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b-AM-Dunford–Pettis Operators on Banach lattices

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Abstract
In our research work, we introduce a new class of operators that we call *b-AM-Dunford–Pettis* operators. Properties of b-AM-Dunford–Pettis operators, the relationship between the b-AM-Dunford–Pettis operators and various classes of operators are investigated. On the other side, our techniques and results will be related to the lattice structure of the b-AM-Dunford–Pettis operators. For instance, it will be proved that under certain conditions, the b-AM-Dunford–Pettis opertors verify the domination properties.

Keywords Banach lattice · b-Order bounded · Order continuous norm

Mathematics Subject Classification 46B42 · 47B60 · 47B65

1 Introduction

Throughout this paper, X, Y will denote Banach spaces, and E, F will express Banach lattices. B_X is the closed unit ball of X. An operator $T : X \to Y$ is said *compact* (resp. *weakly compact*) whenever T maps B_X onto a norm totally bounded (resp. weakly relatively compact) subset of Y. Dunford–Pettis operators were introduced by Grothendieck in [13]. A bounded operator $T : X \to Y$ is called *Dunford–Pettis* if it maps weakly compact subsets of X into norm compact subsets of Y. Alternatively,

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T carries weakly convergent sequences onto norm convergent sequences. Note that each compact operator is Dunford–Pettis. But, the converse is not always true. For instance, the identity of the Banach lattice ℓ^1 is Dunford–Pettis, however, it is not compact (Theorem 4.32 in [1]). These notions coincide when X is reflexive (Theorem 3.40 in [1]). As soon as the order structure is taken into account, some other classes of operators manifest naturally. In the spirit of compact operators, the class of AM-compact operators was introduced by Dodds-Fremlin in [11]. A linear operator T from E to Y is said to be *AM-compact* if it maps order bounded subsets of E to totally bounded subsets of Y. Further T is said to be order weakly compact if it maps order bounded subsets of E into relatively weakly compact subsets of Y (see [10])

Alpay et al. in [3] introduced a new notion of boundedness that generalizes the usual order boundedness. A subset A of a Banach lattice E is said to be *b-order bounded* if it is order bounded in $E^{''}$ (the topological bidual of E). Clearly, order bounded subsets of E are b-order bounded. The converse is not always true (see Examples 1.2 in [3]). This notion enriched the study of operators on Banach lattices. Naturally, an operator T from E into Y is called *b-AM-compact* whenever T maps each b-order bounded subset of E into a relatively compact subset of Y. The class of b-AM-compact operators was elaborated by Aqzzouz.

Referring the reader to Theorem 5.98 in [1], recall that a norm bounded subset A of X is said to be *Dunford–Pettis* set whenever every weakly compact operator from X to Y carries A to a norm relatively compact set of Y. Alternatively, A is a Dunford–Pettis set if and only if every weakly null sequence (f_n) of $X^{'}$ converge uniformly to zero on the set A, that is $\sup_{x\in A} |f_n(x)| \to 0$ (see [4, Theorem 1]). Next, order Dunford–Pettis (resp. *b-order Dunford–Pettis*) operators is defined in [6] (resp. in [14]). An operator $T : E \to X$ is said to be *order Dunford–Pettis* (resp. *b-order Dunford–Pettis*) if it carries each order bounded (resp. b-order bounded) subset of E into a Dunford–Pettis set of X. We denote by:

- $AM(E, F)$ the class of all AM-compact operators from E to F.
- $AM_b(E, F)$ the class of all b-AM-compact operators from E to F.
- $DP_o(E, F)$ the class of all order Dunford–Pettis operators from E to F.
- $DP_b(E, F)$ the class of all b-order Dunford–Pettis operators from E to F. Clearly, $AM_b(E, F) \subseteq AM(E, F) \subseteq DP_o(E, F)$. Since $L_1[0, 1]$ has the Dunford–Pettis property, it follows that $I_{L_1[0,1]}$ is an order Dunford–Pettis operator but not AM-compact (see in [4]). Also, the following inclusions hold.
  $AM_b(E, F) \subseteq DP_b(E, F) \subseteq DP_o(E, F)$.

Let (f_n) be a sequence of $c_0' = \ell^1$ such that $f_n \to 0$ in $\sigma(\ell^1, \ell^\infty)$. Since ℓ^1 has the Schur property, then $\|f_n\| = \sup_{x\in B_{c_0}} |f_n(x)| \to 0$, which implies that I_{c_0} is a b-order Dunford–Pettis operator and not b-AM-compact. In general the inclusion of $DP_b(E, F)$ in $DP_o(E, F)$ is proper (see in Example 3.3 in [14]).

Recently, in [16], Hajji and Mahfoudhi introduced the class of LW-compact operators. An operator $T : E \to Y$ is said to be LW-compact if T maps L-weakly compact subsets of E into relatively compact subsets of Y. It follows from Theorem 2.3 in [16] and Corollary 5.54 in [1] that T is LW-compact if and only if T carries weakly compact intervals of E into relatively compact sets of Y.

Along this line, we define a new class of operators called *b-AM-Dunford–Pettis* that extends the class of Dunford–Pettis and b-AM-compact operators. Precisely, it carries subsets which are both relatively weakly compact and b-order bounded onto relatively compact subsets. We also attempt to determin some of their properties. Moreover, we explore the relationship between this class and various classes like order Dunford–Pettis (resp. b-order Dunford–Pettis) operators. In the same direction, we show that every Dunford–Pettis operator is b-AM-Dunford–Pettis. On the other side, a b-AM-Dunford–Pettis operator does not need to be Dunford–Pettis. Detailed analysis is provided in Example 2.4. In addition, we characterize Banach lattice E on which each b-AM-Dunford–Pettis operator from E into a Banach space Y is b-AM-compact. Finally, we highlight that the class of b-AM-Dunford–Pettis operators between Banach lattices E and F verifies the domination properties, whenever F has an order continuous norm.

We assume readers are already familiar with the notions of a Riesz space and a positive operator. For teminology and concepts not explained in this text we refer to the standard reference [1].

2 *b*-AM-Dunford–Pettis Operators

Based on the concept of Dunford–Pettis operators and order Dunford–Pettis operators, we introduce a new class of operators as follows.

Definition 2.1 An operator T from a Banach lattice E into a Banach space Y is said to be *b-AM-Dunford–Pettis*, if T carries subsets of E which are both relatively weakly compact and b-order bounded onto relatively compact subsets of Y.

Note that an order Dunford–Pettis (resp. b-order Dunford–Pettis) operator is not necessarily b-Dunford–Pettis. And conversely a b-AM-Dunford–Pettis operator is not necessarily order Dunford–Pettis (resp. b-order Dunford–Pettis). The detail follows.

Example 2.2 Since $c_0^{'} = \ell^1$ has the Schur property, then the identity operator I_{c_0} is a b-order Dunford–Pettis operator, and hence order Dunford–Pettis operator. On the other hand, the standard basis $(e_n)_n$ of c_0 is a weakly null sequence and does not have any convergent subsequences. This implies that I_{c_0} is not b-AM-Dunford–Pettis.

Example 2.3 Let J be the naturel embedding from $L^\infty[0, 1]$ into $L^2[0, 1]$. Since J is weakly compact and $L^\infty[0, 1]$ has the Dunford–Pettis property, it follows from Theorem 5.82 in [1] that J is Dunford–Pettis, and hence b-AM-Dunford–Pettis. On the other hand, since J is not AM-compact (see [18, Example on page 222]), it follows that J is not order Dunford–Pettis (since $L^2[0, 1]$ is reflexive), and so it is not b-order Dunford–Pettis.

Every Dunford–Pettis operator is b-AM-Dunford–Pettis. It is noteworthy that, in general, a b-AM-Dunford–Pettis operator is not necessarily Dunford–Pettis. The details follow.

Example 2.4 Let S be an isomorphism from ℓ^2 to a subspace of $L_1[0, 1]$ (see Corollary 2.77 in [18]). Note that S is b-AM compact (see [18] on page 218) and so is b-AM-

Dunford–Pettis. However, the linear operator S is not Dunford–Pettis. Otherwise, S is compact (ℓ^2 is reflexive) which Contradicts the fact that S is an isomorphism.

An easy application of Eberlein-Šmulian theorem ([1], Theorem 3.40) reveals the following result.

Lemma 2.5 *An operator $T : E \to Y$ is b-AM-Dunford–Pettis if and only if $\lim_n \|Tx_n\| = 0$ holds in Y, for every weakly null b-order bounded sequence (x_n) of E.*

Let us recall that an operator $T : E \to X$ is said to be *b-weakly compact* whenever T carries each b-order bounded subset of E into a relatively weakly compact subset of X (see in [3]). By Proposition 2.8 in [3], T is b-weakly compact if and only if each b-order bounded disjoint sequence (x_n) in E satisfies $\lim_n \|Tx_n\| = 0$. The next proposition follows.

Proposition 2.6 *Every continuous b-AM-Dunford–Pettis operator $T : E \to Y$ is b-weakly compact.*

Proof Let (x_n) be a b-order bounded disjoint sequence in E. Then, by [5, Lemma 2.20] we see that (x_n) is weakly null. Since T is b-AM-Dunford–Pettis, it follows that $\lim_n \|Tx_n\| = 0$. The rest of the proof follows from Proposition 2.8 in [3]. □

Notice that the identity operator $Id_{L_1[0,1]} : L_1[0,1] \longrightarrow L_1[0,1]$ is b-weakly compact, but not b-AM-Dunford–Pettis (see Example 2.6 (a) in [3]).

A Banach space X is said to have the *Dunford–Pettis property* (for short *DPP*) if $x_n^{'}(x_n) \to 0$ whenever $x_n \xrightarrow{\sigma(X,X')} 0$ and $x_n' \xrightarrow{\sigma(X',X)} 0$. Recall from [18] that a continuous operator $T : X \to E$ is said to be *L-weakly compact* whenever $\lim \|f_n\| = 0$ for every disjoint sequence $(f_n)_n$ in the solid hull of $T(B_X)$. Note that the solid hull of a subset S of E is the following.

$$\mathrm{Sol}(S) = \{x \in E : \exists a \in A \;\text{ with }\; |x| \leq |a|\}.$$

Note that every b-AM-compact operator is b-AM-Dunford–Pettis. However, the converse is not always true, as follows from the next example.

Example 2.7 Consider the linear operator $T : C[0,1] \to c_0$ defined for each $f \in C[0,1]$ by

$$Tf = \left(\int_0^1 f(t) r_n(t) dt\right)_{n=1}^{+\infty},$$

where r_n is the n'th Rademacher function on $[0,1]$. Note that T is a weakly compact operator (Example 4.4 in [7]). Since $C[0,1]$ has the DPP, it follows from Theorem 5.82 in [1] that T is Dunford–Pettis. Therefore, it is b-AM-Dunford–Pettis. On the other hand, T is not L-weakly compact (refer back to Example 4.4 in [7]), see Definition 5.59 in [1] for the notion of an L-weakly compact operator. Since c_0 has an order continuous norm, it follows from [8] that T is not compact, and consequently it is not b-AM-compact because $C[0,1]$ is an AM-space with unit, see Definition 4.20 in [1] for the notion of an AM-space.

A useful characterization of b-AM-Dunford–Pettis operator is exhibited in what follows.

Theorem 2.8 *An operator T from a Banach lattice E into a Banach space X is a b-AM-Dunford–Pettis operator if and only if T carries b-order bounded weakly Cauchy sequences of E to norm convergent sequences of Y.*

Proof Suppose that T is b-AM-Dunford–Pettis, and let $(x_n)_n$ be a weakly Cauchy sequence of E satisfying $0 \leq x_n \leq x''$ for all $n \in \mathbb{N}$ and for some $x'' \in E''$. If $(Tx_n)_n$ is not a norm Cauchy sequence of Y, then there is a subsequence $(z_n)_n$ of $(x_n)_n$ and $\epsilon > 0$ such that $\|T(z_{2n+1} - z_{2n})\| \geq \epsilon$ for all n. Next, since $z_{2n+1} - z_{2n} \xrightarrow{\sigma(E,E')} 0$ and $0 \leq |z_{2n+1} - z_{2n}| \leq 2x''$ for all $n \in \mathbb{N}$, it follows from our hypothesis that $\|Tz_{2n+1} - Tz_{2n}\| \to 0$, which turns out to be contradictory. Hence, $(Tx_n)_n$ is a norm Cauchy sequence, and thus it is norm convergent. □

A linear map T between two Banach lattices E and F is said *b-order bounded* if it maps b-order bounded subsets of E into b-order bounded subsets of F ([3]). The following lemma asserts that a regular operator between two Banach lattices is b-order bounded.

Lemma 2.9 *Let E and F be two Banach lattices, then every regular operator $T : E \to F$ is b-order bounded.*

Proof Let A be a b-order bounded subset of E. Now, we claim that $T(A)$ is b-order bounded subset of F. For this, using the fact that T is regular, it is easy see that $T^{''} : F^{''} \to E^{''}$ is also regular (see [1, Theorem 1.73]). This shown that $T^{''}(A)$ is order bounded on $F^{''}$, and hence $T(A)$ is b-order bounded on F, as claimed. □

However, as Example 2.4 in [3] depicts, the converse is not true. Recall from Theorem 4.3 in [1], that every positive operator from a Banach lattice to a normed Riesz space is continuous. In a more general context, we have the following.

Proposition 2.10 *Every b-order bounded operator from a Banach lattice E to a Banach lattice F is continuous.*

Proof Let $T : E \to F$ be a b-order bounded operator from a Banach lattice E into a Banach lattice F. Suppose that T is not continuous. Then, there exists a sequence $(x_n) \subseteq B_E$ such that $\|Tx_n\| \geq n^3$ for all $n \in \mathbb{N}$. Since E is a Banach lattice, it follows that $x := \sum_{n=1}^{\infty} \frac{|x_n|}{n^2} \in E$. Obviously, $-x \leq \frac{x_n}{n^2} \leq x, \quad$ for $n \in \mathbb{N}^*$. Hence, we infer that from the b-order boundedness of the operator T, there exists $0 \leq y^{''} \in F^{''}$ such that $|T(\frac{x_n}{n^2})| \leq y^{''}$ holds for each $n \in \mathbb{N}^*$. As a matter of fact,

$$n \leq \| \frac{T(x_n)}{n^2} \Big\| \leq \|y^{''}\Big\| < \infty$$

holds for each $n \in \mathbb{N}^*$, which is a contradiction. Thus, T must be continuous. □

To continue our discussion we need the next definition.

Definition 2.11 A linear operator from a Banach space X into a Banach lattice F is said to be *norm-b-order bounded* if it maps norm bounded subsets of X into b-order bounded subsets in F.

Also, we need the following Lemma.

Lemma 2.12 *Let E be a Banach lattice. If $(x_n)_n$ is a b-order bounded sequence of E, then the operator $S : \ell^1 \to E$ defined by*

$$S((\alpha_n)_n) = \sum_{n=1}^{\infty} \alpha_n x_n$$

is norm-b-order bounded.

Proof Departing from our hypotheseis, the sequence $(x_n)_n$ is b-order bounded on E. Therefore, there exists $x^{''} \in E^{''}_{+}$ such that $|x_n| \leq x^{''}$ for each n. In addition, it is not difficult to trace that S is well defined. Indeed, for $\alpha = (\alpha_1, \alpha_2, \ldots) \in \ell^1$,

$$\|S\alpha\| = \left\| \sum_{i=1}^{+\infty} \alpha_i x_i \right\| \leq \left\| \sum_{i=1}^{+\infty} |\alpha_i||x_i| \right\| \leq \left\| \sum_{i=1}^{+\infty} |\alpha_i| \right\| \|x^{''}\| \leq \|\alpha\|_1 \|x^{''}\|.$$

To elaborate the proof, we need to show that S is norm-b-order bounded.
Let A be a norm bounded subset of ℓ^1, then there exists a real $M > 0$ such that for each $\alpha = (\alpha_1, \alpha_2, \ldots) \in A$, we get

$$\|\alpha\|_1 = \sum_{i=1}^{+\infty} |\alpha_i| \leq M.$$

This yields

$$|S\alpha| = \left| \sum_{i=1}^{+\infty} \alpha_i x_i \right| \leq \sum_{i=1}^{+\infty} |\alpha_i||x_i| \leq \left(\sum_{i=1}^{+\infty} |\alpha_i| \right) x^{''} \leq Mx^{''} \in E^{''}.$$

Thus, S is a norm-b-order bounded operator as desired. □

A similar result entailed by Theorem 5.81 in [1] is presented in what follows

Theorem 2.13 *Let $T : E \to F$ be a bounded operator between Banach lattices. Then, the following statements are equivalent.*

(1) *T is b-AM-Dunford–Pettis.*
(2) *For an arbitrary Banach space X and for every norm-b-order bounded weakly compact operator $S : X \to E$, the operator TS is a compact operator.*

Proof (1) $\Rightarrow$ (2) Let $S : X \to E$ be both weakly compact and norm-b-order bounded (where X is a Banach space), and let W be a norm bounded subset of X. We infer that the subset $S(W)$ is a relatively weakly compact and b-order bounded in E. Hence, grounded on our hypotheseis, we realize that $TS(W)$ is a relatively compact subset of E. This implies that TS is a compact operator, as desired.

(2) $\Rightarrow$ (1) Let $(x_n)_n$ be a b-order bounded sequence such that $x_n \xrightarrow{\sigma(E,E')} 0$. Lemma 2.12 combined with Theorem 5.26 in [1] is suggestive that the linear operator $S : \ell^1 \to E$ defined by $S(\alpha)_n = \sum_{n=1}^{+\infty} \alpha_n x_n$ is both weakly compact and norm-b-order bounded. Resting upon our hypothesis, we detect that TS is a compact operator. Note that if $(e_n)_n$ is the sequence of the basic unit vectors of ℓ^1, then $S(e_n) = x_n$ holds for each n. Therefore, in view of the compactness of the operator TS, we notice that $(TS(e_n))_n = (Tx_n)_n$ is relatively compact. Since $(x_n)_n$ is a weakly null sequence and T is continuous then $(Tx_n)_n$ is weakly null. It follows that $\|Tx_n\| \to 0$. Thus, we deduce that T is a b-AM-Dunford–Pettis operator, and the proof holds. □

The collection of b-AM-Dunford–Pettis operators from E to Y will be denoted by $DP_b(E, Y)$.

Proposition 2.14 *Let E and F be Banach lattices, and let Y be a Banach space.*

(1) *$DP_b(E, Y)$ is a norm closed vector subspace of $L(E, Y)$.*
(2) *If $S : E \to F$ is a b-Dunford Pettis operator, then for each bounded operator $T : F \to Y$, the composed operator $TS : E \to Y$ is a b-Dunford Pettis operator.*
(3) *If $T : F \to Y$ is a b-AM-Dunford–Pettis operator, then for each b-order bounded operator $S : E \to F$, the composed operator $TS : E \to Y$ is a b-Dunford Pettis operator.*

Proof (1) To check that the vector subspace of all b-AM-Dunford–Pettis operators from E to Y is closed, assume that a sequence $(T_n)_n$ of b-AM-Dunford–Pettis operators from E to Y satisfies $T_n \to T$ in $L(E, Y)$. Let A be a subset of E which is both b-order bounded and relatively weakly compact, and let $\epsilon > 0$. Observe that for n big enough,

$$T(A) \subset T_n(A) + \epsilon B_Y.$$

Since $T_n(A)$ is a norm relatively compact subset of Y, it follows that $T(A)$ is also a relatively compact subset of Y. This is indicative that T is b-Dunford Pettis.

(2) Let A be a b-order bounded relatively weakly compact subset of E. Since S is b-AM-Dunford–Pettis and hence $S(A)$ is norm relatively compact in F and T is continuous, then $TS(A)$ is also a norm relatively compact subset in Y. Hence, TS is b-AM-Dunford–Pettis.

(3) Let A be a b-order bounded subset of E, which is relatively weakly compact. Since S is b-order bounded, then by the continuity of S (Proposition 2.10), we obtain that $S(A)$ is a $b-$order bounded and relatively weakly compact subset in F. Now, as T is b-AM-Dunford–Pettis, $TS(A)$ is norm relatively compact. Therefore, TS is b-AM-Dunford–Pettis.

□

3 *b*-AM-Compactness of *b*-AM-Dunford–Pettis Operators

The square of a b-order bounded b-AM-Dunford–Pettis operator is b-AM-compact. The details are included in the next proposition.

Proposition 3.1 *Let E be a Banach lattice and $T : E \to E$ be a b-order bounded operator. If T is b-AM-Dunford–Pettis, then T^2 is b-AM-compact.*

Proof Since $T : E \to E$ is a b-AM-Dunford–Pettis operator, then T is a b-weakly compact operator (see Proposition 2.6). Let A be a b-order bounded subset of E, then $T(A)$ is both b-order bounded and weakly relatively compact subset of E. Thus, resting on the fact that T is a b-AM-Dunford–Pettis operator, we deduce that $T^2(A)$ is a relatively compact subset of A. In this way, the proof is finished. □

We are now in a sound position to present a criterion for b-AM-compactness of b-AM-Dunford–Pettis operators. Specifically, we shall provide a sufficient condition for which each b-AM-Dunford–Pettis operator is b-AM-compact.

Theorem 3.2 *Let E be a Banach lattice. If E has an order continuous norm, then each b-AM-Dunford–Pettis operator from E into Y is b-AM-compact for every Banach space Y.*

Proof Assume that $T : E \to Y$ is a b-AM-Dunford–Pettis operator. Let A be a b-order bounded subset of E_+, and let (w_n) be a disjoint sequence in the solid hull of A. Since (w_n) is also a b-order bounded sequence, it follows from Proposition 2.8 in [3] that $\lim_n \|Tx_n\| = 0$. Thus, relying on Theorem 4.36 in [1], for all $\epsilon > 0$, there exists some $u_\epsilon \in E_+$ such that

$$T(A) \subseteq T([-u_\epsilon, u_\epsilon]) + \epsilon B_Y.$$

Since E has an order continuous norm, then $[-u_\epsilon, u_\epsilon]$ is a weakly compact subset of E (Theorem 4.9 in [1]). Thus, resting upon the fact that T is b-AM-Dunford–Pettis, we infer that $T([-u_\epsilon, u_\epsilon])$ is a relatively compact subset of Y. This easily implies that $T(A)$ is a relatively compact subset of Y (Theorem 3.1 in [1]), and therefore the proof is finished. □

A Riesz space E is said to be *discrete* if it admits a complete disjoint system of discrete elements, where we say a non zero element $x \in E$ is discrete whenever the ideal generated by x coincide with the vector subspace generated by x. To continue our discussion we need the following Lemma.

Lemma 3.3 *Let E be an infinite dimensional Banach lattice. Then, there exists a positive b-Dunfor-Pettis operator $T : \ell^\infty \to E$ which is not b-AM-compact.*

Proof Since $(\ell^\infty)'$ is not discrete, it follows from Theorem 2.8 in [2] that there exists a positive Dunford–Pettis operator T from ℓ^∞ into E which is not AM-compact. This implies that T is b-Dunfor-Pettis but not b-AM-compact. □

Theorem 3.4 *Let E be a Dedekind σ-complete Banach lattice. Then, the following assertions are equivalent.*

(1) *The norm of E is order continuous.*
(2) *Every b-AM-Dunford–Pettis operator $T : E \to E$ is b-AM-compact.*
(3) *Every positive b-AM-Dunford–Pettis operator $T : E \to E$ is b-AM-compact.*

Proof (1) $\Longrightarrow$ (2) Follows directly from Theorem 3.2.
(2) $\Longrightarrow$ (3) Obvious.
(3) $\Longrightarrow$ (1) Suppose that the norm of E is not order continuous. Since E is Dedekind σ-complete, it follows from Theorem 4.56 in [1] that there exist a sublattice $H \subset E$ and a lattice isomorphism ψ from H onto ℓ^∞. Let $\hat{\psi} : E \to \ell^\infty$ be a positive extension of ψ to all E (see Exercice 1 page 50 in [18]). By Lemma 3.3, there is a positive b-Dunfor-Pettis operator $S : \ell^\infty \to E$ which is not b-AM-compact. Consider the product $T = S\hat{\psi}$. Since $\hat{\psi}$ is b-order bounded and S is b-Dunfor-Pettis, then T is a b-AM-Dunford–Pettis operator. On the other hand if T is b-AM-compact, then so is $S\psi$. Since ψ is lattice isomorphism, it follows that S is b-AM-compact, which is a contradiction. This argument shows that T is not b-AM-compact.

□

4 Domination by *b*-AM-Dunford–Pettis Operators

Consider two operators $0 \leq S \leq T : E \to F$. The issue concerning finding conditions under which properties of T, related to the norm topology, will be inherited by S, is called the domination problem. As far as compact operators are concerned, Dodds and Fremlin tackled the domination problem in [11]. It was proven that if both $E^{'}$ and E have order continuous norms, every positive operator S on a Banach lattice dominated by a compact operator, is itself compact. The problem of domination in the class of weakly compact operators has been handled by Wickstead in [19, 20]. It was shown that if $E^{'}$ or F is order continuous and T is weakly compact, then so is S. In addition, Kalton and Saab confirmed in [17] that, if F has an order continuous norm, and T is Dunford–Pettis, then S is also Dunford–Pettis.

Otherwise, in terms of b-order boundedness, the authors in [3] argued that if T is b-weakly compact, then S is also b-weakly compact. As for the class of b-AM-compact operators, Cheng and Chen [9] reported that, if the norm of E is order continuous or $E^{'}$ is discrete, and T is b-AM-compact; then so is S. Within this framework, we are basically concerned with the domination problem for b-AM-Dunford–Pettis operators. For this reason, we need to introduce the following approximation result incorporated in [17] by Kalton and Saab. Let $L_r(E, F)$ denote the class of regular operators from E into F.

Theorem 4.1 *Let E and F be Banach lattices, each with a quasi-interior positive element. Let T be a positive operator $T : E \to F$ and let $A \subset E$, $B \subset F^{'}$ be solid bounded sets. Suppose that whenever $(a_n)_n$ and $(b_n)_n$ are sequences of disjoint positive elements in A and B respectively, then*

(i) *$T(a_n)$ converges weakly to 0.*
(ii) *$T^{'}(b_n)$ converges weakly star to 0.*
(iii) *(Ta_n, b_n) converges to 0.*

Suppose further that $R, S \in L_r(E, F)$ satisfy $|R| \le |S| \le T$ in $L_r(E, F^{''})$. Then, given $\epsilon > 0$, there exist $M_1, \ldots, M_k \in L_r(E)$ and $L_1, \ldots, L_k \in L_r(F)$ such that operator $R_\epsilon = \sum_{i=1}^{k} L_i S M_i$ satisfies

$$|(Ra - R_\epsilon a, b)| \le \epsilon, \qquad \forall a \in A,\ b \in B.$$

Let $R : E \to F$ be a positive operator between two Banach lattices dominated by a b-AM-Dunford–Pettis operator T. Is R then necessarily b-AM-Dunford–Pettis ? The answer is negative in general. The details are set afterwards.

Example 4.2 There exist two operators $0 \le R \le T : L_1[0, 1] \longrightarrow \ell^\infty$ such that T is b-AM-Dunford–Pettis but R is not b-AM-Dunford–Pettis.

Proof Let $(r_n)_n$ be the sequence of Rademacher functions on $[0, 1]$. That is, $r_n(t) = \text{Sign}\,(\sin(2^n \pi t))$ for each $t \in [0, 1]$. Let $0 \le R \le T : L_1[0, 1] \longrightarrow \ell^\infty$ be the positive operators defined in Example 3.1 of [1] by

$$Rf = \left(\int_0^1 f(t) r_1^+(t)dt, \int_0^1 f(t) r_2^+(t)dt, \ldots\right)$$

and

$$Tf = \left(\int_0^1 f(t)dt, \int_0^1 f(t)dt, \ldots\right).$$

Obviously, T is compact (has rank one) and hence b-AM-Dunford–Pettis. On the other hand, for each $n \in \mathbb{N}$, we record that $-1 \le r_n \le 1$. Hence, $(r_n)_n$ is a b-order bounded sequence in $L_1[0, 1]$. Since for each $n \in \mathbb{N}$, we have $r_n \xrightarrow{\sigma(L_1[0,1], (L_1[0,1])^{'})} 0$ (see [1] on page 345); then, investing the fact that $\|Rr_n\|_\infty \ge \int_0^1 r_n(t) r_n^+(t)dt = \frac{1}{2}$, entails that R is not b-AM-Dunford–Pettis. □

The following result offers a sufficent condition on the Banach lattice, under which the b-AM-Dunford–Pettis property of a positive operator T will be inherited by any positive operator smaller than T.

Theorem 4.3 *Let E and F be Banach lattices such that F has an order continuous norm, and let $T : E \to F$ be a positive b-AM-Dunford–Pettis operator. If $S : E \to F$ satisfies $0 \le S \le T$, then S itself is b-AM-Dunford–Pettis.*

Proof Let $(x_n)_n$ be a weakly null sequence of E satisfying $0 \le x_n \le x^{''}$, for all $n \in \mathbb{N}$ and for some $x^{''} \in E^{''}$. Let $x = \sum_{n=1}^{+\infty} \frac{|x_n|}{2^n}$, and consider E_x (resp F_{Tx}) the ideal generated by $x \in E$ (resp Tx in F). Clearly, S and T carry E_x into F_{Tx}. This means that we can assume, without loss of generality, that each E and F has a quasi-interior point. Consider A= Sol $(\{x_n;\ n\})$ and $B = B_{F'}$. Let $(a_n)_n$ and $(b_n)_n$ be two positive disjoint sequences in A and B, respectively. Based upon Proposition 3.10 in [15], $\lim_n \|Ta_n\| = 0$, and grounded on Corollary 2.4.3 in [18], $T^{'} b_n \xrightarrow{\sigma(E', E)} 0$. Let $\epsilon > 0$

be fixed. Referring to Theorem 4.1, there exist operators $M_1, \ldots, M_n \in L_r(E)$, and positive operators $L_1, \ldots, L_n \in L_r(F)$ such that

$$S_\epsilon = \sum_{i=1}^{k} L_i T M_i,$$

and

$$\mid (Sx_n - S_\epsilon x_n, b) \mid \leq \epsilon \quad \text{for every} \quad b \in B.$$

Thus,

$$\sup\{\mid (Sx_n - S_\epsilon x_n, b) \mid\} \leq \epsilon.$$

This implies that $\|(Sx_n - S_\epsilon x_n\| \leq \epsilon$. Using Lemma 2.9, we see that S_ϵ is b-AM-Dunford–Pettis, then it is easy to trace that $\|S_\epsilon x_n\| \to 0$. Consequently, $\|Sx_n\| \to 0$ and the proof holds. □

As a consequence, we get the following.

Corollary 4.4 *Let E be a Banach lattice, and consider operators $0 \leq R \leq T : E \to E$. If T is b-AM-Dunford–Pettis, then R^2 is b-AM-Dunford–Pettis.*

Proof Since T is b-AM-Dunford–Pettis, it follows from in [15, Proposition 3.10] that T is order weakly compact. According to Theorem I.2 in [12], there exist an order continuous Banach lattice G, a lattice homomorphism $\phi : E \longrightarrow G$ and operators $0 \leq R^G \leq T^G : G \longrightarrow E$, with $R = R^G\phi$ and $T = T^G\phi$. Note that

$$0 \leq \phi R \leq \phi T : E \to G.$$

Since G is order continuous and ϕT is b-AM-Dunford–Pettis, it follows from Theorem 4.3 that ϕR is b-AM-Dunford–Pettis and consequently $R^2 = R^G\phi R$ is b-AM-Dunford–Pettis. □

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Declarations

Conflict of Interest The authors declare no conflict of interest.

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