \text{and} \qquad \delta\mathbf{B}_{2n}\mathsf{D}_{2n}^{\dagger} = L_n^{\dagger} \tag{13.11}$$

hold true. Furthermore, Lemma 13.2 provides

$$\ddot{\mathbf{p}}_{4n-1}(z) = -(\beta - z)(z-\alpha)\ddot{\mathbf{x}}_{n-1}(z), \qquad \ddot{\mathbf{p}}_{4n+3}(z) = -(\beta - z)(z-\alpha)\ddot{\mathbf{x}}_{n}(z) \tag{13.12}$$

and

$$\ddot{\mathbf{p}}_{4n}(z) = \ddot{\mathbf{r}}_n(z),\ \ \ddot{\mathbf{p}}_{4n+4}(z) = \ddot{\mathbf{r}}_{n+1}(z),\ \ \ddot{\mathbf{q}}_{4n}(z) = \ddot{\mathbf{o}}_n(z),\ \ \ddot{\mathbf{q}}_{4n+4}(z) = \ddot{\mathbf{o}}_{n+1}(z) \tag{13.13}$$

as well as

$$\ddot{\mathbf{q}}_{4n+3}(z) + [(\alpha+\beta-z)s_0 - s_1]\ddot{\mathbf{x}}_n(z) = -\ddot{\mathbf{y}}_n(z). \tag{13.14}$$

**Part 2:** In the next step, we are going to prove the validity of the equation

$$\ddot{\mathbf{q}}_{4n-1}(z) + [(\alpha+\beta-z)s_0 - s_1]\ddot{\mathbf{x}}_{n-1}(z) = -\ddot{\mathbf{y}}_{n-1}(z). \tag{13.15}$$

First, we consider the case $n = 1$. Notation 8.3 provides

$$\ddot{\mathbf{q}}_3(z) = -(\beta - z)\ddot{\mathbf{q}}_1(z)\mathbf{A}_1 + (z-\alpha)\ddot{\mathbf{q}}_2(z)\mathbf{B}_1, \quad \ddot{\mathbf{q}}_1(z) = \mathfrak{f}_0, \quad \ddot{\mathbf{q}}_2(z) = \mathfrak{f}_0. \tag{13.16}$$

Moreover, according to Lemma 13.4, the inequality $\det \mathsf{D}_0 \neq 0$ and the equations $\mathbf{A}_1 = \mathsf{D}_0^{-1}\mathfrak{f}_2$ and $\mathbf{B}_1 = \mathsf{D}_0^{-1}\mathfrak{f}_1$ hold true. By virtue of (5.9) and (5.8), we obtain then $\det(\delta s_0) \neq 0$ and $\mathfrak{f}_0\mathsf{D}_0^{-1} = s_0(\delta s_0)^{-1} = \frac{1}{\delta}I_q$. Taking additionally into account (13.16) and (5.9), we get then $\ddot{\mathbf{q}}_1(z)\mathbf{A}_1 = \mathfrak{f}_0\mathsf{D}_0^{-1}\mathfrak{f}_2 = \frac{1}{\delta}(\beta s_0 - s_1)$ and $\ddot{\mathbf{q}}_2(z)\mathbf{B}_1 = \mathfrak{f}_0\mathsf{D}_0^{-1}\mathfrak{f}_1 = \frac{1}{\delta}(s_1 - \alpha s_0)$. Thus, using (13.16), we conclude

$$\begin{aligned}\ddot{\mathbf{q}}_3(z) &= -(\beta - z)\frac{1}{\delta}(\beta s_0 - s_1) + (z-\alpha)\frac{1}{\delta}(s_1 - \alpha s_0)\\ &= -[(\alpha+\beta-z)s_0 - s_1].\end{aligned} \tag{13.17}$$

Applying Lemma 8.6, we infer $\ddot{\mathbf{p}}_3(z) = -(\beta - z)(z-\alpha)\mathbf{A}_1 - (z-\alpha)(\beta-z)\mathbf{B}_1 = -(\beta-z)(z-\alpha)I_q$. Consequently, Remark 8.9(c) delivers $\ddot{\mathbf{x}}_0(z) = I_q$. Therefore, using Notations 13.1 and 6.11, we obtain $\ddot{\mathbf{y}}_0(z) = \ddot{\mathbf{x}}_0^{[\![c]\!]}(z) = I_q^{[\![c]\!]}(z) = O_{q\times q}$. Taking additionally into account (13.17), we get then $\ddot{\mathbf{q}}_3(z) + [(\alpha+\beta-z)s_0 - s_1]\ddot{\mathbf{x}}_0(z) = O_{q\times q} = -\ddot{\mathbf{y}}_0(z)$. Hence, equation (13.15) is verified in the case $n = 1$. If $n \geq 2$, then Remark 5.9 and Lemma 13.2 yield (13.15).

**Part 3:** In the following, we want to verify (13.1). Because of $\det \mathsf{D}_{2n} \neq 0$, we have

$$\mathsf{D}_{2n}^{\dagger}\mathsf{D}_{2n}\mathbf{A}_{2n+1} = \mathbf{A}_{2n+1}. \tag{13.18}$$

Using Notation 13.3, (13.18), (13.9), (13.5), (13.8), (13.4), and (13.11), we observe that

$$\begin{aligned}&\ddot{\mathbf{V}}_{2n}(z)\mathbf{U}_{c,2n}(z)\\ &= \begin{bmatrix} -(\beta-z)\ddot{\mathbf{q}}_{4n+1}(z)\mathsf{D}_{2n}^{\dagger} & -\delta^{2n-1}\ddot{\mathbf{q}}_{4n+2}(z)\\ (\beta-z)\ddot{\mathbf{p}}_{4n+1}(z)\mathsf{D}_{2n}^{\dagger} & \delta^{2n-1}\ddot{\mathbf{p}}_{4n+2}(z)\end{bmatrix}\\ &\quad\times\begin{bmatrix}\delta^{2n-1}I_q & -\delta^{2n-1}\mathsf{D}_{2n}\mathbf{A}_{2n+1}\\ (z-\alpha)\mathsf{D}_{2n}^{\dagger} & (z-\alpha)\mathbf{B}_{2n+1}\end{bmatrix}\\ &= \delta^{2n-1}\begin{bmatrix}-\delta\ddot{\mathbf{q}}_{4n-1}(z)\mathbf{A}_{2n}\mathsf{D}_{2n}^{\dagger} & -\ddot{\mathbf{q}}_{4n+3}(z)\\ \delta\ddot{\mathbf{p}}_{4n+1}(z)\mathbf{A}_{2n}\mathsf{D}_{2n}^{\dagger} & \ddot{\mathbf{p}}_{4n+3}(z)\end{bmatrix}\\ &= \delta^{2n-1}\begin{bmatrix}-\ddot{\mathbf{q}}_{4n-1}(z)L_{\alpha,n-1,\beta}^{\dagger} & -\ddot{\mathbf{q}}_{4n+3}(z)\\ \ddot{\mathbf{p}}_{4n-1}(z)L_{\alpha,n-1,\beta}^{\dagger} & \ddot{\mathbf{p}}_{4n+3}(z)\end{bmatrix}\end{aligned}$$

is valid. Therefore, using additionally (13.12), (13.15), (13.14) as well as Notation 13.3, we get

$$\Delta(z)\ddot{\mathbf{V}}_{2n}(z)\mathbf{U}_{c,2n}(z) = \Delta(z)\delta^{2n-1}\begin{bmatrix}-\ddot{\mathbf{q}}_{4n-1}(z)L_{\alpha,n-1,\beta}^{\dagger} & -\ddot{\mathbf{q}}_{4n+3}(z)\\ \ddot{\mathbf{p}}_{4n-1}(z)L_{\alpha,n-1,\beta}^{\dagger} & \ddot{\mathbf{p}}_{4n+3}(z)\end{bmatrix}$$

$$= -\delta^{2n-1}(\beta - z)(z - \alpha)\begin{bmatrix} Y_{n-1}(z)L^{\dagger}_{\alpha,n-1,\beta} & Y_n(z) \\ \ddot{\mathbf{x}}_{n-1}(z)L^{\dagger}_{\alpha,n-1,\beta} & \ddot{\mathbf{x}}_n(z) \end{bmatrix}$$

$$= -\delta^{2n-1}(\beta - z)(z - \alpha)\begin{bmatrix} -\ddot{\mathbf{y}}_{n-1}(z)L^{\dagger}_{\alpha,n-1,\beta} & -\ddot{\mathbf{y}}_n(z) \\ \ddot{\mathbf{x}}_{n-1}(z)L_{\alpha,n-1,\beta} & \ddot{\mathbf{x}}_n(z) \end{bmatrix}$$

$$= -\delta^{2n-1}(\beta - z)(z - \alpha)\mathbf{V}_{c,2n-1}(z),$$

where $Y_\ell(z) := \ddot{\mathbf{q}}_{4\ell+3}(z) + [(\alpha + \beta - z)s_0 - s_1]\ddot{\mathbf{x}}_\ell(z)$ for $\ell = n-1, n$.

**Part 4:** Finally, we want to verify (13.2). Using Notation 13.3, (13.18), (13.10), (13.7), (13.10), (13.6), (13.11), (13.13), and (13.13), we see that

$$\ddot{\mathbf{V}}_{2n}(z)\mathbf{U}_{2n}(z)$$

$$= \begin{bmatrix} -(\beta - z)\ddot{\mathbf{q}}_{4n+1}(z)\mathsf{D}^{\dagger}_{2n} & -\delta^{2n-1}\ddot{\mathbf{q}}_{4n+2}(z) \\ (\beta - z)\ddot{\mathbf{p}}_{4n+1}(z)\mathsf{D}^{\dagger}_{2n} & \delta^{2n-1}\ddot{\mathbf{p}}_{4n+2}(z) \end{bmatrix}$$

$$\times \begin{bmatrix} \delta^{2n-1}I_q & \delta^{2n-1}\mathsf{D}_{2n}\mathbf{A}_{2n+1} \\ -(\beta - z)\mathsf{D}^{\dagger}_{2n} & (\beta - z)\mathbf{B}_{2n+1} \end{bmatrix}$$

$$= \delta^{2n-1}(\beta - z)\begin{bmatrix} -[\ddot{\mathbf{q}}_{4n+1}(z) - \ddot{\mathbf{q}}_{4n+2}(z)]\mathsf{D}^{\dagger}_{2n} & -Q_n(z) \\ [\ddot{\mathbf{p}}_{4n+1}(z) - \ddot{\mathbf{p}}_{4n+2}(z)]\mathsf{D}^{\dagger}_{2n} & P_n(z) \end{bmatrix}$$

$$= \delta^{2n-1}(\beta - z)\begin{bmatrix} -\delta\ddot{\mathbf{q}}_{4n}(z)\mathbf{B}_{2n}\mathsf{D}^{\dagger}_{2n} & -\ddot{\mathbf{q}}_{4n+4}(z) \\ \delta\ddot{\mathbf{p}}_{4n}(z)\mathbf{B}_{2n}\mathsf{D}^{\dagger}_{2n} & \ddot{\mathbf{p}}_{4n+4}(z) \end{bmatrix}$$

$$= \delta^{2n-1}(\beta - z)\begin{bmatrix} -\ddot{\mathbf{q}}_{4n}(z)L^{\dagger}_n & -\ddot{\mathbf{q}}_{4n+4}(z) \\ \ddot{\mathbf{p}}_{4n}(z)L^{\dagger}_n & \ddot{\mathbf{p}}_{4n+4}(z) \end{bmatrix}$$

$$= \delta^{2n-1}(\beta - z)\begin{bmatrix} -\ddot{\mathbf{o}}_n(z)L^{\dagger}_n & -\ddot{\mathbf{o}}_{n+1}(z) \\ \ddot{\mathbf{r}}_n(z)L^{\dagger}_n & \ddot{\mathbf{r}}_{n+1}(z) \end{bmatrix} = \delta^{2n-1}(\beta - z)\mathbf{V}_{2n+1}(z),$$

where $Q_n(z) := \ddot{\mathbf{q}}_{4n+1}(z)\mathbf{A}_{2n+1} + \ddot{\mathbf{q}}_{4n+2}(z)\mathbf{B}_{2n+1}$ and $P_n(z) := \ddot{\mathbf{p}}_{4n+1}(z)\mathbf{A}_{2n+1} + \ddot{\mathbf{p}}_{4n+2}(z)\mathbf{B}_{2n+1}$. □

**Lemma 13.6** *Suppose that* $\kappa \geq 1$ *and that* $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{\succ}_{q,\kappa,\alpha,\beta}$. *For all* $z \in \mathbb{C}$, *then*

$$-\delta^{-1}(\beta - z)(z - \alpha)I_q = \Delta(z)\ddot{\mathbf{V}}_0(z)\mathbf{U}_{c,0}(z) \tag{13.19}$$

*and, in particular,* $\det \mathbf{U}_{c,0}(z) \neq 0$, *and further*

$$\delta^{-1}(\beta - z)\mathbf{V}_1(z) = \ddot{\mathbf{V}}_0(z)\mathbf{U}_0(z), \tag{13.20}$$

*where* $\Delta(z)$ *is given via (13.3).*

***Proof*** Since $(s_j)_{j=0}^{\kappa}$ belongs to $\mathcal{F}^{\succ}_{q,\kappa,\alpha,\beta}$, from Proposition 5.10 and the definition of the set $\mathcal{F}^{\succ}_{q,0,\alpha,\beta}$ we see that $s_0$ belongs to $\mathbb{C}^{q\times q}_{\succ}$. In particular, $\det s_0 \neq 0$. Thus, because of (5.8), we have $\mathsf{D}_0 = \delta s_0$ and $\mathsf{D}^{\dagger}_0 = \delta^{-1}s_0^{-1}$. Using (8.5) and (5.9), we conclude moreover $\mathbf{A}_1 = \delta^{-1}s_0^{-1}(\beta s_0 - s_1)$ and $\mathbf{B}_1 = \delta^{-1}s_0^{-1}(s_1 - \alpha s_0)$. We consider now an arbitrary $z \in \mathbb{C}$. Thus, in view of Notation 13.3 as well as (8.3) and (8.4), it follows

$$\ddot{\mathbf{V}}_0(z) = -\delta^{-1}\begin{bmatrix} (\beta - z)I_q & s_0 \\ -(\beta - z)(z-\alpha)s_0^{-1} & (\beta - z)I_q \end{bmatrix}, \tag{13.21}$$

$$\mathbf{U}_{c,0}(z) = \delta^{-1}\begin{bmatrix} I_q & -(\beta s_0 - s_1) \\ (z-\alpha)s_0^{-1} & (z-\alpha)s_0^{-1}(s_1 - \alpha s_0) \end{bmatrix}, \tag{13.22}$$

and

$$\mathbf{U}_0(z) = \delta^{-1}\begin{bmatrix} I_q & \beta s_0 - s_1 \\ -(\beta - z)s_0^{-1} & (\beta - z)s_0^{-1}(s_1 - \alpha s_0) \end{bmatrix}. \tag{13.23}$$

Combining (13.21) and (13.22), we see that

$$\ddot{\mathbf{V}}_0(z)\mathbf{U}_{c,0}(z) = -\delta^{-1}\begin{bmatrix} I_q & (z-\alpha-\beta)s_0 + s_1 \\ O & (\beta - z)(z-\alpha)I_q \end{bmatrix}$$

holds true. Hence, (13.19) can be easily checked, which implies $\det \mathbf{U}_{c,0}(z) \neq 0$. From Lemma 13.2, (5.9), and (8.3) we infer $\ddot{\mathbf{o}}_0(z)\mathfrak{f}_0^\dagger = \ddot{\mathbf{q}}_0(z)s_0^\dagger = O$ and, taking additionally into account (8.9) and (8.4), moreover $\ddot{\mathbf{o}}_1(z) = \ddot{\mathbf{q}}_4(z) = \ddot{\mathbf{q}}_1(z)\mathbf{A}_1 + \ddot{\mathbf{q}}_2(z)\mathbf{B}_1 = \delta^{-1}(\beta s_0 - s_1) + \delta^{-1}(s_1 - \alpha s_0) = s_0$. By virtue of Remark 8.9, (8.8), (8.3), (8.4), and (5.9), we obtain $\ddot{\mathbf{r}}_0(z) = \ddot{\mathbf{p}}_0(z) = I_q$ and

$$\begin{aligned}\ddot{\mathbf{r}}_1(z) &= \ddot{\mathbf{p}}_4(z) = \ddot{\mathbf{p}}_1(z)\mathbf{A}_1 + \ddot{\mathbf{p}}_2(z)\mathbf{B}_1 \\ &= \delta^{-1}(z-\alpha)(\beta I_q - s_0^{-1}s_1) - \delta^{-1}(\beta - z)(s_0^{-1}s_1 - \alpha I_q) = zI_q - s_0^{-1}s_1.\end{aligned}$$

Thus, in view of Notation 13.3 and (5.9), we conclude

$$\mathbf{V}_1(z) = \begin{bmatrix} -\ddot{\mathbf{o}}_0(z)\mathfrak{f}_0^\dagger & -\ddot{\mathbf{o}}_1(z) \\ \ddot{\mathbf{r}}_0(z)\mathfrak{f}_0^\dagger & \ddot{\mathbf{r}}_1(z) \end{bmatrix} = \begin{bmatrix} O & -s_0 \\ s_0^{-1} & zI_q - s_0^{-1}s_1 \end{bmatrix}. \tag{13.24}$$

Multiplying the matrices stated in (13.21) and (13.23) and comparing with (13.24) yields (13.20). □

***Remark 13.7*** Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n+1} \in \mathcal{F}_{q,2n+1,\alpha,\beta}^{\succ}$. Since Lemma 13.4 provides $\det \mathsf{D}_{2n} \neq 0$, Lemma 8.6 and straightforward calculations provide

$$\mathbf{U}_{2n}(z) = \begin{bmatrix} \delta^{2n-1}I_q & O_{q\times q} \\ O_{q\times q} & (\beta - z)I_q \end{bmatrix}\begin{bmatrix} I_q & O_{q\times q} \\ -\mathsf{D}_{2n}^\dagger & I_q \end{bmatrix}\begin{bmatrix} I_q & \mathsf{D}_{2n}\mathbf{A}_{2n+1} \\ O_{q\times q} & I_q \end{bmatrix},$$

$$\mathbf{U}_{c,2n}(z) = \begin{bmatrix} \delta^{2n-1}I_q & O_{q\times q} \\ O_{q\times q} & (z-\alpha)I_q \end{bmatrix}\begin{bmatrix} I_q & O_{q\times q} \\ \mathsf{D}_{2n}^\dagger & I_q \end{bmatrix}\begin{bmatrix} I_q & -\mathsf{D}_{2n}\mathbf{A}_{2n+1} \\ O_{q\times q} & I_q \end{bmatrix},$$

and, in particular, $\det \mathbf{U}_{2n}(z) = \delta^{(2n-1)q}(\beta - z)^q$ and $\det \mathbf{U}_{c,2n}(z) = \delta^{(2n-1)q}(z-\alpha)^q$ for all $z \in \mathbb{C}$.

The matrix polynomials $\mathbf{U}_{a,2n+1}$ and $\mathbf{U}_{b,2n+1}$ admit special decompositions as well:

***Remark 13.8*** Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n+2} \in \mathcal{F}^{\succ}_{q,2n+2,\alpha,\beta}$. Since Lemma 13.4 provides $\det \mathsf{D}_{2n+1} \neq 0$, Lemma 8.6 and straightforward calculations yield

$$\mathbf{U}_{a,2n+1}(z) = \begin{bmatrix} \delta^{2n} I_q & O_{q\times q} \\ O_{q\times q} & (z-\alpha)I_q \end{bmatrix} \begin{bmatrix} I_q & O_{q\times q} \\ \mathsf{D}_{2n+1}^\dagger & I_q \end{bmatrix} \begin{bmatrix} I_q & -\mathsf{D}_{2n+1}\mathbf{A}_{2n+2} \\ O_{q\times q} & I_q \end{bmatrix},$$
$$\mathbf{U}_{b,2n+1}(z) = \begin{bmatrix} \delta^{2n} I_q & O_{q\times q} \\ O_{q\times q} & (\beta - z)I_q \end{bmatrix} \begin{bmatrix} I_q & O_{q\times q} \\ -\mathsf{D}_{2n+1}^\dagger & I_q \end{bmatrix} \begin{bmatrix} I_q & \mathsf{D}_{2n+1}\mathbf{A}_{2n+2} \\ O_{q\times q} & I_q \end{bmatrix},$$

and, in particular, $\det \mathbf{U}_{a,2n+1}(z) = \delta^{2nq}(z-\alpha)^q$ and $\det \mathbf{U}_{b,2n+1}(z) = \delta^{2nq}(\beta - z)^q$ for all $z \in \mathbb{C}$.

**Lemma 13.9** *Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n+2} \in \mathcal{F}^{\succ}_{q,2n+2,\alpha,\beta}$. For all $z \in \mathbb{C}$, then*

$$\delta^{2n}(z-\alpha)\mathbf{V}_{a,2n+1}(z) = \begin{bmatrix} (z-\alpha)I_q & s_0 \\ O_{q\times q} & I_q \end{bmatrix} \ddot{\mathbf{V}}_{2n+1}(z)\mathbf{U}_{a,2n+1}(z) \tag{13.25}$$

*and*

$$\delta^{2n}(\beta - z)\mathbf{V}_{b,2n+1}(z) = \begin{bmatrix} (\beta - z)I_q & -s_0 \\ O_{q\times q} & I_q \end{bmatrix} \ddot{\mathbf{V}}_{2n+1}(z)\mathbf{U}_{b,2n+1}(z). \tag{13.26}$$

***Proof.*** Our proof is divided into four parts. Let $z \in \mathbb{C}$.
**Part 1:** Lemma 13.4 yields $\det \mathsf{D}_{2n+1} \neq 0$. Notation 8.3 provides

$$\ddot{\mathbf{p}}_{4n+5}(z) = \ddot{\mathbf{p}}_{4n+3}(z)\mathbf{A}_{2n+2} + (z-\alpha)\ddot{\mathbf{p}}_{4n+4}(z)\mathbf{B}_{2n+2}, \tag{13.27}$$
$$\ddot{\mathbf{q}}_{4n+5}(z) = \ddot{\mathbf{q}}_{4n+3}(z)\mathbf{A}_{2n+2} + (z-\alpha)\ddot{\mathbf{q}}_{4n+4}(z)\mathbf{B}_{2n+2}, \tag{13.28}$$
$$\ddot{\mathbf{p}}_{4n+6}(z) = \ddot{\mathbf{p}}_{4n+3}(z)\mathbf{A}_{2n+2} - (\beta - z)\ddot{\mathbf{p}}_{4n+4}(z)\mathbf{B}_{2n+2}, \tag{13.29}$$

and

$$\ddot{\mathbf{q}}_{4n+6}(z) = \ddot{\mathbf{q}}_{4n+3}(z)\mathbf{A}_{2n+2} - (\beta - z)\ddot{\mathbf{q}}_{4n+4}(z)\mathbf{B}_{2n+2}, \tag{13.30}$$

whereas Remark 8.5 delivers

$$(z-\alpha)\ddot{\mathbf{p}}_{4n+4}(z) - \ddot{\mathbf{p}}_{4n+3}(z) = \delta\ddot{\mathbf{p}}_{4n+1}(z)\mathbf{A}_{2n+1}, \tag{13.31}$$
$$(z-\alpha)\ddot{\mathbf{q}}_{4n+4}(z) - \ddot{\mathbf{q}}_{4n+3}(z) = \delta\ddot{\mathbf{q}}_{4n+1}(z)\mathbf{A}_{2n+1}, \tag{13.32}$$
$$(\beta - z)\ddot{\mathbf{p}}_{4n+4}(z) + \ddot{\mathbf{p}}_{4n+3}(z) = \delta\ddot{\mathbf{p}}_{4n+2}(z)\mathbf{B}_{2n+1}, \tag{13.33}$$

and

$$(\beta - z)\ddot{\mathbf{q}}_{4n+4}(z) + \ddot{\mathbf{q}}_{4n+3}(z) = \delta\ddot{\mathbf{q}}_{4n+2}(z)\mathbf{B}_{2n+1}. \tag{13.34}$$

Let $(\mathfrak{f}_k)_{k=0}^{4n+4}$ be the $\mathcal{F}_{\alpha,\beta}$-parameter sequence of $(s_j)_{j=0}^{2n+2}$. Lemma 8.7 shows that $\delta\mathfrak{f}_{4n+1}\mathbf{A}_{2n+1} = \mathsf{D}_{2n+1}$ and $\delta\mathfrak{f}_{4n+2}\mathbf{B}_{2n+1} = \mathsf{D}_{2n+1}$. In view of $\det \mathsf{D}_{2n+1} \neq 0$, then $\det \mathfrak{f}_{4n+1} \neq 0$ as well as $\det \mathfrak{f}_{4n+2} \neq 0$ and, thus,

$$\delta\mathbf{A}_{2n+1}\mathsf{D}_{2n+1}^{\dagger} = \mathfrak{f}_{4n+1}^{\dagger} \qquad \text{and} \qquad \delta\mathbf{B}_{2n+1}\mathsf{D}_{2n+1}^{\dagger} = \mathfrak{f}_{4n+2}^{\dagger} \tag{13.35}$$

follow. Furthermore, from Lemma 13.2 we know that

$$\ddot{\mathbf{p}}_{4n+1}(z) = (z-\alpha)\ddot{\mathbf{t}}_n(z), \qquad \ddot{\mathbf{p}}_{4n+5}(z) = (z-\alpha)\ddot{\mathbf{t}}_{n+1}(z), \tag{13.36}$$

$$\ddot{\mathbf{p}}_{4n+2}(z) = -(\beta-z)\ddot{\mathbf{v}}_n(z), \qquad \ddot{\mathbf{p}}_{4n+6}(z) = -(\beta-z)\ddot{\mathbf{v}}_{n+1}(z), \tag{13.37}$$

$$\ddot{\mathbf{q}}_{4n+5}(z) = \ddot{\mathbf{u}}_{n+1}(z) + s_0\ddot{\mathbf{t}}_{n+1}(z), \quad \ddot{\mathbf{q}}_{4n+6}(z) = -[\ddot{\mathbf{w}}_{n+1}(z) - s_0\ddot{\mathbf{v}}_{n+1}(z)]. \tag{13.38}$$

**Part 2:** In the next step of the proof, we are going to prove that

$$\ddot{\mathbf{q}}_{4n+1}(z) - s_0\ddot{\mathbf{t}}_n(z) = \ddot{\mathbf{u}}_n(z) \qquad \text{and} \qquad \ddot{\mathbf{q}}_{4n+2}(z) - s_0\ddot{\mathbf{v}}_n(z) = -\ddot{\mathbf{w}}_n(z) \tag{13.39}$$

are fulfilled. First, we consider the case $n = 0$. In view of Notation 8.3, we have $\ddot{\mathbf{p}}_1(z) = (z-\alpha)I_q$ and $\ddot{\mathbf{p}}_2(z) = -(\beta-z)I_q$ as well as $\ddot{\mathbf{q}}_1(z) = \mathfrak{f}_0$ and $\ddot{\mathbf{q}}_2(z) = \mathfrak{f}_0$. Consequently, Notation 8.3 and Remark 8.9(b) deliver $\ddot{\mathbf{t}}_0(z) = I_q$ and $\ddot{\mathbf{v}}_0(z) = I_q$. Thus, using Notations 13.1 and 6.11, we conclude $\ddot{\mathbf{u}}_0(z) = \ddot{\mathbf{t}}_0^{[\![a]\!]}(z) = I_q^{[\![a]\!]}(z) = O_{q\times q}$ and $\ddot{\mathbf{w}}_0(z) = \ddot{\mathbf{v}}_0^{[\![b]\!]}(z) = I_q^{[\![b]\!]}(z) = O_{q\times q}$. Consequently, taking additionally into account (5.9), we get $\ddot{\mathbf{q}}_1(z) - s_0\ddot{\mathbf{t}}_0(z) = \mathfrak{f}_0 - s_0I_q = O_{q\times q} = \ddot{\mathbf{u}}_0(z)$ and $\ddot{\mathbf{q}}_2(z) - s_0\ddot{\mathbf{v}}_0(z) = \mathfrak{f}_0 - s_0I_q = O_{q\times q} = -\ddot{\mathbf{w}}_0(z)$. Hence, the equations in (13.39) are verified in the case $n = 0$.

Now assume $n \geq 1$. Because of Remark 5.9, the sequence $(s_j)_{j=0}^{2n+2}$ belongs in particular to $\mathcal{F}_{q,2n+2,\alpha,\beta}^{\succcurlyeq}$. Therefore, Lemma 13.2(d) yields (13.39).

**Part 3:** In the following, we want to verify (13.25). Using Notation 13.3, $\det \mathsf{D}_{2n+1} \neq 0$, (13.32), (13.28), (13.31), (13.27), and (13.35), we get

$$\begin{aligned}
&\ddot{\mathbf{V}}_{2n+1}(z)\mathbf{U}_{\mathrm{a},2n+1}(z)\\
&= \begin{bmatrix} \ddot{\mathbf{q}}_{4n+3}(z)\mathsf{D}_{2n+1}^{\dagger} & -\delta^{2n}\ddot{\mathbf{q}}_{4n+4} \\ -\ddot{\mathbf{p}}_{4n+3}(z)\mathsf{D}_{2n+1}^{\dagger} & \delta^{2n}\ddot{\mathbf{p}}_{4n+4} \end{bmatrix} \begin{bmatrix} \delta^{2n}I_q & -\delta^{2n}\mathsf{D}_{2n+1}\mathbf{A}_{2n+2} \\ (z-\alpha)\mathsf{D}_{2n+1}^{\dagger} & (z-\alpha)\mathbf{B}_{2n+2} \end{bmatrix}\\
&= \delta^{2n}\begin{bmatrix} [\ddot{\mathbf{q}}_{4n+3}(z) - (z-\alpha)\ddot{\mathbf{q}}_{4n+4}(z)]\mathsf{D}_{2n+1}^{\dagger} & -Q_n(z) \\ -[\ddot{\mathbf{p}}_{4n+3}(z) - (z-\alpha)\ddot{\mathbf{p}}_{4n+4}(z)]\mathsf{D}_{2n+1}^{\dagger} & P_n(z) \end{bmatrix}\\
&= \delta^{2n}\begin{bmatrix} -\delta\ddot{\mathbf{q}}_{4n+1}(z)\mathbf{A}_{2n+1}\mathsf{D}_{2n+1}^{\dagger} & -\ddot{\mathbf{q}}_{4n+5}(z) \\ \delta\ddot{\mathbf{p}}_{4n+1}(z)\mathbf{A}_{2n+1}\mathsf{D}_{2n+1}^{\dagger} & \ddot{\mathbf{p}}_{4n+5}(z) \end{bmatrix}\\
&= \delta^{2n}\begin{bmatrix} -\ddot{\mathbf{q}}_{4n+1}(z)L_{\alpha,n,\bullet}^{\dagger} & -\ddot{\mathbf{q}}_{4n+5}(z) \\ \ddot{\mathbf{p}}_{4n+1}(z)L_{\alpha,n,\bullet}^{\dagger} & \ddot{\mathbf{p}}_{4n+5}(z) \end{bmatrix},
\end{aligned}$$

where $Q_n(z) := \ddot{\mathbf{q}}_{4n+3}(z)\mathbf{A}_{2n+2} + (z-\alpha)\ddot{\mathbf{q}}_{4n+4}(z)\mathbf{B}_{2n+2}$ and where $P_n(z) := \ddot{\mathbf{p}}_{4n+3}(z)\mathbf{A}_{2n+2} + (z-\alpha)\ddot{\mathbf{p}}_{4n+4}(z)\mathbf{B}_{2n+2}$. Thus, using additionally (13.36), (13.39), (13.38), and Notation 13.3, we obtain then

$$\begin{bmatrix} (z-\alpha)I_q & s_0 \\ O_{q\times q} & I_q \end{bmatrix} \ddot{\mathbf{V}}_{2n+1}(z)\mathbf{U}_{\mathrm{a},2n+1}(z)$$

$$= \delta^{2n} \begin{bmatrix} (z-\alpha)I_q & s_0 \\ O_{q\times q} & I_q \end{bmatrix} \begin{bmatrix} -\ddot{\mathbf{q}}_{4n+1}(z)\mathfrak{f}_{4n+1}^\dagger & -\ddot{\mathbf{q}}_{4n+5}(z) \\ \ddot{\mathbf{p}}_{4n+1}(z)\mathfrak{f}_{4n+1}^\dagger & \ddot{\mathbf{p}}_{4n+5}(z) \end{bmatrix}$$

$$= \delta^{2n}(z-\alpha) \begin{bmatrix} -[\ddot{\mathbf{q}}_{4n+1}(z) - s_0\ddot{\mathbf{t}}_n(z)]\mathfrak{f}_{4n+1}^\dagger & -[\ddot{\mathbf{q}}_{4n+5}(z) - s_0\ddot{\mathbf{t}}_{n+1}(z)] \\ \ddot{\mathbf{t}}_n(z)\mathfrak{f}_{4n+1}^\dagger & \ddot{\mathbf{t}}_{n+1}(z) \end{bmatrix}$$

$$= \delta^{2n}(z-\alpha) \begin{bmatrix} -\ddot{\mathbf{u}}_n(z)\mathfrak{f}_{4n+1}^\dagger & -\ddot{\mathbf{u}}_{n+1}(z) \\ \ddot{\mathbf{t}}_n(z)\mathfrak{f}_{4n+1}^\dagger & \ddot{\mathbf{t}}_{n+1}(z) \end{bmatrix} = \delta^{2n}(z-\alpha)\mathbf{V}_{\mathrm{a},2n+1}(z).$$

**Part 4:** Finally, we want to check (13.26). Setting $R_n(z) := \ddot{\mathbf{q}}_{4n+3}(z)\mathbf{A}_{2n+2} - (\beta - z)\ddot{\mathbf{q}}_{4n+4}(z)\mathbf{B}_{2n+2}$ and $S_n(z) := \ddot{\mathbf{p}}_{4n+3}(z)\mathbf{A}_{2n+2} - (\beta - z)\ddot{\mathbf{p}}_{4n+4}(z)\mathbf{B}_{2n+2}$ and using Notation 13.3, $\det \mathsf{D}_{2n+1} \neq 0$, (13.34), (13.30), (13.33), (13.29), and (13.35), we observe that

$$\ddot{\mathbf{V}}_{2n+1}(z)\mathbf{U}_{\mathrm{b},2n+1}(z)$$

$$= \begin{bmatrix} \ddot{\mathbf{q}}_{4n+3}(z)\mathsf{D}_{2n+1}^\dagger & -\delta^{2n}\ddot{\mathbf{q}}_{4n+4}(z) \\ -\ddot{\mathbf{p}}_{4n+3}(z)\mathsf{D}_{2n+1}^\dagger & \delta^{2n}\ddot{\mathbf{p}}_{4n+4}(z) \end{bmatrix} \begin{bmatrix} \delta^{2n} I_q & \delta^{2n}\mathsf{D}_{2n+1}\mathbf{A}_{2n+2} \\ -(\beta - z)\mathsf{D}_{2n+1}^\dagger & (\beta - z)\mathbf{B}_{2n+2} \end{bmatrix}$$

$$= \delta^{2n} \begin{bmatrix} [\ddot{\mathbf{q}}_{4n+3}(z) + (\beta - z)\ddot{\mathbf{q}}_{4n+4}(z)]\mathsf{D}_{2n+1}^\dagger & R_n(z) \\ -[\ddot{\mathbf{p}}_{4n+3}(z) + (\beta - z)\ddot{\mathbf{p}}_{4n+4}(z)]\mathsf{D}_{2n+1}^\dagger & -S_n(z) \end{bmatrix}$$

$$= \delta^{2n} \begin{bmatrix} \delta\ddot{\mathbf{q}}_{4n+2}(z)\mathbf{B}_{2n+1}\mathsf{D}_{2n+1}^\dagger & \ddot{\mathbf{q}}_{4n+6}(z) \\ -\delta\ddot{\mathbf{p}}_{4n+2}(z)\mathbf{B}_{2n+1}\mathsf{D}_{2n+1}^\dagger & -\ddot{\mathbf{p}}_{4n+6}(z) \end{bmatrix}$$

$$= \delta^{2n} \begin{bmatrix} \ddot{\mathbf{q}}_{4n+2}(z)\mathfrak{f}_{4n+2}^\dagger & \ddot{\mathbf{q}}_{4n+6}(z) \\ -\ddot{\mathbf{p}}_{4n+2}(z)\mathfrak{f}_{4n+2}^\dagger & -\ddot{\mathbf{p}}_{4n+6}(z) \end{bmatrix}.$$

Thus, applying additionally (13.37), (13.39), and (13.38) as well as Notation 13.3, we conclude finally

$$\begin{bmatrix} (\beta - z)I_q & -s_0 \\ O_{q\times q} & I_q \end{bmatrix} \ddot{\mathbf{V}}_{2n+1}(z)\mathbf{U}_{\mathrm{b},2n+1}(z)$$

$$= \delta^{2n} \begin{bmatrix} (\beta - z)I_q & -s_0 \\ O_{q\times q} & I_q \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_{4n+2}(z)\mathfrak{f}_{4n+2}^\dagger & \ddot{\mathbf{q}}_{4n+6}(z) \\ -\ddot{\mathbf{p}}_{4n+2}(z)\mathfrak{f}_{4n+2}^\dagger & -\ddot{\mathbf{p}}_{4n+6}(z) \end{bmatrix}$$

$$= \delta^{2n}(\beta - z) \begin{bmatrix} [\ddot{\mathbf{q}}_{4n+2}(z) - s_0\ddot{\mathbf{v}}_n(z)]\mathfrak{f}_{4n+2}^\dagger & \ddot{\mathbf{q}}_{4n+6}(z) - s_0\ddot{\mathbf{v}}_{n+1}(z) \\ \ddot{\mathbf{v}}_n(z)\mathfrak{f}_{4n+2}^\dagger & \ddot{\mathbf{v}}_{n+1}(z) \end{bmatrix}$$

$$= \delta^{2n}(\beta - z) \begin{bmatrix} -\ddot{\mathbf{w}}_n(z)\mathfrak{f}_{4n+2}^\dagger & -\ddot{\mathbf{w}}_{n+1}(z) \\ \ddot{\mathbf{v}}_n(z)\mathfrak{f}_{4n+2}^\dagger & \ddot{\mathbf{v}}_{n+1}(z) \end{bmatrix} = \delta^{2n}(\beta - z)\mathbf{V}_{\mathrm{b},2n+1}(z). \quad \square$$

We continue our investigations with an interesting result regarding the sequences $(a_j)_{j=0}^{2\kappa}$ and $(b_j)_{j=0}^{2\kappa}$ given via (3.2) built from a $[\alpha,\beta]$-positive definite sequence $(s_j)_{j=0}^{2\kappa+1}$ of complex $q\times q$ matrices:

**Lemma 13.10** *Assume $\kappa\ge 1$ and let $(s_j)_{j=0}^{2\kappa+1}\in\mathcal{F}_{q,2\kappa+1,\alpha,\beta}^{\succ}$. Then the sequence $(a_j)_{j=0}^{2\kappa}$ belongs to $\mathcal{H}_{q,2\kappa}^{\succ}$ and $(\ddot{\mathbf{t}}_\ell)_{\ell=0}^{\kappa}$ is the uniquely determined MROS of matrix polynomials with respect to $(a_j)_{j=0}^{2\kappa}$. Moreover, the sequence $(b_j)_{j=0}^{2\kappa}$ belongs to $\mathcal{H}_{q,2\kappa}^{\succ}$ and $(\ddot{\mathbf{v}}_\ell)_{\ell=0}^{\kappa}$ is the uniquely determined MROS of matrix polynomials with respect to $(b_j)_{j=0}^{2\kappa}$.*

***Proof*** The representation of the class $\mathcal{F}_{q,2\kappa+1,\alpha,\beta}^{\succ}$ given in (5.5) delivers $\{(a_j)_{j=0}^{2\kappa},(b_j)_{j=0}^{2\kappa}\}\subseteq\mathcal{H}_{q,2\kappa}^{\succ}$. Because of Remark 5.9, we have $(s_j)_{j=0}^{2\kappa+1}\in\mathcal{F}_{q,2\kappa+1,\alpha,\beta}^{\succcurlyeq}$. Theorem 8.14 provides that the sequence $(\ddot{\mathbf{t}}_\ell)_{\ell=0}^{\kappa}$ (resp. $(\ddot{\mathbf{v}}_\ell)_{\ell=0}^{\kappa}$) forms an MROS of matrix polynomials with respect to $(a_j)_{j=0}^{2\kappa}$ (resp. $(b_j)_{j=0}^{2\kappa}$). Applying [15, Prop. 5.6(b)] completes the proof. □

**Lemma 13.11** *Let $\tau\in\mathbb{N}_0\cup\{\infty\}$ be such that $\kappa\le\tau$ and let $(s_j)_{j=0}^{2\tau}\in\mathcal{F}_{q,2\tau,\alpha,\beta}^{\succ}$.*

(a) *Let $[(\mathfrak{a}_k)_{k=0}^{\kappa},(\mathfrak{b}_k)_{k=0}^{\kappa},(\mathfrak{c}_k)_{k=0}^{\kappa},(\mathfrak{d}_k)_{k=0}^{\kappa}]$ be the $\mathbb{R}$-QMP associated with $(s_j)_{j=0}^{2\kappa}$. Then $\mathfrak{b}_k(z)=\ddot{\mathbf{r}}_k(z)$ and $\mathfrak{a}_k(z)=\ddot{\mathbf{o}}_k(z)$ for all $k\in\mathbb{Z}_{0,\kappa}$ and all $z\in\mathbb{C}$.*
(b) *Suppose $\kappa\ge 1$. Let $[(\mathfrak{a}_{c,k})_{k=0}^{\kappa-1},(\mathfrak{b}_{c,k})_{k=0}^{\kappa-1},(\mathfrak{c}_{c,k})_{k=0}^{\kappa-1},(\mathfrak{d}_{c,k})_{k=0}^{\kappa-1}]$ be the $\mathbb{R}$-QMP associated with $(c_j)_{j=0}^{2\kappa-2}$. Then $\mathfrak{b}_{c,j}(z)=\ddot{\mathbf{x}}_j(z)$ and $\mathfrak{a}_{c,j}(z)=\ddot{\mathbf{y}}_j(z)$ for all $j\in\mathbb{Z}_{0,\kappa-1}$ and all $z\in\mathbb{C}$.*

***Proof*** Proposition 5.10 yields $(s_j)_{j=0}^{m}\in\mathcal{F}_{q,m,\alpha,\beta}^{\succ}$ for all $m\in\mathbb{Z}_{0,2\tau}$ and, in view of (5.3) and Remark 5.1, consequently, $(s_j)_{j=0}^{2m}\in\mathcal{H}_{q,2m}^{\succ}\subseteq\mathcal{H}_{q,2m}^{\succcurlyeq,\mathrm{e}}\subseteq\mathcal{H}_{q,2m}^{\succcurlyeq}$ for all $m\in\mathbb{Z}_{0,\tau}$.

(a) From Theorem 8.14 we know that $(\ddot{\mathbf{r}}_k)_{k=0}^{\kappa}$ forms an MROS of matrix polynomials with respect to $(s_j)_{j=0}^{2\kappa}$. On the other hand, Proposition 6.10 yields that $(\mathfrak{b}_k)_{k=0}^{\kappa}$ is an MROS of matrix polynomials with respect to $(s_j)_{j=0}^{2\kappa}$. Applying [15, Prop. 5.6(b)], then we can conclude $\mathfrak{b}_k(z)=\ddot{\mathbf{r}}_k(z)$ for all $k\in\mathbb{Z}_{0,\kappa}$ and all $z\in\mathbb{C}$. Using Proposition 6.13 and Notation 13.1, then we get $\mathfrak{a}_k=\mathfrak{b}_k^{[\![s]\!]}=\ddot{\mathbf{r}}_k^{[\![s]\!]}=\ddot{\mathbf{o}}_k$ for all $k\in\mathbb{Z}_{0,\kappa}$.

(b) According to Theorem 8.14, the sequence $(\ddot{\mathbf{x}}_k)_{k=0}^{\kappa-1}$ forms an MROS of matrix polynomials with respect to $(c_j)_{j=0}^{2\kappa-2}$. Since $(s_j)_{j=0}^{2\kappa}$ belongs to $\mathcal{F}_{q,2\kappa,\alpha,\beta}^{\succ}$, from (5.3) we see that $(c_j)_{j=0}^{2\kappa-2}$ belongs to $\mathcal{H}_{q,2\kappa-2}^{\succ}$. Thus, Remark 5.1 provides $(c_j)_{j=0}^{2\kappa-2}\in\mathcal{H}_{q,2\kappa-2}^{\succcurlyeq,\mathrm{e}}\subseteq\mathcal{H}_{q,2\kappa-2}^{\succcurlyeq}$. Consequently, Proposition 6.10 yields that $(\mathfrak{b}_{c,k})_{k=0}^{\kappa-1}$ is an MROS of matrix polynomials with respect to $(c_j)_{j=0}^{2\kappa-2}$. Applying [15, Prop. 5.6(b)], we obtain $\mathfrak{b}_{c,k}(z)=\ddot{\mathbf{x}}_k(z)$ for every choice of $k\in\mathbb{Z}_{0,\kappa-1}$ and $z\in\mathbb{C}$. Using Proposition 6.13 and Notation 13.1, we get finally $\mathfrak{a}_{c,k}=\mathfrak{b}_{c,k}^{[\![c]\!]}=\ddot{\mathbf{x}}_k^{[\![c]\!]}=\ddot{\mathbf{y}}_k$ for all $k\in\mathbb{Z}_{0,\kappa-1}$. □

**Lemma 13.12** *Let $\tau\in\mathbb{N}_0\cup\{\infty\}$ be such that $\kappa\le\tau$ and let $(s_j)_{j=0}^{2\tau+1}\in\mathcal{F}_{q,\tau+1,\alpha,\beta}^{\succ}$. Further, let $[(\mathfrak{a}_{a,k})_{k=0}^{\kappa},(\mathfrak{b}_{a,k})_{k=0}^{\kappa},(\mathfrak{c}_{a,k})_{k=0}^{\kappa},(\mathfrak{d}_{a,k})_{k=0}^{\kappa}]$ be the $\mathbb{R}$-QMP associated with*

$(a_j)_{j=0}^{2\kappa}$ *and let* $[(\mathfrak{a}_{b,k})_{k=0}^{\kappa}, (\mathfrak{b}_{b,k})_{k=0}^{\kappa}, (\mathfrak{c}_{b,k})_{k=0}^{\kappa}, (\mathfrak{d}_{b,k})_{k=0}^{\kappa}]$ *be the* $\mathbb{R}$*-QMP associated with* $(b_j)_{j=0}^{2\kappa}$*. For every choice of* $k \in \mathbb{Z}_{0,\kappa}$ *and* $z \in \mathbb{C}$*, then* $\mathfrak{b}_{a,k}(z) = \ddot{\mathbf{t}}_k(z)$ *and* $\mathfrak{a}_{a,k}(z) = \ddot{\mathbf{u}}_k(z)$ *as well as* $\mathfrak{b}_{b,k}(z) = \ddot{\mathbf{v}}_k(z)$ *and* $\mathfrak{a}_{b,k}(z) = \ddot{\mathbf{w}}_k(z)$*.*

***Proof*** Proposition 5.10 yields $(s_j)_{j=0}^{m} \in \mathcal{F}_{q,m,\alpha,\beta}^{\succ}$ for all $m \in \mathbb{Z}_{0,2\tau+1}$. From Lemma 13.10 we know that $\{(a_j)_{j=0}^{2\kappa}, (b_j)_{j=0}^{2\kappa}\} \subseteq \mathcal{H}_{q,2\kappa}^{\succ}$, that $(\ddot{\mathbf{t}}_\ell)_{\ell=0}^{\kappa}$ is the uniquely determined MROS of matrix polynomials with respect to $(a_j)_{j=0}^{2\kappa}$, and that $(\ddot{\mathbf{v}}_\ell)_{\ell=0}^{\kappa}$ is the uniquely determined MROS of matrix polynomials with respect to $(b_j)_{j=0}^{2\kappa}$. Remark 5.1 yields then $\{(a_j)_{j=0}^{2\kappa}, (b_j)_{j=0}^{2\kappa}\} \subseteq \mathcal{H}_{q,2\kappa}^{\succcurlyeq,e} \subseteq \mathcal{H}_{q,2\kappa}^{\succcurlyeq}$, which, by virtue of Proposition 6.10, implies that $(\mathfrak{b}_{a,k})_{k=0}^{\kappa}$ is an MROS of matrix polynomials with respect to $(a_j)_{j=0}^{\kappa}$ and that $(\mathfrak{b}_{b,k})_{k=0}^{\kappa}$ is an MROS of matrix polynomials with respect to $(b_j)_{j=0}^{\kappa}$. Consequently, $\mathfrak{b}_{a,k}(z) = \ddot{\mathbf{t}}_k(z)$ and $\mathfrak{b}_{b,k}(z) = \ddot{\mathbf{v}}_k(z)$ for all $k \in \mathbb{Z}_{0,\kappa}$ and all $z \in \mathbb{C}$. Using Proposition 6.13, we get $\mathfrak{a}_{a,k} = \mathfrak{b}_{a,k}^{[\![a]\!]} = \ddot{\mathbf{t}}_k^{[\![a]\!]}$ and $\mathfrak{a}_{b,k} = \mathfrak{b}_{b,k}^{[\![b]\!]} = \ddot{\mathbf{v}}_k^{[\![b]\!]}$ for all $k \in \mathbb{Z}_{0,\kappa}$. Thus, from Notation 13.1 we see that $\mathfrak{a}_{a,k}(z) = \ddot{\mathbf{u}}_k(z)$ and $\mathfrak{a}_{b,k}(z) = \ddot{\mathbf{w}}_k(z)$ are proved for all $k \in \mathbb{Z}_{0,\kappa}$ and all $z \in \mathbb{C}$. □

**Lemma 13.13** *Suppose* $\kappa \geq 1$*. Let* $(s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q,\kappa}^{\succcurlyeq,e}$*, let* $[(\mathfrak{a}_k)_{k=0}^{\langle\!\langle\kappa]\!]}, (\mathfrak{b}_k)_{k=0}^{\langle\!\langle\kappa]\!]}, (\mathfrak{c}_k)_{k=0}^{\langle\!\langle\kappa]\!]}, (\mathfrak{d}_k)_{k=0}^{\langle\!\langle\kappa]\!]}]$ *be the* $\mathbb{R}$*-QMP associated with* $(s_j)_{j=0}^{\kappa}$*, and let* $n \in \mathbb{N}_0$ *be such that* $2n+1 \leq \kappa$*. Furthermore, let* $S, T \in \mathbb{C}^{q\times q}$ *be such that the three conditions*

(I) $\operatorname{rank}\left[\begin{smallmatrix} S \\ T \end{smallmatrix}\right] = q$,
(II) $\Im(T^*S) \succcurlyeq O_{q\times q}$,
(III) $\mathcal{R}(S) \subseteq \mathcal{R}(L_n)$

*are fulfilled. For all* $w \in \Pi_+$*, then* $\det[\mathfrak{b}_n(w)L_n^{\dagger}S + \mathfrak{b}_{n+1}(w)T] \neq 0$.

***Proof*** Let $w \in \Pi_+$. Consider an arbitrary $v \in \mathcal{N}(\mathfrak{b}_n(w)L_n^{\dagger}S + \mathfrak{b}_{n+1}(w)T)$. Then

$$[\mathfrak{b}_n(w)L_n^{\dagger}S + \mathfrak{b}_{n+1}(w)T]v = O. \tag{13.40}$$

**Part 1:** Lemma 6.18 provides $\det \mathfrak{b}_n(w) \neq 0$ and $\det \mathfrak{b}_{n+1}(w) \neq 0$. Combining Remarks 7.4 and 5.7 and using the notations given there, we get $\chi_{2n+1}(w) = L_n[\mathfrak{b}_n(w)]^{-1}[\mathfrak{b}_{n+1}(w)]$. Thus, the multiplication of $L_n[\mathfrak{b}_n(w)]^{-1}$ to the left side of (13.40) implies $L_nL_n^{\dagger}Sv + L_n[\mathfrak{b}_n(w)]^{-1}\mathfrak{b}_{n+1}(w)Tv = O$, which is equivalent to $L_nL_n^{\dagger}Sv + \chi_{2n+1}(w)Tv = O$. Taking into account condition (III) and applying Remark A.1, we can rewrite the last equation to

$$Sv + \chi_{2n+1}(w)Tv = O. \tag{13.41}$$

The multiplication of (13.41) with $v^*T^*$ from the left yields $v^*T^*Sv + v^*T^*\chi_{2n+1}(w)Tv = 0$. By virtue of Remark A.2, consequently,

$$v^*\Im(T^*S)v + v^*T^*\Im(\chi_{2n+1}(w))Tv = \Im(v^*T^*Sv + v^*T^*\chi_{2n+1}(w)Tv) = 0. \tag{13.42}$$

**Part 2:** Our next goal is to prove, for all $n \in \mathbb{N}_0$ fulfilling $2n+1 \leq \kappa$, that

$$\Im(\chi_{2n+1}(w)) \succcurlyeq \Im(w)L_n \succcurlyeq O. \tag{13.43}$$

If $n = 0$, then (7.3) provides $\chi_1(w) = ws_0 - s_1$. From Notation 5.2, $(s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q,\kappa}^{\succcurlyeq,\mathrm{e}}$, and Remark 5.3, we get $L_0 = H_0 = s_0 \succcurlyeq O$ and $s_1^* = s_1$, which implies $\Im(s_1) = O$ and, consequently, $\Im(\chi_1(w)) = \Im(ws_0 - s_1) = \Im(w)s_0 = \Im(w)L_0 \succcurlyeq O$. Now assume $n \geq 1$. From Remarks 5.7 and 6.14 and [14, Cor. 7.22] we conclude then

$$\begin{aligned}\Im(\chi_{2n+1}(w)) = [\Im(w)]L_n + L_n[\mathfrak{b}_n(\overline{w})]^{-1}\mathfrak{b}_{n-1}(\overline{w})L_{n-1}^{\dagger}[\Im\chi_{2n-1}(w)]\\ \times L_{n-1}^{\dagger}[\mathfrak{b}_{n-1}(\overline{w})]^*[\mathfrak{b}_n(\overline{w})]^{-*}L_n.\end{aligned}$$

Because of $(s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q,\kappa}^{\succcurlyeq,\mathrm{e}}$, from [15, Propositions 2.10(b) and 2.15(b)] and Remark 5.7 we get $L_n \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ and, according to Remark A.9, then $L_n^{\dagger} \in \mathbb{C}_{\succcurlyeq}^{q\times q} \subseteq \mathbb{C}_{\mathrm{H}}^{q\times q}$. Since $w$ belongs to $\Pi_+$, Remark 7.5 provides $\Im(\chi_{2n-1}(w)) \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. Consequently, (13.43) holds true.
**Part 3:** Under guidance of condition (II), from (13.42) and (13.43) we infer

$$\begin{aligned}0 = v^*\Im(T^*S)v + v^*T^*\Im(\chi_{2n+1}(w))Tv \geq v^*T^*\Im(\chi_{2n+1}(w))Tv\\ \geq v^*T^*[\Im(w)L_n]Tv \geq 0.\end{aligned}$$

and, hence, $\Im(w)v^*T^*L_nTv = v^*T^*[\Im(w)L_n]Tv = 0$. Consequently, in view of $w \in \Pi_+$ and $L_n \in \mathbb{C}_{\succcurlyeq}^{q\times q}$, we get $L_nTv = O$, i.e., $Tv \in \mathcal{N}(L_n)$. Since the combination of [14, Prop. 7.19] and Remark 5.7 provides $\mathcal{N}(\chi_{2n+1}(w)) = \mathcal{N}(\mathfrak{h}_{2n}) = \mathcal{N}(L_n)$, this implies $\chi_{2n+1}(w)Tv = O$. Thus, from (13.41) we observe $Sv = O$.
**Part 4:** Regarding $Sv = O$, we see that (13.40) simplifies to $\mathfrak{b}_{n+1}(w)Tv = O$ and, since $\det \mathfrak{b}_{n+1}(w) \neq 0$ is valid as well, we conclude $Tv = O$. Using additionally $Sv = O$, then $\left[\begin{smallmatrix} S \\ T \end{smallmatrix}\right]v = O$ follows. Because of condition (I), this implies $v = O$. The proof is finished. □

Now we introduce particular matrix balls (see also (7.2)).

**Notation 13.14** Let $(s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq}$ and let $w \in \Pi_+$. For each $n \in \mathbb{N}_0$ such that $2n \leq \kappa$, let $\mathscr{A}_{2n}$, $\mathscr{B}_{2n}$, and $\mathscr{C}_{2n}$ be given by Notation 7.7(a) and let

$$\mathscr{K}_{2n}(w) := \mathfrak{K}(\mathscr{C}_{2n}(w); (w-\overline{w})^{-1}\mathscr{A}_{2n}(w), \mathscr{B}_{2n}(w)).$$

If $\kappa \geq 1$, then, for all $n \in \mathbb{N}_0$ fulfilling $2n+1 \leq \kappa$, let $\mathscr{A}_{\alpha,2n,\bullet}, \mathscr{B}_{\alpha,2n,\bullet}, \mathscr{C}_{\alpha,2n,\bullet} : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{q\times q}$ and $\mathscr{A}_{\bullet,2n,\beta}, \mathscr{B}_{\bullet,2n,\beta}, \mathscr{C}_{\bullet,2n,\beta} : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{q\times q}$ be defined by Notations 3.4 and 7.7(a) and let

$$\mathscr{K}_{\mathrm{a},2n}(w) := \frac{1}{w-\alpha}\left[\mathfrak{K}(\mathscr{C}_{\alpha,2n,\bullet}(w); (w-\overline{w})^{-1}\mathscr{A}_{\alpha,2n,\bullet}(w), \mathscr{B}_{\alpha,2n,\bullet}(w)) - s_0\right]$$

and

$$\mathscr{K}_{\mathrm{b},2n}(w) := \frac{1}{\beta - w}\left[\mathfrak{K}(\mathscr{C}_{\bullet,2n,\beta}(w); (w-\overline{w})^{-1}\mathscr{A}_{\bullet,2n,\beta}(w), \mathscr{B}_{\bullet,2n,\beta}(w)) + s_0\right].$$

If $\kappa \geq 2$, then, for all $n \in \mathbb{N}_0$ such that $2n+2 \leq \kappa$, let $\mathscr{A}_{\alpha,2n,\beta}, \mathscr{B}_{\alpha,2n,\beta}, \mathscr{C}_{\alpha,2n,\beta} \colon \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{q\times q}$ be defined by Notations 3.4 and 7.7(a) and let

$$\begin{aligned}\mathscr{K}_{\mathrm{c},2n}(w) \\ := \frac{1}{(\beta - w)(w-\alpha)}\Big\{&\mathfrak{K}(\mathscr{C}_{\alpha,2n,\beta}(w); (w-\overline{w})^{-1}\mathscr{A}_{\alpha,2n,\beta}(w), \mathscr{B}_{\alpha,2n,\beta}(w)) \\ &-(\alpha+\beta-w)s_0 + s_1\Big\}.\end{aligned}$$

**Lemma 13.15** *Let $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{\succcurlyeq}_{q,\kappa,\alpha,\beta}$ and let $F \in \mathcal{R}_q[[\alpha,\beta]; (s_j)_{j=0}^{\kappa}, =]$. Furthermore, let $n \in \mathbb{N}_0$ be such that $2n \leq \kappa$. For all $w \in \Pi_+$, then $(s_j)_{j=0}^{2n} \in \mathcal{H}^{\succcurlyeq,\mathrm{e}}_{q,2n}$ and $F(w) \in \mathscr{K}_{2n}(w)$.*

***Proof*** Proposition 5.10 delivers $(s_j)_{j=0}^{2n} \in \mathcal{F}^{\succcurlyeq}_{q,2n,\alpha,\beta}$. Hence, according to [19, Prop. 7.10], the sequence $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{H}^{\succcurlyeq,\mathrm{e}}_{q,2n}$. Since $F$ belongs to $\mathcal{R}_q[[\alpha,\beta]; (s_j)_{j=0}^{\kappa}, =]$, because of Theorem 4.3 and the definition of the set $\mathcal{R}_q[[\alpha,\beta]; (s_j)_{j=0}^{\kappa}, =]$, we have $\ddot{\sigma}_F \in \mathcal{M}_q^{\succcurlyeq}[[\alpha,\beta]; (s_j)_{j=0}^{\kappa}, =]$. Let $f$ be the restriction of $F$ onto $\Pi_+$. Then according to $F \in \mathcal{R}_q(\mathbb{C} \setminus [\alpha,\beta])$ and [23, Def. 4.4, Lem. 4.12], we know that $f \in \mathcal{R}_{0,q}(\Pi_+)$ and that the $\mathbb{R}$-Stieltjes measure $\sigma_f$ of $f$ fulfills $\sigma_f(\mathbb{R}\setminus[\alpha,\beta]) = O_{q\times q}$ and $\sigma_f(B) = \ddot{\sigma}_F(B)$ for all $B \in \mathfrak{B}_{[\alpha,\beta]}$. Thus, in view of $\ddot{\sigma}_F \in \mathcal{M}_q^{\succcurlyeq}[[\alpha,\beta]; (s_j)_{j=0}^{\kappa}, =]$, we obtain $s_j = \int_{[\alpha,\beta]} x^j \ddot{\sigma}_F(\mathrm{d}x) = \int_{\mathbb{R}} x^j \sigma_f(\mathrm{d}x)$ for all $j \in \mathbb{Z}_{0,\kappa}$. Consequently, $\sigma_f \in \mathcal{M}_q^{\succcurlyeq}[\mathbb{R}; (s_j)_{j=0}^{\kappa}, =]$. Hence, in particular, we get $\sigma_f \in \mathcal{M}_q^{\succcurlyeq}[\mathbb{R}; (s_j)_{j=0}^{2n}, \preccurlyeq]$. Therefore, $f \in \mathcal{R}_{0,q}[\Pi_+; (s_j)_{j=0}^{2n}, \preccurlyeq]$. Thus, taking into account $(s_j)_{j=0}^{2n} \in \mathcal{H}^{\succcurlyeq,\mathrm{e}}_{q,2n}$, Theorem 7.11 yields $f(w) \in \mathfrak{K}(\mathscr{C}_{2n}(w); (w-\overline{w})^{-1}\mathscr{A}_{2n}(w), \mathscr{B}_{2n}(w))$, i.e., $f(w) \in \mathscr{K}_{2n}(w)$ for all $w \in \Pi_+$. Since $f$ is the restriction of $F$ onto $\Pi_+$, the proof is complete. □

**Theorem 13.16** *Suppose $\kappa \geq 1$. Let $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{\succcurlyeq}_{q,\kappa,\alpha,\beta}$, let $F \in \mathcal{R}_q[[\alpha,\beta]; (s_j)_{j=0}^{\kappa}, =]$, and let $w \in \Pi_+$.*

(a) *Let $n \in \mathbb{N}_0$ be such that $2n+1 \leq \kappa$. Then the sequences $(a_j)_{j=0}^{2n}$ and $(b_j)_{j=0}^{2n}$ given by (3.2) both belong to $\mathcal{H}^{\succcurlyeq,\mathrm{e}}_{q,2n}$ and the matrix $F(w)$ belongs to $\mathscr{K}_{\mathrm{a},2n}(w) \cap \mathscr{K}_{\mathrm{b},2n}(w)$.*
(b) *Let $\kappa \geq 2$ and let $n \in \mathbb{N}_0$ be such that $2n+2 \leq \kappa$. Then the sequence $(c_j)_{j=0}^{2n}$ given by (3.3) belongs to $\mathcal{H}^{\succcurlyeq,\mathrm{e}}_{q,2n}$ and the matrix $F(w)$ belongs to $\mathscr{K}_{\mathrm{c},2n}(w)$.*

***Proof*** Since $F$ belongs to $\mathcal{R}_q[[\alpha,\beta]; (s_j)_{j=0}^{\kappa}, =]$, the corresponding $\mathcal{R}[\alpha,\beta]$-measure $\ddot{\sigma}_F$ fulfills $\ddot{\sigma}_F([\alpha,\beta]) = s_0$ and $\int_{[\alpha,\beta]} x \ddot{\sigma}_F(\mathrm{d}x) = s_1$.

(a) In view of $\ddot{\sigma}_F([\alpha, \beta]) = s_0$, Remark 4.4 delivers

$$F(w) = \frac{1}{w - \alpha}[F_{\mathrm{a}}(w) - s_0] \quad \text{and} \quad F(w) = \frac{1}{\beta - w}[F_{\mathrm{b}}(w) + s_0]. \tag{13.44}$$

We know from Proposition 5.10 that $(s_j)_{j=0}^{2n+1} \in \mathcal{F}_{q,2n+1,\alpha,\beta}^{\succcurlyeq}$. Moreover, by assumption and $\mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^{\kappa}, =] \subseteq \mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^{2n+1}, =]$, we have $F \in \mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^{2n+1}, =]$. Therefore, Remarks 4.4 and 4.5 yield $F_{\mathrm{a}} \in \mathcal{R}_q[[\alpha, \beta]; (a_j)_{j=0}^{2n}, =]$ and $F_{\mathrm{b}} \in \mathcal{R}_q[[\alpha, \beta]; (b_j)_{j=0}^{2n}, =]$. Furthermore, taking into account $(s_j)_{j=0}^{2n+1} \in \mathcal{F}_{q,2n+1,\alpha,\beta}^{\succcurlyeq}$, Proposition 5.11 shows that $\{(a_j)_{j=0}^{2n}, (b_j)_{j=0}^{2n}\} \subseteq \mathcal{F}_{q,2n,\alpha,\beta}^{\succcurlyeq}$. In view of Notation 13.14, the application of Lemma 13.15 provides $\{(a_j)_{j=0}^{2n}, (b_j)_{j=0}^{2n}\} \subseteq \mathcal{H}_{q,2n}^{\succcurlyeq,\mathrm{e}}$ as well as $F_{\mathrm{a}}(w) \in \mathfrak{K}(\mathscr{C}_{\alpha,2n,\bullet}(w); (w - \overline{w})^{-1}\mathscr{A}_{\alpha,2n,\bullet}(w), \mathscr{B}_{\alpha,2n,\bullet}(w))$ and $F_{\mathrm{b}}(w) \in \mathfrak{K}(\mathscr{C}_{\bullet,2n,\beta}(w); (w - \overline{w})^{-1} \mathscr{A}_{\bullet,2n,\beta}(w), \mathscr{B}_{\bullet,2n,\beta}(w))$. Thus, (13.44) yields $F(w) \in \frac{1}{w-\alpha}[\mathfrak{K}(\mathscr{C}_{\alpha,2n,\bullet}(w); (w - \overline{w})^{-1}\mathscr{A}_{\alpha,2n,\bullet}(w), \mathscr{B}_{\alpha,2n,\bullet}(w)) - s_0]$ and $F(w) \in \frac{1}{\beta-w}[\mathfrak{K}(\mathscr{C}_{\bullet,2n,\beta}(w); (w - \overline{w})^{-1} \mathscr{A}_{\bullet,2n,\beta}(w), \mathscr{B}_{\bullet,2n,\beta}(w)) + s_0]$. In view of Notation 13.14, part (a) is verified.

(b) In view of $\ddot{\sigma}_F([\alpha, \beta]) = s_0$ and $\int_{[\alpha,\beta]} x\ddot{\sigma}_F(\mathrm{d}x) = s_1$, Remark 4.4 provides

$$F(w) = \frac{1}{(\beta - w)(w - \alpha)}[F_{\mathrm{c}}(w) - (\alpha + \beta - w)s_0 + s_1]. \tag{13.45}$$

We know from Proposition 5.10 that $(s_j)_{j=0}^{2n+2} \in \mathcal{F}_{q,2n+2,\alpha,\beta}^{\succcurlyeq}$. By assumption and $\mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^{\kappa}, =] \subseteq \mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^{2n+2}, =]$, we have $F \in \mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^{2n+2}, =]$. Therefore, Remarks 4.4 and 4.5 deliver $F_{\mathrm{c}} \in \mathcal{R}_q[[\alpha, \beta]; (c_j)_{j=0}^{2n}, =]$. Furthermore, taking into account $(s_j)_{j=0}^{2n+2} \in \mathcal{F}_{q,2n+2,\alpha,\beta}^{\succcurlyeq}$, Proposition 5.11 shows $(c_j)_{j=0}^{2n} \in \mathcal{F}_{q,2n,\alpha,\beta}^{\succcurlyeq}$. In view of Notation 13.14, applying Lemma 13.15 provides thus $(c_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\succcurlyeq,\mathrm{e}}$ as well as $F_{\mathrm{c}}(w) \in \mathfrak{K}(\mathscr{C}_{\alpha,2n,\beta}(w); (w-\overline{w})^{-1}\mathscr{A}_{\alpha,2n,\beta}(w), \mathscr{B}_{\alpha,2n,\beta}(w))$. Finally, (13.45) yields then $F(w) \in \frac{1}{(\beta-w)(w-\alpha)}[\mathfrak{K}(\mathscr{C}_{\alpha,2n,\beta}(w); (w-\overline{w})^{-1}\mathscr{A}_{\alpha,2n,\beta}(w), \mathscr{B}_{\alpha,2n,\beta}(w)) - (\alpha + \beta - w)s_0 + s_1]$. In view of Notation 13.14, part (b) is verified as well. □

## 14 Weyl Sets in the Case of an Even Number of Prescribed Matricial Moments

In this section, we work out an explicit representation of the set $\left\{F(w)\colon F \in \mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^{\kappa}, =]\right\}$ in the case of an even number of prescribed matricial moments, i. e., in the situation that $\kappa$ is an odd positive integer.

**Proposition 14.1** *Let $n \in \mathbb{N}_0$, let $(s_j)_{j=0}^{2n+1} \in \mathcal{F}_{q,2n+1,\alpha,\beta}^{\succ}$ and let $w \in \Pi_+$. Furthermore, let $X \in \mathscr{K}_{\mathrm{a},2n}(w) \cap \mathscr{K}_{\mathrm{b},2n}(w)$. Then there is a rational matrix-valued function $F$ belonging to $\mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^{2n+1}, =]$ such that $X = F(w)$.*

***Proof*** Our proof is divided into seven parts.

**Part 1:** By virtue of [19, Prop. 11.12], there exists a sequence $(s_j)_{j=2n+2}^{\infty}$ of complex $q \times q$ matrices such that $(s_j)_{j=0}^{\infty} \in \mathcal{F}_{q,\infty,\alpha,\beta}^{\succ}$. Consequently, $(s_j)_{j=0}^{m} \in \mathcal{F}_{q,m,\alpha,\beta}^{\succ}$ for all $m \in \mathbb{N}_0$ and, in view of Remark 5.9, then $(s_j)_{j=0}^{m} \in \mathcal{F}_{q,m,\alpha,\beta}^{\succcurlyeq}$ for all $m \in \mathbb{N}_0$. Lemma 13.10 yields $\{(a_j)_{j=0}^{2m}, (b_j)_{j=0}^{2m}\} \subseteq \mathcal{H}_{q,2m}^{\succ}$ for all $m \in \mathbb{N}_0$. Thus, Remark 5.1 provides $\{(a_j)_{j=0}^{2m-1}, (b_j)_{j=0}^{2m-1}\} \subseteq \mathcal{H}_{q,2m-1}^{\succcurlyeq,\mathrm{e}}$ for all $m \in \mathbb{N}$ and $\{(a_j)_{j=0}^{2m}, (b_j)_{j=0}^{2m}\} \subseteq \mathcal{H}_{q,2m}^{\succcurlyeq,\mathrm{e}} \subseteq \mathcal{H}_{q,2m}^{\succcurlyeq}$ for all $m \in \mathbb{N}_0$.

**Part 2:** In view of Part 1, Lemma 7.9 provides $\mathscr{A}_{\alpha,2n,\bullet}(w) = \mathscr{A}_{\alpha,2n+1,\bullet}(w)$ as well as $\mathscr{B}_{\alpha,2n,\bullet}(w) = \mathscr{B}_{\alpha,2n+1,\bullet}(w)$, and $\mathscr{C}_{\alpha,2n,\bullet}(w) = \mathscr{C}_{\alpha,2n+1,\bullet}(w)$. Consequently, we see then from the assumption $X \in \mathscr{K}_{\mathrm{a},2n}(w)$, Notation 13.14, and the definition of a matrix ball that there exists a contractive complex $q \times q$ matrix $C_\mathrm{a}$ such that

$$(w-\alpha)X + s_0 = \mathscr{C}_{\alpha,2n+1,\bullet}(w) + (w-\overline{w})^{-1}\mathscr{A}_{\alpha,2n+1,\bullet}(w)C_\mathrm{a}\mathscr{B}_{\alpha,2n+1,\bullet}(w). \quad (14.1)$$

In view of Definition 7.1 and the notion of orthogonal projection matrices introduced in Remark A.4, let

$$E_\mathrm{a} := -[\chi_{\alpha,2n+1,\bullet}(w)]^*, \quad B_\mathrm{a} := \Im(E_\mathrm{a}), \quad P_\mathrm{a} := \mathbb{P}_{\mathcal{R}(B_\mathrm{a})}, \quad Q_\mathrm{a} := \mathbb{P}_{\mathcal{N}(B_\mathrm{a})}. \quad (14.2)$$

Taking into account Part 1, Remark 7.5 provides $\frac{1}{\Im(w)}\Im(\chi_{\alpha,2n+1,\bullet}(w)) \succcurlyeq O$. By virtue of (14.2) and Remark A.2, we have

$$B_\mathrm{a} = \Im(E_\mathrm{a}) = \Im(\chi_{\alpha,2n+1,\bullet}(w)). \quad (14.3)$$

Therefore, from (14.3) and $w \in \Pi_+$, we conclude $B_\mathrm{a} \succcurlyeq O$. Let

$$S_\mathrm{a} := E_\mathrm{a}\sqrt{B_\mathrm{a}}^{\dagger} - E_\mathrm{a}^*\sqrt{B_\mathrm{a}}^{\dagger}C_\mathrm{a}P_\mathrm{a} \quad \text{and} \quad T_\mathrm{a} := \sqrt{B_\mathrm{a}}^{\dagger} - \sqrt{B_\mathrm{a}}^{\dagger}C_\mathrm{a}P_\mathrm{a} + Q_\mathrm{a}. \quad (14.4)$$

Under guidance of Part 1, [14, Prop. 7.19] and Remark 5.7 yield the validity of

$$\mathcal{R}([\chi_{\alpha,2n+1,\bullet}(w)]^*) = \mathcal{R}(\chi_{\alpha,2n+1,\bullet}(w)) = \mathcal{R}(\Im(\chi_{\alpha,2n+1,\bullet}(w))) = \mathcal{R}(L_{\alpha,n,\bullet})$$

and

$$\mathcal{N}([\chi_{\alpha,2n+1,\bullet}(w)]^*) = \mathcal{N}(\chi_{\alpha,2n+1,\bullet}(w)) = \mathcal{N}(\Im(\chi_{\alpha,2n+1,\bullet}(w))) = \mathcal{N}(L_{\alpha,n,\bullet}),$$

which, in compliance with (14.2) and (14.3), implies

$$\mathcal{R}(E_\mathrm{a}) = \mathcal{R}(E_\mathrm{a}^*) = \mathcal{R}(B_\mathrm{a}) = \mathcal{R}(L_{\alpha,n,\bullet}) \quad (14.5)$$

and

$$\mathcal{N}(E_\mathrm{a}) = \mathcal{N}(E_\mathrm{a}^*) = \mathcal{N}(B_\mathrm{a}) = \mathcal{N}(L_{\alpha,n,\bullet}). \quad (14.6)$$

Hence, in view of (14.2), $B_{\mathrm{a}} \succcurlyeq O$, (14.5), (14.4), and $C_{\mathrm{a}} \in \mathbb{K}_{q\times q}$, we can apply [14, Lem. A.26] and get

$$\operatorname{rank}\begin{bmatrix} S_{\mathrm{a}} \\ T_{\mathrm{a}} \end{bmatrix} = q, \qquad P_{\mathrm{a}} S_{\mathrm{a}} = S_{\mathrm{a}} = S_{\mathrm{a}} P_{\mathrm{a}}, \qquad T_{\mathrm{a}} P_{\mathrm{a}} = T_{\mathrm{a}} - Q_{\mathrm{a}}, \tag{14.7}$$

$$\sqrt{B_{\mathrm{a}}}^{\dagger}(S_{\mathrm{a}} - E_{\mathrm{a}} T_{\mathrm{a}})(S_{\mathrm{a}} - E_{\mathrm{a}}^{*} T_{\mathrm{a}})^{\dagger}\sqrt{B_{\mathrm{a}}} = P_{\mathrm{a}} C_{\mathrm{a}} P_{\mathrm{a}}, \text{ and } \Im(T_{\mathrm{a}}^{*} S_{\mathrm{a}}) \succcurlyeq O. \tag{14.8}$$

Because of (14.2), (14.5), and (14.6), we obtain

$$P_{\mathrm{a}} = \mathbb{P}_{\mathcal{R}(B_{\mathrm{a}})} = \mathbb{P}_{\mathcal{R}(L_{\alpha,n,\bullet})} \qquad \text{and} \qquad Q_{\mathrm{a}} = \mathbb{P}_{\mathcal{N}(B_{\mathrm{a}})} = \mathbb{P}_{\mathcal{N}(L_{\alpha,n,\bullet})}. \tag{14.9}$$

Thus, taking into account (14.7), (14.9), Remark A.5, and (14.5), we infer $\mathcal{R}(S_{\mathrm{a}}) = \mathcal{R}(P_{\mathrm{a}} S_{\mathrm{a}}) \subseteq \mathcal{R}(P_a) = \mathcal{R}(\mathbb{P}_{\mathcal{R}(L_{\alpha,n,\bullet})}) = \mathcal{R}(L_{\alpha,n,\bullet})$. Taking additionally into account Part 1, Lemma 13.12, (14.7), (14.8) as well as Lemma 13.13, we conclude that $R_{\mathrm{a}} := \ddot{\mathbf{t}}_n(w) L_{\alpha,n,\bullet}^{\dagger} S_{\mathrm{a}} + \ddot{\mathbf{t}}_{n+1}(w) T_{\mathrm{a}}$ fulfills $\det R_{\mathrm{a}} \neq 0$. Because of Part 1, Remark 5.7, (14.9), (14.8), (14.7) and under guidance of Lemma 13.12, now we can apply [14, Prop. 8.18] and get

$$\begin{aligned} &-[\ddot{\mathbf{u}}_n(w) L_{\alpha,n,\bullet}^{\dagger} S_{\mathrm{a}} + \ddot{\mathbf{u}}_{n+1}(w) T_{\mathrm{a}}] R_{\mathrm{a}}^{-1} \\ &\quad = \mathscr{C}_{\alpha,2n+1,\bullet}(w) + (w - \overline{w})^{-1} \mathscr{A}_{\alpha,2n+1,\bullet}(w) K_{\mathrm{a}} \mathscr{B}_{\alpha,2n+1,\bullet}(w), \end{aligned} \tag{14.10}$$

where the matrix $\Im(\chi_{\alpha,2n+1,\bullet}(w))$ is non-negative Hermitian and where

$$\begin{aligned} K_{\mathrm{a}} \;&:= \sqrt{\Im(\chi_{\alpha,2n+1,\bullet}(w))}^{\dagger} (S_{\mathrm{a}} + [\chi_{\alpha,2n+1,\bullet}(w)]^{*} T_{\mathrm{a}}) \\ &\quad \times (S_{\mathrm{a}} + \chi_{\alpha,2n+1,\bullet}(w) T_{\mathrm{a}})^{\dagger} \sqrt{\Im(\chi_{\alpha,2n+1,\bullet}(w))} \end{aligned}$$

is a contractive matrix. Using (14.3) and (14.2), we see that $K_{\mathrm{a}} = \sqrt{B_{\mathrm{a}}}^{\dagger}(S_{\mathrm{a}} - E_{\mathrm{a}} T_{\mathrm{a}})(S_{\mathrm{a}} - E_{\mathrm{a}}^{*} T_{\mathrm{a}})^{\dagger}\sqrt{B_{\mathrm{a}}}$. From Part 1, Proposition 5.8, and Remark 5.7 we know that $\det L_{\alpha,n,\bullet} \neq 0$, which, applying (14.9), delivers $P_{\mathrm{a}} = I_q$. Hence, (14.8) yields to $K_{\mathrm{a}} = P_{\mathrm{a}} C_{\mathrm{a}} P_{\mathrm{a}} = C_{\mathrm{a}}$. Taking additionally into account (14.1), equation (14.10) simplifies to

$$\begin{aligned} &-\left[\ddot{\mathbf{u}}_n(w) L_{\alpha,n,\bullet}^{\dagger} S_{\mathrm{a}} + \ddot{\mathbf{u}}_{n+1}(w) T_{\mathrm{a}}\right] R_{\mathrm{a}}^{-1} \\ &\quad = \mathscr{C}_{\alpha,2n+1,\bullet}(w) + (w - \overline{w})^{-1} \mathscr{A}_{\alpha,2n+1,\bullet}(w) C_{\mathrm{a}} \mathscr{B}_{\alpha,2n+1,\bullet}(w) \\ &\quad = (w - \alpha) X + s_0. \end{aligned} \tag{14.11}$$

Let $(\mathfrak{f}_k)_{k=0}^{4n+2}$ be the $\mathcal{F}_{\alpha,\beta}$-parameter sequence of $(s_j)_{j=0}^{2n+1}$. In view of (14.11), Remark 5.22, and Notation 13.3, one can see that

$$\begin{aligned} \begin{bmatrix} (w-\alpha)X + s_0 \\ I_q \end{bmatrix} &= \begin{bmatrix} -[\ddot{\mathbf{u}}_n(w) L_{\alpha,n,\bullet}^{\dagger} S_{\mathrm{a}} + \ddot{\mathbf{u}}_{n+1}(w) T_{\mathrm{a}}] R_{\mathrm{a}}^{-1} \\ R_{\mathrm{a}} R_{\mathrm{a}}^{-1} \end{bmatrix} \\ &= \begin{bmatrix} -[\ddot{\mathbf{u}}_n(w) L_{\alpha,n,\bullet}^{\dagger} S_{\mathrm{a}} + \ddot{\mathbf{u}}_{n+1}(w) T_{\mathrm{a}}] \\ \ddot{\mathbf{t}}_n(w) L_{\alpha,n,\bullet}^{\dagger} S_{\mathrm{a}} + \ddot{\mathbf{t}}_{n+1}(w) T_{\mathrm{a}} \end{bmatrix} R_{\mathrm{a}}^{-1} \end{aligned}$$

$$= \begin{bmatrix} -\ddot{\mathbf{u}}_n(w)\mathfrak{f}_{4n+1}^\dagger & -\ddot{\mathbf{u}}_{n+1}(w) \\ \ddot{\mathbf{t}}_n(w)\mathfrak{f}_{4n+1}^\dagger & \ddot{\mathbf{t}}_{n+1}(w) \end{bmatrix} \begin{bmatrix} S_\mathrm{a} \\ T_\mathrm{a} \end{bmatrix} R_\mathrm{a}^{-1} = \mathbf{V}_{\mathrm{a},2n+1}(w) \begin{bmatrix} S_\mathrm{a} \\ T_\mathrm{a} \end{bmatrix} R_\mathrm{a}^{-1}. \tag{14.12}$$

**Part 3:** Since $\delta > 0$ and $w-\alpha \neq 0$ as well as $\det R_\mathrm{a} \neq 0$ are valid, using Notation 13.3, we can define

$$W := (w-\alpha)^{-1}\delta^{-2n}\mathbf{U}_{\mathrm{a},2n+1}(w) \begin{bmatrix} S_\mathrm{a} \\ T_\mathrm{a} \end{bmatrix} R_\mathrm{a}^{-1}. \tag{14.13}$$

Let $W = \begin{bmatrix} Y \\ Z \end{bmatrix}$ be the $q \times q$ block representation of $W$. Applying the representation of $\mathbf{U}_{\mathrm{a},2n+1}$ stated in Remark 13.8, we observe

$$\begin{bmatrix} Y \\ Z \end{bmatrix} = W = \begin{bmatrix} (w-\alpha)^{-1}I_q & O_{q\times q} \\ O_{q\times q} & \delta^{-2n}I_q \end{bmatrix} \begin{bmatrix} I_q & O_{q\times q} \\ \mathsf{D}_{2n+1}^\dagger & I_q \end{bmatrix} \times \begin{bmatrix} I_q & -\mathsf{D}_{2n+1}\mathbf{A}_{2n+2} \\ O_{q\times q} & I_q \end{bmatrix} \begin{bmatrix} S_\mathrm{a} \\ T_\mathrm{a} \end{bmatrix} R_\mathrm{a}^{-1}. \tag{14.14}$$

Combining (14.14) and (14.7) results in

$$\operatorname{rank} \begin{bmatrix} Y \\ Z \end{bmatrix} = \operatorname{rank} \begin{bmatrix} S_\mathrm{a} \\ T_\mathrm{a} \end{bmatrix} = q. \tag{14.15}$$

In view of Part 1, Lemma 13.4 delivers $\det \mathsf{D}_{2n+1} \neq 0$ and $\mathbf{A}_{2n+2} = \mathsf{D}_{2n+1}^{-1}\mathfrak{f}_{4n+4}$. Moreover, Part 1 and Proposition 5.20 yield $\mathfrak{f}_{4n+4} \in \mathbb{C}_{\succ}^{q\times q}$. Consequently, in particular

$$\pm \mathsf{D}_{2n+1}\mathbf{A}_{2n+2} \in \mathbb{C}_\mathrm{H}^{q\times q}. \tag{14.16}$$

Moreover, since $(s_j)_{j=0}^{2n+1}$ belongs to $\mathcal{F}_{q,2n+1,\alpha,\beta}^{\succ}$, from Proposition 5.16 we know that $\mathsf{D}_{2n+1} \in \mathbb{C}_{\succ}^{q\times q} \subseteq \mathbb{C}_{\succcurlyeq}^{q\times q}$. By virtue of Remark A.9, hence, we obtain $\mathsf{D}_{2n+1}^\dagger \in \mathbb{C}_{\succcurlyeq}^{q\times q} \subseteq \mathbb{C}_\mathrm{H}^{q\times q}$. Taking into account $\mathsf{D}_{2n+1}^\dagger \in \mathbb{C}_\mathrm{H}^{q\times q}$ and Remark 10.2, it follows

$$\begin{bmatrix} I_q & O_{q\times q} \\ \pm\mathsf{D}_{2n+1}^\dagger & I_q \end{bmatrix}^* \tilde{J}_q \begin{bmatrix} I_q & O_{q\times q} \\ \pm\mathsf{D}_{2n+1}^\dagger & I_q \end{bmatrix} = \tilde{J}_q, \tag{14.17}$$

where $\tilde{J}_q$ is given by (10.1), and, because of (14.16), furthermore

$$\begin{bmatrix} I_q & \mp\mathsf{D}_{2n+1}\mathbf{A}_{2n+2} \\ O_{q\times q} & I_q \end{bmatrix}^* \tilde{J}_q \begin{bmatrix} I_q & \mp\mathsf{D}_{2n+1}\mathbf{A}_{2n+2} \\ O_{q\times q} & I_q \end{bmatrix} = \tilde{J}_q. \tag{14.18}$$

In view of (10.1), we conclude

$$\begin{bmatrix} I_q & O_{q\times q} \\ O_{q\times q} & \delta^{-2n}I_q \end{bmatrix}^* \tilde{J}_q \begin{bmatrix} I_q & O_{q\times q} \\ O_{q\times q} & \delta^{-2n}I_q \end{bmatrix} = \delta^{-2n}\tilde{J}_q. \tag{14.19}$$

Moreover, one can easily see from (14.14) that

$$\begin{bmatrix}(w-\alpha)Y\\Z\end{bmatrix}=\begin{bmatrix}I_q & O_{q\times q}\\O_{q\times q} & \delta^{-2n}I_q\end{bmatrix}\begin{bmatrix}I_q & O_{q\times q}\\\mathsf{D}_{2n+1}^{\dagger} & I_q\end{bmatrix}\times\begin{bmatrix}I_q & -\mathsf{D}_{2n+1}\mathbf{A}_{2n+2}\\O_{q\times q} & I_q\end{bmatrix}\begin{bmatrix}S_{\mathrm{a}}\\T_{\mathrm{a}}\end{bmatrix}R_{\mathrm{a}}^{-1} \tag{14.20}$$

holds true. Now, using (14.20), (14.19), (14.17), and (14.18), we conclude

$$\begin{bmatrix}(w-\alpha)Y\\Z\end{bmatrix}^*(-\tilde{J}_q)\begin{bmatrix}(w-\alpha)Y\\Z\end{bmatrix}=\delta^{-2n}R_{\mathrm{a}}^{-*}\begin{bmatrix}S_{\mathrm{a}}\\T_{\mathrm{a}}\end{bmatrix}^*(-\tilde{J}_q)\begin{bmatrix}S_{\mathrm{a}}\\T_{\mathrm{a}}\end{bmatrix}R_{\mathrm{a}}^{-1}.$$

Consequently, by virtue of Remarks 10.1 and A.2, (14.8), and $\delta>0$, we get

$$\Im((w-\alpha)Z^*Y)=\delta^{-2n}R_{\mathrm{a}}^{-*}[\Im(T_{\mathrm{a}}^*S_{\mathrm{a}})]R_{\mathrm{a}}^{-1}\succcurlyeq O. \tag{14.21}$$

In view of Part 1, Lemma 13.9 provides

$$\delta^{2n}(w-\alpha)\mathbf{V}_{\mathrm{a},2n+1}(w)=\begin{bmatrix}(w-\alpha)I_q & s_0\\O_{q\times q} & I_q\end{bmatrix}\ddot{\mathbf{V}}_{2n+1}(w)\mathbf{U}_{\mathrm{a},2n+1}(w). \tag{14.22}$$

Taking into account (14.13), (14.22), and (14.12), we infer

$$\begin{aligned}&\begin{bmatrix}(w-\alpha)I_q & s_0\\O_{q\times q} & I_q\end{bmatrix}\ddot{\mathbf{V}}_{2n+1}(w)\begin{bmatrix}Y\\Z\end{bmatrix}\\&=\begin{bmatrix}(w-\alpha)I_q & s_0\\O_{q\times q} & I_q\end{bmatrix}\ddot{\mathbf{V}}_{2n+1}(w)((w-\alpha)^{-1}\delta^{-2n}\mathbf{U}_{\mathrm{a},2n+1}(w)\begin{bmatrix}S_{\mathrm{a}}\\T_{\mathrm{a}}\end{bmatrix}R_{\mathrm{a}}^{-1})\\&=\mathbf{V}_{\mathrm{a},2n+1}(w)\begin{bmatrix}S_{\mathrm{a}}\\T_{\mathrm{a}}\end{bmatrix}R_{\mathrm{a}}^{-1}=\begin{bmatrix}(w-\alpha)X+s_0\\I_q\end{bmatrix}=\begin{bmatrix}(w-\alpha)I_q & s_0\\O_{q\times q} & I_q\end{bmatrix}\begin{bmatrix}X\\I_q\end{bmatrix},\end{aligned}$$

which implies

$$\ddot{\mathbf{V}}_{2n+1}(w)\begin{bmatrix}Y\\Z\end{bmatrix}=\begin{bmatrix}X\\I_q\end{bmatrix}. \tag{14.23}$$

**Part 4:** In view of Part 1, Lemma 7.9 provides $\mathscr{A}_{\bullet,2n,\beta}(w)=\mathscr{A}_{\bullet,2n+1,\beta}(w)$ as well as $\mathscr{B}_{\bullet,2n,\beta}(w)=\mathscr{B}_{\bullet,2n+1,\beta}(w)$, and $\mathscr{C}_{\bullet,2n,\beta}(w)=\mathscr{C}_{\bullet,2n+1,\beta}(w)$. Thus, we see from the assumption $X\in\mathscr{K}_{\mathrm{b},2n}(w)$, Notation 13.14, and the definition of a matrix ball that there is a contractive complex $q\times q$ matrix $C_{\mathrm{b}}$ such that $(\beta-w)X-s_0=\mathscr{C}_{\bullet,2n+1,\beta}(w)+(w-\overline{w})^{-1}\mathscr{A}_{\bullet,2n+1,\beta}(w)C_{\mathrm{b}}\mathscr{B}_{\bullet,2n+1,\beta}(w)$. Let $E_{\mathrm{b}}:=-[\chi_{\bullet,2n+1,\beta}(w)]^*$, $B_{\mathrm{b}}:=\Im(E_{\mathrm{b}})$, $P_{\mathrm{b}}:=\mathbb{P}_{\mathcal{R}(B_{\mathrm{b}})}$, and $Q_{\mathrm{b}}:=\mathbb{P}_{\mathcal{N}(B_{\mathrm{b}})}$. Taking into account Part 1, Remarks 7.5 and A.2

yield $B_\mathrm{b} \succcurlyeq O$. Let $S_\mathrm{b} := E_\mathrm{b}\sqrt{B_\mathrm{b}}^\dagger - E_\mathrm{b}^*\sqrt{B_\mathrm{b}}^\dagger C_\mathrm{b} P_\mathrm{b}$ and $T_\mathrm{b} := \sqrt{B_\mathrm{b}}^\dagger - \sqrt{B_\mathrm{b}}^\dagger C_\mathrm{b} P_\mathrm{b} + Q_\mathrm{b}$. Analogous to Part 2 of the proof we get then

$$\Im(T_\mathrm{b}^* S_\mathrm{b}) \succcurlyeq O_{q\times q}, \tag{14.24}$$

that the matrix $R_\mathrm{b} := \ddot{\mathbf{v}}_n(w) L_{\bullet,n,\beta}^\dagger S_\mathrm{b} + \ddot{\mathbf{v}}_{n+1}(w) T_\mathrm{b}$ is invertible, and that

$$\begin{bmatrix} (\beta - w)X - s_0 \\ I_q \end{bmatrix} = \mathbf{V}_{\mathrm{b},2n+1}(w) \begin{bmatrix} S_\mathrm{b} \\ T_\mathrm{b} \end{bmatrix} R_\mathrm{b}^{-1} \tag{14.25}$$

holds true.

**Part 5:** We know from Part 1 and Lemma 13.9 that

$$\delta^{2n}(\beta - w)\mathbf{V}_{\mathrm{b},2n+1}(w) = \begin{bmatrix} (\beta - w)I_q & -s_0 \\ O_{q\times q} & I_q \end{bmatrix} \ddot{\mathbf{V}}_{2n+1}(w)\mathbf{U}_{\mathrm{b},2n+1}(w) \tag{14.26}$$

is fulfilled and, in view of $\delta > 0$ and $\beta - w \neq 0$, from the assumption $(s_j)_{j=0}^{2n+1} \in \mathcal{F}_{q,2n+1,\alpha,\beta}^{\succ}$ and Remark 13.8 that $\det \mathbf{U}_{\mathrm{b},2n+1}(w) = \delta^{2nq}(\beta - w)^q \neq 0$ is valid. Thus, using (14.25), (14.23), and (14.26), we observe

$$\begin{aligned} \mathbf{V}_{\mathrm{b},2n+1}(w) \begin{bmatrix} S_\mathrm{b} \\ T_\mathrm{b} \end{bmatrix} R_\mathrm{b}^{-1} &= \begin{bmatrix} (\beta - w)X - s_0 \\ I_q \end{bmatrix} \\ &= \begin{bmatrix} (\beta - w)I_q & -s_0 \\ O_{q\times q} & I_q \end{bmatrix} \begin{bmatrix} X \\ I_q \end{bmatrix} = \begin{bmatrix} (\beta - w)I_q & -s_0 \\ O_{q\times q} & I_q \end{bmatrix} \ddot{\mathbf{V}}_{2n+1}(w) \begin{bmatrix} Y \\ Z \end{bmatrix} \\ &= \delta^{2n}(\beta - w)\mathbf{V}_{\mathrm{b},2n+1}(w)[\mathbf{U}_{\mathrm{b},2n+1}(w)]^{-1} \begin{bmatrix} Y \\ Z \end{bmatrix}. \end{aligned} \tag{14.27}$$

Let $P_{n,-1}^\circ := I_q$, $P_{n,l}^\circ := \mathbb{P}_{\mathcal{R}(L_{\bullet,l,\beta})}$ for all $l \in \mathbb{Z}_{0,n}$, and $P_{n,n+1}^\circ := O_{q\times q}$. According to Part 1, Proposition 5.8, and Remark 5.7, for all $l \in \mathbb{Z}_{0,n}$, we have $L_{\bullet,l,\beta} \in \mathbb{C}_{\succ}^{q\times q}$ and, consequently, $\mathbb{P}_{\mathcal{R}(L_{\bullet,l,\beta})} = I_q$. Consequently, $P_{n,l}^\circ = I_q$ for all $l \in \mathbb{Z}_{-1,n}$ and $P_{n,n+1}^\circ = O_{q\times q}$. Summarizing, we get

$$\begin{bmatrix} P_{n,n}^\circ & O \\ O & \sum_{k=0}^{n+1} z^k (P_{n,n-k}^\circ - P_{n,n-k+1}^\circ) \end{bmatrix} = I_{2q} \tag{14.28}$$

for all $z \in \mathbb{C}$. From Notation 13.3 and Remark 5.22 we conclude

$$\mathbf{V}_{\mathrm{b},2n+1}(z) = \begin{bmatrix} -\ddot{\mathbf{w}}_n(z) L_{\bullet,n,\beta}^\dagger & -\ddot{\mathbf{w}}_{n+1}(z) \\ \ddot{\mathbf{v}}_n(z) L_{\bullet,n,\beta}^\dagger & \ddot{\mathbf{v}}_{n+1}(z) \end{bmatrix} \tag{14.29}$$

for all $z \in \mathbb{C}$. Taking into account Part 1, Remark 5.7, Lemma 13.12, (14.29), and [14, Prop. 6.24], we see from [13, Lem. 7.6] and (14.28) that there is a complex

$2q \times 2q$ matrix $\mathbf{W}$ such that $\mathbf{W}\mathbf{V}_{\mathrm{b},2n+1}(w) = I_{2q}$. Taking into account (14.27), then

$$\begin{bmatrix} Y \\ Z \end{bmatrix} = (\beta - w)^{-1}\delta^{-2n}\mathbf{U}_{\mathrm{b},2n+1}(w)\begin{bmatrix} S_{\mathrm{b}} \\ T_{\mathrm{b}} \end{bmatrix} R_{\mathrm{b}}^{-1} \tag{14.30}$$

follows. Using (14.30), Part 1, and the representation of $\mathbf{U}_{\mathrm{b},2n+1}$ stated in Remark 13.8, we infer

$$\begin{bmatrix} (\beta - w)Y \\ Z \end{bmatrix} = \begin{bmatrix} I_q & O_{q\times q} \\ O_{q\times q} & \delta^{-2n}I_q \end{bmatrix}\begin{bmatrix} I_q & O_{q\times q} \\ -\mathsf{D}_{2n+1}^{\dagger} & I_q \end{bmatrix} \\ \times \begin{bmatrix} I_q & \mathsf{D}_{2n+1}\mathbf{A}_{2n+2} \\ O_{q\times q} & I_q \end{bmatrix}\begin{bmatrix} S_{\mathrm{b}} \\ T_{\mathrm{b}} \end{bmatrix} R_{\mathrm{b}}^{-1}. \tag{14.31}$$

Applying (14.31), (14.19), (14.17), and (14.18), we observe that

$$\begin{bmatrix} (\beta - w)Y \\ Z \end{bmatrix}^* (-\tilde{J}_q)\begin{bmatrix} (\beta - w)Y \\ Z \end{bmatrix} \\ = \delta^{-2n}R_{\mathrm{b}}^{-*}\begin{bmatrix} S_{\mathrm{b}} \\ T_{\mathrm{b}} \end{bmatrix}^* (-\tilde{J}_q)\begin{bmatrix} S_{\mathrm{b}} \\ T_{\mathrm{b}} \end{bmatrix} R_{\mathrm{b}}^{-1}.$$

Thus, from Remark 10.1, (14.24), Remark A.2, and $\delta > 0$ we get

$$\Im((\beta - w)Z^*Y) = \delta^{-2n}R_{\mathrm{b}}^{-*}[\Im(T_{\mathrm{b}}^* S_{\mathrm{b}})]R_{\mathrm{b}}^{-1} \in \mathbb{C}_{\succcurlyeq}^{q\times q}. \tag{14.32}$$

Consequently, using additionally (14.21), it follows

$$\Im(Z^*Y) = \frac{1}{\delta}[\Im((\beta - w)Z^*Y) + \Im((w - \alpha)Z^*Y)] \in \mathbb{C}_{\succcurlyeq}^{q\times q}. \tag{14.33}$$

**Part 6:** Let $H := (Y^{\dagger})^*[\Im(Z^*Y)]Y^{\dagger}$. Furthermore, let $\eta, \theta\colon \mathbb{C} \to \mathbb{C}^{q\times q}$ be given by $\eta(z) := Y$ and $\theta(z) := \frac{w-z}{\Im(w)}HY + Z$ for all $z \in \mathbb{C}$. In view of (14.33), then [36, Lem. 8.5] provides that $\eta$ and $\theta$ are holomorphic in $\mathbb{C}$, that

$$\eta(w) = Y \qquad \text{and} \qquad \theta(w) = Z \tag{14.34}$$

hold true, and that

$$\operatorname{rank}\begin{bmatrix} \eta(z) \\ \theta(z) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} Y \\ Z \end{bmatrix}, \tag{14.35}$$

$$\Im((z - \alpha)[\theta(z)]^*\eta(z)) = \frac{\Im(z)}{\Im(w)}\Im((w - \alpha)Z^*Y), \tag{14.36}$$

and

$$\Im((z - \beta)[\theta(z)]^*\eta(z)) = \frac{\Im(z)}{\Im(w)}\Im((w - \beta)Z^*Y) \tag{14.37}$$

are valid for all $z \in \mathbb{C}$. Then the functions $\phi, \psi\colon \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}^{q\times q}$ given by $\phi(z) := \eta(z)$ and $\psi(z) := \theta(z)$ are holomorphic in $\mathbb{C} \setminus [\alpha, \beta]$ and fulfill $\operatorname{rank}\left[\begin{smallmatrix}\phi(z)\\ \psi(z)\end{smallmatrix}\right] = \operatorname{rank}\left[\begin{smallmatrix}\eta(z)\\ \theta(z)\end{smallmatrix}\right] = \operatorname{rank}\left[\begin{smallmatrix}Y\\ Z\end{smallmatrix}\right] = q$ for all $z \in \mathbb{C} \setminus [\alpha, \beta]$, by virtue of (14.35) and (14.15), whereas (14.36) and (14.21) show that

$$\frac{\Im((z-\alpha)[\psi(z)]^*\phi(z))}{\Im(z)} = \frac{\Im((z-\alpha)[\theta(z)]^*\eta(z))}{\Im(z)} = \frac{\Im((w-\alpha)Z^*Y)}{\Im(w)} \in \mathbb{C}^{q\times q}_{\succcurlyeq},$$

and further (14.37) and (14.32) deliver

$$\frac{\Im((\beta-z)[\psi(z)]^*\phi(z))}{\Im(z)} = \frac{\Im((\beta-z)[\theta(z)]^*\eta(z))}{\Im(z)} = \frac{\Im((\beta-w)Z^*Y)}{\Im(w)} \in \mathbb{C}^{q\times q}_{\succcurlyeq}$$

for all $z \in \mathbb{C}\setminus\mathbb{R}$. Consequently, Notation 10.5 yields $[\phi; \psi] \in \mathcal{PR}_q(\mathbb{C}\setminus[\alpha, \beta])$. Since $\det \mathsf{D}_{2n+1} \neq 0$ implies $\mathbb{P}_{\mathcal{R}(\mathsf{D}_{2n+1})} = I_q$ and, therefore, $\mathbb{P}_{\mathcal{R}(\mathsf{D}_{2n+1})}\phi = \phi$, Lemma 10.8 shows then that $[\phi; \psi] \in \ddot{\mathcal{P}}[\mathsf{D}_{2n+1}]$. Since $(s_j)_{j=0}^{2n+1}$ belongs to $\mathcal{F}^{\succ}_{q,2n+1,\alpha,\beta}$, then Remark 11.19 provides that $(s_j)_{j=0}^{2n+1} \in \mathcal{F}^{\succcurlyeq,\mathrm{ld}}_{q,2n+1,\alpha,\beta}$ and, by virtue of Lemma 11.21, then $(s_j)_{j=0}^{2n+1} \in \mathcal{F}^{\succcurlyeq,\mathrm{c}}_{q,2n+1,\alpha,\beta}$.

**Part 7:** Now we can finish the proof with a final step. Let $\tilde{\mathbf{p}}_{4n+3}$, $\tilde{\mathbf{q}}_{4n+3}$, $\tilde{\mathbf{p}}_{4n+4}$, and $\tilde{\mathbf{q}}_{4n+4}$ be the restrictions of $\ddot{\mathbf{p}}_{4n+3}$, $\ddot{\mathbf{q}}_{4n+3}$, $\ddot{\mathbf{p}}_{4n+4}$, and $\ddot{\mathbf{q}}_{4n+4}$ onto $\mathbb{C}\setminus[\alpha, \beta]$, respectively. Using (14.23), (14.34), and Notation 13.3, we obtain

$$\begin{aligned}\begin{bmatrix}X\\ I_q\end{bmatrix} &= \ddot{\mathbf{V}}_{2n+1}(w)\begin{bmatrix}Y\\ Z\end{bmatrix} = \ddot{\mathbf{V}}_{2n+1}(w)\begin{bmatrix}\eta(w)\\ \theta(w)\end{bmatrix} = \ddot{\mathbf{V}}_{2n+1}(w)\begin{bmatrix}\phi(w)\\ \psi(w)\end{bmatrix}\\ &= \begin{bmatrix}\tilde{\mathbf{q}}_{4n+3}(w)\mathsf{D}^{\dagger}_{2n+1}\phi(w) - \delta^{2n}\tilde{\mathbf{q}}_{4n+4}(w)\psi(w)\\ -\tilde{\mathbf{p}}_{4n+3}(w)\mathsf{D}^{\dagger}_{2n+1}\phi(w) + \delta^{2n}\tilde{\mathbf{p}}_{4n+4}(w)\psi(w)\end{bmatrix}.\end{aligned} \tag{14.38}$$

Because of $(s_j)_{j=0}^{2n+1} \in \mathcal{F}^{\succcurlyeq,\mathrm{c}}_{q,2n+1,\alpha,\beta}$ and $[\phi; \psi] \in \ddot{\mathcal{P}}[\mathsf{D}_{2n+1}]$, we know from Theorem 11.29 that

$$F := -(\tilde{\mathbf{q}}_{4n+3}\mathsf{D}^{\dagger}_{2n+1}\phi - \delta^{2n}\tilde{\mathbf{q}}_{4n+4}\psi)(\tilde{\mathbf{p}}_{4n+3}\mathsf{D}^{\dagger}_{2n+1}\phi - \delta^{2n}\tilde{\mathbf{p}}_{4n+4}\psi)^{-1} \tag{14.39}$$

belongs to $\mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^{2n+1}, =]$. Finally, the combination (14.39) and (14.38) shows $F(w) = XI_q^{-1} = X$. □

Finally, now we are able to prove the main result of this section. It contains a description of the set of all values of the functions attained from the reformulated version $\mathsf{FP}[[\alpha, \beta]; (s_j)_{j=0}^{\kappa}, =]$ of the matricial Hausdorff moment problem $\mathsf{MP}[[\alpha, \beta]; (s_j)_{j=0}^{\kappa}, =]$ in the case that the number of given matricial moments is even. As already mentioned, the set admits a representation as intersection of matrix balls.

**Theorem 14.2** *Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha < \beta$, let $n \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{2n+1} \in \mathcal{F}_{q,2n+1,\alpha,\beta}^{\succ}$. For all $w \in \Pi_+$, then*

$$\left\{F(w)\colon F \in \mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^{2n+1}, =]\right\} = \mathscr{K}_{\mathrm{a},2n}(w) \cap \mathscr{K}_{\mathrm{b},2n}(w),$$

*where $\mathscr{K}_{\mathrm{a},2n}(w)$ and $\mathscr{K}_{\mathrm{b},2n}(w)$ are given by Notation 13.14, respectively.*

***Proof*** Apply Theorem 13.16(a) and Proposition 14.1. □

## 15 Weyl Sets in the Case of an Odd Number of Prescribed Matricial Moments

In the last sections, we worked out an explicit representation of the set $\left\{F(w)\colon F \in \mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^{\kappa}, =]\right\}$ in the case that $\kappa$ is an odd positive integer (see Theorem 14.2). Now we turn our attention to the case that $\kappa$ is an even positive integer. We again use Notation 13.14, in order to formulate a result which is similar to Proposition 14.1. The proof is similar as well, however, a couple of details differ. That's why, we also state the proof.

**Proposition 15.1** *Let $n \in \mathbb{N}$, let $(s_j)_{j=0}^{2n} \in \mathcal{F}_{q,2n,\alpha,\beta}^{\succ}$, and let $w \in \Pi_+$. Furthermore, let $X \in \mathscr{K}_{2n}(w) \cap \mathscr{K}_{\mathrm{c},2n-2}(w)$. Then there is a rational matrix-valued function $F$ belonging to $\mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^{2n}, =]$ such that $F(w) = X$.*

***Proof*** Our proof is divided into seven parts. In the first three parts of the proof, we consider the case that $n$ is an arbitrary non-negative integer.

**Part 1:** According to [19, Prop. 11.12], there is a sequence $(s_j)_{j=2n+1}^{\infty}$ of complex $q \times q$ matrices such that $(s_j)_{j=0}^{\infty}$ belongs to $\mathcal{F}_{q,\infty,\alpha,\beta}^{\succ}$, i. e., such that $(s_j)_{j=0}^{m} \in \mathcal{F}_{q,m,\alpha,\beta}^{\succ}$ for all $m \in \mathbb{N}_0$. In view of Remark 5.9, in particular, $(s_j)_{j=0}^{m} \in \mathcal{F}_{q,m,\alpha,\beta}^{\succcurlyeq}$ for all $m \in \mathbb{N}_0$. From Definition 3.5 as well as (5.3) and Remark 5.1, for all $m \in \mathbb{N}_0$, we get then $\{(s_j)_{j=0}^{2m}, (c_j)_{j=0}^{2m}\} \subseteq \mathcal{H}_{q,2m}^{\succ} \subseteq \mathcal{H}_{q,2m}^{\succcurlyeq}$ and, consequently, $\{(s_j)_{j=0}^{m}, (c_j)_{j=0}^{m}\} \subseteq \mathcal{H}_{q,m}^{\succcurlyeq,\mathrm{e}}$.

**Part 2:** Using the assumption $X \in \mathscr{K}_{2n}(w)$, Notation 13.14, and the definition of a matrix ball, we see that there is a contractive complex $q \times q$ matrix $C$ such that $X = \mathscr{C}_{2n}(w) + (w - \overline{w})^{-1}\mathscr{A}_{2n}(w)C\mathscr{B}_{2n}(w)$. By virtue of Part 1 and Lemma 7.9 then

$$X = \mathscr{C}_{2n+1}(w) + (w - \overline{w})^{-1}\mathscr{A}_{2n+1}(w)C\mathscr{B}_{2n+1}(w) \tag{15.1}$$

follows. In view of Definition 7.1 and Remark A.4, we set

$$E := -[\chi_{2n+1}(w)]^*, \quad B := \Im E, \quad P := \mathbb{P}_{\mathcal{R}(B)}, \quad \text{and} \quad Q := \mathbb{P}_{\mathcal{N}(B)}. \tag{15.2}$$

Taking into account Part 1, Remark 7.5 yields $(\Im w)^{-1}\Im\chi_{2n+1}(w) \succcurlyeq O$. Obviously, Remark A.2 provides

$$B = \Im E = \Im\chi_{2n+1}(w), \tag{15.3}$$

which, because of $w \in \Pi_+$, implies $B \succcurlyeq O$. Let

$$S := E\sqrt{B}^{\dagger} - E^*\sqrt{B}^{\dagger}CP \qquad \text{and} \qquad T := \sqrt{B}^{\dagger} - \sqrt{B}^{\dagger}CP + Q. \tag{15.4}$$

In view of Part 1, [14, Prop. 7.19], and Remark 5.7 as well as (15.2) and (15.3), we conclude

$$\mathcal{R}(E) = \mathcal{R}(E^*) = \mathcal{R}(B) = \mathcal{R}(L_n), \quad \mathcal{N}(E) = \mathcal{N}(E^*) = \mathcal{N}(B) = \mathcal{N}(L_n). \tag{15.5}$$

Consequently, taking into account (15.2), $B \succcurlyeq O$, (15.4), (15.5), and $C \in \mathbb{K}_{q\times q}$, applying [14, Lem. A.26] provides

$$\operatorname{rank}\begin{bmatrix} S \\ T \end{bmatrix} = q, \qquad PS = S = SP, \qquad TP = T - Q, \tag{15.6}$$

$$\sqrt{B}^{\dagger}(S - ET)(S - E^*T)\sqrt{B} = PCP, \qquad \text{and} \qquad \Im(T^*S) \succcurlyeq O. \tag{15.7}$$

Because of (15.2) and (15.5), we have

$$P = \mathbb{P}_{\mathcal{R}(B)} = \mathbb{P}_{\mathcal{R}(L_n)} \qquad \text{and} \qquad Q = \mathbb{P}_{\mathcal{N}(B)} = \mathbb{P}_{\mathcal{N}(L_n)}. \tag{15.8}$$

Thus, using (15.6), (15.8), Remark A.5, and (15.5), we get $\mathcal{R}(S) = \mathcal{R}(PS) \subseteq \mathcal{R}(P) = \mathcal{R}(\mathbb{P}_{\mathcal{R}(B)}) = \mathcal{R}(B) = \mathcal{R}(L_n)$. Taking additionally into account Part 1, Lemma 13.11, (15.6), (15.7), and Lemma 13.13, we see that $R := \ddot{\mathbf{r}}_n(w)L_n^{\dagger}S + \ddot{\mathbf{r}}_{n+1}(w)T$ fulfills $\det R \neq 0$. Because of Part 1, Remark 5.7, (15.8), $w \in \Pi_+$, (15.7), (15.6) as well as $\det R \neq 0$ and Lemma 13.11, now we can apply [14, Prop. 8.18] in order to obtain

$$\begin{aligned} &-[\ddot{\mathbf{o}}_n(w)L_n^{\dagger}S + \ddot{\mathbf{o}}_{n+1}(w)T]R^{-1} \\ &\quad = \mathscr{C}_{2n+1}(w) + (w - \overline{w})^{-1}\mathscr{A}_{2n+1}(w)K\mathscr{B}_{2n+1}(w) \end{aligned} \tag{15.9}$$

where the matrix $\Im\chi_{2n+1}(w)$ is non-negative Hermitian and where the matrix

$$K := \sqrt{\Im\chi_{2n+1}(w)}^{\dagger}(S + [\chi_{2n+1}(w)]^*T)(S + \chi_{2n+1}(w)T)^{\dagger}\sqrt{\Im\chi_{2n+1}(w)}$$

is contractive. Using (15.3) and (15.2), it follows

$$K = \sqrt{B}^{\dagger}(S - ET)(S - E^*T)^{\dagger}\sqrt{B}. \tag{15.10}$$

From Part 1, Proposition 5.8, and Remark 5.7 we know that $\det L_n \neq 0$ holds true, which, because of (15.8) implies $P = I_q$. Thus, (15.7) and (15.10) give $K = PCP = C$. Combining (15.1) and (15.9), we get then $-[\ddot{\mathbf{o}}_n(w)L_n^{\dagger}S + \ddot{\mathbf{o}}_{n+1}(w)T]R^{-1} = X$.

Consequently, from Remark 5.22 and Notation 13.3(b) we see that

$$\begin{bmatrix} X \\ I_q \end{bmatrix} = \begin{bmatrix} -[\ddot{\mathbf{o}}_n(w)L_n^{\dagger}S + \ddot{\mathbf{o}}_{n+1}(w)T] \\ \ddot{\mathbf{r}}_n(w)L_n^{\dagger}S + \ddot{\mathbf{r}}_{n+1}(w)T \end{bmatrix} R^{-1}$$

$$= \begin{bmatrix} -\ddot{\mathbf{o}}_n(w)\mathfrak{f}_{4n}^{\dagger} & -\ddot{\mathbf{o}}_{n+1}(w) \\ \ddot{\mathbf{r}}_n(w)\mathfrak{f}_{4n}^{\dagger} & \ddot{\mathbf{r}}_{n+1}(w) \end{bmatrix} \begin{bmatrix} S \\ T \end{bmatrix} R^{-1} = \mathbf{V}_{2n+1}(w) \begin{bmatrix} S \\ T \end{bmatrix} R^{-1} \quad (15.11)$$

is fulfilled, where $(\mathfrak{f}_j)_{j=0}^{\infty}$ is the $\mathcal{F}_{\alpha,\beta}$-parameter sequence of $(s_j)_{j=0}^{\infty}$.
**Part 3:** Since $\delta > 0$ and $\beta - w \neq 0$ as well as $\det R \neq 0$ are valid, taking into account Notation 13.3(b), we can define

$$W := \delta^{-2n+1}(\beta - w)^{-1}\mathbf{U}_{2n}(w) \begin{bmatrix} S \\ T \end{bmatrix} R^{-1} \quad (15.12)$$

and choose the $q \times q$ block representation $W = \left[\begin{smallmatrix} Y \\ Z \end{smallmatrix}\right]$ of $W$. In view of Remark 13.7, we conclude then

$$\begin{bmatrix} Y \\ Z \end{bmatrix} = W = \begin{bmatrix} (\beta - w)^{-1}I_q & O \\ O & \delta^{-2n+1}I_q \end{bmatrix} M \quad (15.13)$$

where

$$M := \begin{bmatrix} I_q & O \\ -\mathsf{D}_{2n}^{\dagger} & I_q \end{bmatrix} \begin{bmatrix} I_q & \mathsf{D}_{2n}\mathbf{A}_{2n+1} \\ O & I_q \end{bmatrix} \begin{bmatrix} S \\ T \end{bmatrix} R^{-1}. \quad (15.14)$$

Combining (15.14), (15.13), and (15.6), we see that $\operatorname{rank}\left[\begin{smallmatrix} Y \\ Z \end{smallmatrix}\right] = \operatorname{rank}\left[\begin{smallmatrix} S \\ T \end{smallmatrix}\right] = q$ holds true. Because of Part 1, Lemma 13.4 provides $\det \mathsf{D}_{2n} \neq 0$ and $\mathbf{A}_{2n+1} = \mathsf{D}_{2n}^{-1}\mathfrak{f}_{4n+2}$, whereas Part 1 and Proposition 5.20 show that $\mathfrak{f}_{4n+2}$ belongs to $\mathbb{C}_{\succ}^{q\times q}$. In view of (15.7), the same arguments as in Part 3 of the proof of Proposition 14.1 yield $\{\mathsf{D}_{2n}, \mathsf{D}_{2n}^{\dagger}, \mathsf{D}_{2n}\mathbf{A}_{2n+1}\} \subseteq \mathbb{C}_{\mathrm{H}}^{q\times q}$,

$$\begin{bmatrix} I_q & O \\ \pm\mathsf{D}_{2n}^{\dagger} & I_q \end{bmatrix}^{*} \tilde{J}_q \begin{bmatrix} I_q & O \\ \pm\mathsf{D}_{2n}^{\dagger} & I_q \end{bmatrix} = \tilde{J}_q, \quad (15.15)$$

$$\begin{bmatrix} I_q & \mp\mathsf{D}_{2n}\mathbf{A}_{2n+1} \\ O & I_q \end{bmatrix}^{*} \tilde{J}_q \begin{bmatrix} I_q & \mp\mathsf{D}_{2n}\mathbf{A}_{2n+1} \\ O & I_q \end{bmatrix} = \tilde{J}_q, \quad (15.16)$$

$$\begin{bmatrix} I_q & O \\ O & \delta^{-2n+1}I_q \end{bmatrix}^{*} \tilde{J}_q \begin{bmatrix} I_q & O \\ O & \delta^{-2n+1}I_q \end{bmatrix} = \delta^{-2n+1}\tilde{J}_q, \quad (15.17)$$

and, consequently,

$$\Im((\beta - w)Z^*Y) = \delta^{-2n+1}R^{-*}[\Im(T^*S)]R^{-1} \succcurlyeq O. \quad (15.18)$$

Because of Part 1, using Lemma 13.6 in the case $n = 0$ and Lemma 13.5 in the case $n \geq 1$, we have $\delta^{2n-1}(\beta - w)\mathbf{V}_{2n+1}(w) = \ddot{\mathbf{V}}_{2n}(w)\mathbf{U}_{2n}(w)$. Combining the foregoing equation with (15.11) and (15.12) yields

$$\begin{aligned}\begin{bmatrix} X \\ I_q \end{bmatrix} &= \delta^{-2n+1}(\beta - w)^{-1}\ddot{\mathbf{V}}_{2n}(w)\mathbf{U}_{2n}(w)\begin{bmatrix} S \\ T \end{bmatrix} R^{-1} \\ &= \ddot{\mathbf{V}}_{2n}(w)W = \ddot{\mathbf{V}}_{2n}(w)\begin{bmatrix} Y \\ Z \end{bmatrix}. \end{aligned} \tag{15.19}$$

In the following Parts 4–7, we suppose that $n \geq 1$.

**Part 4:** In view of Part 1, from Lemma 7.9 we get $\mathscr{A}_{\alpha,2n-2,\beta}(w) = \mathscr{A}_{\alpha,2n-1,\beta}(w)$, $\mathscr{B}_{\alpha,2n-2,\beta}(w) = \mathscr{B}_{\alpha,2n-1,\beta}(w)$, and $\mathscr{C}_{\alpha,2n-2,\beta}(w) = \mathscr{C}_{\alpha,2n-1,\beta}(w)$. Hence, from the assumption $X \in \mathscr{K}_{c,2n-2}(w)$, Notation 13.14, and the definition of a matrix ball we see that there exists a contractive complex $q \times q$ matrix $C_c$ such that

$$\begin{aligned}&(\beta - w)(w - \alpha)X + (\alpha + \beta - w)s_0 - s_1 \\ &\quad = \mathscr{C}_{\alpha,2n-1,\beta}(w) + (w - \overline{w})^{-1}\mathscr{A}_{\alpha,2n-1,\beta}(w)C_c\mathscr{B}_{\alpha,2n-1,\beta}(w).\end{aligned}$$

Let $E_c := -[\chi_{\alpha,2n-1,\beta}(w)]^*$, $B_c := \Im E_c$, $P_c := \mathbb{P}_{\mathcal{R}(B_c)}$, and $Q_c := \mathbb{P}_{\mathcal{N}(B_c)}$. Taking into account Part 1, $w \in \Pi_+$, and Remarks 7.5 and A.2, we infer $B_c \succcurlyeq O$. Let $S_c := E_c\sqrt{B_c}^{\dagger} - E_c^*\sqrt{B_c}^{\dagger}C_cP_c$ and $T_c := \sqrt{B_c}^{\dagger} - \sqrt{B_c}^{\dagger}C_cP_c + Q_c$. Analogous to Part 2 of the proof we get then $\Im(T_c^*S_c) \succcurlyeq O$, that the matrix $R_c := \ddot{\mathbf{x}}_{n-1}(w)L_{\alpha,n-1,\beta}^{\dagger}S_c + \ddot{\mathbf{x}}_n(w)T_c$ is invertible, and that

$$\begin{bmatrix} (\beta - w)(w - \alpha)X + (\alpha + \beta - w)s_0 - s_1 \\ I_q \end{bmatrix} = \mathbf{V}_{c,2n-1}(w)\begin{bmatrix} S_c \\ T_c \end{bmatrix} R_c^{-1}. \tag{15.20}$$

**Part 5:** In view of Part 1, from Lemma 13.5 we get that (13.1) holds true for $z = w$, whereas Remark 13.7 shows that $\det \mathbf{U}_{c,2n}(w) = \delta^{(2n-1)q}(w - \alpha)^q \neq 0$. Using additionally $\det R_c \neq 0$, (15.20), and (15.19), we conclude then

$$\begin{aligned}\mathbf{V}_{c,2n-1}(w)\begin{bmatrix} S_c \\ T_c \end{bmatrix} R_c^{-1} &= \left[\begin{array}{c:c} (\beta - w)(w - \alpha)I_q & (\alpha + \beta - w)s_0 - s_1 \\ \hdashline O & I_q \end{array}\right]\begin{bmatrix} X \\ I_q \end{bmatrix} \\ &= \left[\begin{array}{c:c} (\beta - w)(w - \alpha)I_q & (\alpha + \beta - w)s_0 - s_1 \\ \hdashline O & I_q \end{array}\right]\ddot{\mathbf{V}}_{2n}(w)\begin{bmatrix} Y \\ Z \end{bmatrix} \\ &= -\delta^{2n-1}(\beta - w)(w - \alpha)\mathbf{V}_{c,2n-1}(w)[\mathbf{U}_{c,2n}(w)]^{-1}\begin{bmatrix} Y \\ Z \end{bmatrix}.\end{aligned} \tag{15.21}$$

Let $P_{n-1,-1}^{\diamond} := I_q$, $P_{n-1,\ell}^{\diamond} := \mathbb{P}_{\mathcal{R}(L_{\alpha,\ell,\beta})}$ for all $\ell \in \mathbb{Z}_{0,n-1}$, and $P_{n-1,n}^{\diamond} := O_{q\times q}$. From Part 1, Proposition 5.20, and Remark 5.22 we know that $L_{\alpha,\ell,\beta} \in \mathbb{C}_{\succ}^{q\times q}$ and, consequently, $\mathbb{P}_{\mathcal{R}(L_{\alpha,\ell,\beta})} = I_q$ hold true for all $\ell \in \mathbb{Z}_{0,n-1}$. Thus,

$$\begin{bmatrix} P_{n-1,n-1}^{\diamond} & O \\ O & \sum_{k=0}^{n} z^k(P_{n-1,n-1-k}^{\diamond} - P_{n-1,n-k}^{\diamond}) \end{bmatrix} = I_{2q} \tag{15.22}$$

for all $z \in \mathbb{C}$. From Notations 13.3 and 3.3 and Remarks 5.7 and 5.22 we get

$$\mathbf{V}_{c,2n-1}(z) = \begin{bmatrix} -\ddot{\mathbf{y}}_{n-1}(z)L^{\dagger}_{\alpha,n-1,\beta} & -\ddot{\mathbf{y}}_{n}(z) \\ \ddot{\mathbf{x}}_{n-1}(z)L^{\dagger}_{\alpha,n-1,\beta} & \ddot{\mathbf{x}}_{n}(z) \end{bmatrix}$$

for all $z \in \mathbb{C}$. Hence, in view of Part 1, Lemma 13.11, and [14, Prop. 6.24], using [13, Lem. 7.6] and (15.22), we see that there exists a complex $2q \times 2q$ matrix $\mathbf{W}_c$ such that $\mathbf{W}_c\mathbf{V}_{c,2n-1}(w) = I_{2q}$. Thus, from (15.21) we see that

$$\begin{bmatrix} Y \\ Z \end{bmatrix} = -\delta^{-(2n-1)}(\beta - w)^{-1}(w-\alpha)^{-1}\mathbf{U}_{c,2n}(w)\begin{bmatrix} S_c \\ T_c \end{bmatrix} R_c^{-1} \tag{15.23}$$

is valid. Using (15.23) and Remark 13.7, we obtain

$$\begin{aligned}
&\begin{bmatrix} (w-\alpha)Y \\ Z \end{bmatrix} \\
&= -\delta^{-(2n-1)}(\beta - w)^{-1}(w-\alpha)^{-1}\begin{bmatrix} (w-\alpha)I_q & O \\ O & I_q \end{bmatrix} \\
&\quad\times \begin{bmatrix} \delta^{2n-1}I_q & O \\ O & (w-\alpha)I_q \end{bmatrix}\begin{bmatrix} I_q & O \\ \mathsf{D}^{\dagger}_{2n} & I_q \end{bmatrix}\begin{bmatrix} I_q & -\mathsf{D}_{2n}\mathbf{A}_{2n+1} \\ O & I_q \end{bmatrix}\begin{bmatrix} S_c \\ T_c \end{bmatrix} R_c^{-1} \\
&= -(\beta - w)^{-1}\begin{bmatrix} I_q & O \\ O & \delta^{-2n+1}I_q \end{bmatrix}\begin{bmatrix} I_q & O \\ \mathsf{D}^{\dagger}_{2n} & I_q \end{bmatrix}\begin{bmatrix} I_q & -\mathsf{D}_{2n}\mathbf{A}_{2n+1} \\ O & I_q \end{bmatrix}\begin{bmatrix} S_c \\ T_c \end{bmatrix} R_c^{-1}. \qquad (15.24)
\end{aligned}$$

Taking into account (15.24), (15.17), (15.15), and (15.16), we can conclude

$$\begin{aligned}
&\begin{bmatrix} (w-\alpha)Y \\ Z \end{bmatrix}^{*}(-\tilde{J}_q)\begin{bmatrix} (w-\alpha)Y \\ Z \end{bmatrix} \\
&\quad= |\beta - w|^{-2}\delta^{-2n+1}R_c^{-*}\begin{bmatrix} S_c \\ T_c \end{bmatrix}^{*}(-\tilde{J}_q)\begin{bmatrix} S_c \\ T_c \end{bmatrix} R_c^{-1}.
\end{aligned}$$

Because of Remark 10.1, $\Im(T_c^* S_c) \succcurlyeq O$, Remark A.2, and $\delta > 0$, then

$$\Im((w-\alpha)Z^*Y) = |\beta - w|^{-2}\delta^{-2n+1}R_c^{-*}[\Im(T_c^* S_c)]R_c^{-1} \succcurlyeq O \tag{15.25}$$

follows. Since

$$\delta^{-1}[\Im((\beta - w)Z^*Y) + \Im((w-\alpha)Z^*Y)] = \Im(Z^*Y) \tag{15.26}$$

holds true, from (15.18) and (15.25) then we infer $\Im(Z^*Y) \succcurlyeq O$.

**Part 6:** Because of Part 1, Remark 11.19, and Lemma 11.21, we have $(s_j)_{j=0}^{2n} \in \mathcal{F}^{\succcurlyeq,\mathrm{ld}}_{q,2n,\alpha,\beta} \subseteq \mathcal{F}^{\succcurlyeq,\mathrm{c}}_{q,2n,\alpha,\beta}$. Let $\phi, \psi\colon \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}^{q\times q}$ be given by $\phi(z) := Y$ and $\psi(z) := \frac{w-z}{\Im(w)}HY + Z$, where $H := (Y^{\dagger})^*[\Im(Z^*Y)]Y^{\dagger}$. Using $\Im(Z^*Y) \succcurlyeq O$ and [36, Lem. 8.5], one can check analogous to Part 6 of the proof of Proposition 14.1 that

$\phi(w) = Y$ and $\psi(w) = Z$ hold true and that the pair $[\phi; \psi]$ belongs to the class $\ddot{\mathcal{P}}[\mathsf{D}_{2n}]$.

**Part 7:** Let $\varepsilon_{2n} : \mathbb{C} \to \mathbb{C}$ be given by $\varepsilon_{2n}(z) := z - \beta$. Let $\tilde{\mathbf{p}}_{4n+1}$, $\tilde{\mathbf{q}}_{4n+1}$, $\tilde{\mathbf{p}}_{4n+2}$, and $\tilde{\mathbf{q}}_{4n+2}$ be the restrictions of the functions $\mathbf{p}^{\diamond}_{4n+1} := \varepsilon_{2n}\ddot{\mathbf{p}}_{4n+1}$, $\mathbf{q}^{\diamond}_{4n+1} := \varepsilon_{2n}\ddot{\mathbf{q}}_{4n+1}$, $\ddot{\mathbf{p}}_{4n+2}$, and $\ddot{\mathbf{q}}_{4n+2}$ onto $\mathbb{C} \setminus [\alpha, \beta]$, respectively. By virtue of (15.19), $\phi(w) = Y$, $\psi(w) = Z$, and Notation 13.3, we can conclude

$$\begin{bmatrix} X \\ I_q \end{bmatrix} = \ddot{\mathbf{V}}_{2n}(w) \begin{bmatrix} Y \\ Z \end{bmatrix} = \ddot{\mathbf{V}}_{2n}(w) \begin{bmatrix} \phi(w) \\ \psi(w) \end{bmatrix}$$
$$= \begin{bmatrix} \tilde{\mathbf{q}}_{4n+1}(w)\mathsf{D}^{\dagger}_{2n}\phi(w) - \delta^{2n-1}\tilde{\mathbf{q}}_{4n+2}(w)\psi(w) \\ -\tilde{\mathbf{p}}_{4n+1}(w)\mathsf{D}^{\dagger}_{2n}\phi(w) + \delta^{2n-1}\tilde{\mathbf{p}}_{4n+2}(w)\psi(w) \end{bmatrix}. \tag{15.27}$$

Because of $(s_j)_{j=0}^{2n} \in \mathcal{F}^{\succcurlyeq,\mathrm{c}}_{q,2n,\alpha,\beta}$ and $[\phi; \psi] \in \ddot{\mathcal{P}}[\mathsf{D}_{2n}]$, Theorem 11.29 shows that

$$F := -\left(\tilde{\mathbf{q}}_{4n+1}\mathsf{D}^{\dagger}_{2n}\phi - \delta^{2n-1}\tilde{\mathbf{q}}_{4n+2}\psi\right)\left(\tilde{\mathbf{p}}_{4n+1}\mathsf{D}^{\dagger}_{2n}\phi - \delta^{2n-1}\tilde{\mathbf{p}}_{4n+2}\psi\right)^{-1} \tag{15.28}$$

belongs to $\mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^{2n}, =]$. Combining (15.27) and (15.28) yields finally $X = XI_q^{-1} = F(w)$. □

Now we obtain a description of the set of all values of the functions attained from the reformulated version $\mathsf{FP}[[\alpha, \beta]; (s_j)_{j=0}^{\kappa}, =]$ of the matricial Hausdorff moment problem in the case that $\kappa$ is an even positive integer.

**Theorem 15.2** *Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha < \beta$. Let $n \in \mathbb{N}$ and let $(s_j)_{j=0}^{2n} \in \mathcal{F}^{\succ}_{q,2n,\alpha,\beta}$. For each $w \in \Pi_+$, then*

$$\left\{F(w) : F \in \mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^{2n}, =]\right\} = \mathscr{K}_{2n}(w) \cap \mathscr{K}_{\mathrm{c},2n-2}(w),$$

*where $\mathscr{K}_{2n}(w)$ and $\mathscr{K}_{\mathrm{c},2n-2}(w)$ are given by Notation 13.14.*

***Proof*** Combine Lemma 13.15, Theorem 13.16(b), and Proposition 15.1. □

Finally, now we turn our attention to case that only the matrix moment $s_0$ is prescribed to obtain a result analogous to Theorems 14.2 and 15.2. In other words, a sequence $(s_j)_{j=0}^{2n} \in \mathcal{F}^{\succ}_{q,2n,\alpha,\beta}$ with $n = 0$ is given. We will see that the result is different in comparison to Theorem 15.2 where it is supposed that $n$ is a positive integer. The corresponding proofs are similar. However, several details are different as well. We again use the notation given in Notation 13.14. Furthermore, for every choice of $s_0 \in \mathbb{C}^{q\times q}$ and $z \in \mathbb{C}$, let

$$\mathscr{F}_{\bullet}(z) := \left\{X \in \mathbb{C}^{q\times q} : \Im((\beta - z)(z - \alpha)X - zs_0) \succcurlyeq O_{q\times q}\right\}. \tag{15.29}$$

**Proposition 15.3** *Let $(s_j)_{j=0}^{\kappa} \in \mathcal{F}^{\succcurlyeq}_{q,\kappa,\alpha,\beta}$ and let $F \in \mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^{\kappa}, =]$. For each $w \in \Pi_+$, then $F(w) \in \mathscr{K}_0(w) \cap \mathscr{F}_{\bullet}(w)$.*

***Proof*** We consider an arbitrary $w \in \Pi_+$. From Lemma 13.15 we know that $F(w)$ belongs to $\mathscr{K}_0(w)$. Since $F \in \mathcal{R}_q[[\alpha,\beta];(s_j)_{j=0}^{\kappa},=]$ is supposed, the $\mathcal{R}[\alpha,\beta]$-measure $\ddot{\sigma}_F$ of $F$ fulfills $\ddot{\sigma}_F([\alpha,\beta]) = s_0$. Remark 4.4 yields that $F_c$ given by (4.4) belongs to $\mathcal{R}_q(\mathbb{C}\setminus[\alpha,\beta])$. In particular, $\Im F_c(w) \succcurlyeq O$. Moreover, in view of (4.4), we can conclude

$$\Im F_c(w) = \Im\left((\beta-w)(w-\alpha)F(w) + (\alpha+\beta-w)s_0 - \int_{[\alpha,\beta]} t\ddot{\sigma}_F(\mathrm{d}t)\right).$$

Because of $(s_j)_{j=0}^{\kappa} \in \mathcal{F}_{q,\kappa,\alpha,\beta}^{\succcurlyeq}$, we have $s_0 \in \mathbb{C}_{\mathrm{H}}^{q\times q}$. Furthermore, [18, Rem. A.4] provides $\Im(\int_{[\alpha,\beta]} t\ddot{\sigma}_F(\mathrm{d}t)) = O$. Consequently, we obtain $\Im F_c(w) = \Im((\beta-w)(w-\alpha)F(w) - ws_0)$. Thus, $F(w)$ belongs to $\mathscr{F}_\bullet(w)$ as well. □

**Proposition 15.4** *Let $(s_j)_{j=0}^{0} \in \mathcal{F}_{q,0,\alpha,\beta}^{\succ}$, let $w \in \Pi_+$, and let $X \in \mathscr{K}_0(w) \cap \mathscr{F}_\bullet(w)$. Then there exists a rational matrix-valued function $F \in \mathcal{R}_q[[\alpha,\beta];(s_j)_{j=0}^{0},=]$ such that $F(w) = X$.*

***Proof*** We use parts of the proof of Proposition 15.1 and the notations (with $n = 0$) applied there. The proof of Proposition 15.1 shows that there exists a sequence $(s_j)_{j=1}^{\infty}$ of complex $q\times q$ matrices such that $(s_j)_{j=0}^{m} \in \mathcal{F}_{q,m,\alpha,\beta}^{\succ}$ and $(s_j)_{j=0}^{2m} \in \mathcal{H}_{q,2m}^{\succ}$ are valid for all $m \in \mathbb{N}_0$ and that

$$\Im((\beta-w)Z^*Y) \succcurlyeq O \tag{15.30}$$

and

$$\begin{bmatrix} X \\ I_q \end{bmatrix} = \ddot{\mathbf{V}}_0(w)\begin{bmatrix} Y \\ Z \end{bmatrix} \tag{15.31}$$

hold true. Let $S_\bullet := (\beta-w)(w-\alpha)X + (\alpha+\beta-w)s_0 - s_1$ and let $T_\bullet := I_q$. Obviously, $\operatorname{rank}\left[\begin{smallmatrix} S_\bullet \\ T_\bullet \end{smallmatrix}\right] = q$. Since $(s_j)_{j=0}^{2}$ belongs to $\mathcal{H}_{q,2}^{\succ}$, we have $\{s_0, s_1\} \subseteq \mathbb{C}_{\mathrm{H}}^{q\times q}$. Consequently, $\Im((\alpha+\beta-w)s_0) = \Im(-ws_0)$ and $\Im s_1 = O$. Thus, the assumption $X \in \mathscr{F}_\bullet(w)$ provides

$$\Im(T_\bullet^* S_\bullet) = \Im((\beta-w)(w-\alpha)X - ws_0) \succcurlyeq O. \tag{15.32}$$

Using (13.3), (15.31), and Lemma 13.6, we conclude

$$\begin{aligned}\begin{bmatrix} S_\bullet \\ T_\bullet \end{bmatrix} &= \Delta(w)\begin{bmatrix} X \\ I_q \end{bmatrix} \\ &= \Delta(w)\ddot{\mathbf{V}}_0(w)\begin{bmatrix} Y \\ Z \end{bmatrix} = -\delta^{-1}(\beta-w)(w-\alpha)[\mathbf{U}_{c,0}(w)]^{-1}\begin{bmatrix} Y \\ Z \end{bmatrix},\end{aligned}$$

which implies

$$\begin{bmatrix} Y \\ Z \end{bmatrix} = -\delta(\beta-w)^{-1}(w-\alpha)^{-1}\mathbf{U}_{c,0}(w)\begin{bmatrix} S_\bullet \\ T_\bullet \end{bmatrix} \tag{15.33}$$

and, in particular, $\operatorname{rank}\left[\begin{smallmatrix} Y \\ Z \end{smallmatrix}\right] = \operatorname{rank}\left[\begin{smallmatrix} S_\bullet \\ T_\bullet \end{smallmatrix}\right] = q$. Taking into account (15.33) and Remark 13.7, we get

$$\begin{bmatrix} (w-\alpha)Y \\ Z \end{bmatrix} = -\delta(\beta - w)^{-1}(w-\alpha)^{-1} \begin{bmatrix} (w-\alpha)I_q & O \\ O & I_q \end{bmatrix} \mathbf{U}_{c,0}(w) \begin{bmatrix} S_\bullet \\ T_\bullet \end{bmatrix}$$
$$= -(\beta - w)^{-1} \begin{bmatrix} I_q & O \\ O & \delta I_q \end{bmatrix} \begin{bmatrix} I_q & O \\ \mathsf{D}_0^\dagger & I_q \end{bmatrix} \begin{bmatrix} I_q & -\mathsf{D}_0 \mathbf{A}_1 \\ O & I_q \end{bmatrix} \begin{bmatrix} S_\bullet \\ T_\bullet \end{bmatrix}$$

and, because of $\left[\begin{smallmatrix} I_q & O \\ O & \delta I_q \end{smallmatrix}\right]^* \tilde{J}_q \left[\begin{smallmatrix} I_q & O \\ O & \delta I_q \end{smallmatrix}\right] = \delta \tilde{J}_q$ as well as (15.15) and (15.16), therefore

$$\begin{bmatrix} (w-\alpha)Y \\ Z \end{bmatrix}^* \tilde{J}_q \begin{bmatrix} (w-\alpha)Y \\ Z \end{bmatrix} = \delta|\beta - w|^{-2} \begin{bmatrix} S_\bullet \\ T_\bullet \end{bmatrix}^* \tilde{J}_q \begin{bmatrix} S_\bullet \\ T_\bullet \end{bmatrix}.$$

Consequently, by virtue of Remark 10.2 and (15.32), we obtain then

$$\Im((w-\alpha)Z^*Y) = \delta|\beta - w|^{-2} \Im(T_\bullet^* S_\bullet) \succcurlyeq O. \tag{15.34}$$

Because of Part 1, Remark 11.19 and Lemma 11.21, we have $(s_j)_{j=0}^0 \in \mathcal{F}_{q,0,\alpha,\beta}^{\succcurlyeq,\mathrm{ld}} \subseteq \mathcal{F}_{q,0,\alpha,\beta}^{\succcurlyeq,\mathrm{c}}$. Let $\phi, \psi \colon \mathbb{C} \setminus [\alpha, \beta] \to \mathbb{C}^{q \times q}$ be given by $\phi(z) := Y$ and $\psi(z) := \frac{w-z}{\Im(w)} HY + Z$, where $H := (Y^\dagger)^*[\Im(Z^*Y)]Y^\dagger$. Since (15.26) is valid, from (15.30) and (15.34) we see that $\Im(Z^*Y) \succcurlyeq O$ holds true. Proposition 5.16 yields $\det \mathsf{D}_0 \neq 0$. Using [36, Lem. 8.5], then one can check analogous to Part 6 of the proof of Proposition 14.1 that $\phi(w) = Y$ and $\psi(w) = Z$ are fulfilled and that the pair $[\phi; \psi]$ belongs to $\ddot{\mathcal{P}}[\mathsf{D}_0]$. Let $\varepsilon_0 \colon \mathbb{C} \to \mathbb{C}$ be given by $\varepsilon_0(z) := z - \beta$, and let $\tilde{\mathbf{p}}_1$, $\tilde{\mathbf{q}}_1$, $\tilde{\mathbf{p}}_2$, and $\tilde{\mathbf{q}}_2$ be the restrictions of the functions $\mathbf{p}_1^\diamond := \varepsilon_0 \ddot{\mathbf{p}}_1$, $\mathbf{q}_1^\diamond := \varepsilon_0 \ddot{\mathbf{q}}_1$, $\ddot{\mathbf{p}}_2$, and $\ddot{\mathbf{q}}_2$ onto $\mathbb{C} \setminus [\alpha, \beta]$, respectively. By virtue of (15.31), $\phi(w) = Y$, $\psi(w) = Z$, and Notation 13.14, we conclude that (15.27) holds true for $n = 0$. In view of $(s_j)_{j=0}^0 \in \mathcal{F}_{q,0,\alpha,\beta}^{\succcurlyeq,\mathrm{c}}$ and $[\phi; \psi] \in \ddot{\mathcal{P}}[\mathsf{D}_0]$, Theorem 11.29 shows that $F := -(\tilde{\mathbf{q}}_1 \mathsf{D}_0^\dagger \phi - \delta^{-1} \tilde{\mathbf{q}}_2 \psi)(\tilde{\mathbf{p}}_1 \mathsf{D}_0^\dagger \phi - \delta^{-1} \tilde{\mathbf{p}}_2 \psi)^{-1}$ belongs to $\mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^0, =]$. Thus, (15.27) for $n = 0$ provides $X = X I_q^{-1} = F(w)$. □

Now we are able to prove the announced result in the case that exactly one matrix moment is prescribed.

**Theorem 15.5** *Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha < \beta$ and let $(s_j)_{j=0}^0 \in \mathcal{F}_{q,0,\alpha,\beta}^{\succ}$. For each $w \in \Pi_+$, then*

$$\left\{ F(w) \colon F \in \mathcal{R}_q[[\alpha, \beta]; (s_j)_{j=0}^0, =] \right\} = \mathscr{K}_0(w) \cap \mathscr{F}_\bullet(w),$$

*where $\mathscr{K}_0(w)$ and $\mathscr{F}_\bullet(w)$ are given by Notation 13.14 and (15.29), respectively.*

***Proof*** Use Propositions 15.3 and 15.4. □

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## Appendix A: Some Facts on Matrix Theory

***Remark A.1*** ([10]) Let $A \in \mathbb{C}^{p\times q}$.

(a) For $B \in \mathbb{C}^{p\times m}$, then $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ if and only if $AA^\dagger B = B$.
(b) For $C \in \mathbb{C}^{n\times q}$, then $\mathcal{N}(A) \subseteq \mathcal{N}(C)$ if and only if $CA^\dagger A = C$.

***Remark A.2*** Let $A \in \mathbb{C}^{q\times q}$. Then $\Re(A^*) = \Re(A)$ and $\Im(A^*) = -\Im(A)$. Moreover, $\Re(BAB^*) = B[\Re(A)]B^*$ and $\Im(BAB^*) = B[\Im(A)]B^*$, for all $B \in \mathbb{C}^{p\times q}$.

***Remark A.3*** Let $A \in \mathbb{C}^{p\times q}$. Then $\mathcal{R}(A^*) = [\mathcal{N}(A)]^\perp$ and $\mathcal{N}(A^*) = [\mathcal{R}(A)]^\perp$.

***Remark A.4*** Let $\mathcal{U}$ be a subspace of $\mathbb{C}^q$. Then there exists a unique matrix $\mathbb{P}_\mathcal{U} \in \mathbb{C}^{q\times q}$ such that both $\mathbb{P}_\mathcal{U} x \in \mathcal{U}$ and $x - \mathbb{P}_\mathcal{U} x \in \mathcal{U}^\perp$ are fulfilled for each $x \in \mathbb{C}^q$. This matrix $\mathbb{P}_\mathcal{U}$ is called the orthogonal projection matrix onto $\mathcal{U}$. In particular, $\mathbb{P}_\mathcal{U} u = u$ for all $u \in \mathcal{U}$. A complex $q \times q$ matrix $P$ is said to be an orthogonal projection matrix, if there exists a subspace $\mathcal{U} \in \mathbb{C}^q$ such that $P = \mathbb{P}_\mathcal{U}$. In this case, the subspace $\mathcal{U}$ is uniquely determined.

***Remark A.5*** Let $\mathcal{U}$ be a subspace of $\mathbb{C}^q$ and let $P \in \mathbb{C}^{q\times q}$. Then $P = \mathbb{P}_\mathcal{U}$ if and only if the three conditions $P^2 = P$, $P^* = P$, and $\mathcal{R}(P) = \mathcal{U}$ are fulfilled. Moreover, $\mathcal{N}(\mathbb{P}_\mathcal{U}) = \mathcal{U}^\perp$ is valid.

***Remark A.6*** Let $A \in \mathbb{C}^{p\times q}$. Then $(A^\dagger)^\dagger = A$, $(A^\dagger)^* = (A^*)^\dagger$, $(AA^*)^\dagger = (A^\dagger)^* A^\dagger$, $(A^*A)^\dagger = A^\dagger (A^*)^\dagger$, $A^\dagger = A^*(AA^*)^\dagger$, $A^\dagger = (A^*A)^\dagger A^*$, $\mathcal{N}(A^\dagger) = \mathcal{N}(A^*)$, and $\mathcal{R}(A^\dagger) = \mathcal{R}(A^*)$.

***Remark A.7*** Let $A \in \mathbb{C}^{q\times q}$. Then $A^* = A$ if and only if $(A^\dagger)^* = A^\dagger$. Furthermore, if $A^* = A$, then $AA^\dagger = A^\dagger A$.

***Remark A.8*** Let $A \in \mathbb{C}^{p\times q}$. Then $\mathbb{P}_{\mathcal{R}(A)} = AA^\dagger$, $\mathbb{P}_{\mathcal{N}(A)} = I_q - A^\dagger A$, $\mathbb{P}_{\mathcal{R}(A^*)} = A^\dagger A$, and $\mathbb{P}_{\mathcal{N}(A^*)} = I_p - AA^\dagger$.

***Remark A.9*** Let $A \in \mathbb{C}^{q\times q}$. Then $A \in \mathbb{C}_{\succcurlyeq}^{q\times q}$ if and only if $A^\dagger \in \mathbb{C}_{\succcurlyeq}^{q\times q}$. If $A \in \mathbb{C}_{\succcurlyeq}^{q\times q}$, then $\sqrt{A}^\dagger = \sqrt{A^\dagger}$ and $A\sqrt{A}^\dagger = \sqrt{A} = \sqrt{A}^\dagger A$.

**Lemma A.10** *Let $M = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ be the block representation of a complex $(p+q)\times(r+s)$ matrix $M$ with $p\times r$ block $A$ fulfilling $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $\mathcal{N}(A) \subseteq \mathcal{N}(C)$ and let $M/A := D - CA^\dagger B$. Then $M\left[\begin{smallmatrix} -X \\ I_s \end{smallmatrix}\right] = \left[\begin{smallmatrix} O_{p\times s} \\ M/A \end{smallmatrix}\right]$ holds true for all $X \in \mathbb{C}^{r\times s}$ such that $AX = B$. Furthermore, $[-Z, I_q]M = [O_{q\times r}, M/A]$ is valid for all $Z \in \mathbb{C}^{q\times p}$ such that $ZA = C$.*

***Proof***. Remark A.1 yields $AA^\dagger B = B$ and $CA^\dagger A = C$. For all $X \in \mathbb{C}^{r\times s}$ with $AX = B$, we get then

$$M\begin{bmatrix} -X \\ I_s \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} -X \\ I_s \end{bmatrix} = \begin{bmatrix} B - AX \\ D - CX \end{bmatrix} = \begin{bmatrix} B - AX \\ D - CA^\dagger AX \end{bmatrix}$$
$$= \begin{bmatrix} B - B \\ D - CA^\dagger B \end{bmatrix} = \begin{bmatrix} \mathrm{O}p\times s \\ M/A \end{bmatrix}.$$

Analogously, for all $Z \in \mathbb{C}^{q\times p}$ such that $ZA = C$, we obtain

$$[-Z, I_q]M = [C - ZA, D - ZB] = [C - ZA, D - ZAA^\dagger B] = [O_{q\times r}, M/A]. \quad \square$$

## Appendix B: Some Remarks on Non-negative Hermitian Measures

For the convenience of the reader, we give some facts on the integration theory of non-negative Hermitian measures, which was developed by Kats [26] and Rosenberg [31–33] (see also [11, Sec. 2.2]). Let $\mathfrak{B}_{p\times q}$ be the $\sigma$-algebra of all Borel subsets of $\mathbb{C}^{p\times q}$. Let $(\Omega, \mathfrak{A})$ be a measurable space and let $\mu = [\mu_{jk}]_{j,k=1}^q \in \mathcal{M}_q^{\succcurlyeq}(\Omega, \mathfrak{A})$. For every choice of $j, k \in \mathbb{Z}_{1,q}$, then the complex measure $\mu_{jk}$ is absolutely continuous with respect to the trace measure $\tau := \operatorname{tr}\mu$ of $\mu$. The matrix-valued function $\mu_\tau' := [\frac{\mathrm{d}\mu_{jk}}{\mathrm{d}\tau}]_{j,k=1}^q$ built by the corresponding Radon–Nikodym derivatives of $\mu_{jk}$ with respect to $\tau$ is well defined up to sets of zero $\tau$-measure. Let $\Phi\colon \Omega \to \mathbb{C}^{p\times q}$ be an $\mathfrak{A}$-$\mathfrak{B}_{p\times q}$-measurable function and let $\Psi\colon \Omega \to \mathbb{C}^{r\times q}$ be an $\mathfrak{A}$-$\mathfrak{B}_{r\times q}$-measurable function. Then the pair $[\Phi; \Psi]$ is said to be left-integrable with respect to $\mu$ if $\Phi\mu_\tau'\Psi^*$ belongs to $[\mathcal{L}^1(\Omega, \mathfrak{A}, \tau; \mathbb{C})]^{p\times r}$. In this case, let $\int_A \Phi\mathrm{d}\mu\Psi^* := \int_A \Phi\mu_\tau'\Psi^*\mathrm{d}\tau$ for all $A \in \mathfrak{A}$.

**Proposition B.1** *Let $(\Omega, \mathfrak{A})$ be a measurable space and let $\mu \in \mathcal{M}_q^{\succcurlyeq}(\Omega, \mathfrak{A})$. Let $f \in \mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{C})$ be such that $f(\Omega) \subseteq [0, \infty)$. Then $\mu_f\colon \mathfrak{A} \to \mathbb{C}^{q\times q}$ given by $\mu_f(A) := \int_A f\mathrm{d}\mu$ belongs to $\mathcal{M}_q^{\succcurlyeq}(\Omega, \mathfrak{A})$ and the following statements hold true:*

(a) *Let $\Phi\colon \Omega \to \mathbb{C}^{p\times q}$ be an $\mathfrak{A}$-$\mathfrak{B}_{p\times q}$-measurable function and let $\Psi\colon \Omega \to \mathbb{C}^{r\times q}$ be an $\mathfrak{A}$-$\mathfrak{B}_{r\times q}$-measurable function. Then the following statements are equivalent:*

   (i) *The pair $[\Phi;\Psi]$ is left-integrable with respect to $\mu_f$.*
   (ii) *The pair $[f\Phi;\Psi]$ is left-integrable with respect to $\mu$.*
   (iii) *The pair $[\Phi; f\Psi]$ is left-integrable with respect to $\mu$.*

(b) *If (i) holds true, then*

$$\int_A \Phi \mathrm{d}\mu_f \Psi^* = \int_A (f\Phi)\mathrm{d}\mu\Psi^* = \int_A \Phi\mathrm{d}\mu(f\Psi)^* \qquad \text{for all } A\in\mathfrak{A}.$$

Proposition B.1 can be easily proved by application of [24, Prop. 7.4]. We omit the details.

## Appendix C: Matrix Polynomials of Second Kind

***Remark C.1*** ([22, Rem. E.3]). Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q\times q$ matrices and let $P$ and $Q$ be two complex $q\times q$ matrix polynomials, each having degree not greater than $\kappa+1$. Then $(P+Q)^{[\![s]\!]} = P^{[\![s]\!]} + Q^{[\![s]\!]}$. Furthermore, $(PA)^{[\![s]\!]} = P^{[\![s]\!]}A$ for all $A\in\mathbb{C}^{q\times q}$ and, in particular, $(\lambda P)^{[\![s]\!]} = \lambda P^{[\![s]\!]}$ for all $\lambda\in\mathbb{C}$.

**Lemma C.2** ([22, Lem. E.4]) *Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q\times q$ matrices, let $k\in\mathbb{N}$ with $2k-1\le\kappa$, and let $P$ be a complex $q\times q$ matrix polynomial of degree $k$ and with leading coefficient matrix $I_q$ such that $Y_k(P)$ admits the representation $Y_k(P) = \begin{bmatrix} -X_k \\ I_q \end{bmatrix}$ with some matrix $X_k$ fulfilling $H_{k-1}X_k = y_{k,2k-1}$ (see also (6.1)). Then the matrix polynomial $Q\colon\mathbb{C}\to\mathbb{C}^{q\times q}$ given by $Q(w) := wP(w)$ fulfills $Q^{[\![s]\!]}(z) = zP^{[\![s]\!]}(z)$ for all $z\in\mathbb{C}$.*

**Lemma C.3** (cf. [22, Lem. E.5]) *Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q\times q$ matrices and let $P$ be a complex $q\times q$ matrix polynomial such that $k := \deg P$ fulfills $k\le\kappa$. Then:*

(a) *Let $Q\colon\mathbb{C}\to\mathbb{C}^{q\times q}$ be defined by $Q(z) := (z-\alpha)P(z)$. Then $\deg Q\le\kappa+1$. If $k\le 0$, then $Q^{[\![s]\!]} = s_0P$. If $k\ge 1$, then $Q^{[\![s]\!]} = P^{[\![a]\!]} + s_0P$ where $(a_j)_{j=0}^{\kappa-1}$ is defined in (3.2).*
(b) *Let $R\colon\mathbb{C}\to\mathbb{C}^{q\times q}$ be defined by $R(z) := (\beta-z)P(z)$. Then $\deg R\le\kappa+1$. If $k\le 0$, then $R^{[\![s]\!]} = -s_0P$. If $k\ge 1$, then $R^{[\![s]\!]} = P^{[\![b]\!]} - s_0P$ where $(b_j)_{j=0}^{\kappa-1}$ is defined in (3.2).*

The proof of Lemma C.3 consists in straightforward calculations. We omit the details.

**Lemma C.4** *Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q\times q$ matrices and let $P$ be a complex matrix polynomial such that $k := \deg P$ fulfills $k\le\kappa-1$. Furthermore, let $S\colon\mathbb{C}\to\mathbb{C}^{q\times q}$ be given by $S(z) := (\beta-z)(z-\alpha)P(z)$. Then $\deg S\le\kappa+1$ and the following statements hold true:*

(a) *If* $k = -\infty$*, then* $S^{[\![s]\!]} = O_{q\times q}$ *for all* $z \in \mathbb{C}$.

(b) *If* $k = 0$*, then* $S^{[\![s]\!]} = ([\alpha + \beta - z]s_0 - s_1)P(z)$ *for all* $z \in \mathbb{C}$.

(c) *If* $k \geq 1$*, then* $S^{[\![s]\!]} = P^{[\![c]\!]}(z) + ([\alpha + \beta - z]s_0 - s_1)P(z)$ *for all* $z \in \mathbb{C}$*, where the sequence* $(c_j)_{j=0}^{\kappa-2}$ *is defined by (3.3).*

Lemma C.4 can be proved analogous to [8, Lem. 11.1]. We omit the details.

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