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A Potapov-Type Approach to a Particular Truncated Stieltjes Moment Problem in the Case of an Odd Number of Prescribed Matricial Moments

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Abstract

The paper gives a parametrization of the solution set of a matricial Stieltjes-type truncated power moment problem in the non-degenerate and degenerate cases. The problem will be reformulated as an interpolation problem for a distinguished class of holomorphic matrix-valued functions. An essential role plays a solution of the corresponding system of Potapov's fundamental matrix inequalities.

Keywords Matricial Stieltjes moment problem · Potapov's fundamental matix inequality

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Dedicated to the memory of V. E. Katsnelson

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1 Introduction

The topic of this paper was inspired by a lot of discussions with V. E. Katsnelson during his many stays in Leipzig. He was strongly interested in moment problems of Stieltjes type and related interpolation problems for special classes of holomorphic matrix-valued functions. This topic was already considered in the phd thesis of his student Yu. M. Dyukarev (see [8, 11, 12]). Studying these problems, V. E. Katsnelson drew Yu. M. Dyukarev's attention to V. P. Potapov's method of fundamental matrix inequalities. It is very remarkable that V. E. Katsnelson succeeded in extending V. P. Potapov's method to the treatment of continuous problems of analysis (see [24–26]).

In this paper, we use a general method to solve Potapov's fundamental matrix inequalities if the information block is degenerate. This method has its origin in the work of V. K. Dubovoy [5] on his treatment of the degenerate matricial Schur problem. Dubovoy's idea was taken up and modified by V. A. Bolotnikov [2, 3] to handle the degenerate matricial Stieltjes and Hamburger moment problems. In particular, V. A. Bolotnikov observed that Dubovoy's construction is essentially connected with the use of a special type of generalized inverses of complex matrices. In the last decade, the first three authors (together with several co-authors) have discussed various matricial moment problems of Stieltjes type (see [14–23]). Roughly speaking, the methods used there can be classified into two types, namely into Schur analysis methods and Potapov's method. As in [21] and [22], Potapov's method is the essential instrument in this paper as well.

Our approach is similar to [21]. The basic difference is now that the two fundamental matrix inequalities of Potapov type have different size. This requires new techniques to realize a coupling between them. This phenomenon was already observed in the paper [2] by V. A. Bolotnikov.

2 Preliminaries

In view of formulating the problems, we are going to consider, first we state some notations. Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}_0$, and $\mathbb{N}$ be the set of all complex numbers, the set of all real numbers, the set of all integers, the set of all non-negative integers, and the set of all positive integers, respectively. Further, for all $\rho, \tau \in \mathbb{R} \cup \{-\infty, \infty\}$, let $\mathbb{Z}_{\rho,\tau}$ be the set of all integers k for which $\rho \leq k \leq \tau$ holds true. Throughout this paper, let $p, q \in \mathbb{N}$. If $\mathcal{X}$ is a non-empty set, then $\mathcal{X}^{p \times q}$ represents the set of all $p \times q$ matrices each entry of which belongs to $\mathcal{X}$, and $\mathcal{X}^p$ is abbreviating $\mathcal{X}^{p \times 1}$. The notation $\mathbb{C}_H^{q \times q}$ is used to denote the set of all Hermitian complex $q \times q$ matrices. Moreover, we write $\mathbb{C}_{\geq}^{q \times q}$ and $\mathbb{C}_{>}^{q \times q}$ to designate the set of all non-negative Hermitian complex $q \times q$ matrices and the set of all positive Hermitian complex $q \times q$ matrices, respectively.

If $(\Omega, \mathfrak{A})$ is a measurable space, then each countably additive mapping defined on $\mathfrak{A}$ with values in $\mathbb{C}_{\geq}^{q \times q}$ is called a non-negative Hermitian $q \times q$ measure on $(\Omega, \mathfrak{A})$. Let $\mathcal{M}_{\geq}^{q}(\Omega, \mathfrak{A})$ be the set of all non-negative Hermitian $q \times q$ measures on $(\Omega, \mathfrak{A})$. If $\mu = [\mu_{jk}]_{j,k=1}^{q} \in \mathcal{M}_{\geq}^{q}(\Omega, \mathfrak{A})$, then we use $\mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{C})$ to designate the set of all Borel-measurable functions $f : \Omega \to \mathbb{C}$ for which $\int_{\Omega} |f| \, d\tilde{\mu}_{jk} < \infty$ holds true for every choice of j and k in $\mathbb{Z}_{1,q}$, where $\tilde{\mu}_{jk}$ is the variation of the complex measure μ_{jk} (see also Lemma A.8 for equivalent formulations). If $f \in \mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{C})$, then $\int_A f d\mu := \left[\int_A f d\mu_{jk}\right]_{j,k=1}^q$ for each $A \in \mathfrak{A}$, and we also write $\int_A f(w)\mu(dw)$ for this integral. Let $\mathfrak{B}_\mathbb{R}$ be the σ-algebra of all Borel subsets of $\mathbb{R}$. For all $\Omega \in \mathfrak{B}_\mathbb{R}\setminus\{\emptyset\}$, let $\mathfrak{B}_\Omega$ be the σ-algebra of all Borel subsets of Ω and let $\mathcal{M}_\geq^q(\Omega)$ be the set of all non-negative Hermitian $q \times q$ measures on $(\Omega, \mathfrak{B}_\Omega)$. For each $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $\mathcal{M}_{\geq,\kappa}^q(\Omega)$ be the set of all $\sigma \in \mathcal{M}_\geq^q(\Omega)$ such that, for all $j \in \mathbb{Z}_{0,\kappa}$, the function $f_j : \Omega \to \mathbb{C}$ defined by $f_j(t) := t^j$ belongs to $\mathcal{L}^1(\Omega, \mathfrak{B}_\Omega, \sigma; \mathbb{C})$. If $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and if σ belongs to $\mathcal{M}_{\geq,\kappa}^q(\Omega)$, then the integral $s_j^{(\sigma)} := \int_\Omega t^j \sigma(dt)$ is well defined for each $j \in \mathbb{Z}_{0,\kappa}$. Obviously, $\mathcal{M}_{\geq,\infty}^q(\Omega) \subseteq \mathcal{M}_{\geq,l}^q(\Omega) \subseteq \mathcal{M}_{\geq,k}^q(\Omega) \subseteq \mathcal{M}_{\geq,0}^q(\Omega) = \mathcal{M}_\geq^q(\Omega)$ for every choice of non-negative integers k and l with $k \leq l$. If Ω is a bounded set belonging to $\mathfrak{B}_\mathbb{R}\setminus\{\emptyset\}$, then $\mathcal{M}_{\geq,\infty}^q(\Omega) = \mathcal{M}_\geq^q(\Omega)$.

The following two types of matricial power moment problems are in the background of our considerations:

$\mathsf{M}[\Omega; (s_j)_{j=0}^m, \leq]$: Let $\Omega \in \mathfrak{B}_\mathbb{R}\setminus\{\emptyset\}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m$ be a sequence of complex $q \times q$ matrices. Parametrize the set $\mathcal{M}_\geq^q[\Omega; (s_j)_{j=0}^m, \leq]$ of all $\sigma \in \mathcal{M}_{\geq,m}^q(\Omega)$ for which the matrix $s_m - s_m^{(\sigma)}$ is non-negative Hermitian and, in case $m \geq 1$, for which $s_j^{(\sigma)} = s_j$ is additionally fulfilled for all $j \in \mathbb{Z}_{0,m-1}$.

$\mathsf{M}[\Omega; (s_j)_{j=0}^\kappa, =]$: Let $\Omega \in \mathfrak{B}_\mathbb{R}\setminus\{\emptyset\}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $q \times q$ matrices. Parametrize the set $\mathcal{M}_\geq^q[\Omega; (s_j)_{j=0}^\kappa, =]$ of all $\sigma \in \mathcal{M}_{\geq,\kappa}^q(\Omega)$ for which $s_j^{(\sigma)} = s_j$ is fulfilled for all $j \in \mathbb{Z}_{0,\kappa}$.

In [27, Chapter 1], M. G. Kreĭn was the first who studied problems of the form $\mathsf{M}[\Omega; (s_j)_{j=0}^m, \leq]$, i.e., truncated power moment problems with a condition that the highest prescribed moment is restricted via an inequality. This was done by transferring the method of Chebyshev systems to semi-infinite intervals. Examining scalar moment problems on bounded closed intervals opened up the access to scalar moment problems on the closed half-axis. This was explained in detail in the monograph [28, Chapter 5]. The considerations of this paper are mostly concentrated on a Stieltjes-type moment problem, i.e., we will consider the case $\Omega = [\alpha, \infty)$, where α is an arbitrarily given real number. The moment problem $\mathsf{M}[[\alpha, \infty); (s_j)_{j=0}^m, =]$, where m is an arbitrarily given non-negative integer, is discussed in [16] and [17], where a parametrization of the set $\mathcal{M}_\geq^q[[\alpha, \infty); (s_j)_{j=0}^m, =]$ is stated and where the approach is based on a Schur-type algorithm. Modifying the Schur-type algorithm used there, a parametrization of the solution set $\mathcal{M}_\geq^q[[\alpha, \infty); (s_j)_{j=0}^m, \leq]$ of Problem $\mathsf{M}[[\alpha, \infty); (s_j)_{j=0}^m, \leq]$, where m is an arbitrarily given non-negative integer, is given in [23]. In the case that m is an odd positive integer, an alternative description of the set $\mathcal{M}_\geq^q[[\alpha, \infty); (s_j)_{j=0}^m, \leq]$ can be found in [21]. The approach chosen there is based on solving the corresponding system of V. P. Potapov's fundamental matrix inequalities. We extend this strategy by obtaining a description of the set $\mathcal{M}_\geq^q[[\alpha, \infty); (s_j)_{j=0}^m, \leq]$, where m is an even non-negative integer.

For our further considerations, we introduce certain sets of sequences of complex $q \times q$ matrices, which are determined by properties of particular block Hankel matrices constructed by them. If $n \in \mathbb{N}_0$ and if $\left(s_j\right)_{j=0}^{2n}$ is a sequence of complex $q \times q$

matrices, then $(s_j)_{j=0}^{2n}$ is called $\mathbb{R}$-*non-negative definite* or (*Hankel non-negative definite*) if the block Hankel matrix

$$H_n := [s_{j+k}]_{j,k=0}^{n} \tag{2.1}$$

is non-negative Hermitian. For all $n \in \mathbb{N}_0$, we will write $\mathcal{H}_{q,2n}^{\geq}$ for the set of all sequences $(s_j)_{j=0}^{2n}$ of complex $q \times q$ matrices which are $\mathbb{R}$-non-negative definite. If $n \in \mathbb{N}$ and if $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq}$, then, for each $m \in \mathbb{Z}_{0,n}$, the sequence $(s_j)_{j=0}^{2m}$ obviously belongs to $\mathcal{H}_{q,2m}^{\geq}$. Thus, let $\mathcal{H}_{q,\infty}^{\geq}$ be the set of all sequences $(s_j)_{j=0}^{\infty}$ of complex $q \times q$ matrices such that, for all $n \in \mathbb{N}_0$, the sequence $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{H}_{q,2n}^{\geq}$. It is wellknown (see, e.g., [4, Theorem 3.2] or [10, Theorem 4.16]) that, for every choice of $n \in \mathbb{N}_0$ and each sequence $(s_j)_{j=0}^{2n}$ of complex $q \times q$ matrices, the set $\mathcal{M}_{\geq}^{q}[\mathbb{R}; (s_j)_{j=0}^{2n}, \leq]$ is non-empty if and only if $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{H}_{q,2n}^{\geq}$. For all $n \in \mathbb{N}_0$, let $\mathcal{H}_{q,2n}^{\geq,\mathrm{e}}$ be the set of all sequences $(s_j)_{j=0}^{2n}$ of complex $q \times q$ matrices for which there exist complex $q \times q$ matrices s_{2n+1} and s_{2n+2} such that $(s_j)_{j=0}^{2(n+1)}$ belongs to $\mathcal{H}_{q,2(n+1)}^{\geq}$. Furthermore, for all $n \in \mathbb{N}_0$, we will use $\mathcal{H}_{q,2n+1}^{\geq,\mathrm{e}}$ to denote the set of all sequences $(s_j)_{j=0}^{2n+1}$ of complex $q \times q$ matrices for which there exists a complex $q \times q$ matrix s_{2n+2} such that $(s_j)_{j=0}^{2(n+1)}$ belongs to $\mathcal{H}_{q,2(n+1)}^{\geq}$. For technical reasons, let $\mathcal{H}_{q,\infty}^{\geq,\mathrm{e}} := \mathcal{H}_{q,\infty}^{\geq}$. For each $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, the elements of the set $\mathcal{H}_{q,\kappa}^{\geq,e}$ are called $\mathbb{R}$-*non-negative definite extendable sequences* or (*Hankel non-negative definite extendable sequences*). If $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, then the meaning of the set $\mathcal{H}_{q,\kappa}^{\geq,e}$ consists in the fact that $\mathcal{M}_{\geq}^{q}[\mathbb{R}; (s_j)_{j=0}^{\kappa}, =]$ is non-empty if and only if the given sequence $(s_j)_{j=0}^{\kappa}$ of complex $q \times q$ matrices belongs to $\mathcal{H}_{q,\kappa}^{\geq,e}$ (see [3, Lemma 2.10], [4, Theorem 3.1], [10, Theorem 4.17], [13, Theorem 6.6]).

In order to describe a solvablity criterion for matricial Stieltjes moment problems, it is useful that one considers further classes of sequences of comples $q \times q$ matrices. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Then, for all $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$, we also introduce the block Hankel matrix $K_n := [s_{j+k+1}]_{j,k=0}^{n}$. If $\kappa \geq 1$, then let the sequence $(s_{\alpha\triangleright j})_{j=0}^{\kappa-1}$ be given by

$$s_{\alpha\triangleright j} := -\alpha s_j + s_{j+1} \qquad \text{for all } j \in \mathbb{Z}_{0,\kappa-1} \tag{2.2}$$

and we introduce the block Hankel matrices $H_{\alpha\triangleright n-1} := [s_{\alpha\triangleright j+k}]_{j,k=0}^{n-1}$ for all $n \in \mathbb{N}$ such that $2n - 1 \leq \kappa$. Note that $(s_{\alpha\triangleright j})_{j=0}^{\kappa-1}$ is called the sequence generated from $(s_j)_{j=0}^{\kappa}$ by (right) α-shifting. (An analogous left-sided version is discussed in [14, Definition 2.1].) Let $\mathcal{K}_{q,0,\alpha}^{\geq} := \mathcal{H}_{q,0}^{\geq}$, and, for all $n \in \mathbb{N}$, let $\mathcal{K}_{q,2n,\alpha}^{\geq}$ be the set of all sequences $(s_j)_{j=0}^{2n}$ of complex $q \times q$ matrices for which the block Hankel matrices H_n and $-\alpha H_{n-1} + K_{n-1}$ both are non-negative Hermitian, i.e., let $\mathcal{K}_{q,2n,\alpha}^{\geq} := \left\{ (s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq} \,\middle|\, (s_{\alpha\triangleright j})_{j=0}^{2(n-1)} \in \mathcal{H}_{q,2(n-1)}^{\geq} \right\}$. Furthermore, for all $n \in \mathbb{N}_0$, let $\mathcal{K}_{q,2n+1,\alpha}^{\geq}$ be the set of all sequences $(s_j)_{j=0}^{2n+1}$ of complex $q \times q$ matrices for which the block Hankel matrices H_n and $-\alpha H_n + K_n$ both are non-negative Hermitian, i.e., $\mathcal{K}_{q,2n+1,\alpha}^{\geq}$ is the set of all sequences $(s_j)_{j=0}^{2n+1}$ from $\mathbb{C}^{q\times q}$ such that $\{(s_j)_{j=0}^{2n}, (s_{\alpha\triangleright j})_{j=0}^{2n}\} \subseteq \mathcal{H}_{q,2n}^{\geq}$. Hence, the sets $\mathcal{K}_{q,2n,\alpha}^{\geq}$ and $\mathcal{K}_{q,2n+1,\alpha}^{\geq}$ are determined by two conditions. The condition $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq}$ ensures that a particular Hamburger moment problem associated with the sequence $(s_j)_{j=0}^{2n}$ is solvable (see [4, Theorem 3.2], [10, Theorem 4.16]). The second condition $(s_{\alpha\triangleright j})_{j=0}^{2(n-1)} \in \mathcal{H}_{q,2(n-1)}^{\geq}$ (respectively, $(s_{\alpha\triangleright j})_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq}$) controls that the original sequences $(s_j)_{j=0}^{2n}$ and $(s_j)_{j=0}^{2n+1}$ are well adapted to the interval $[\alpha, \infty)$. Using the sets introduced above, we recall a solvability criterion for the Problem $\mathsf{M}[[\alpha, \infty); (s_j)_{j=0}^{m}, \leq]$:

Theorem 2.1 ([9, Theorem 1.4]) *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m}$ be a sequence of complex $q \times q$ matrices. Then $\mathcal{M}_{\geq}^{q}[[\alpha, \infty); (s_j)_{j=0}^{m}, \leq] \neq \emptyset$ if and only if $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\geq}$.*

In order to formulate a necessary and sufficient condition for the solvability of Problem $\mathsf{M}[[\alpha, \infty); (s_j)_{j=0}^{\kappa}, =]$, further classes of sequences of complex matrices are of interest. Let $\alpha \in \mathbb{R}$ and let $m \in \mathbb{N}_0$. Then let $\mathcal{K}_{q,m,\alpha}^{\geq,\mathrm{e}}$ be the set of all sequences $(s_j)_{j=0}^{m}$ of complex $q \times q$ matrices for which there exists a complex $q \times q$ matrix s_{m+1} such that $(s_j)_{j=0}^{m+1}$ belongs to $\mathcal{K}_{q,m+1,\alpha}^{\geq}$. Obviously, $\mathcal{K}_{q,0,\alpha}^{\geq,\mathrm{e}} = \mathcal{K}_{q,0,\alpha}^{\geq}$. For all $n \in \mathbb{N}$, we have $\mathcal{K}_{q,2n,\alpha}^{\geq,\mathrm{e}} = \left\{(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq} \middle| (s_{\alpha\triangleright j})_{j=0}^{2n-1} \in \mathcal{H}_{q,2n-1}^{\geq,\mathrm{e}}\right\}$ and, for all $n \in \mathbb{N}_0$, furthermore $\mathcal{K}_{q,2n+1,\alpha}^{\geq,\mathrm{e}} = \left\{(s_j)_{j=0}^{2n+1} \in \mathcal{H}_{q,2n+1}^{\geq,\mathrm{e}} \middle| (s_{\alpha\triangleright j})_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq}\right\}$. If $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\geq}$ (resp. $\mathcal{K}_{q,m,\alpha}^{\geq,\mathrm{e}}$), then $(s_j)_{j=0}^{l} \in \mathcal{K}_{q,l,\alpha}^{\geq}$ (resp. $\mathcal{K}_{q,l,\alpha}^{\geq,\mathrm{e}}$) for all $l \in \mathbb{Z}_{0,m}$. Thus, let $\mathcal{K}_{q,\infty,\alpha}^{\geq}$ be the set of all sequences $(s_j)_{j=0}^{\infty}$ of complex $q \times q$ matrices such that $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\geq}$ holds true for all $m \in \mathbb{N}_0$. Furthermore, let $\mathcal{K}_{q,\infty,\alpha}^{\geq,\mathrm{e}} := \mathcal{K}_{q,\infty,\alpha}^{\geq}$. Obviously, $\mathcal{K}_{q,\kappa,\alpha}^{\geq,e} \subseteq \mathcal{K}_{q,\kappa,\alpha}^{\geq}$ for all $\kappa \in \mathbb{N}_0 \cup \{\infty\}$. If $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and if $\left(s_j\right)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq}$, then $\left(s_j\right)_{j=0}^{m} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$ for all $m \in \mathbb{Z}_{0,\kappa-1}$. If $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, then a sequence $(s_j)_{j=0}^{\kappa}$ of complex $q \times q$ matrices is called $[\alpha, \infty)$ *-Stieltjes non-negative definite* (resp. $[\alpha, \infty)$ *–Stieltjes non-negative definite extendable)* if it belongs to $\mathcal{K}_{q,\kappa,\alpha}^{\geq}$ (resp. $\mathcal{K}_{q,\kappa,\alpha}^{\geq,\mathrm{e}}$).

Remark 2.2 ([9, Remark 4.14]) Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\geq,e}$. For all $\kappa \in \mathbb{Z}_{m+1} \cup \{\infty\}$, then there is a sequence $(s_j)_{j=m+1}^{\kappa}$ of complex $q \times q$ matrices such that $(s_j)_{j=0}^{\kappa}$ belongs to $\mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$.

We now recall the announced solvability criterion for Problem $\mathsf{M}[[\alpha, \infty); (s_j)_{j=0}^{\kappa}, =]$:

Theorem 2.3 ([9, Theorem 1.3], [14, Theorem 1.6]) *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. Then $\mathcal{M}_{\geq}^{q}[[\alpha, \infty); (s_j)_{j=0}^{\kappa}, =] \neq \emptyset$ if and only if $(s_j)_{j=0}^{\kappa}$ belongs to $\mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$.*

It should be noted that the $q \times q$ matrix sequences, characterizing the solvability of the moment problems in Theorem 2.1 and Theorem 2.3, are investigated in detail in [9, 14, 15].

For a parametrization of the solution set $\mathcal{M}_{\geq}^{q}[[\alpha, \infty); (s_j)_{j=0}^{m}, \leq]$ of Problem $\mathsf{M}[[\alpha, \infty); (s_j)_{j=0}^{m}, \leq]$, it is essential that one can suppose extendable data without loss of generality:

Theorem 2.4 ([9, Theorem 5.2]) *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\geq}$. Then there is a unique sequence $(\tilde{s}_j)_{j=0}^{m} \in \mathcal{K}_{q,m,\alpha}^{\geq,e}$ such that $\mathcal{M}_{\geq}^{q}[[\alpha, \infty); (\tilde{s}_j)_{j=0}^{m}, \leq] = \mathcal{M}_{\geq}^{q}[[\alpha, \infty); (s_j)_{j=0}^{m}, \leq]$.*

For each $A \in \mathbb{C}^{p \times q}$, let $\mathcal{N}(A)$ be the null space of A, let $\mathcal{R}(A)$ be the column space of A, let rank A be the rank of A, and let A^{+} be the Moore-Penrose inverse of A. For each $A \in \mathbb{C}^{q \times q}$, let $\Re A := \frac{1}{2}(A + A^*)$ and $\Im A := \frac{1}{2i}(A - A^*)$ be real part of A and the imaginary part of A, respectively. Furthermore, for each $A \in \mathbb{C}^{p \times q}$, let $\|A\|_F$ be the Frobenius norm of A and let $\|A\|_S$ be the operator norm of A. A complex $p \times q$ matrix A is said to be contractive if $\|A\|_S \leq 1$. If $A \in \mathbb{C}^{q \times q}$, then $\det A$ denotes the determinant of A and $\operatorname{tr} A$ stands for the trace of A. For each $x \in \mathbb{C}^{q}$, we write $\|x\|_E$ for the Euclidean norm of x. For each $x, y \in \mathbb{C}^{q}$, by $\langle x, y \rangle_E$ we denote the (left-hand side) Euclidean inner product of x and y, i.e., we have $\langle x, y \rangle_E := y^* x$. If $\mathcal{M}$ is a non-empty subset of $\mathbb{C}^{q}$, then let $\mathcal{M}^{\perp}$ be the set of all vectors in $\mathbb{C}^{q}$ which are orthogonal to $\mathcal{M}$ (with respect to the Euclidean inner product $\langle ., . \rangle_E$), i.e., $\mathcal{M}^{\perp}$ is the set of all $x \in \mathbb{C}^{q}$ which fulfill $\langle x, y \rangle_E = 0$ for all $y \in \mathcal{M}$.

Now we are going to reformulate Problem $\mathsf{M}[[\alpha, \infty); (s_j)_{j=0}^{m}, \leq]$ as an equivalent interpolation problem for an appropriately chosen class of holomorphic matrix-valued functions. For this reason, we consider the class $\mathcal{S}_{q,[\alpha,\infty)}$ of all matrix-valued functions $S : \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q \times q}$ which are holomorphic in $\mathbb{C} \setminus [\alpha, \infty)$ and which fulfill $\Im[S(\Pi_+)] \subseteq \mathbb{C}_{\geq}^{q \times q}$ as well as $S((-\infty, \alpha)) \subseteq \mathbb{C}_{\geq}^{q \times q}$ where $\Pi_+ := \{z \in \mathbb{C} : \Im z \in (0, \infty)\}$. In [18, Theorems 3.1 and 3.6, Proposition 2.16, Section 4], several characterizations and, in particular, integral representations of functions belonging to $\mathcal{S}_{q,[\alpha,\infty)}$ are proved. Furthermore, several characterizations of the class $\mathcal{S}_{q,[\alpha,\infty)}$ are given in [18, Section 4]. A key role plays the subclass $\mathcal{S}_{0,q,[\alpha,\infty)}$ of all $F \in \mathcal{S}_{q,[\alpha,\infty)}$ which fulfill $\sup_{y \in (0,\infty)} y \|F(iy)\|_S < \infty$. The functions belonging to $\mathcal{S}_{0,q,[\alpha,\infty)}$ admit a special integral representation:

Theorem 2.5 *([18, Theorem 5.1]) Let $\alpha \in \mathbb{R}$. If $S \in \mathcal{S}_{0,q,[\alpha,\infty)}$, then there is a unique $\sigma \in \mathcal{M}_{\geq}^{q}([\alpha, \infty))$ such that*

$$S(z) = \int_{[\alpha,\infty)} \frac{1}{t - z} \sigma(dt) \qquad \text{for each } z \in \mathbb{C} \setminus [\alpha, \infty). \tag{2.3}$$

Conversely, if $\sigma \in \mathcal{M}_{\geq}^{q}([\alpha, \infty))$ is such that $S : \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q \times q}$ can be represented via (2.3), then S belongs to $\mathcal{S}_{0,q,[\alpha,\infty)}$.

If $F \in \mathcal{S}_{0,q,[\alpha,\infty)}$ is given, then the unique $\sigma \in \mathcal{M}_{\geq}^{q}([\alpha, \infty))$ which fulfills (2.3) is called the $[\alpha, \infty)$-*Stieltjes measure of* F. If $\sigma \in \mathcal{M}_{\geq}^{q}([\alpha, \infty))$ is given, then

$F : \mathbb{C} \setminus [\alpha, \infty) \to \mathbb{C}^{q \times q}$ defined by (2.3) is said to be the $[\alpha, \infty)$-*Stieltjes transform of* σ. In view of Theorem 2.5, the moment problems $\mathsf{M}[[\alpha, \infty); (s_j)_{j=0}^m, \leq]$ and $\mathsf{M}[[\alpha, \infty); (s_j)_{j=0}^\kappa, =]$ admit reformulations in the language of $[\alpha, \infty)$-Stieltjes transforms:

$\mathsf{S}[[\alpha, \infty); (s_j)_{j=0}^m, \leq]$: Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m$ be a sequence of complex $q \times q$ matrices. Parametrize the set $\mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^m, \leq]$ of all $F \in \mathcal{S}_{0,q,[\alpha,\infty)}$ the $[\alpha, \infty)$-Stieltjes measure of which belongs to $\mathcal{M}_{\geq}^q[[\alpha, \infty); (s_j)_{j=0}^m, \leq]$.

$\mathsf{S}[[\alpha, \infty); (s_j)_{j=0}^\kappa, =]$: Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $q \times q$ matrices. Parametrize the set $\mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^\kappa, =]$ of all $F \in \mathcal{S}_{0,q,[\alpha,\infty)}$ the $[\alpha, \infty)$-Stieltjes measure of which belongs to $\mathcal{M}_{\geq}^q[[\alpha, \infty); (s_j)_{j=0}^\kappa, =]$.

Following the classical line started by Stieltjes [30, 31], we study the reformulated problems in the sequel.

3 On the Equivalence of the Stieltjes Moment Problem to a System of Two Fundamental Inequalities of Potapov Type

In [22, Notation 4.3, Theorem 6.20] a special system of Potapov's fundamental inequalities is introduced in order to give conditions which yield a powerful tool for parametrizing all solutions of the moment problem $\mathsf{S}[[\alpha, \infty); (s_j)_{j=0}^m, \leq]$. Since that case of an odd positive integer m is worked out completely in [21], we consider here only the situation that $m = 2n$ with some non-negative integer n.

We introduce some further notation. First, we observe that, throughout the following, we consider an arbitrarily given real number α and an arbitrarily given $\kappa \in \mathbb{N}_0 \cup \{\infty\}$. We will write I_q to denote the identity matrix belonging to $\mathbb{C}^{q \times q}$, whereas $0_{p \times q}$ is the null matrix belonging to $\mathbb{C}^{p \times q}$. If the size of an identity matrix or a null matrix is obvious, then we also omit the indices.

If $n \in \mathbb{N}$, if $(p_j)_{j=1}^n$ is a sequence of positive integers, and if $x_j \in \mathbb{C}^{p_j \times q}$ for each $j \in \mathbb{Z}_{1,n}$, then let $\operatorname{col}(x_j)_{j=1}^n := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. If $n \in \mathbb{N}$, if $(q_k)_{k=1}^n$ is a sequence of positive integers, and if $y_k \in \mathbb{C}^{p \times q_k}$ for each $k \in \mathbb{Z}_{1,n}$, then let $\operatorname{row}(y_k)_{k=1}^n := [y_1, y_2, \ldots, y_n]$. Given $n \in \mathbb{N}$ and complex matrices $A_1, A_2, \ldots, A_n$, we use $\operatorname{diag}(A_j)_{j=1}^n$ or $\operatorname{diag}(A_1, A_2, \ldots, A_n)$ to denote the corresponding block diagonal matrix with diagonal blocks $A_1, A_2, \ldots, A_n$. For each $n \in \mathbb{N}$ and each $A \in \mathbb{C}^{p \times q}$, we will write $I_n \otimes A$ for $\operatorname{diag}(A)_{j=1}^n$ as well.

To introduce further notation, let a sequence $(s_j)_{j=0}^\kappa$ of complex $p \times q$ matrices be given. Due to technical reasons, let $s_{-1} := 0_{p \times q}$. For every choice of $l \in \mathbb{Z}_{-1,\kappa}$ and $m \in \mathbb{N}_0$ with $l \leq m \leq \kappa$, let $y_{l,m} := \operatorname{col}(s_j)_{j=l}^m$ and $z_{l,m} := \operatorname{row}(s_j)_{j=l}^m$.

For all $n \in \mathbb{N}_0$ with $n \leq \kappa + 1$, let $u_n := \operatorname{col}(-s_{j-1})_{j=0}^n$ and $w_n := \operatorname{row}(-s_{j-1})_{j=0}^n$. Obviously, we have $u_0 = 0_{p \times q}$, $w_0 = 0_{p \times q}$, and, for all $n \in \mathbb{N}_0$ with

$n \leq \kappa + 1$, moreover $u_n = -y_{-1,n-1}$ and $w_n = -z_{-1,n-1}$. We set $\mathfrak{u}_0 := 0_{p\times q}$, $\mathfrak{w}_0 := 0_{p\times q}$, and, for each $n \in \mathbb{N}_0$ with $2n \leq \kappa$, furthermore $\mathfrak{u}_n := \begin{bmatrix} -y_{n+1,2n} \\ 0_{p\times q} \end{bmatrix}$ and $\mathfrak{w}_n := \left[-z_{n+1,2n}, 0_{p\times q}\right]$. We use $\delta_{j,k}$ to denote the Kronecker delta, i.e., $\delta_{j,k}:=1$ if $j = k$ and $\delta_{j,k}:=0$ if $j \neq k$. For each $n \in \mathbb{N}_0$, let $T_{q,n} := \left[\delta_{j,k+1} I_q\right]_{j,k=0}^{n}$, $v_{q,n} := \operatorname{col}(\delta_{j,0} I_q)_{j=0}^{n}$, and $\mathfrak{v}_{q,n} := \operatorname{col}(\delta_{n-j,0} I_q)_{j=0}^{n}$. We easily see that $T_{q,n}^* = (\delta_{j+1,k} I_q)_{j,k=0}^{n}$ is valid for each $n \in \mathbb{N}_0$. If $n \geq 1$, we then set $V_{q,n} := (\delta_{j,k} I_q)_{\substack{j=0,\dots,n \\ k=0,\dots,n-1}}$ and $\mathfrak{V}_{q,n} := (\delta_{j,k+1} I_q)_{\substack{j=0,\dots,n \\ k=0,\dots,n-1}}$.

Remark 3.1 For each $n \in \mathbb{N}$, it is readily checked that $V_{q,n}^* v_{q,n} = v_{q,n-1}$, $T_{q,n} V_{q,n} = \mathfrak{V}_{q,n}$, $V_{q,n} V_{q,n}^* T_{q,n}^* = T_{q,n}^*$, $V_{q,n}^* V_{q,n} = I_{nq}$, $T_{q,n}^* V_{q,n} = V_{q,n} T_{q,n-1}^*$, and $T_{q,n}^* \mathfrak{V}_{q,n} - \mathfrak{V}_{q,n} T_{q,n-1}^* = v_{q,n} v_{q,n-1}^*$ hold true.

We will use the following well-known statement concerning the resolvent of the block shift operator $T_{q,n}$:

Remark 3.2 For each $n \in \mathbb{N}_0$ and each $z \in \mathbb{C}$, we have $\det(I_{(n+1)q} - zT_{q,n}) = 1 \neq 0$ and $\det(I_{(n+1)q} - zT_{q,n}^*) = 1 \neq 0$. Hence, for each $n \in \mathbb{N}_0$, the matrix-valued functions $R_{T_{q,n}} : \mathbb{C} \to \mathbb{C}^{(n+1)q\times(n+1)q}$ and $R_{T_{q,n}^*} : \mathbb{C} \to \mathbb{C}^{(n+1)q\times(n+1)q}$ given by

$$R_{T_{q,n}}(z) := (I_{(n+1)q} - zT_{q,n})^{-1} \quad \text{and} \quad R_{T_{q,n}^*}(z) := (I_{(n+1)q} - zT_{q,n}^*)^{-1} \tag{3.1}$$

are well-defined matrix polynomials of degree n, which, for each $z \in \mathbb{C}$, can be represented via $R_{T_{q,n}}(z) = \sum_{j=0}^{n} z^j T_{q,n}^j$, and $R_{T_{q,n}^*}(z) = \sum_{j=0}^{n} z^j (T_{q,n}^*)^j$, respectively. In particular, $R_{T_{q,n}^*}(z) = [R_{T_{q,n}}(\overline{z})]^*$ and $v_{q,n}^* \left[R_{T_{q,n}}(z)\right]^{-1} = v_{q,n}^*$ for all $z \in \mathbb{C}$. For each $n \in \mathbb{N}_0$, let the matrix-valued functions $E_{q,n} : \mathbb{C} \to \mathbb{C}^{(n+1)q\times q}$ and $\widetilde{E}_{q,n} : \mathbb{C} \to \mathbb{C}^{(n+1)q\times q}$ be defined by $E_{q,n}(z) := \operatorname{col}(z^j I_q)_{j=0}^{n}$ and $\widetilde{E}_{q,n}(z) := \operatorname{col}(z^{n-j} I_q)_{j=0}^{n}$. Obviously, $R_{T_{q,n}}(z) v_{q,n} = E_{q,n}(z)$ and $v_{q,n}^* R_{T_{q,n}^*}(z) = E_{q,n}^*(\overline{z})$ for each $n \in \mathbb{N}_0$ and each $z \in \mathbb{C}$.

Notation 3.3 Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. Further, let $S : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{q\times q}$ be a matrix-valued function. For each $n \in \mathbb{N}_0$ such that $2n \leq \kappa$, then let $P_{2n}^{[S]} : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{(n+2)\times(n+2)}$ be defined by

$$P_{2n}^{[S]}(z) := \begin{bmatrix} H_n & b_{2n}^{[S]}(z) \\ \left[b_{2n}^{[S]}(z)\right]^* & \frac{S(z)-S^*(z)}{z-\overline{z}} \end{bmatrix}, \tag{3.2}$$

where $b_{2n}^{[S]} : \mathbb{C}\setminus[\alpha,\infty) \to \mathbb{C}^{(n+1)q\times q}$ is given by $b_{2n}^{[S]}(z) := R_{T_{q,n}}(z)[v_{q,n} S(z) - u_n]$. Furthermore, let $P_{-1}^{[S]} : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{q\times q}$ be defined by

$$P_{-1}^{[S]}(z) := \left[\frac{(z-\alpha)S(z) - [(z-\alpha)S(z)]^*}{z-\overline{z}}\right] \tag{3.3}$$

and, if $\kappa \geq 1$, then let $P^{[S]}_{2n-1} : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{(n+1)q\times(n+1)q}$ be given by

$$P^{[S]}_{2n-1}(z) := \begin{bmatrix} H_{\alpha\rhd n-1} & b^{[S]}_{2n-1}(z) \\ \left[b^{[S]}_{2n-1}(z)\right]^* & P^{[S]}_{-1}(z) \end{bmatrix} \tag{3.4}$$

for each $n \in \mathbb{N}$ such that $2n-1 \leq \kappa$, where $b^{[S]}_{2n-1} : \mathbb{C}\setminus[\alpha,\infty) \to \mathbb{C}^{nq\times q}$ is defined by $b^{[S]}_{2n-1}(z) := R_{T_{q,n-1}}(z)\left[v_{q,n-1}[(z-\alpha)S(z)] - (-\alpha u_{n-1} - y_{0,n-1})\right]$.

With respect to the Stieltjes moment problem $\mathsf{M}[[\alpha,\infty); (s_j)_{j=0}^m, \leq]$, the functions (3.2) and (3.4) are called the Potapov fundamental matrix-valued functions connected to the Stieltjes moment problem (generated by S).

As usual, if $\mathcal{G}$ is a non-empty open subset of $\mathbb{C}$, then we will call a subset $\mathcal{D}$ of $\mathcal{G}$ a discrete subset of $\mathcal{G}$ if $\mathcal{D}$ does not have an accumulation point in $\mathcal{G}$. If f is a meromorphic function defined on a non-empty open subset of the complex plane, then we use $\mathbb{H}_f$ to denote the set of all points w at which f is holomorphic, the notation $\mathbb{P}_f$ to indicate the set of all poles of f, and we set $\mathcal{N}_f := \{w \in \mathbb{H}_f : f(w) = 0\}$.

Now we recall the characterization of the solution set of the matricial truncated Stieltjes moment problem by the so-called Potapov's Fundamental Matrix Inequalities.

Theorem 3.4 ([22, Theorem 6.20]) *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $q \times q$ matrices. Let $\mathcal{D}$ be a discrete subset of Π_+ and let $S : \mathbb{C}\setminus[\alpha,\infty) \to \mathbb{C}^{q\times q}$ be a holomorphic matrix-valued function. Then:*

(a) *Let $n \in \mathbb{N}_0$ be such that $2n \leq \kappa$. Then S belongs to $\mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^{2n}, \leq]$ if and only if*

$$P^{[S]}_{2n-1}(z) \in \mathbb{C}^{(n+1)q\times(n+1)q}_{\geq} \quad \text{and} \quad P^{[S]}_{2n}(z) \in \mathbb{C}^{(n+2)q\times(n+2)q}_{\geq} \tag{3.5}$$

are fulfilled for all $z \in \Pi_+ \setminus \mathcal{D}$.

(b) *Suppose $\kappa \geq 1$ and let $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$. Then S belongs to $\mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^{2n+1}, \leq]$ if and only if $\left\{P^{[S]}_{2n}(z), P^{[S]}_{2n+1}(z)\right\} \subseteq \mathbb{C}^{(n+2)q\times(n+2)q}_{\geq}$ holds true for all $z \in \Pi_+ \setminus \mathcal{D}$.*

Theorem 3.4 already indicates that the block Hankel matrices requiered for the moment problem of Stieltjes type we are considering here do not have the same number of rows. This also applies to the associated generalized inverses, which we will study below. To enable a coupling of these matrices, we will use special non-square matrices. This technical procedure, we will present in the following sections was also used by V. A. Bolotnikov in [2, Chapter 6] in the case $\alpha = 0$. Observe that the meaning of each of the two Fundamental Matrix Inequalities stated in Theorem 3.4 can be interpreted in the context of matricial Hamburger moment problems (see [21, Remarks 4.4 and 4.5]).

Corollary 3.5 *Let $n \in \mathbb{N}$, let $(s_j)_{j=0}^{2n} \in \mathcal{K}_{q,2n,\alpha}^{\geq,e}$, and let $S : \mathbb{C}\backslash[\alpha,\infty) \to \mathbb{C}^{q\times q}$ be holomorphic. Let $\Sigma_{2n}^{[S]}, \Sigma_{2n-1}^{[S]} : \mathbb{C}\backslash\mathbb{R} \to \mathbb{C}^{q\times q}$ be defined by*

$$\Sigma_{2n}^{[S]}(z) := \frac{S(z) - [S(z)]^*}{z - \overline{z}} - \left[b_{2n}^{[S]}(z)\right]^* H_n^+ \left[b_{2n}^{[S]}(z)\right], \tag{3.6}$$

and

$$\Sigma_{2n-1}^{[S]}(z) := \frac{(z-\alpha)S(z) - [(z-\alpha)S(z)]^*}{z - \overline{z}} - \left[b_{2n-1}^{[S]}(z)\right]^* H_{\alpha\triangleright n-1}^+ \left[b_{2n-1}^{[S]}(z)\right], \tag{3.7}$$

respectively. Furthermore, let $\mathcal{D}$ be a discrete subset Π_+. Then S belongs to $\mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^{2n}, \leq]$ if and only if the four conditions $(I - H_n^+ H_n) b_{2n}^{[S]}(z) = 0$ and $\Sigma_{2n}^{[S]}(z) \in \mathbb{C}_{\geq}^{q\times q}$ as well as $(I - H_{\alpha\triangleright n-1}^+ H_{\alpha\triangleright n-1}) b_{2n-1}^{[S]}(z) = 0$ and $\Sigma_{2n-1}^{[S]}(z) \in \mathbb{C}_{\geq}^{q\times q}$ are fulfilled for all $z \in \Pi_+\backslash\mathcal{D}$.

Proof Apply Theorem 3.4 (a), the definition of the set $\mathcal{K}_{q,2n,\alpha}^{\geq,e}$, and a well-known characterization of non-negative Hermitian block matrices (see, e.g., [6, Lemma 1.1.9] and [21, Lemmata 4.6 and 4.7]). □

We now state some more or less known identities.

Remark 3.6 Let $n \in \mathbb{N}$ and let $z \in \mathbb{C}$. Then $V_{q,n}^* R_{T_{q,n}}(z) = R_{T_{q,n-1}}(z) V_{q,n}^*$ and $R_{T_{q,n}^*}(z) V_{q,n} = V_{q,n} R_{T_{q,n-1}^*}(z)$. In view of Remarks 3.2 and 3.1, $\left[R_{T_{q,n}}(z)\right]^{-*} V_{q,n} = V_{q,n}\left[R_{T_{q,n-1}}(z)\right]^{-*}$, and $V_{q,n}^* T_{q,n} R_{T_{q,n}}(z) V_{q,n} = T_{q,n-1} V_{q,n}^* R_{T_{q,n}}(z) V_{q,n} = T_{q,n-1} R_{T_{q,n-1}}(z) V_{q,n}^* V_{q,n} = T_{q,n-1} R_{T_{q,n-1}}(z)$.

Remark 3.7 (*see, e.g.* [21, Remark 17.10]) Let $n \in \mathbb{N}_0$. For all $w, z \in \mathbb{C}$, then one can easily see that $(z-w) T_{q,n}^* R_{T_{q,n}^*}(z) = \left[R_{T_{q,n}^*}(z)\right]^{-1} R_{T_{q,n}^*}(w) - I$, $(z - w) T_{q,n} R_{T_{q,n}}(z) = R_{T_{q,n}}(z)\left[R_{T_{q,n}}(w)\right]^{-1} - I$, and, for every choice of $l \in \mathbb{N}_0$ and $w, z \in \mathbb{C}$, moreover $T_{q,n}^l R_{T_{q,n}}(z) = R_{T_{q,n}}(z) T_{q,n}^l$, and $R_{T_{q,n}^*}(z)\left(T_{q,n}^*\right)^l R_{T_{q,n}^*}(w) = R_{T_{q,n}^*}(w)\left(T_{q,n}^*\right)^l R_{T_{q,n}^*}(z)$.

Remark 3.8 Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Then $H_n = \operatorname{col}(z_{j,n+j})_{j=0}^n$ and $H_n = \operatorname{row}(y_{k,n+k})_{k=0}^n$ for all $n \in \mathbb{N}_0$ such that $2n \leq \kappa$. If $\kappa \geq 2$, then $H_n = \begin{bmatrix} H_{n-1} & y_{n,2n-1} \\ z_{n,2n-1} & s_{2n} \end{bmatrix}$ and $H_n = \begin{bmatrix} z_{0,n-1} & s_{2n} \\ K_{n-1} & y_{n+1,2n} \end{bmatrix}$ for all $n \in \mathbb{N}$ with $2n \leq \kappa$.

Remark 3.9 Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. In view of Remark 3.8, then the following more or less well-known equalities hold true:

(a) $H_n v_{q,n} = y_{0,n}$, $v_{p,n}^* H_n = z_{0,n}$, and $-T_{q,n} H_n v_{q,n} = u_n$ as well as $[R_{T_{p,n}}(\alpha)]^{-1} H_n v_{q,n} = \alpha u_n + y_{0,n}$ and $v_{p,n}^* H_n [R_{T_{q,n}}(\alpha)]^{-1} = \alpha w_n + z_{0,n}$ for all $n \in \mathbb{N}_0$ such that $2n \leq \kappa$.

(b) If $\kappa \geq 1$, then $v_{p,n} v_{p,n}^* = [R_{T_{p,n}}(\alpha)]^{-1} H_n - T_{p,n} H_{\alpha \triangleright n}$ for all $n \in \mathbb{N}_0$ such that $2n + 1 \leq \kappa$. If $\kappa \geq 2$, then $z_{0,n} V_{q,n} = z_{0,n-1}$, $v_{q,n}^* H_n V_{q,n} = z_{0,n-1}$, $V_{q,n}^* y_{0,n} = y_{0,n-1}$, $V_{q,n}^* H_n v_{q,n} = y_{0,n-1}$, $T_{q,n} H_n V_{q,n} = \mathfrak{V}_{q,n} H_{n-1}$ as well as $v_{p,n} v_{p,n}^* H_n V_{q,n} = H_n V_{q,n} - \mathfrak{V}_{p,n} K_{n-1}$ and $v_{p,n} v_{p,n}^* H_n V_{q,n} = [R_{T_{p,n}}(\alpha)]^{-1} H_n V_{q,n} - \mathfrak{V}_{p,n} H_{\alpha \triangleright n-1}$ hold true for all $n \in \mathbb{N}$ such that $2n \leq \kappa$. Moreover, if $\kappa \geq 3$, then $T_{q,n} K_n V_{q,n} = \mathfrak{V}_{q,n} K_{n-1}$ for all $n \in \mathbb{N}$ fulfilling $2n + 1 \leq \kappa$.

Certain Schur complements play an essential role for our following considerations if a sequence $(s_j)_{j=0}^{\kappa}$ of complex $p \times q$ matrices is given: Let $L_0 := s_0$ and, for each $n \in \mathbb{N}$ such that $2n \leq \kappa$, moreover $L_n := s_{2n} - z_{n,2n-1} H_{n-1}^{+} y_{n,2n-1}$. For every choice of integers m and n such that $0 \leq m \leq n \leq \kappa - 1$, let $y_{\alpha \triangleright m,n} := \operatorname{col}(s_{\alpha \triangleright m+j})_{j=0}^{n-m}$ and $z_{\alpha \triangleright m,n} := \operatorname{row}(s_{\alpha \triangleright m+k})_{k=0}^{n-m}$. Let $L_{\alpha \triangleright 0} := s_{\alpha \triangleright 0}$ and, for each $n \in \mathbb{N}_0$ such that $2n+1 \leq \kappa$, furthermore $L_{\alpha \triangleright n} := s_{\alpha \triangleright 2n} - z_{\alpha \triangleright n,2n-1} H_{\alpha \triangleright n-1}^{+} y_{\alpha \triangleright n,2n-1}$.

If $\mathcal{U}$ and $\mathcal{W}$ are subspaces of $\mathbb{C}^q$, then we write $\mathcal{U} \dot{+} \mathcal{W}$ if the Minkowski sum $\mathcal{U} + \mathcal{W}$ of $\mathcal{U}$ and $\mathcal{W}$ is a direct one, i.e., if $\mathcal{U} \cap \mathcal{W} = \{0\}$ holds true. In [5], V. K. Dubovoj studied particular invariant subspaces to discuss the matricial Schur problem. Keeping this in mind, we call a subspace $\mathcal{D}$ of $\mathbb{C}^p$ a *Dubovoj subspace corresponding to a given pair* $[H, T]$ of complex $p \times p$ matrices if both $T^*(\mathcal{D}) \subseteq \mathcal{D}$ and $\mathcal{N}(H) \dot{+} \mathcal{D} = \mathbb{C}^p$ are fulfilled. We are going to consider Dubovoj subspaces in connection to non-negative Hermitian block Hankel matrices. If $n \in \mathbb{N}_0$ and if $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq,e}$, then the existence of a Dubovoj subspace corresponding to $[H_n, T_{q,n}]$ was proved in [3, Lemma 3.2], [7], and [32, Satz 1.24]. If the sequence $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{H}_{q,2n}^{\geq} \setminus \mathcal{H}_{q,2n}^{\geq,e}$ the existence of such a subspace is not ensured. For instance, in [32, Abschnitt 1.4] it is observed that $(s_j)_{j=0}^{2}$ given by $s_0 := 0$, $s_1 := 0$, and $s_2 := 1$ describes a sequence belonging to $\mathcal{H}_{1,2}^{\geq}$ for which there does not exist a Dubovoj subspace corresponding to $[H_1, T_{1,1}]$.

Notation 3.10 Let $\kappa \geq 1$ and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. For each $n \in \mathbb{N}_0$ with $2n \leq \kappa$, let $\mathcal{D}_n := \mathcal{R}(\operatorname{diag}(L_0, L_1, \ldots, L_n))$. Furthermore, if $\kappa \geq 1$, then, for every choice of $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$, let $\mathcal{D}_{\alpha \triangleright n} := \mathcal{R}(\operatorname{diag}(L_{\alpha \triangleright 0}, L_{\alpha \triangleright 1}, \ldots, L_{\alpha \triangleright n}))$.

Remark 3.11 ([21, Remark 6.6]) Let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$. For each $n \in \mathbb{N}_0$ such that $2n \leq \kappa$, the subspace $\mathcal{D}_n$ of $\mathbb{C}^{(n+1)q}$ is a Dubovoj subspace corresponding to $[H_n, T_{q,n}]$, the so-called *canonical Dubovoj subspace corresponding to* $[H_n, T_{q,n}]$. Furthermore, in case $\kappa \geq 1$, then for each $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$, the subspace $\mathcal{D}_{\alpha \triangleright n}$ of $\mathbb{C}^{(n+1)q}$ is a Dubovoj subspace corresponding to $[H_{\alpha \triangleright n}, T_{q,n}]$, the so-called *canonical Dubovoj subspace corresponding to* $[H_{\alpha \triangleright n}, T_{q,n}]$.

Observe that already the assumption $(s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q,\kappa}^{\geq,e}$ implies that, for each $n \in \mathbb{N}_0$ such that $2n \leq \kappa$, the subspace $\mathcal{D}_n$ is a Dubovoj subspace corresponding to $[H_n, T_{q,n}]$ (see [21, Remark 5.8 and Proposition 6.4]).

We now turn our attention to particular generalized inverses of a complex matrix.

Remark 3.12 (*see, e.g.,* [1, Chapter 2, Theorem 12 (c)]) If $A \in \mathbb{C}^{p \times q}$ and if $\mathcal{U}$ and $\mathcal{V}$ are subspaces of $\mathbb{C}^q$ and $\mathbb{C}^p$, respectively, such that $\mathcal{N}(A) \dot{+} \mathcal{U} = \mathbb{C}^q$ and $\mathcal{R}(A) \dot{+} \mathcal{V} = \mathbb{C}^p$ are fulfilled, then there is a unique $X \in \mathbb{C}^{q \times p}$ such that the four equations

$$AXA = A, \qquad XAX = X, \qquad \mathcal{R}(X) = \mathcal{U}, \qquad \text{and} \qquad \mathcal{N}(X) = \mathcal{V}$$

hold true, and we will use $A^{(1,2)}_{\mathcal{U},\mathcal{V}}$ to denote this matrix X. In particular, if A is a Hermitian complex $q \times q$-matrix and if $\mathcal{U}$ is a subspace of $\mathbb{C}^q$ with $\mathcal{N}(A) \dotplus \mathcal{U} = \mathbb{C}^q$, then one can easily check that $\mathcal{R}(A) \dotplus \mathcal{U}^\perp = \mathbb{C}^q$ holds true and we will also write $A^-_{\mathcal{U}}$ for $A^{(1,2)}_{\mathcal{U},\mathcal{U}^\perp}$.

Observe that, in the case of an arbitrary given non-negative Hermitian matrix A, its general inverse according to Remark 3.12 can be expressed by the construction of the generalized inverse applied by V. A. Bolotnikov [2, 3]. This connection is stated in Lemma A.6 below.

Notation 3.13 Let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}^{\geq,e}_{q,\kappa,\alpha}$. For each $n \in \mathbb{N}_0$ such that $2n \leq \kappa$, let $H_n^- := (H_n)^-_{\mathcal{D}_n}$, where $\mathcal{D}_n$ is the canonical Dubovoj subspace corresponding to $[H_n, T_{q,n}]$. If $\kappa \geq 1$ and if $\alpha \in \mathbb{R}$, then, for each $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$, let $H^-_{\alpha\triangleright n} := (H_{\alpha\triangleright n})^-_{\mathcal{D}_{\alpha\triangleright n}}$, where $\mathcal{D}_{\alpha\triangleright n}$ is the canonical Dubovoj subspace corresponding to $[H_{\alpha\triangleright n}, T_{q,n}]$.

Lemma 3.14 *Suppose $\kappa \geq 2$. Let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}^{\geq,e}_{q,\kappa,\alpha}$ and let $n \in \mathbb{N}_0$ be such that $2n \leq \kappa$. Then*

$$H_n^- H_n V_{q,n} H^-_{\alpha\triangleright n-1} = V_{q,n} H^-_{\alpha\triangleright n-1} \quad \text{and} \quad H^-_{\alpha\triangleright n-1} V^*_{q,n} = H^-_{\alpha\triangleright n-1} V^*_{q,n} H_n H_n^-, \tag{3.8}$$

$$H_n^- R_{T_{q,n}}(\alpha) \left(v_{q,n} v^*_{q,n} H_n V_{q,n} H^-_{\alpha\triangleright n-1} V^*_{q,n} + T_{q,n} \right) = V_{q,n} H^-_{\alpha\triangleright n-1} V^*_{q,n} \tag{3.9}$$

as well as

$$V_{q,n} H^-_{\alpha\triangleright n-1} V^*_{q,n} \left(I_{(n+1)q} - H_n v_{q,n} v^*_{q,n} R_{T^*_{q,n}}(\alpha) H_n^- \right) = T^*_{q,n} R_{T^*_{q,n}}(\alpha) H_n^-. \tag{3.10}$$

Moreover,

$$H_n^- T^l_{q,n} V_{q,n} (I_{nq} - H_{\alpha\triangleright n-1} H^-_{\alpha\triangleright n-1}) = 0 \tag{3.11}$$

for all $l \in \mathbb{N}$ and

$$H_n^- R_{T_{q,n}}(\zeta) T^l_{q,n} V_{q,n} (I_{nq} - H_{\alpha\triangleright n-1} H^-_{\alpha\triangleright n-1}) = 0 \tag{3.12}$$

as well as

$$H_n^- R_{T_{q,n}}(\zeta) T^l_{q,n} \left(I_{(n+1)q} - V_{q,n} H_{\alpha\triangleright n-1} H^-_{\alpha\triangleright n-1} V^*_{q,n} \right) = 0 \tag{3.13}$$

for every choice of $l \in \mathbb{N}$ and $\zeta \in \mathbb{C}$.

Proof Because of $(s_j)_{j=0}^{\kappa} \in \mathcal{K}^{\geq,e}_{q,\kappa,\alpha}$, the matrices H_n and $H^-_{\alpha\triangleright n-1}$ are Hermitian. From Notation 3.10 and [21, Proposition 6.7] we know that $\mathcal{D}_n$ is a subspace of $\mathbb{C}^{(n+1)q}$, that $\mathcal{D}_{\alpha\triangleright n-1}$ is a subspace of $\mathbb{C}^{nq}$, and that $T^*_{q,n}(\mathcal{D}_n) \subseteq \mathcal{D}_{\alpha\triangleright n} \subseteq \mathcal{D}_n$ as well as

$$\mathcal{N}(H_n) \dotplus \mathcal{D}_n = \mathbb{C}^{(n+1)q}, \qquad \mathcal{N}(H_{\alpha\triangleright n}) \dotplus \mathcal{D}_{\alpha\triangleright n} = \mathbb{C}^{(n+1)q},$$

$$\mathfrak{V}_{q,n}^*(\mathcal{D}_n) \subseteq \mathcal{D}_{\alpha \triangleright n-1}, \qquad V_{q,n}(\mathcal{D}_{\alpha \triangleright n-1}) \subseteq \mathcal{D}_n, \tag{3.14}$$

$$\mathcal{N}(H_n) \dotplus \mathcal{D}_n = \mathbb{C}^{(n+1)q}, \qquad \text{and} \qquad \mathcal{N}(H_{\alpha \triangleright n-1}) \dotplus \mathcal{D}_{\alpha \triangleright n-1} = \mathbb{C}^{nq} \tag{3.15}$$

hold true. Consequently, [21, Lemma 17.3] and Notation 3.13 provide $\mathcal{R}(H_{\alpha \triangleright n-1}^-) = \mathcal{D}_{\alpha \triangleright n-1}$. Since [21, Lemma 17.6] shows that $\mathcal{N}(I_{(n+1)q} - H_n^- H_n) = \mathcal{D}_n$, from the second inclusion in (3.14) we then infer $V_{q,n}\mathcal{R}(H_{\alpha \triangleright n-1}^-) = V_{q,n}(\mathcal{D}_{\alpha \triangleright n-1}) \subseteq \mathcal{D}_n = \mathcal{N}(I_{(n+1)q} - H_n^- H_n)$. Hence, for each $x \in \mathbb{C}^{nq}$, we obtain $V_{q,n} H_{\alpha \triangleright n-1}^- x - H_n^- H_n V_{q,n} H_{\alpha \triangleright n-1}^- x = \left(I_{(n+1)q} - H_n^- H_n\right) V_{q,n} H_{\alpha \triangleright n-1}^- x = 0$. Consequently, $H_n^- H_n V_{q,n} H_{\alpha \triangleright n-1}^- = V_{q,n} H_{\alpha \triangleright n-1}^-$. Using additionally [21, Lemma 6.12], we then get

$$\begin{aligned}
H_{\alpha \triangleright n-1}^- V_{q,n}^* &= \left(H_{\alpha \triangleright n-1}^-\right)^* V_{q,n}^* = \left(V_{q,n} H_{\alpha \triangleright n-1}^-\right)^* = \left(H_n^- H_n V_{q,n} H_{\alpha \triangleright n-1}^-\right)^* \\
&= \left(H_{\alpha \triangleright n-1}^-\right)^* V_{q,n}^* H_n^* (H_n^-)^* = H_{\alpha \triangleright n-1}^- V_{q,n}^* H_n H_n^- .
\end{aligned}$$

Thus, (3.8) is proved. Remark 3.1 yields $T_{q,n} V_{q,n} = \mathfrak{V}_{q,n}$. We now apply Lemma A.7, using $A = H_n$, $B = H_{\alpha \triangleright n-1}$, $\mathcal{U} = \mathcal{D}_n$, $\mathcal{V} = \mathcal{D}_{\alpha \triangleright n-1}$, $T = T_{q,n}$, $V = V_{q,n}$, and $\mathfrak{V} = \mathfrak{V}_{q,n}$. Since Remark 3.12 and Notation 3.13 provide $A_{\mathcal{U}}^- = H_n^-$ and $B_{\mathcal{V}}^- = H_{\alpha \triangleright n-1}$, from Lemma A.7 we then get (3.11) for all $l \in \mathbb{N}$. For each $l \in \mathbb{N}$ and each $\zeta \in \mathbb{C}$, then Remark 3.2 and (3.11) yield

$$\begin{aligned}
&H_n^- R_{T_{q,n}}(\zeta) T_{q,n}^l V_{q,n}(I_{nq} - H_{\alpha \triangleright n-1} H_{\alpha \triangleright n-1}^-) \\
&\quad = H_n^- \left(\sum_{j=0}^{n} \zeta^j T_{q,n}^j \right) T_{q,n}^l V_{q,n}(I_{nq} - H_{\alpha \triangleright n-1} H_{\alpha \triangleright n-1}^-) \\
&\quad = \sum_{j=0}^{n} \zeta^j H_n^- T_{q,n}^{j+l} V_{q,n}(I_{nq} - H_{\alpha \triangleright n-1} H_{\alpha \triangleright n-1}^-) = 0.
\end{aligned}$$

Thus, (3.12) is also proved. Obviously, $I_{(n+1)q} = V_{q,n} V_{q,n}^* + \operatorname{diag}\left(0_{nq \times nq}, \ I_q\right)$ and $T_{q,n} \cdot \operatorname{diag}\left(0_{nq \times nq}, \ I_q\right) = 0$. Consequently, for every choice of $l \in \mathbb{N}$ and $\zeta \in \mathbb{C}$, from (3.12) we conclude

$$\begin{aligned}
&H_n^- R_{T_{q,n}}(\zeta) T_{q,n}^l \left(I_{(n+1)q} - V_{q,n} H_{\alpha \triangleright n-1} H_{\alpha \triangleright n-1}^- V_{q,n}^*\right) \\
&\quad = H_n^- R_{T_{q,n}}(\zeta) T_{q,n}^l \left(V_{q,n} V_{q,n}^* + \operatorname{diag}\left(0_{nq \times nq}, \ I_q\right) - V_{q,n} H_{\alpha \triangleright n-1} H_{\alpha \triangleright n-1}^- V_{q,n}^*\right) \\
&\quad = H_n^- R_{T_{q,n}}(\zeta) T_{q,n}^l \left(V_{q,n} V_{q,n}^* - V_{q,n} H_{\alpha \triangleright n-1} H_{\alpha \triangleright n-1}^- V_{q,n}^*\right) \\
&\qquad + H_n^- R_{T_{q,n}}(\zeta) T_{q,n}^l \cdot \operatorname{diag}\left(0_{nq \times nq}, \ I_q\right) \\
&\quad = H_n^- R_{T_{q,n}}(\zeta) T_{q,n}^l V_{q,n} \left(I_{nq} - H_{\alpha \triangleright n-1} H_{\alpha \triangleright n-1}^-\right) V_{q,n}^* = 0
\end{aligned}$$

and, hence, (3.13). In view of Remarks 3.9 (b) and 3.1 as well as (3.8) and (3.13), equation (3.9) is checked by

$$
\begin{aligned}
&H_n^- R_{T_{q,n}}(\alpha)\left(v_{q,n}v_{q,n}^* H_n V_{q,n} H_{\alpha\triangleright n-1}^- V_{q,n}^* + T_{q,n}\right)\\
&= H_n^- R_{T_{q,n}}(\alpha)\left[\left(\left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n V_{q,n} - \mathfrak{V}_{q,n} H_{\alpha\triangleright n-1}\right) H_{\alpha\triangleright n-1}^- V_{q,n}^* + T_{q,n}\right]\\
&= H_n^- R_{T_{q,n}}(\alpha)\\
&\quad\cdot\left(\left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n V_{q,n} H_{\alpha\triangleright n-1}^- V_{q,n}^* - \mathfrak{V}_{q,n} H_{\alpha\triangleright n-1} H_{\alpha\triangleright n-1}^- V_{q,n}^* + T_{q,n}\right)\\
&= H_n^- R_{T_{q,n}}(\alpha)\left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n V_{q,n} H_{\alpha\triangleright n-1}^- V_{q,n}^*\\
&\quad + H_n^- R_{T_{q,n}}(\alpha)\left[T_{q,n} - T_{q,n} V_{q,n} H_{\alpha\triangleright n-1} H_{\alpha\triangleright n-1}^- V_{q,n}^*\right]\\
&= H_n^- H_n V_{q,n} H_{\alpha\triangleright n-1}^- V_{q,n}^* + H_n^- R_{T_{q,n}}(\alpha) T_{q,n}\left(I - V_{q,n} H_{\alpha\triangleright n-1} H_{\alpha\triangleright n-1}^- V_{q,n}^*\right)\\
&= V_{q,n} H_{\alpha\triangleright n-1}^- V_{q,n}^*.
\end{aligned}
$$

Obviously, (3.9) yields

$$\left(I_{(n+1)q} - H_n^- R_{T_{q,n}}(\alpha) v_{q,n} v_{q,n}^* H_n\right) V_{q,n} H_{\alpha\triangleright n-1}^- V_{q,n}^* = H_n^- R_{T_{q,n}}(\alpha) T_{q,n}. \quad (3.16)$$

Keeping in mind $H_n^* = H_n$ and $H_{\alpha\triangleright n-1}^* = H_{\alpha\triangleright n-1}$, from [21, Lemma 6.12] we get $(H_n^-)^* = H_n^-$ and $(H_{\alpha\triangleright n-1})^* = H_{\alpha\triangleright n-1}$. This, together with $[R_{T_{q,n}}(\alpha)]^* = [R_{T_{q,n}}(\bar{\alpha})]^* = R_{T_{q,n}^*}(\alpha)$, and (3.16), delivers (3.10). □

Remark 3.15 The matrix

$$\tilde{J}_q := \begin{bmatrix} 0 & -iI_q \\ iI_q & 0 \end{bmatrix} \quad (3.17)$$

is a $2q \times 2q$ signature matrix, i.e., $\tilde{J}_q^* = \tilde{J}_q$ and $\tilde{J}_q^2 = I_{2q}$ hold true. Obviously, for all $A, B \in \mathbb{C}^{q\times q}$, then $\begin{bmatrix} A \\ B \end{bmatrix}^*(-\tilde{J}_q)\begin{bmatrix} A \\ B \end{bmatrix} = 2\Im(B^*A)$. In particular, $\begin{bmatrix} A \\ I \end{bmatrix}^*(-\tilde{J}_q)\begin{bmatrix} A \\ I \end{bmatrix} = 2\Im(A)$ for all $A \in \mathbb{C}^{q\times q}$.

We continue to use the notation given in (3.17).

Remark 3.16 Let $A \in \mathbb{C}_H^{q\times q}$. Then the matrices $B := \begin{bmatrix} I & 0 \\ A & I \end{bmatrix}$ and $C := \begin{bmatrix} I & A \\ 0 & I \end{bmatrix}$ fulfill $B^*\tilde{J}_q B = \tilde{J}_q$, $B\tilde{J}_q B^* = \tilde{J}_q$, $C^*\tilde{J}_q C = \tilde{J}_q$, and $C\tilde{J}_q C^* = \tilde{J}_q$.

Similar to [29] and [21], respectively, we now modify Bolotnikov's [2] approach, who considered the particular case $\alpha = 0$. We start with a remark analogue to [29, Lemma 8.10].

Lemma 3.17 *Suppose $\kappa \geq 2$. Let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$. For each $n \in \mathbb{N}$ with $2n \leq \kappa$, then the $2q \times 2q$ block matrices*

$$C_{n,\alpha} := \begin{bmatrix} I & v_{q,n}^* H_n V_{q,n} H_{\alpha \triangleright n-1}^- V_{q,n}^* H_n v_{q,n} \\ 0 & I \end{bmatrix} \tag{3.18}$$

and

$$\tilde{C}_{n,\alpha} := \begin{bmatrix} I & 0 \\ -v_{q,n}^* R_{T_{q,n}^*}(\alpha) H_n^- R_{T_{q,n}}(\alpha) v_{q,n} & I \end{bmatrix} \tag{3.19}$$

fulfill $C_{n,\alpha} \tilde{J}_q C_{n,\alpha}^* = \tilde{J}_q$, $C_{n,\alpha}^* \tilde{J}_q C_{n,\alpha} = \tilde{J}_q$, $\widetilde{C}_{n,\alpha} \tilde{J}_q \widetilde{C}_{n,\alpha}^* = \tilde{J}_q$, and $\widetilde{C}_{n,\alpha}^* \tilde{J}_q \widetilde{C}_{n,\alpha} = \tilde{J}_q$.

Proof Let $n \in \mathbb{N}$ such that $2n \leq \kappa$. From [21, Lemma 6.12] we get $H_n^- \in \mathbb{C}_{\geq}^{q \times q}$, $H_n H_n^- H_n = H_n$, and $H_n^- H_n H_n^- = H_n^-$. Because of $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$, we have $H_n^* = H_n$ and $H_{\alpha \triangleright n-1}^* = H_{\alpha \triangleright n-1}$. Again from [21, Lemma 6.12] we get then $(H_n^-)^* = H_n^-$ and $(H_{\alpha \triangleright n-1}^-)^* = H_{\alpha \triangleright n-1}^-$. Consequently, $\left(v_{q,n}^* H_n V_{q,n} H_{\alpha \triangleright n-1}^- V_{q,n}^* H_n v_{q,n}\right)^* = v_{q,n}^* H_n V_{q,n} H_{\alpha \triangleright n-1}^- V_{q,n}^* H_n v_{q,n}$ and, in view of Remark 3.2, furthermore $\left(-v_{q,n}^* R_{T_{q,n}^*}(\alpha) H_n^- R_{T_{q,n}}(\alpha) v_{q,n}\right)^* = -v_{q,n}^* R_{T_{q,n}^*}(\alpha) H_n^- R_{T_{q,n}}(\alpha) v_{q,n}$. Applying Remark 3.16 completes the proof. □

Remark 3.18 Suppose $\kappa \geq 2$ and $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$. In view of (3.18) and (3.19), for each $n \in \mathbb{N}$ with $2n \leq \kappa$, then

$$\begin{aligned} &\left[I_{(n+1)q},\ T_{q,n} H_n\right] \left(I_2 \otimes v_{q,n}\right) C_{n,\alpha} \\ &\quad = \left[I_{(n+1)q},\ \left(v_{q,n} v_{q,n}^* H_n V_{q,n} H_{\alpha \triangleright n-1}^- V_{q,n}^* + T_{q,n}\right) H_n\right] (I_2 \otimes v_{q,n}) \end{aligned}$$

and

$$\begin{aligned} &\left[I_{(n+1)q},\ \left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n\right] (I_2 \otimes v_{q,n}) \widetilde{C}_{n,\alpha} \\ &\quad = \left[R_{T_{q,n}}(\alpha)\right]^{-1} \left[\left(I_{(n+1)q} - H_n v_{q,n} v_{q,n}^* R_{T_{q,n}^*}(\alpha) H_n^-\right) R_{T_{q,n}}(\alpha),\ H_n\right] \\ &\quad \cdot (I_2 \otimes v_{q,n}). \end{aligned}$$

Remark 3.19 Suppose $\kappa \geq 2$ and $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$. Combining Remark 3.18 and Lemma 3.14, for each $n \in \mathbb{N}$ with $2n \leq \kappa$, one easily sees that $H_n^- R_{T_{q,n}}(\alpha) \left[I_{(n+1)q},\ T_{q,n} H_n\right] (I_2 \otimes v_{q,n}) C_{n,\alpha}$ $= \left[H_n^- R_{T_{q,n}}(\alpha),\ V_{q,n} H_{\alpha \triangleright n-1}^- V_{q,n}^* H_n\right] (I_2 \otimes v_{q,n})$.

Remark 3.20 ([21, Remark 7.4]) Suppose $\kappa \geq 2$ and let $(s_j)_{j=0}^{\kappa} \in \mathcal{H}_{q,\kappa}^{\geq,e}$. For each $n \in \mathbb{N}$ with $2n \leq \kappa$, then Remark 3.2 shows that $U_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2q \times 2q}$ defined by

$$U_{n,\alpha}(\zeta) := I_{2q} + (\zeta - \alpha)(I_2 \otimes v_{q,n})^* \left[T_{q,n} H_n,\ -I_{(n+1)q}\right]^*$$

$$\cdot R_{T_{q,n}^*}(\zeta) H_n^- R_{T_{q,n}}(\alpha) \left[I_{(n+1)q},\ T_{q,n} H_n \right] (I_2 \otimes v_{q,n}). \tag{3.20}$$

is a $2q \times 2q$ matrix polynomial of degree not greater than $n + 1$.

We now modify [21, Remark 7.7].

Remark 3.21 Suppose $\kappa \geq 2$ and let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$. For each $n \in \mathbb{N}$ with $2n \leq \kappa$, Remark 3.2 shows that $\widehat{U}_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2q \times 2q}$ defined by

$$\begin{aligned} &\widehat{U}_{n,\alpha}(\zeta) \\ &:= I_{2q} + (\zeta - \alpha) \left(I_2 \otimes v_{q,n} \right)^* \left[\left[R_{T_{q,n}}(\alpha) \right]^{-1} H_n,\ -I_{(n+1)q} \right]^* R_{T_{q,n}^*}(\zeta) \\ &\quad \cdot V_{q,n} H_{\alpha \triangleright n-1}^- V_{q,n}^* R_{T_{q,n}}(\alpha) \left[I_{(n+1)q},\ \left[R_{T_{q,n}}(\alpha) \right]^{-1} H_n \right] \left(I_2 \otimes v_{q,n} \right) \end{aligned} \tag{3.21}$$

is a $2q \times 2q$ matrix polynomial of degree of at most $n + 1$.

Remark 3.22 ([21, Remark 7.14]) Let $n \in \mathbb{N}$. According to Remark 3.2, the matrix-valued functions $\Omega_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2(n+1)q \times 2(n+1)q}$ and $\widetilde{\Omega}_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2(n+1)q \times 2(n+1)q}$ given by

$$\Omega_{n,\alpha}(\zeta) := \begin{bmatrix} (\zeta - \alpha) T_{q,n}^* & \left[R_{T_{q,n}^*}(\alpha) \right]^{-1} \\ -(\zeta - \alpha) I_{(n+1)q} & -(\zeta - \alpha) I_{(n+1)q} \end{bmatrix} \left(I_2 \otimes R_{T_{q,n}^*}(\zeta) \right) \tag{3.22}$$

and

$$\widetilde{\Omega}_{n,\alpha}(\zeta) := \begin{bmatrix} (\zeta - \alpha) T_{q,n}^* & (\zeta - \alpha) \left[R_{T_{q,n}^*}(\alpha) \right]^{-1} \\ -I_{(n+1)q} & -(\zeta - \alpha) I_{(n+1)q} \end{bmatrix} \left(I_2 \otimes R_{T_{q,n}^*}(\zeta) \right), \tag{3.23}$$

respectively, both are matrix polynomials of degree $n + 1$.

Lemma 3.23 *Suppose $\kappa \geq 2$ and $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$. Let $n \in \mathbb{N}$ be such that $2n \leq \kappa$. Let $\Xi_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2q \times 2q}$ and $\widetilde{\Xi}_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2q \times 2q}$ be given by*

$$\begin{aligned} \Xi_{n,\alpha}(\zeta) := &I_{2q} + (I_2 \otimes v_{q,n})^* \cdot \operatorname{diag}\left(H_n,\ I_{(n+1)q} \right) \cdot \Omega_{n,\alpha}(\zeta) \\ &\cdot \operatorname{diag}\left(H_n^-,\ V_{q,n} H_{\alpha \triangleright n-1}^- V_{q,n}^* \right) \cdot \operatorname{diag}\left(R_{T_{q,n}}(\alpha),\ H_n \right) \cdot (I_2 \otimes v_{q,n}) \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} \widetilde{\Xi}_{n,\alpha}(\zeta) := &I_{2q} + (I_2 \otimes v_{q,n})^* \cdot \operatorname{diag}\left(H_n,\ I_{(n+1)q} \right) \cdot \widetilde{\Omega}_{n,\alpha}(\zeta) \\ &\cdot \operatorname{diag}\left(H_n^-,\ V_{q,n} H_{\alpha \triangleright n-1}^- V_{q,n}^* \right) \cdot \operatorname{diag}\left(R_{T_{q,n}}(\alpha),\ H_n \right) \cdot (I_2 \otimes v_{q,n}). \end{aligned} \tag{3.25}$$

Then $\Xi_{n,\alpha}$ *and* $\widetilde{\Xi}_{n,\alpha}$ *are matrix polynomials of degree not greater than* $n+1$ *and, for each* $\zeta \in \mathbb{C}$*, the representations*

$$\Xi_{n,\alpha}(\zeta) = U_{n,\alpha}(\zeta)\, C_{n,\alpha} \quad \text{and} \quad \widetilde{\Xi}_{n,\alpha}(\zeta) = \widehat{U}_{n,\alpha}(\zeta)\, \widetilde{C}_{n,\alpha} \tag{3.26}$$

hold true. If $\Xi_{n,\alpha} = \left(\Xi_{n,\alpha}^{(j,k)}\right)_{j,k=1}^{2}$ *and* $\widetilde{\Xi}_{n,\alpha} = \left(\widetilde{\Xi}_{n,\alpha}^{(j,k)}\right)_{j,k=1}^{2}$ *are the* $q \times q$ *block representations of* $\Xi_{n,\alpha}$ *and* $\widetilde{\Xi}_{n,\alpha}$*, respectively, for all* $\zeta \in \mathbb{C}$*, then*

$$\Xi_{n,\alpha}^{(1,1)}(\zeta) = I_q + (\zeta-\alpha) v_{q,n}^* H_n T_{q,n}^* R_{T_{q,n}^*}(\zeta)\, H_n^- R_{T_{q,n}}(\alpha) v_{q,n},$$
$$\Xi_{n,\alpha}^{(1,2)}(\zeta) = v_{q,n}^* H_n \left[R_{T_{q,n}^*}(\alpha)\right]^{-1} R_{T_{q,n}^*}(\zeta)\, V_{q,n} H_{\alpha \triangleright n-1}^- V_{q,n}^* H_n v_{q,n},$$
$$\Xi_{n,\alpha}^{(2,1)}(\zeta) = -(\zeta-\alpha) v_{q,n}^* R_{T_{q,n}^*}(\zeta)\, H_n^- R_{T_{q,n}}(\alpha) v_{q,n},$$
$$\Xi_{n,\alpha}^{(2,2)}(\zeta) = I_q - (\zeta-\alpha) v_{q,n}^* R_{T_{q,n}^*}(\zeta)\, V_{q,n} H_{\alpha \triangleright n-1}^- V_{q,n}^* H_n v_{q,n},$$
$$\widetilde{\Xi}_{n,\alpha}^{(1,1)}(\zeta) = I_q + (\zeta-\alpha) v_{q,n}^* H_n T_{q,n}^* R_{T_{q,n}^*}(\zeta)\, H_n^- R_{T_{q,n}}(\alpha) v_{q,n},$$
$$\widetilde{\Xi}_{n,\alpha}^{(1,2)}(\zeta) = (\zeta-\alpha) v_{q,n}^* H_n \left[R_{T_{q,n}^*}(\alpha)\right]^{-1} R_{T_{q,n}^*}(\zeta)\, V_{q,n} H_{\alpha \triangleright n-1}^- V_{q,n}^* H_n v_{q,n},$$
$$\widetilde{\Xi}_{n,\alpha}^{(2,1)}(\zeta) = -v_{q,n}^* R_{T_{q,n}^*}(\zeta)\, H_n^- R_{T_{q,n}}(\alpha) v_{q,n},$$

and

$$\widetilde{\Xi}_{n,\alpha}^{(2,2)}(\zeta) = I_q - (\zeta-\alpha) v_{q,n}^* R_{T_{q,n}^*}(\zeta)\, V_{q,n} H_{\alpha \triangleright n-1}^- V_{q,n}^* H_n v_{q,n}.$$

Proof Lemma 3.23 can be proved analogous to [21, Lemma 7.15]. □

Remark 3.24 Let $s_0 \in \mathbb{C}^{q\times q}$. Then $\Xi_{0,\alpha} : \mathbb{C} \to \mathbb{C}^{q\times q}$ defined by

$$\Xi_{0,\alpha}(\zeta) := I_{2q} + (\zeta-\alpha)\left[0_{q\times q}, -I_q\right]^* s_0^+ \left[I_q, 0_{q\times q}\right] \tag{3.27}$$

is a $2q \times 2q$ matrix polynomial of degree d with $d \le 1$ fulfilling $\det \Xi_{0,\alpha}(\zeta) = 1 \neq 0$ and $\Xi_{0,\alpha}(\zeta) = \begin{bmatrix} I_q & 0_{q\times q} \\ -(\zeta-\alpha)s_0^+ & I_q \end{bmatrix}$ and $\Xi_{0,\alpha}^{-1}(\zeta) = \begin{bmatrix} I_q & 0_{q\times q} \\ (\zeta-\alpha)s_0^+ & I_q \end{bmatrix}$ for each $\zeta \in \mathbb{C}$.

Lemma 3.25 *Suppose* $\kappa \ge 2$ *and let* $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\ge,e}$*. Let* $n \in \mathbb{N}$ *be such that* $2n \le \kappa$*. Then* $\widetilde{U}_{n-1,\alpha} = \widehat{U}_{n,\alpha}$*, where* $\widetilde{U}_{n-1,\alpha} : \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ *is defined by*

$$\begin{aligned}\widetilde{U}_{n-1,\alpha}(\zeta) &:= I_{2q} + (\zeta-\alpha)(I_2 \otimes v_{q,n-1})^* \left[[R_{T_{q,n}}(\alpha)]^{-1} H_{n-1}, -I_{nq}\right]^* R_{T_{q,n-1}^*}(\zeta) \\ &\quad \cdot H_{\alpha\triangleright n-1}^- R_{T_{q,n-1}}(\alpha) \left[I_{nq}, [R_{T_{q,n-1}}(\alpha)]^{-1} H_{n-1}\right](I_2 \otimes v_{q,n-1}).\end{aligned} \tag{3.28}$$

Proof. From $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\ge,e}$ we get $H_m^* = H_m$ and $y_{0,m}^* = z_{0,m}$ for all $m \in \mathbb{Z}_{0,n}$ (see, e.g. [21, Remark 5.7]). Let $\zeta \in \mathbb{C}$. By virtue of Remark 3.9 (a), for each $m \in \mathbb{Z}_{0,n}$, we obtain

$$\left[I_{(m+1)q}, \left[R_{T_{q,m}}(\zeta)\right]^{-1} H_m\right]\left(I_2 \otimes v_{q,m}\right) = \left[v_{q,m}, \left[R_{T_{q,m}}(\zeta)\right]^{-1} y_{0,m}\right] \tag{3.29}$$

and, in view of $H_m^* = H_m$ and $y_{0,m}^* = z_{0,m}$, consequently,

$$\left(I_2 \otimes v_{q,m}\right)^* \left[\left[R_{T_{q,m}}(\zeta)\right]^{-1} H_m, -I_{(m+1)q}\right]^* = \begin{bmatrix} z_{0,m}\left[R_{T_{q,m}^*}\left(\overline{\zeta}\right)\right]^{-1} \\ -v_{q,m}^* \end{bmatrix}. \tag{3.30}$$

Moreover, (3.1), Remark 3.1, and Remark 3.9 (a) show that

$$\begin{aligned} z_{0,n}\left[R_{T_{q,n}^*}(\alpha)\right]^{-1} V_{q,n} &= z_{0,n}\left(V_{q,n} - \alpha T_{q,n}^* V_{q,n}\right) = z_{0,n}\left(V_{q,n} - \alpha V_{q,n} T_{q,n-1}^*\right) \\ &= z_{0,n} V_{q,n}\left(I_{nq} - \alpha T_{q,n-1}^*\right) = z_{0,n-1}\left[R_{T_{q,n-1}^*}(\alpha)\right]^{-1} \end{aligned} \tag{3.31}$$

holds true. Hence, because of $z_{0,n}^* = y_{0,n}$ and $z_{0,n-1}^* = y_{0,n-1}$, we also conclude

$$V_{q,n}^*\left[R_{T_{q,n}}(\alpha)\right]^{-1} y_{0,n} = \left[R_{T_{q,n-1}}(\alpha)\right]^{-1} y_{0,n-1}. \tag{3.32}$$

Taking into account (3.30), (3.31), Remark 3.1, and (3.30) again, we get

$$\begin{aligned} &(I_2 \otimes v_{q,n})^* \left[\left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n, -I_{(n+1)q}\right]^* V_{q,n} \\ &= \begin{bmatrix} z_{0,n}\left[R_{T_{q,n}^*}(\alpha)\right]^{-1} V_{q,n} \\ -v_{q,n}^* V_{q,n} \end{bmatrix} = \begin{bmatrix} z_{0,n-1}\left[R_{T_{q,n-1}^*}(\alpha)\right]^{-1} \\ -v_{q,n-1}^* \end{bmatrix} \\ &= \left(I_2 \otimes v_{q,n-1}\right)^* \left[\left[R_{T_{q,n-1}}(\alpha)\right]^{-1} H_{n-1}, -I_{nq}\right]^*. \end{aligned} \tag{3.33}$$

Analogously, from (3.29), (3.32), Remark 3.1, and (3.29), we infer

$$\begin{aligned} &V_{q,n}^*\left[I_{(n+1)q}, \left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n\right](I_2 \otimes v_{q,n}) \\ &= \left[V_{q,n}^* v_{q,n}, V_{q,n}^*\left[R_{T_{q,n}}(\alpha)\right]^{-1} y_{0,n}\right] = \left[v_{q,n-1}, \left[R_{T_{q,n-1}}(\alpha)\right]^{-1} y_{0,n-1}\right] \\ &= \left[I_{nq}, \left[R_{T_{q,n-1}}(\alpha)\right]^{-1} H_{n-1}\right]\left(I_2 \otimes v_{q,n-1}\right). \end{aligned}$$

Thus, using additionally (3.21), Remark 3.6, (3.33), and (3.28), we get finally

$$\begin{aligned}\widehat{U}_{n,\alpha}(\zeta) &= I_{2q} + (\zeta-\alpha)\left(I_2 \otimes v_{q,n}\right)^* \left[\left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n,\ -I_{(n+1)q}\right]^* R_{T^*_{q,n}}(\zeta)\\ &\quad\cdot V_{q,n} H^-_{\alpha \triangleright n-1} V^*_{q,n} R_{T_{q,n}}(\alpha)\left[I_{(n+1)q},\ \left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n\right]\left(I_2 \otimes v_{q,n}\right)\\ &= I_{2q} + (\zeta-\alpha)\left(I_2 \otimes v_{q,n}\right)^* \left[\left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n,\ -I_{(n+1)q}\right]^* V_{q,n} R_{T^*_{q,n-1}}(\zeta)\\ &\quad\cdot H^-_{\alpha \triangleright n-1} R_{T_{q,n-1}}(\alpha)\, V^*_{q,n}\left[I_{(n+1)q},\ \left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n\right]\left(I_2 \otimes v_{q,n}\right)\\ &= \widetilde{U}_{n-1,\alpha}(\zeta).\end{aligned}$$

□

Now we modify [21, Proposition 7.16].

Lemma 3.26 *Suppose $\kappa \geq 2$ and let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}^{\geq,e}_{q,\kappa,\alpha}$. Let $n \in \mathbb{N}$ be such that $2n \leq \kappa$. Then the functions $\Xi_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ given by (3.24) and $\widetilde{\Xi}_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ given by (3.25) fulfill, for each $\zeta \in \mathbb{C}\setminus\{\alpha\}$, the identity*

$$\widetilde{\Xi}_{n,\alpha}(\zeta) = \operatorname{diag}\left((\zeta-\alpha)I_q,\ I_q\right) \cdot \Xi_{n,\alpha}(\zeta) \cdot \operatorname{diag}\left((\zeta-\alpha)^{-1}I_q,\ I_q\right). \tag{3.34}$$

Proof In view of (3.22), (3.23), (3.24), and (3.25), equation (3.34) follows by straightforward computation. We omit the details. □

Lemma 3.27 *Suppose $\kappa \geq 2$ and let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}^{\geq,e}_{q,\kappa,\alpha}$. For each $n \in \mathbb{N}$ with $2n \leq \kappa$ and for every choice of $z \in \mathbb{C}$ and $w \in \mathbb{C}$, then*

$$\begin{aligned}\tilde{J}_q - \Xi_{n,\alpha}(z)\tilde{J}_q \Xi^*_{n,\alpha}(w) &= -i(z-\overline{w})(I_2 \otimes v_{q,n})^*\left[T_{q,n}H_n,\ -I_{(n+1)q}\right]^*\\ &\quad\cdot R_{T^*_{q,n}}(z)\, H_n^- \left[R_{T^*_{q,n}}(w)\right]^* \left[T_{q,n}H_n,\ -I_{(n+1)q}\right](I_2 \otimes v_{q,n})\end{aligned}$$

and

$$\begin{aligned}&\tilde{J}_q - \widetilde{\Xi}_{n,\alpha}(z)\tilde{J}_q \widetilde{\Xi}^*_{n,\alpha}(w)\\ &= -i(z-\overline{w})(I_2 \otimes v_{q,n-1})^*\left[\left[R_{T_{q,n-1}}(\alpha)\right]^{-1} H_{n-1},\ -I_{nq}\right]^* R_{T^*_{q,n-1}}(z)\\ &\quad\cdot H^-_{\alpha\triangleright n-1}\left[R_{T^*_{q,n-1}}(w)\right]^*\left[\left[R_{T_{q,n-1}}(\alpha)\right]^{-1} H_{n-1},\ -I_{nq}\right]^*(I_2 \otimes v_{q,n-1}).\end{aligned}$$

Proof Taking into account (3.26), Lemma 3.17, Remark 3.2, Lemma 3.25, and [21, Lemmata 7.5 and 7.8], the proof of Lemma 3.27 is straightforward. □

Remark 3.28 Suppose $\kappa \geq 2$ and let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}^{\geq,e}_{q,\kappa,\alpha}$. Analogous to [21, Remark 7.22], using [21, Lemma 6.12] and Lemma 3.27, one can easily see that $\frac{1}{2\Im z}(\tilde{J}_q - \Xi_{n,\alpha}(z)\tilde{J}_q\Xi^*_{n,\alpha}(z)) \in \mathbb{C}^{2q\times 2q}_{\geq}$ and $\frac{1}{2\Im z}(\tilde{J}_q - \widetilde{\Xi}_{n,\alpha}(z)\tilde{J}_q\widetilde{\Xi}^*_{n,\alpha}(z)) \in \mathbb{C}^{2q\times 2q}_{\geq}$ for each $n \in \mathbb{N}$ with $2n \leq \kappa$ and for each $z \in \mathbb{C}\setminus\mathbb{R}$.

Remark 3.29 Suppose $\kappa \geq 2$ and let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$. Analogous to [21, Lemma 7.19], for each $n \in \mathbb{N}$ with $2n \leq \kappa$, one can easily prove the following statements:

(a) $\tilde{J}_q - \Xi_{n,\alpha}(x)\tilde{J}_q\Xi_{n,\alpha}^*(x) = 0$ and $\tilde{J}_q - \widetilde{\Xi}_{n,\alpha}(x)\tilde{J}_q\widetilde{\Xi}_{n,\alpha}^*(x) = 0$ for all $x \in \mathbb{R}$.
(b) For all $z \in \mathbb{C}$, the matrices $\Xi_{n,\alpha}(z)$ and $\widetilde{\Xi}_{n,\alpha}(z)$ both are non-singular and fulfill $\Xi_{n,\alpha}^{-1}(z) = \tilde{J}_q\Xi_{n,\alpha}^*(\bar{z})\tilde{J}_q$ and $\widetilde{\Xi}_{n,\alpha}^{-1}(z) = \tilde{J}_q\widetilde{\Xi}_{n,\alpha}^*(\bar{z})\tilde{J}_q$.
(c) $\tilde{J}_q - \Xi_{n,\alpha}^{-*}(z)\tilde{J}_q\Xi_{n,\alpha}^{-1}(w) = \tilde{J}_q(\tilde{J}_q - \Xi_{n,\alpha}(\bar{z})\tilde{J}_q\Xi_{n,\alpha}^*(\overline{w}))\tilde{J}_q$ and $\tilde{J}_q - \widetilde{\Xi}_{n,\alpha}^{-*}(z)\tilde{J}_q\widetilde{\Xi}_{n,\alpha}^{-1}(w) = \tilde{J}_q(\tilde{J}_q - \widetilde{\Xi}_{n,\alpha}(\bar{z})\tilde{J}_q\widetilde{\Xi}_{n,\alpha}^*(\overline{w}))\tilde{J}_q$ for every choice of z and w in $\mathbb{C}$.

Lemma 3.30 *Suppose $\kappa \geq 2$ and let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$. For each $n \in \mathbb{N}$ with $2n \leq \kappa$ and every choice of z and w in $\mathbb{C}$, then*

$$\tilde{J}_q - \Xi_{n,\alpha}^{-*}(z)\tilde{J}_q\Xi_{n,\alpha}^{-1}(w) = -i(\bar{z} - w)\left(I_2 \otimes v_{q,n}\right)^*\left[I_{(n+1)q},\, T_{q,n}H_n\right]^* \cdot R_{T_{q,n}^*}(\bar{z})\, H_n^- R_{T_{q,n}}(w)\left[I_{(n+1)q},\, T_{q,n}H_n\right](I_2 \otimes v_{q,n}) \tag{3.35}$$

and

$$\tilde{J}_q - \widetilde{\Xi}_{n,\alpha}^{-*}(z)\tilde{J}_q\widetilde{\Xi}_{n,\alpha}^{-1}(w) = -i(\bar{z} - w)\left(I_2 \otimes v_{q,n-1}\right)^*\left[I_{nq},\, \left[R_{T_{q,n-1}}(\alpha)\right]^{-1} H_{n-1}\right]^* R_{T_{q,n-1}^*}(\bar{z}) \cdot H_{\alpha\triangleright n-1}^- R_{T_{q,n-1}}(w)\left[I_{nq},\, \left[R_{T_{q,n-1}}(\alpha)\right]^{-1} H_{n-1}\right]\left(I_2 \otimes v_{q,n-1}\right). \tag{3.36}$$

Proof Using parts (b) and (c) of Remark 3.29, Lemma 3.27, Remark 3.2, and Remark [21, Remark 7.3], one can prove Lemma 3.30 analogous to [21, Lemma 7.20]. □

We now state a result similar to [21, Lemma 7.21].

Remark 3.31 Suppose $\kappa \geq 2$ and let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$. For each $n \in \mathbb{N}$ with $2n \leq \kappa$ and for all $z, w \in \mathbb{C}$, then Lemma 3.17 and Lemma 3.23 yield

$$\begin{aligned}\tilde{J}_q &- \Xi_{n,\alpha}^*(w)\tilde{J}_q\Xi_{n,\alpha}(z) \\ &= i(\overline{w} - z)C_{n,\alpha}^*(I_2 \otimes v_{q,n})^*\left[I_{(n+1)q},\, T_{q,n}H_n\right]^*\left[R_{T_{q,n}}(\alpha)\right]^* \\ &\quad \cdot H_n^-\left[R_{T_{q,n}^*}(w)\right]^*\left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n\left[R_{T_{q,n}^*}(w)\right]^{-1} R_{T_{q,n}^*}(z)\, H_n^- \\ &\quad \cdot R_{T_{q,n}}(\alpha)\left[I_{(n+1)q},\, T_{q,n}H_n\right](I_2 \otimes v_{q,n})C_{n,\alpha}.\end{aligned}$$

Let $\mathcal{G}$ be a non-empty open subset of $\mathbb{C}$ and let $f = \left[f_{jk}\right]_{\substack{j=1,\dots,p\\k=1,\dots,q}}$ be a $p \times q$ matrix-valued function which is meromorphic in $\mathcal{G}$. Let $\mathbb{H}_f := \bigcap_{j=1}^{p}\bigcap_{k=1}^{q}\mathbb{H}_{f_{jk}}$ and let $\mathbb{P}_f := \bigcup_{j=1}^{p}\bigcup_{k=1}^{q}\mathbb{P}_{f_{jk}}$. Furthermore, we write $B \leq A$ or $A \geq B$ to indicate that A and B are Hermitian complex $q \times q$ matrices such that the matrix $B - A$ is non-negative Hermitian.

Notation 3.32 (*see* [21, Notation 9.1]) By $\widetilde{\mathfrak{W}}_{\tilde{J}_q,\alpha}$ we denote the set of all $2q \times 2q$ matrix-valued functions Ξ which are meromorphic in $\mathbb{C}\setminus[\alpha, \infty)$ and for which there exists a discrete subset $\mathcal{D}$ of $\mathbb{C}\setminus[\alpha, \infty)$ fulfilling the following three conditions:

(i) Ξ is holomorphic in $\mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$.
(ii) $\Xi(z)\tilde{J}_q \Xi^*(z) \leq \tilde{J}_q$ for each $z \in \Pi_+ \setminus \mathcal{D}$.
(iii) $\Xi(x)\tilde{J}_q \Xi^*(x) = \tilde{J}_q$ for each $x \in (-\infty, \alpha) \setminus \mathcal{D}$.

If $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are non-empty sets with $\mathcal{Z} \subseteq \mathcal{X}$ and if $f : \mathcal{X} \to \mathcal{Y}$ is a mapping, then $\operatorname{Rstr}_{\mathcal{Z}} f$ marks the restriction of f onto $\mathcal{Z}$.

Remark 3.33 Similar to [21, Remark 9.2], from Lemma 3.23, Remark 3.28, and Remark 3.29 we get the following: Suppose $\kappa \geq 2$ and let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$. For each $n \in \mathbb{N}$ with $2n \leq \kappa$, then $\widehat{\Xi}_{n,\alpha} := \operatorname{Rstr}_{\mathbb{C}\setminus[\alpha,\infty)} \Xi_{n,\alpha}$ and $\widehat{\widetilde{\Xi}}_{n,\alpha} := \operatorname{Rstr}_{\mathbb{C}\setminus[\alpha,\infty)} \widetilde{\Xi}_{n,\alpha}$ are holomorphic in $\mathbb{C}\setminus[\alpha, \infty)$ and belong to $\mathfrak{W}_{\tilde{J}_q,\alpha}$.

Notation 3.34 (*see* [21, Notation 9.3]) Let $P_\alpha : \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ be defined by $P_\alpha(z) := \operatorname{diag}\left((z-\alpha)I_q,\ I_q\right)$. Then let $\mathfrak{W}_{\tilde{J}_q,\alpha}$ be the set of all $\Xi \in \widetilde{\mathfrak{W}}_{\tilde{J}_q,\alpha}$ for which $\widetilde{\Xi} := P_\alpha \Xi P_\alpha^{-1}$ belongs to $\widetilde{\mathfrak{W}}_{\tilde{J}_q,\alpha}$.

Analogous to [21, Remark 9.4], we see the following:

Lemma 3.35 *Let* $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$. *For each* $n \in \mathbb{N}_0$ *with* $2n \leq \kappa$, *the function* $\operatorname{Rstr}_{\mathbb{C}\setminus[\alpha,\infty)} \Xi_{n,\alpha}$ *is holomorphic in* $\mathbb{C}\setminus[\alpha, \infty)$ *and belongs to* $\mathfrak{W}_{\tilde{J}_q,\alpha}$.

Proof Use Notation 3.34, Lemmata 3.23 and 3.26, and Remark 3.33 in the case $n \geq 1$ and $s_0 \in \mathbb{C}_{\geq}^{q\times q}$, Remarks 3.24 and A.2, and Notations 3.34 and 3.32 in the case $n = 0$. □

Remark 3.36 ([29, Bemerkung 8.2 and Bemerkung 8.3]) Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. Furthermore, let $S : \mathbb{C}\setminus[\alpha, \infty) \to \mathbb{C}^{q\times q}$ be a matrix-valued function. Regarding Remarks 3.9 (a), 3.15, 3.12 as well as Notation 3.13, one can easily see that, for each $n \in \mathbb{N}_0$ such that $2n \leq \kappa$, the matrix-valued function $\Sigma_{2n}^{[S]} : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{q\times q}$ given by (3.6) admits, for each $z \in \mathbb{C}\setminus\mathbb{R}$, the representation

$$\Sigma_{2n}^{[S]}(z) = \frac{1}{i(z-\bar{z})} \begin{bmatrix} S(z) \\ I_q \end{bmatrix}^* \Big\{ \tilde{J}_q - i(z-\bar{z})(I_2 \otimes v_{q,n})^* \left[I_{(n+1)q},\ T_{q,n}H_n\right]^* \cdot R_{T_{q,n}^*}(\bar{z}) H_n^- R_{T_{q,n}}(z) \left[I_{(n+1)q},\ T_{q,n}H_n\right] (I_2 \otimes v_{q,n}) \Big\} \begin{bmatrix} S(z) \\ I_q \end{bmatrix}. \quad (3.37)$$

If $\kappa \geq 1$, then, for each $n \in \mathbb{N}$ with $2n - 1 \leq \kappa$, using Remarks 3.9 (a), 3.15, 3.12 as well as Notation 3.13, it is readily checked that $\Sigma_{2n-1}^{[S]} : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{q\times q}$ given by (3.7) can be represented for all $z \in \mathbb{C}\setminus\mathbb{R}$, via

$$\begin{aligned} \Sigma_{2n-1}^{[S]}(z) = \frac{1}{i(z-\bar{z})} \begin{bmatrix} (z-\alpha)S(z) \\ I_q \end{bmatrix}^* \Big\{ &\tilde{J}_q - i(z-\bar{z})(I_2 \otimes v_{q,n-1})^* \\ &\cdot \left[I_{nq},\ \left[R_{T_{q,n-1}}(\alpha)\right]^{-1} H_{n-1}\right]^* R_{T_{q,n-1}^*}(\bar{z}) H_{\alpha\triangleright n-1}^- R_{T_{q,n-1}}(z) \\ &\cdot \left[I_{nq},\ \left[R_{T_{q,n-1}}(\alpha)\right]^{-1} H_{n-1}\right] (I_2 \otimes v_{q,n-1}) \Big\} \begin{bmatrix} (z-\alpha)S(z) \\ I_q \end{bmatrix}. \end{aligned} \quad (3.38)$$

Lemma 3.37 *Suppose $\kappa \geq 2$. Let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$ and let $S : \mathbb{C}\setminus[\alpha, \infty) \to \mathbb{C}^{q\times q}$ be a matrix-valued function. For all $n \in \mathbb{N}$ such that $2n \leq \kappa$, then $\Sigma_{2n}^{[S]} : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{q\times q}$ given by (3.6) can be represented via*

$$\Sigma_{2n}^{[S]}(z) = \begin{bmatrix} S(z) \\ I_q \end{bmatrix}^* \Xi_{n,\alpha}^{-*}(z) \left(\frac{-\tilde{J}_q}{2\Im z} \right) \Xi_{n,\alpha}^{-1}(z) \begin{bmatrix} S(z) \\ I_q \end{bmatrix}$$

for each $z \in \mathbb{C}\setminus\mathbb{R}$. Moreover, for all $n \in \mathbb{N}$ with $2n-1 \leq \kappa$, the matrix-valued function $\Sigma_{2n-1}^{[S]} : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{q\times q}$ defined by (3.7) admits, for each $z \in \mathbb{C}\setminus\mathbb{R}$, the representation

$$\Sigma_{2n-1}^{[S]}(z) = \begin{bmatrix} (z-\alpha)S(z) \\ I_q \end{bmatrix}^* \Xi_{n,\alpha}^{-*}(z) \left(\frac{-\tilde{J}_q}{2\Im z} \right) \Xi_{n,\alpha}^{-1}(z) \begin{bmatrix} (z-\alpha)S(z) \\ I_q \end{bmatrix}.$$

Proof Remark 3.36 provides (3.37) and (3.38) for all $z \in \mathbb{C}\setminus\mathbb{R}$. Lemma 3.30 yields (3.35) and (3.36) for all $z \in \mathbb{C}$ and $w = z$. Combining (3.37) and (3.35) (resp. (3.38) and (3.36)) completes the proof. □

Lemma 3.38 *Suppose $\kappa \geq 2$ and let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$. For each $n \in \mathbb{N}$ with $2n \leq \kappa$ and for all $z \in \mathbb{C}$, then*

$$\begin{aligned} &(I_{(n+1)q} - H_n^+ H_n) R_{T_{q,n}}(z) \left[I_{(n+1)q},\ T_{q,n} H_n \right] (I_2 \otimes v_{q,n}) \Xi_{n,\alpha}(z) \\ &= \left[I_{(n+1)q} + (z-\alpha)(I_{(n+1)q} - H_n^+ H_n) T_{q,n} R_{T_{q,n}}(z)(I_{(n+1)q} - H_n H_n^-) \right] \\ &\quad \cdot (I_{(n+1)q} - H_n^+ H_n) R_{T_{q,n}}(\alpha) \left[I_{(n+1)q},\ T_{q,n} H_n \right] (I_2 \otimes v_{q,n}) C_{n,\alpha}. \end{aligned}$$

Proof In view of Remarks 2.2 and 3.2, using Remark 3.7, [21, Remarks 5.4, 5.7, 6.13, and 7.3], as well as (3.26), (3.20), Lemma 3.38 can be proved in the same way as [21, Lemma 12.1]. We omit the details. □

Lemma 3.39 *Suppose $\kappa \geq 2$. Let $n \in \mathbb{N}$ with $2n \leq \kappa$, and let $\zeta \in \mathbb{C}$. Then*

$$\begin{aligned} &V_{q,n}^* R_{T_{q,n}}(\zeta) \left[I_{(n+1)q},\ \left[R_{T_{q,n}}(\alpha) \right]^{-1} H_n \right] (I_2 \otimes v_{q,n}) \\ &\quad = R_{T_{q,n-1}}(\zeta) \left[I_{nq},\ \left[R_{T_{q,n-1}}(\alpha) \right]^{-1} H_{n-1} \right] \left(I_2 \otimes v_{q,n-1} \right) \end{aligned}$$

and

$$\begin{aligned} &(I_2 \otimes v_{q,n})^* \left[\left[R_{T_{q,n}}(\alpha) \right]^{-1} H_n,\ -I_{(n+1)q} \right]^* R_{T_{q,n}^*}(\zeta)\, V_{q,n} \\ &\quad = (I_2 \otimes v_{q,n-1})^* \left[[R_{T_{q,n-1}}(\alpha)]^{-1} H_{n-1},\ -I_{nq} \right]^* R_{T_{q,n-1}^*}(\zeta). \end{aligned}$$

***Proof*.** Obviously, $V_{q,n}^* H_n v_{q,n} = H_{n-1} v_{q,n-1}$. Consequently, in view of Remarks 3.2 and Remark 3.6, we obtain $V_{q,n}^* \left[R_{T_{q,n}}(\alpha) \right]^{-1} H_n v_{q,n} = [R_{T_{q,n-1}}(\alpha)]^{-1} V_{q,n}^* H_n v_{q,n} = [R_{T_{q,n-1}}(\alpha)]^{-1} H_{n-1} v_{q,n-1}$. From Remark 3.6 we get $R_{T_{q,n}^*}(\zeta)\, V_{q,n} = V_{q,n} R_{T_{q,n-1}^*}(\zeta)$

$= \left(\left[R_{T^*_{q,n-1}}(\zeta)\right]^* V^*_{q,n}\right)^*$. Hence, taking into account Remark 3.6 and Remark 3.1, we infer

$$
\begin{aligned}
&V^*_{q,n} R_{T_{q,n}}(\zeta)\left[I_{(n+1)q},\, \left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n\right](I_2 \otimes v_{q,n})\\
&\quad = R_{T_{q,n-1}}(\zeta)\, V^*_{q,n}\left[I_{(n+1)q},\, \left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n\right](I_2 \otimes v_{q,n})\\
&\quad = R_{T_{q,n-1}}(\zeta)\left[V^*_{q,n} v_{q,n},\; V^*_{q,n}\left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n v_{q,n}\right]\\
&\quad = R_{T_{q,n-1}}(\zeta)\left[v_{q,n-1},\, \left[R_{T_{q,n-1}}(\alpha)\right]^{-1} H_{n-1} v_{q,n-1}\right]\\
&\quad = R_{T_{q,n-1}}(\zeta)\left[I_{nq},\, \left[R_{T_{q,n-1}}(\alpha)\right]^{-1} H_{n-1}\right]\left(I_2 \otimes v_{q,n-1}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&(I_2 \otimes v_{q,n})^*\left[\left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n,\, -I_{(n+1)q}\right]^* R_{T^*_{q,n}}(\zeta)\, V_{q,n}\\
&\quad = \left(\left[R_{T^*_{q,n-1}}(\zeta)\right]^* V^*_{q,n}\left[[R_{T_{q,n}}(\alpha)]^{-1} H_n,\, -I_{(n+1)q}\right](I_2 \otimes v_{q,n})\right)^*\\
&\quad = \left(\left[R_{T^*_{q,n-1}}(\zeta)\right]^*\left[V^*_{q,n}[R_{T_{q,n}}(\alpha)]^{-1} H_n v_{q,n},\, -V^*_{q,n} v_{q,n}\right]\right)^*\\
&\quad = \left(\left[R_{T^*_{q,n-1}}(\zeta)\right]^*\left[[R_{T_{q,n-1}}(\alpha)]^{-1} H_{n-1} v_{q,n-1},\, -v_{q,n-1}\right]\right)^*\\
&\quad = (I_2 \otimes v_{q,n-1})^*\left[[R_{T_{q,n-1}}(\alpha)]^{-1} H_{n-1},\, -I_{nq}\right]^* R_{T^*_{q,n-1}}(\zeta).
\end{aligned}
$$

□

Lemma 3.40 *Suppose $\kappa \geq 2$ and let $\left(s_j\right)_{j=0}^{\kappa} \in \mathcal{K}^{\geq,e}_{q,\kappa,\alpha}$. For each $n \in \mathbb{N}$ with $2n \leq \kappa$ and for all $z \in \mathbb{C}$, then*

$$
\begin{aligned}
&(I_{nq} - H^+_{\alpha\triangleright n-1} H_{\alpha\triangleright n-1}) V^*_{q,n} R_{T_{q,n}}(z)\\
&\quad \cdot\left[I_{(n+1)q},\, \left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n\right](I_2 \otimes v_{q,n})\widetilde{\Xi}_{n,\alpha}(z)\\
&\quad = \Big\{I_{nq} + (z-\alpha)(I_{nq} - H^+_{\alpha\triangleright n-1} H_{\alpha\triangleright n-1}) V^*_{q,n} T_{q,n} R_{T_{q,n}}(z) V_{q,n}\\
&\quad \cdot(I_{nq} - H_{\alpha\triangleright n-1} H^-_{\alpha\triangleright n-1})\Big\}(I_{nq} - H^+_{\alpha\triangleright n-1} H_{\alpha\triangleright n-1})\\
&\quad \cdot V^*_{q,n} R_{T_{q,n}}(\alpha)\left[I_{(n+1)q},\, \left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n\right]\left(I_2 \otimes v_{q,n}\right)\widetilde{C}_{n,\alpha}.
\end{aligned}
$$

Proof In view of Remark 2.2, Lemmata 3.39, 3.23, 3.7, and 3.6 as well as [21, Remarks 5.5, 7.3, and 6.13] and (3.21), the proof of Lemma 3.40 is analogous to proof of [21, Lemma 12.1]. We omit the details. □

Lemma 3.41 *Suppose $\kappa \geq 2$ and let $\left(s_j\right)_{j=0}^{\kappa} \in \mathcal{K}^{\geq,e}_{q,\kappa,\alpha}$. In view of (3.18) and (3.19), for each $n \in \mathbb{N}$ such that $2n \leq \kappa$, then*

$$
(I_{(n+1)q} - H^+_n H_n) R_{T_{q,n}}(\alpha)\left[I_{(n+1)q},\, T_{q,n} H_n\right](I_2 \otimes v_{q,n}) C_{n,\alpha}
$$

$$
\begin{aligned}
&= (I_{(n+1)q} - H_n^+ H_n) R_{T_{q,n}}(\alpha) \\
&\quad \cdot \left[I_{(n+1)q},\, T_{q,n} \left(I_{(n+1)q} - V_{q,n} H_{\alpha \triangleright n-1} H_{\alpha \triangleright n-1}^- V_{q,n}^* \right) H_n \right] (I_2 \otimes v_{q,n}) \qquad (3.39)
\end{aligned}
$$

and

$$
\begin{aligned}
&(I_{nq} - H_{\alpha \triangleright n-1}^+ H_{\alpha \triangleright n-1}) V_{q,n}^* R_{T_{q,n}}(\alpha) \left[I_{(n+1)q},\, \left[R_{T_{q,n}}(\alpha) \right]^{-1} H_n \right] \\
&\quad \cdot (I_2 \otimes v_{q,n}) \widetilde{C}_{n,\alpha} \\
&\quad = (I_{nq} - H_{\alpha \triangleright n-1}^+ H_{\alpha \triangleright n-1}) V_{q,n}^* \left[(I_{(n+1)q} - H_n H_n^-) R_{T_{q,n}}(\alpha),\, H_n \right] \\
&\quad \cdot (I_2 \otimes v_{q,n}).
\end{aligned}
$$

Proof. Our proof is a modification of the proof of [21, Lemma 12.2]. Let $n \in \mathbb{N}$ be such that $2n \le \kappa$. Because of Remark 3.9 (b), Remark 3.1, and [21, Remark 6.13], we have

$$
\begin{aligned}
&\left(I - H_n^+ H_n \right) R_{T_{q,n}}(\alpha) \left[v_{q,n} v_{q,n}^* H_n V_{q,n} H_{\alpha \triangleright n-1}^- V_{q,n}^* + T_{q,n} \right] \\
&\quad = \left(I - H_n^+ H_n \right) R_{T_{q,n}}(\alpha) \\
&\quad\quad \cdot \left[\left(\left[R_{T_{q,n}}(\alpha) \right]^{-1} H_n V_{q,n} - \mathfrak{V}_{q,n} H_{\alpha \triangleright n-1} \right) H_{\alpha \triangleright n-1}^- V_{q,n}^* + T_{q,n} \right] \\
&\quad = \left(I - H_n^+ H_n \right) H_n V_{q,n} H_{\alpha \triangleright n-1}^- V_{q,n}^* \\
&\quad\quad - \left(I - H_n^+ H_n \right) R_{T_{q,n}}(\alpha) T_{q,n} V_{q,n} H_{\alpha \triangleright n-1} H_{\alpha \triangleright n-1}^- V_{q,n}^* \\
&\quad\quad + \left(I - H_n^+ H_n \right) R_{T_{q,n}}(\alpha) T_{q,n} \\
&\quad = \left(I - H_n^+ H_n \right) R_{T_{q,n}}(\alpha) T_{q,n} \left(I - V_{q,n} H_{\alpha \triangleright n-1} H_{\alpha \triangleright n-1}^- V_{q,n}^* \right). \qquad (3.40)
\end{aligned}
$$

Applying Remark 3.18 and (3.40), we conclude

$$
\begin{aligned}
&(I_{(n+1)q} - H_n^+ H_n) R_{T_{q,n}}(\alpha) \left[I_{(n+1)q},\, T_{q,n} H_n \right] (I_2 \otimes v_{q,n}) C_{n,\alpha} \\
&\quad = \Big[(I_{(n+1)q} - H_n^+ H_n) R_{T_{q,n}}(\alpha), \\
&\quad\quad (I_{(n+1)q} - H_n^+ H_n) R_{T_{q,n}}(\alpha) \left[v_{q,n} v_{q,n}^* H_n V_{q,n} H_{\alpha \triangleright n-1}^- V_{q,n}^* + T_{q,n} \right] H_n \Big] \\
&\quad\quad \cdot (I_2 \otimes v_{q,n}) \\
&\quad = (I_{(n+1)q} - H_n^+ H_n) R_{T_{q,n}}(\alpha) \\
&\quad\quad \cdot \left[I_{(n+1)q},\, T_{q,n} \left(I_{(n+1)q} - V_{q,n} H_{\alpha \triangleright n-1} H_{\alpha \triangleright n-1}^- V_{q,n}^* \right) H_n \right] (I_2 \otimes v_{q,n}).
\end{aligned}
$$

That means that (3.39) is fulfilled. Again, equation $H_n^* = H_n$, Remark 3.9 (b), $H_{\alpha \triangleright n-1}^* = H_{\alpha \triangleright n-1}$, [21, Remark 6.13], and $\overline{\alpha} = \alpha$ deliver

$$
\left(I_{nq} - H_{\alpha \triangleright n-1}^+ H_{\alpha \triangleright n-1} \right) V_{q,n}^* \left(I_{(n+1)q} - H_n v_{q,n} v_{q,n}^* R_{T_{q,n}^*}(\alpha) H_n^- \right) R_{T_{q,n}}(\alpha)
$$

$$
\begin{aligned}
&= \left(I_{nq} - H^{+}_{\alpha\triangleright n-1} H_{\alpha\triangleright n-1}\right) \left[V^{*}_{q,n} - (v_{q,n} v^{*}_{q,n} H_n V_{q,n})^{*} R_{T^{*}_{q,n}}(\alpha)\, H^{-}_{n}\right] R_{T_{q,n}}(\alpha) \\
&= \left(I_{nq} - H^{+}_{\alpha\triangleright n-1} H_{\alpha\triangleright n-1}\right) \\
&\quad \cdot \left[V^{*}_{q,n} - \left(V^{*}_{q,n} H_n \left[R_{T_{q,n}}(\alpha)\right]^{-*} - H_{\alpha\triangleright n-1} \mathfrak{V}^{*}_{q,n}\right) R_{T^{*}_{q,n}}(\alpha)\, H^{-}_{n}\right] R_{T_{q,n}}(\alpha) \\
&= \left(I_{nq} - H^{+}_{\alpha\triangleright n-1} H_{\alpha\triangleright n-1}\right) V^{*}_{q,n} R_{T_{q,n}}(\alpha) \\
&\quad - \left(I_{nq} - H^{+}_{\alpha\triangleright n-1} H_{\alpha\triangleright n-1}\right) V^{*}_{q,n} H_n \left[R_{T_{q,n}}(\alpha)\right]^{-*} R_{T^{*}_{q,n}}(\alpha)\, H^{-}_{n} R_{T_{q,n}}(\alpha) \\
&\quad + \left(I_{nq} - H^{+}_{\alpha\triangleright n-1} H_{\alpha\triangleright n-1}\right) H_{\alpha\triangleright n-1} \mathfrak{V}^{*}_{q,n} R_{T^{*}_{q,n}}(\alpha)\, H^{-}_{n} R_{T_{q,n}}(\alpha) \\
&= (I_{nq} - H^{+}_{\alpha\triangleright n-1} H_{\alpha\triangleright n-1}) V^{*}_{q,n} (I_{(n+1)q} - H_n H^{-}_{n}) R_{T_{q,n}}(\alpha).
\end{aligned} \tag{3.41}
$$

Moreover, Remark 3.18 and (3.41) provide finally

$$
\begin{aligned}
&(I_{nq} - H^{+}_{\alpha\triangleright n-1} H_{\alpha\triangleright n-1}) V^{*}_{q,n} R_{T_{q,n}}(\alpha) \left[I_{(n+1)q}, \left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n\right] \\
&\quad \cdot (I_2 \otimes v_{q,n}) \widetilde{C}_{n,\alpha} \\
&= (I_{nq} - H^{+}_{\alpha\triangleright n-1} H_{\alpha\triangleright n-1}) V^{*}_{q,n} \\
&\quad \cdot \left[\left(I_{(n+1)q} - H_n v_{q,n} v^{*}_{q,n} R_{T^{*}_{q,n}}(\alpha)\, H^{-}_{n}\right) R_{T_{q,n}}(\alpha), H_n\right] (I_2 \otimes v_{q,n}) \\
&= (I_{nq} - H^{+}_{\alpha\triangleright n-1} H_{\alpha\triangleright n-1}) V^{*}_{q,n} \left[(I_{(n+1)q} - H_n H^{-}_{n}) R_{T_{q,n}}(\alpha), H_n\right] \\
&\quad \cdot (I_2 \otimes v_{q,n}).
\end{aligned}
$$

□

Lemma 3.42 *Suppose $\kappa \geq 2$ and let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}^{\geq,e}_{q,\kappa,\alpha}$. Further, let $n \in \mathbb{N}$ be such that $2n \leq \kappa$ and let the matrix-valued functions $\widetilde{P}_{n,\alpha}$, $\widetilde{Q}_{n,\alpha}$, and $\widetilde{S}_{n,\alpha}$ be defined on $\mathbb{C}$ and, for all $z \in \mathbb{C}$ be given by*

$$
\begin{aligned}
\widetilde{P}_{n,\alpha}(z) &:= I_{(n+1)q} + (z-\alpha)\left(I_{(n+1)q} - H^{+}_{n} H_n\right) T_{q,n} R_{T_{q,n}}(z) \\
&\qquad \cdot \left(I_{(n+1)q} - H_n H^{-}_{n}\right),
\end{aligned} \tag{3.42}
$$

$$
\begin{aligned}
\widetilde{Q}_{n,\alpha}(z) &:= I_{nq} + (z-\alpha)\left(I_{nq} - H^{+}_{\alpha\triangleright n-1} H_{\alpha\triangleright n-1}\right) T_{q,n-1} R_{T_{q,n-1}}(z) \\
&\qquad \cdot \left(I_{nq} - H_{\alpha\triangleright n-1} H^{-}_{\alpha\triangleright n-1}\right),
\end{aligned} \tag{3.43}
$$

and

$$
\begin{aligned}
\widetilde{S}_{n,\alpha}(z) &:= I_{nq} - (z-\alpha)\left(I_{nq} - H^{+}_{\alpha\triangleright n-1} H_{\alpha\triangleright n-1}\right) T_{q,n-1} R_{T_{q,n-1}}(\alpha) \\
&\qquad \cdot \left(I_{nq} - H_{\alpha\triangleright n-1} H^{-}_{\alpha\triangleright n-1}\right).
\end{aligned} \tag{3.44}
$$

For each $z \in \mathbb{C}$, then

$$
\begin{aligned}
&\operatorname{diag}\left(\widetilde{P}_{n,\alpha}(z),\ \widetilde{Q}_{n,\alpha}(z)\right) \\
&\quad \cdot \begin{bmatrix} I_{(n+1)q} & 0 \\ (z-\alpha)(I_{nq} - H^{+}_{\alpha\triangleright n-1} H_{\alpha\triangleright n-1}) V^{*}_{q,n} (I_{(n+1)q} - H_n H^{-}_{n}) & \widetilde{S}_{n,\alpha}(z) \end{bmatrix}
\end{aligned}
$$

$$\cdot \begin{bmatrix} I_{(n+1)q} & (I_{(n+1)q} - H_n^+ H_n) R_{T_{q,n}}(\alpha) T_{q,n} V_{q,n} (I_{nq} - H_{\alpha\triangleright n-1} H_{\alpha\triangleright n-1}^-) \\ 0 & I_{nq} \end{bmatrix}$$
$$\cdot \operatorname{diag}((I_{(n+1)q} - H_n^+ H_n) R_{T_{q,n}}(\alpha) v_{q,n}, \left(I_{nq} - H_{\alpha\triangleright n-1}^+ H_{\alpha\triangleright n-1}\right) V_{q,n}^* H_n v_{q,n})$$
$$= \begin{bmatrix} (I_{(n+1)q} - H_n^+ H_n) R_{T_{q,n}}(z) \left[I_{(n+1)q}, T_{q,n} H_n\right] (I_2 \otimes v_{q,n}) \Xi_{n,\alpha}(z) \\ \hdashline (I_{nq} - H_{\alpha\triangleright n-1}^+ H_{\alpha\triangleright n-1}) V_{q,n}^* R_{T_{q,n}}(z) \\ \cdot \left[I_{(n+1)q}, \left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n\right] (I_2 \otimes v_{q,n}) \\ \cdot \widetilde{\Xi}_{n,\alpha}(z) \cdot \operatorname{diag}\left((z-\alpha) I_q,\ I_q\right) \end{bmatrix}. \tag{3.45}$$

Proof Our proof is a modification of the proof of [21, Lemma 12.3]. Let $z \in \mathbb{C}$. Obviously, the product of matrices on the left-hand side of (3.45) coincides with

$$R_{n,\alpha}(z) := \operatorname{diag}\left(\widetilde{P}_{n,\alpha}(z),\ \widetilde{Q}_{n,\alpha}(z)\right) \begin{bmatrix} \widetilde{\Psi}_{n,\alpha}^{(1,1)}(z) & \widetilde{\Psi}_{n,\alpha}^{(1,2)}(z) \\ \widetilde{\Psi}_{n,\alpha}^{(2,1)}(z) & \widetilde{\Psi}_{n,\alpha}^{(2,2)}(z) \end{bmatrix} (I_2 \otimes v_{q,n}), \tag{3.46}$$

where

$$\widetilde{\Psi}_{n,\alpha}^{(1,1)}(z) := (I_{(n+1)q} - H_n^+ H_n) R_{T_{q,n}}(\alpha), \tag{3.47}$$
$$\widetilde{\Psi}_{n,\alpha}^{(1,2)}(z) := (I_{(n+1)q} - H_n^+ H_n) R_{T_{q,n}}(\alpha) T_{q,n} V_{q,n} \left(I_{nq} - H_{\alpha\triangleright n-1} H_{\alpha\triangleright n-1}^-\right) \cdot (I_{nq} - H_{\alpha\triangleright n-1}^+ H_{\alpha\triangleright n-1}) V_{q,n}^* H_n, \tag{3.48}$$
$$\widetilde{\Psi}_{n,\alpha}^{(2,1)}(z) := (z-\alpha)(I_{nq} - H_{\alpha\triangleright n-1}^+ H_{\alpha\triangleright n-1}) V_{q,n}^* (I_{(n+1)q} - H_n H_n^-) \cdot \left(I_{(n+1)q} - H_n^+ H_n\right) R_{T_{q,n}}(\alpha), \tag{3.49}$$

and

$$\begin{aligned} &\widetilde{\Psi}_{n,\alpha}^{(2,2)}(z) \\ &:= (z-\alpha)(I_{nq} - H_{\alpha\triangleright n-1}^+ H_{\alpha\triangleright n-1}) V_{q,n}^* (I_{(n+1)q} - H_n H_n^-) \left(I_{(n+1)q} - H_n^+ H_n\right) \\ &\quad \cdot R_{T_{q,n}}(\alpha) T_{q,n} V_{q,n} \left(I_{nq} - H_{\alpha\triangleright n-1} H_{\alpha\triangleright n-1}^-\right) (I_{nq} - H_{\alpha\triangleright n-1}^+ H_{\alpha\triangleright n-1}) V_{q,n}^* H_n \\ &\quad + \widetilde{S}_{n,\alpha}(z)(I_{nq} - H_{\alpha\triangleright n-1}^+ H_{\alpha\triangleright n-1}) V_{q,n}^* H_n. \end{aligned} \tag{3.50}$$

Because of (3.48) and [21, Remark 6.13], we have

$$\widetilde{\Psi}_{n,\alpha}^{(1,2)}(z) = (I_{(n+1)q} - H_n^+ H_n) R_{T_{q,n}}(\alpha) T_{q,n} V_{q,n} \cdot \left(I_{nq} - H_{\alpha\triangleright n-1} H_{\alpha\triangleright n-1}^-\right) V_{q,n}^* H_n. \tag{3.51}$$

Furthermore, (3.49) and [21, Remark 6.13] yield

$$\widetilde{\Psi}_{n,\alpha}^{(2,1)}(z) = (z-\alpha)(I_{nq} - H_{\alpha\triangleright n-1}^+ H_{\alpha\triangleright n-1}) V_{q,n}^* (I_{(n+1)q} - H_n H_n^-) R_{T_{q,n}}(\alpha). \tag{3.52}$$

From [21, Remark 6.13], Lemma 3.14, Remark 3.6, and (3.44) we conclude

$$
\begin{aligned}
&(z-\alpha)(I_{nq}-H^{+}_{\alpha\rhd n-1}H_{\alpha\rhd n-1})V^{*}_{q,n}(I_{(n+1)q}-H_nH_n^{-})\left(I_{(n+1)q}-H_n^{+}H_n\right)\\
&\quad\cdot R_{T_{q,n}}(\alpha)T_{q,n}V_{q,n}\left(I_{nq}-H_{\alpha\rhd n-1}H^{-}_{\alpha\rhd n-1}\right)\\
&=(z-\alpha)(I_{nq}-H^{+}_{\alpha\rhd n-1}H_{\alpha\rhd n-1})V^{*}_{q,n}(I_{(n+1)q}-H_nH_n^{-})R_{T_{q,n}}(\alpha)T_{q,n}V_{q,n}\\
&\quad\cdot\left(I_{nq}-H_{\alpha\rhd n-1}H^{-}_{\alpha\rhd n-1}\right)\\
&=(z-\alpha)(I_{nq}-H^{+}_{\alpha\rhd n-1}H_{\alpha\rhd n-1})V^{*}_{q,n}R_{T_{q,n}}(\alpha)T_{q,n}V_{q,n}\\
&\quad\cdot\left(I_{nq}-H_{\alpha\rhd n-1}H^{-}_{\alpha\rhd n-1}\right)\\
&=(z-\alpha)(I_{nq}-H^{+}_{\alpha\rhd n-1}H_{\alpha\rhd n-1})V^{*}_{q,n}T_{q,n}R_{T_{q,n}}(\alpha)V_{q,n}\\
&\quad\cdot\left(I_{nq}-H_{\alpha\rhd n-1}H^{-}_{\alpha\rhd n-1}\right)\\
&=(z-\alpha)(I_{nq}-H^{+}_{\alpha\rhd n-1}H_{\alpha\rhd n-1})T_{q,n-1}R_{T_{q,n-1}}(\alpha)\left(I_{nq}-H_{\alpha\rhd n-1}H^{-}_{\alpha\rhd n-1}\right)\\
&=I_{nq}-\widetilde{S}_{n,\alpha}(z).
\end{aligned}
\tag{3.53}
$$

Combining (3.50) and (3.53), we obtain

$$
\begin{aligned}
&\widetilde{\Psi}^{(2,2)}_{n,\alpha}(z)\\
&\quad=(z-\alpha)(I_{nq}-H^{+}_{\alpha\rhd n-1}H_{\alpha\rhd n-1})V^{*}_{q,n}(I_{(n+1)q}-H_nH_n^{-})\\
&\qquad\cdot\left(I_{(n+1)q}-H_n^{+}H_n\right)R_{T_{q,n}}(\alpha)T_{q,n}V_{q,n}\left(I_{nq}-H_{\alpha\rhd n-1}H^{-}_{\alpha\rhd n-1}\right)\\
&\qquad\cdot(I_{nq}-H^{+}_{\alpha\rhd n-1}H_{\alpha\rhd n-1})V^{*}_{q,n}H_n\\
&\qquad+\widetilde{S}_{n,\alpha}(z)(I_{nq}-H^{+}_{\alpha\rhd n-1}H_{\alpha\rhd n-1})V^{*}_{q,n}H_n\\
&\quad=(I_{nq}-H^{+}_{\alpha\rhd n-1}H_{\alpha\rhd n-1})V^{*}_{q,n}H_n.
\end{aligned}
\tag{3.54}
$$

From Remark 3.1 we infer $T_{q,n}=T_{q,n}V_{q,n}V^{*}_{q,n}$ and $V^{*}_{q,n}V_{q,n}=I_{nq}$. Hence,

$$
\begin{aligned}
&T_{q,n}\left(I_{(n+1)q}-V_{q,n}H_{\alpha\rhd n-1}H^{-}_{\alpha\rhd n-1}V^{*}_{q,n}\right)\\
&\quad=T_{q,n}\left(V_{q,n}V^{*}_{q,n}-V_{q,n}V^{*}_{q,n}V_{q,n}H_{\alpha\rhd n-1}H^{-}_{\alpha\rhd n-1}V^{*}_{q,n}\right)\\
&\quad=T_{q,n}V_{q,n}\left(I_{nq}-H_{\alpha\rhd n-1}H^{-}_{\alpha\rhd n-1}\right)V^{*}_{q,n}.
\end{aligned}
\tag{3.55}
$$

Using Lemma 3.38, Lemma 3.41, (3.55), (3.42), (3.47), and (3.51), we get

$$
\begin{aligned}
&(I_{(n+1)q}-H_n^{+}H_n)R_{T_{q,n}}(z)\left[I_{(n+1)q},T_{q,n}H_n\right](I_2\otimes v_{q,n})\Xi_{n,\alpha}(z)\\
&\quad=\left[I_{(n+1)q}+(z-\alpha)(I_{(n+1)q}-H_n^{+}H_n)T_{q,n}R_{T_{q,n}}(z)(I_{(n+1)q}-H_nH_n^{-})\right]\\
&\qquad\cdot(I_{(n+1)q}-H_n^{+}H_n)R_{T_{q,n}}(\alpha)\left[I_{(n+1)q},T_{q,n}H_n\right](I_2\otimes v_{q,n})C_{n,\alpha}\\
&\quad=\left[I_{(n+1)q}+(z-\alpha)(I_{(n+1)q}-H_n^{+}H_n)T_{q,n}R_{T_{q,n}}(z)(I_{(n+1)q}-H_nH_n^{-})\right]\\
&\qquad\cdot(I_{(n+1)q}-H_n^{+}H_n)R_{T_{q,n}}(\alpha)\\
&\qquad\cdot\left[I_{(n+1)q},T_{q,n}\left(I_{(n+1)q}-V_{q,n}H_{\alpha\rhd n-1}H^{-}_{\alpha\rhd n-1}V^{*}_{q,n}\right)H_n\right](I_2\otimes v_{q,n})
\end{aligned}
$$

$$
\begin{aligned}
&= \left[I_{(n+1)q} + (z-\alpha)(I_{(n+1)q} - H_n^+ H_n) T_{q,n} R_{T_{q,n}}(z)(I_{(n+1)q} - H_n H_n^-)\right] \\
&\quad \cdot (I_{(n+1)q} - H_n^+ H_n) R_{T_{q,n}}(\alpha) \\
&\quad \cdot \left[I_{(n+1)q}, T_{q,n} V_{q,n} (I_{nq} - H_{\alpha \triangleright n-1} H_{\alpha \triangleright n-1}^-) V_{q,n}^* H_n\right] (I_2 \otimes v_{q,n}) \\
&= \widetilde{P}_{n,\alpha}(z) \left[\Psi_{n,\alpha}^{(1,1)}(z), \Psi_{n,\alpha}^{(1,2)}(z)\right] (I_2 \otimes v_{q,n}).
\end{aligned} \tag{3.56}
$$

Similarly, Lemma 3.40, Remark 3.6, Lemma 3.41, (3.43), (3.52), and (3.54) provide

$$
\begin{aligned}
&(I_{nq} - H_{\alpha \triangleright n-1}^+ H_{\alpha \triangleright n-1}) V_{q,n}^* R_{T_{q,n}}(z) \left[I_{(n+1)q}, \left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n\right] \\
&\quad \cdot (I_2 \otimes v_{q,n}) \widetilde{\Xi}_{n,\alpha}(z) \cdot \operatorname{diag}((z-\alpha) I_q, I_q) \\
&= \Big\{ I_{nq} + (z-\alpha)(I_{nq} - H_{\alpha \triangleright n-1}^+ H_{\alpha \triangleright n-1}) V_{q,n}^* T_{q,n} R_{T_{q,n}}(z) V_{q,n} \\
&\quad \cdot (I_{nq} - H_{\alpha \triangleright n-1} H_{\alpha \triangleright n-1}^-) \Big\} (I_{nq} - H_{\alpha \triangleright n-1}^+ H_{\alpha \triangleright n-1}) V_{q,n}^* R_{T_{q,n}}(\alpha) \\
&\quad \cdot \left[I_{(n+1)q}, \left[R_{T_{q,n}}(\alpha)\right]^{-1} H_n\right] (I_2 \otimes v_{q,n}) \widetilde{C}_{n,\alpha} \cdot \operatorname{diag}((z-\alpha) I_q, I_q) \\
&= \Big\{ I_{nq} + (z-\alpha)(I_{nq} - H_{\alpha \triangleright n-1}^+ H_{\alpha \triangleright n-1}) T_{q,n-1} R_{T_{q,n-1}}(z) \\
&\quad \cdot (I_{nq} - H_{\alpha \triangleright n-1} H_{\alpha \triangleright n-1}^-) \Big\} (I_{nq} - H_{\alpha \triangleright n-1}^+ H_{\alpha \triangleright n-1}) V_{q,n}^* \\
&\quad \cdot \left[(I_{(n+1)q} - H_n H_n^-) R_{T_{q,n}}(\alpha), H_n\right] (I_2 \otimes v_{q,n}) \cdot \operatorname{diag}((z-\alpha) I_q, I_q) \\
&= \Big\{ I_{nq} + (z-\alpha)(I_{nq} - H_{\alpha \triangleright n-1}^+ H_{\alpha \triangleright n-1}) T_{q,n-1} R_{T_{q,n-1}}(z) \\
&\quad \cdot (I_{nq} - H_{\alpha \triangleright n-1} H_{\alpha \triangleright n-1}^-) \Big\} (I_{nq} - H_{\alpha \triangleright n-1}^+ H_{\alpha \triangleright n-1}) V_{q,n}^* \\
&\quad \cdot \left[(z-\alpha)(I_{(n+1)q} - H_n H_n^-) R_{T_{q,n}}(\alpha), H_n\right] (I_2 \otimes v_{q,n}) \\
&= \widetilde{Q}_{n,\alpha}(z) \left[\Psi_{n,\alpha}^{(2,1)}(z), \Psi_{n,\alpha}^{(2,2)}(z)\right] (I_2 \otimes v_{q,n}).
\end{aligned} \tag{3.57}
$$

Since $R_{n,\alpha}(z)$ given by (3.46) coincides with the matrix on the left-hand side of (3.45), equation (3.45) follows from (3.56) and (3.57). □

Lemma 3.43 *Suppose $\kappa \geq 2$ and let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$. For each $n \in \mathbb{N}$ such that $2n \leq \kappa$, then the following statements hold true:*

(a) The set $\mathcal{N}_{\det \widetilde{P}_{n,\alpha}} \cup \mathcal{N}_{\det \widetilde{Q}_{n,\alpha}} \cup \mathcal{N}_{\det \widetilde{S}_{n,\alpha}}$ is finite.

(b) Let $x, y \in \mathbb{C}^{q \times q}$. Then the following statements are equivalent:

(i) For all $z \in \mathbb{C} \setminus (\mathcal{N}_{\det \widetilde{P}_{n,\alpha}} \cup \mathcal{N}_{\det \widetilde{Q}_{n,\alpha}} \cup \mathcal{N}_{\det \widetilde{S}_{n,\alpha}})$, the equations

$$
\left(I_{(n+1)q} - H_n^+ H_n\right) R_{T_{q,n}}(z) \left[I_{(n+1)q}, T_{q,n} H_n\right] (I_2 \otimes v_{q,n}) \Xi_{n,\alpha}(z) \begin{bmatrix} x \\ y \end{bmatrix} = 0 \tag{3.58}
$$

and

$$(I_{nq} - H_{\alpha\triangleright n-1}^{+} H_{\alpha\triangleright n-1}) V_{q,n}^{*} R_{T_{q,n}}(z) \left[I_{(n+1)q}, \left[R_{T_{q,n}}(\alpha) \right]^{-1} H_n \right] (I_2 \otimes v_{q,n}) \widetilde{\Xi}_{n,\alpha}(z) \cdot \operatorname{diag}((z-\alpha) I_q, I_q) \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0 \tag{3.59}$$

are fulfilled.

(ii) *There exists a* $z \in \mathbb{C} \setminus (\mathcal{N}_{\det \widetilde{P}_{n,\alpha}} \cup \mathcal{N}_{\det \widetilde{Q}_{n,\alpha}} \cup \mathcal{N}_{\det \widetilde{S}_{n,\alpha}})$ *such that (3.58) and (3.59) hold true.*

(iii)

$$\left(I_{(n+1)q\times q} - H_n^{+} H_n \right) R_{T_{q,n}}(\alpha) v_{q,n} x = 0 \tag{3.60}$$

and

$$\left(I_{nq\times q} - H_{\alpha\triangleright n-1}^{+} H_{\alpha\triangleright n-1} \right) V_{q,n}^{*} H_n v_{q,n} y = 0. \tag{3.61}$$

Proof Our proof is a modification of the proof of [21, Proposition 12.4]. By virtue of Remark 3.2, (3.42), (3.43), and (3.44), we see that $\widetilde{P}_{n,\alpha}$, $\widetilde{Q}_{n,\alpha}$, and $\widetilde{S}_{n,\alpha}$ all are matrix polynomials with $\widetilde{P}_{n,\alpha}(\alpha) = I_{(n+1)q}$, $\widetilde{Q}_{n,\alpha}(\alpha) = I_{nq}$, and $\widetilde{S}_{n,\alpha}(\alpha) = I_{nq}$. In particular, $\det \widetilde{P}_{n,\alpha}$, $\det \widetilde{Q}_{n,\alpha}$, and $\det \widetilde{S}_{n,\alpha}$ are polynomials not vanishing identically. In view of the Fundamental Theorem of Algebra, the proof of (a) is complete. For each $z \in \mathbb{C}$, then Lemma 3.42 provides

$$\begin{aligned}
&\begin{bmatrix} (I_{(n+1)q} - H_n^{+} H_n) R_{T_{q,n}}(z) \left[I_{(n+1)q}, T_{q,n} H_n \right] (I_2 \otimes v_{q,n}) \Xi_{n,\alpha}(z) \begin{bmatrix} x \\ y \end{bmatrix} \\ \hdashline (I_{nq} - H_{\alpha\triangleright n-1}^{+} H_{\alpha\triangleright n-1}) V_{q,n}^{*} R_{T_{q,n}}(z) \left[I_{(n+1)q}, \left[R_{T_{q,n}}(\alpha) \right]^{-1} H_n \right] \\ \cdot (I_2 \otimes v_{q,n}) \widetilde{\Xi}_{n,\alpha}(z) \cdot \operatorname{diag}((z-\alpha) I_q, I_q) \begin{bmatrix} x \\ y \end{bmatrix} \end{bmatrix} \\
&= \operatorname{diag}\left(\widetilde{P}_{n,\alpha}(z), \ \widetilde{Q}_{n,\alpha}(z) \right) \\
&\quad \cdot \begin{bmatrix} I_{(n+1)q} & 0_{(n+1)q\times nq} \\ (z-\alpha)(I_{nq} - H_{\alpha\triangleright n-1}^{+} H_{\alpha\triangleright n-1}) V_{q,n}^{*} (I_{(n+1)q} - H_n H_n^{-}) & \widetilde{S}_{n,\alpha}(z) \end{bmatrix} \\
&\quad \cdot \begin{bmatrix} I_{(n+1)q} & (I_{(n+1)q} - H_n^{+} H_n) R_{T_{q,n}}(\alpha) T_{q,n} V_{q,n} (I_{nq} - H_{\alpha\triangleright n-1} H_{\alpha\triangleright n-1}^{-}) \\ 0_{nq\times(n+1)q} & I_{nq} \end{bmatrix} \\
&\quad \cdot \begin{bmatrix} (I_{(n+1)q} - H_n^{+} H_n) R_{T_{q,n}}(\alpha) v_{q,n} x \\ (I_{nq} - H_{\alpha\triangleright n-1}^{+} H_{\alpha\triangleright n-1}) V_{q,n}^{*} H_n v_{q,n} y \end{bmatrix}.
\end{aligned} \tag{3.62}$$

(i) $\Rightarrow$ (ii): This implication is trivial.
(ii) $\Rightarrow$ (iii): According to (ii), there exists a $z \in \mathbb{C} \setminus (\mathcal{N}_{\det \widetilde{P}_{n,\alpha}} \cup \mathcal{N}_{\det \widetilde{Q}_{n,\alpha}} \cup \mathcal{N}_{\det \widetilde{S}_{n,\alpha}})$ such that (3.58) and (3.59) hold true. Using (3.58), (3.59), and (3.62), we get

$$\begin{bmatrix} 0_{(n+1)q\times q} \\ 0_{nq\times q} \end{bmatrix} = \operatorname{diag}\left(\widetilde{P}_{n,\alpha}(z), \ \widetilde{Q}_{n,\alpha}(z) \right)$$

$$\cdot \begin{bmatrix} I_{(n+1)q} & 0_{(n+1)q\times nq} \\ (z-\alpha)(I_{nq} - H^+_{\alpha\rhd n-1} H_{\alpha\rhd n-1}) V^*_{q,n}(I_{(n+1)q} - H_n H_n^-) & \widetilde{S}_{n,\alpha}(z) \end{bmatrix}$$
$$\cdot \begin{bmatrix} I_{(n+1)q} & (I_{(n+1)q} - H_n^+ H_n) R_{T_{q,n}}(\alpha) T_{q,n} V_{q,n}(I_{nq} - H_{\alpha\rhd n-1} H^-_{\alpha\rhd n-1}) \\ 0_{nq\times(n+1)q} & I_{nq} \end{bmatrix}$$
$$\cdot \begin{bmatrix} (I_{(n+1)q} - H_n^+ H_n) R_{T_{q,n}}(\alpha) v_{q,n} x \\ (I_{nq} - H^+_{\alpha\rhd n-1} H_{\alpha\rhd n-1}) V^*_{q,n} H_n v_{q,n} y \end{bmatrix}, \tag{3.63}$$

where the first three factors of the matrix product on the right-hand side of equation (3.63) all are non-singular matrices. Thus, (3.63) implies (3.60) and (3.61).
(iii) $\Rightarrow$ (i): Taking into account (3.60), (3.61), and (3.62), we conclude that the matrix on the left-hand side of (3.62) is a null matrix and, consequently, that (3.58) and (3.59) hold true for each $z \in \mathbb{C}$. □

Let $\mathcal{U}$ be a subspace of $\mathbb{C}^q$. Then there exists exactly one matrix $P_{\mathcal{U}} \in \mathbb{C}^{q\times q}$ such that $P_{\mathcal{U}} x \in \mathcal{U}$ and $x - P_{\mathcal{U}} x \in \mathcal{U}^\perp$ are fulfilled for each $x \in \mathbb{C}^q$. This matrix $P_{\mathcal{U}}$ is called the orthogonal projection matrix onto $\mathcal{U}$. One can easily check that the matrix $P_{\mathcal{U}}$ coincides with the unique complex $q \times q$ matrix P which fulfills the three conditions $P^2 = P$, $P^* = P$, and $\mathcal{R}(P) = \mathcal{U}$.

Lemma 3.44 *Suppose $\kappa \geq 1$ and let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}^{\geq,e}_{q,\kappa,\alpha}$. For all $n \in \mathbb{N}_0$ with $2n \leq \kappa$, then:*

(a) The sets

$$\mathcal{U}_{n,\alpha} := \left[\mathcal{N}\left(\left(I_{(n+1)q} - H_n^+ H_n\right) R_{T_{q,n}}(\alpha) v_{q,n}\right)\right]^\perp \tag{3.64}$$

and

$$\mathcal{W}_{n,\alpha} := \begin{cases} \left[\mathcal{N}\left((I_{nq} - H^+_{\alpha\rhd n-1} H_{\alpha\rhd n-1}) V^*_{q,n} H_n v_{q,n}\right)\right]^\perp, & \text{if } n \geq 1 \\ \{0_{q\times 1}\}, & \text{if } n = 0 \end{cases} \tag{3.65}$$

are orthogonal subspaces of $\mathbb{C}^q$ with $\dim \mathcal{U}_{n,\alpha} = \operatorname{rank}\left[\left(I - H_n^+ H_n\right) R_{T_{q,n}}(\alpha) v_{q,n}\right]$ and, in the case $n \geq 1$, with $\dim \mathcal{W}_{n,\alpha} = \operatorname{rank}\left[\left(I - H^+_{\alpha\rhd n-1} H_{\alpha\rhd n-1}\right) V^*_{q,n} H_n v_{q,n}\right]$.

(b) Let $A \in \mathbb{C}^{q\times p}$. Then $\left(I - H_n^+ H_n\right) R_{T_{q,n}}(\alpha) v_{q,n} A = 0$ if and only if $P_{\mathcal{U}_{n,\alpha}} A = 0$.

(c) Let $A \in \mathbb{C}^{q\times p}$. If $n \geq 1$, then $\left(I - H^+_{\alpha\rhd n-1} H_{\alpha\rhd n-1}\right) V^*_{q,n} H_n v_{q,n} A = 0$ if and only if $P_{\mathcal{W}_{n,\alpha}} A = 0$.

Proof Lemma 3.44 can be proved analogous to the proof of [21, Lemma 14.1]. We omit the details. □

4 Stieltjes Pairs of Meromorphic Matrix-Valued Functions

In this section, a certain class of pairs of meromorphic matrix-valued functions will be considered, which will be used to parametrize the solution set of the matricial

Stieltjes-type moment problem $\mathsf{M}[[\alpha, \infty); (s_j)_{j=0}^{m}, \leq]$. As we can see from [29] and [21], respectively, it will be irrelevant whether the problem if one considers the even or the odd case. Therefore, in this section, we modify results from [29, Chapter 8] and [21, Chapter 8]. Since the approaches presented in this paper are largely independent of each other, we will repeat the definition of the Stieltjes pairs and some other results at this point.

Definition 4.1 ([21, Definition 8.1]) Let $\alpha \in \mathbb{R}$. Let ϕ and ψ be $q \times q$ matrix-valued functions meromorphic in $\mathbb{C}\setminus[\alpha, \infty)$. Then $[\phi; \psi]$ is called a $q \times q$-*Stieltjes pair* in $\mathbb{C}\setminus[\alpha, \infty)$ if there exists a discrete subset $\mathcal{D}$ of $\mathbb{C}\setminus[\alpha, \infty)$ such that ϕ and ψ are holomorphic in $\mathbb{C}\setminus([\alpha, \infty) \cup \mathcal{D})$ and that

$$\operatorname{rank} \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} = q \tag{4.1}$$

for all $z \in \mathbb{C}\setminus([\alpha, \infty) \cup \mathcal{D})$ is valid and that both inequalities

$$\begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix}^* \left(\frac{-\tilde{J}_q}{2\Im z} \right) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} \geq 0_{q\times q} \tag{4.2}$$

and

$$\begin{bmatrix} (z-\alpha)\phi(z) \\ \psi(z) \end{bmatrix}^* \left(\frac{-\tilde{J}_q}{2\Im z} \right) \begin{bmatrix} (z-\alpha)\phi(z) \\ \psi(z) \end{bmatrix} \geq 0_{q\times q} \tag{4.3}$$

hold true for all $z \in \mathbb{C}\setminus(\mathbb{R} \cup \mathcal{D})$. The set of all $q \times q$-Stieltjes pairs in $\mathbb{C}\setminus[\alpha, \infty)$ will be denoted by $\mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C}\setminus[\alpha, \infty))$.

Note that $q \times q$-Stieltjes pairs in $\mathbb{C}\setminus[\alpha, \infty)$ also are used [21, 23], and [33].

Remark 4.2 In view of Remark 3.15, it is readily checked that (4.2) and (4.3) are fulfilled for all $z \in \mathbb{C}\setminus(\mathbb{R} \cup \mathcal{D})$ if and only if $\frac{1}{\Im z}\Im[\psi^*(z)\phi(z)] \in \mathbb{C}_{\geq}^{q\times q}$ and $\frac{1}{\Im z}\Im[(z-\alpha)\psi^*(z)\phi(z)] \in \mathbb{C}_{\geq}^{q\times q}$ for all $z \in \mathbb{C}\setminus(\mathbb{R} \cup \mathcal{D})$.

Remark 4.3 ([21, Remarks 8.2 and 8.3]) Let $[\phi; \psi] \in \mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C}\setminus[\alpha, \infty))$ and let g be a $q \times q$ matrix-valued function which is meromorphic in $\mathbb{C}\setminus[\alpha, \infty)$ such that $\det g$ does not vanish identically. Then it is readily checked that $[\phi g; \psi g]$ belongs to $\mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C}\setminus[\alpha, \infty))$ as well. Two $q \times q$-Stieltjes pairs $[\phi_1; \psi_1]$ and $[\phi_2; \psi_2]$ in $\mathbb{C}\setminus[\alpha, \infty)$ are said to be equivalent if there are a $q \times q$ matrix-valued function g being meromorphic in $\mathbb{C}\setminus[\alpha, \infty)$ and a discrete subset $\mathcal{D}$ of $\mathbb{C}\setminus[\alpha, \infty)$ such that $\phi_1, \phi_2, \psi_1, \psi_2$, and g are holomorphic in $\mathbb{C}\setminus([\alpha, \infty) \cup \mathcal{D})$ and that $\det g(z) \neq 0$ and $\phi_2(z) = \phi_1(z)g(z)$ as well as $\psi_2(z) = \psi_1(z)g(z)$ hold true for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D})$. One can easily see that this causes an equivalence relation on $\mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C}\setminus[\alpha, \infty))$. For each $[\phi; \psi] \in \mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C}\setminus[\alpha, \infty))$, by $\langle[\phi; \psi]\rangle$ we denote the equivalence class generated by $[\phi; \psi]$. Furthermore, let $\langle\mathcal{M}\rangle := \{\langle[\phi; \psi]\rangle : [\phi; \psi] \in \mathcal{M}\}$ for each subset $\mathcal{M}$ of $\mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C}\setminus[\alpha, \infty))$.

Remark 4.4 ([23, Proposition 7.7]) Let $f \in \mathcal{S}_{q,[\alpha,\infty)}$. Then $\left[f; I_q\right]$ belongs to $\mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C}\backslash[\alpha,\infty))$. In particular, $\left[0_{q\times q}; I_q\right] \in \mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C}\backslash[\alpha,\infty))$. Furthermore, if $f, g \in \mathcal{S}_{q,[\alpha,\infty)}$ are such that the pairs $\left[f; I_q\right]$ and $\left[g; I_q\right]$ are equivalent, then $f = g$.

Remark 4.5 Let ϕ and ψ be $q \times q$ matrix-valued functions which are meromorphic in $\mathbb{C}\backslash[\alpha,\infty)$. Then one can see (compare with [23, Proposition 7.6]) that $[\phi; \psi]$ belongs to $\mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C}\backslash[\alpha,\infty))$ if and only if there exists a discrete subset $\mathcal{D}$ of $\mathbb{C}\backslash[\alpha,\infty)$ such that ϕ and ψ are holomorphic in $\mathbb{C}\backslash([\alpha,\infty)\cup\mathcal{D})$ and fulfill (4.1) and (4.2) for each $z \in \mathbb{C}\backslash(\mathbb{R}\cup\mathcal{D})$ as well as $\Re[\psi^*(z)\phi(z)] \in \mathbb{C}^{q\times q}_{\geq}$ for each $z \in \mathbb{C}_{\alpha,-}\backslash\mathcal{D}$, where $\mathbb{C}_{\alpha,-}:=\{z \in \mathbb{C} : \Re z \in (-\infty,\alpha)\}$.

Lemma 4.6 *Let $\left(s_j\right)_{j=0}^{\kappa} \in \mathcal{K}^{\geq,e}_{q,\kappa,\alpha}$. Let ϕ and ψ be $q \times q$ matrix-valued functions which are meromorphic in $\mathbb{C} \setminus [\alpha,\infty)$. Let $n \in \mathbb{N}_0$ be such that $2n \leq \kappa$ and let $\Xi_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ be defined by (3.27) in the case $n = 0$ and by (3.24) in the case $n \geq 1$. Let $\widehat{\Xi}_{n,\alpha}:= \operatorname{Rstr}_{\mathbb{C}\backslash[\alpha,\infty)} \Xi_{n,\alpha}$ and let*

$$\widehat{\Xi}_{n,\alpha} = [\widehat{\Xi}^{(j,k)}_{n,\alpha}]^2_{j,k=1} \tag{4.4}$$

be the $q \times q$ block representation of $\widehat{\Xi}_{n,\alpha}$. Let

$$\widetilde{\phi}:=\widehat{\Xi}^{(1,1)}_{n,\alpha}\phi + \widehat{\Xi}^{(1,2)}_{n,\alpha}\psi \quad \text{and} \quad \widetilde{\psi}:=\widehat{\Xi}^{(2,1)}_{n,\alpha}\phi + \widehat{\Xi}^{(2,2)}_{n,\alpha}\psi. \tag{4.5}$$

Furthermore, let $z \in (\mathbb{H}_\phi \cap \mathbb{H}_\psi) \setminus \mathbb{R}$ be such that (4.2) and

$$(I_{(n+1)q} - H_n^+ H_n) R_{T_{q,n}}(\alpha) v_{q,n}\phi(z) = 0_{(n+1)q\times q} \tag{4.6}$$

hold true. Then $\mathcal{N}(\widetilde{\psi}(z)) \subseteq \mathcal{N}(\widetilde{\phi}(z))$. Moreover, if (4.1) is valid, then $\det \widetilde{\psi}(z) \neq 0$.

Proof Our proof modifies the proof of [21, Lemma 13.1]. By virtue of $\mathcal{K}^{\geq,e}_{q,\kappa,\alpha} \subseteq \mathcal{K}^{\geq}_{q,\kappa,\alpha}$ as well as [21, Lemma 6.12] and Remark 2.2, we have $H_n^- \in \mathbb{C}^{q\times q}_{\geq}$, $H_n H_n^- H_n = H_n$, and $H_n^- H_n H_n^- = H_n^-$. We consider an arbitrary $y \in \mathcal{N}(\widetilde{\psi}(z))$. Because of Remark 3.15, we get then

$$y^* \begin{bmatrix} \widetilde{\phi}(z) \\ \widetilde{\psi}(z) \end{bmatrix}^* \tilde{J}_q \begin{bmatrix} \widetilde{\phi}(z) \\ \widetilde{\psi}(z) \end{bmatrix} y = i y^* (\widetilde{\psi}^*(z)\widetilde{\phi}(z) - \widetilde{\phi}^*(z)\widetilde{\psi}(z)) y = 0. \tag{4.7}$$

Obviously, $\Xi_{n,\alpha}(z) = \widehat{\Xi}_{n,\alpha}(z)$. From (4.4) and (4.5) we infer

$$\widehat{\Xi}_{n,\alpha}(z) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} = \begin{bmatrix} \widetilde{\phi}(z) \\ \widetilde{\psi}(z) \end{bmatrix}. \tag{4.8}$$

Using (4.8) and (4.7), straightforward calculation yields

$$y^* \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix}^* \left(\tilde{J}_q - \Xi^*_{n,\alpha}(z)\tilde{J}_q \Xi_{n,\alpha}(z)\right) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} y = -y^* \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix}^* (-\tilde{J}_q) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} y. \tag{4.9}$$

Now we consider the case $n \geq 1$. Because of [21, Lemma 6.12], Remark 3.2, Remark 3.31, (4.9), $i(\bar{z}-z) = 2\Im z$, and (4.2), for $B := \sqrt{H_n}[R_{T^*_{q,n}}(\alpha)]^{-1} R_{T^*_{q,n}} H_n^- R_{T_{q,n}}(\alpha) \left[I_{(n+1)q}, T_{q,n}H_n\right](I_2 \otimes v_{q,n})C_{n,\alpha}[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}]y$, we obtain

$$\begin{aligned}
0 &\leq \|B(z)\|_E^2\\
&= y^* \begin{bmatrix}\phi(z)\\ \psi(z)\end{bmatrix}^* C^*_{n,\alpha}(I_2\otimes v_{q,n})^* \left[I_{(n+1)q}, T_{q,n}H_n\right]^* [R_{T_{q,n}}(\alpha)]^* H_n^-\\
&\quad \cdot [R_{T^*_{q,n}}(z)]^*[R_{T_{q,n}}(\alpha)]^{-1} H_n [R_{T^*_{q,n}}(\alpha)]^{-1} R_{T^*_{q,n}}(z) H_n^-\\
&\quad \cdot R_{T_{q,n}}(\alpha)\left[I_{(n+1)q}, T_{q,n}H_n\right](I_2\otimes v_{q,n})C_{n,\alpha}\begin{bmatrix}\phi(z)\\ \psi(z)\end{bmatrix} y\\
&= y^* \begin{bmatrix}\phi(z)\\ \psi(z)\end{bmatrix}^* \frac{1}{i(\bar{z}-z)}\left(\tilde{J}_q - \Xi^*_{n,\alpha}(z)\tilde{J}_q \Xi_{n,\alpha}(z)\right)\begin{bmatrix}\phi(z)\\ \psi(z)\end{bmatrix} y\\
&= -y^* \begin{bmatrix}\phi(z)\\ \psi(z)\end{bmatrix}^* \left(\frac{-\tilde{J}_q}{2\Im z}\right)\begin{bmatrix}\phi(z)\\ \psi(z)\end{bmatrix} y \leq 0
\end{aligned}$$

and, consequently, $B(z) = 0$. Thus, using Remark 3.19, we conclude

$$\begin{aligned}
&H_n[R_{T^*_{q,n}}(\alpha)]^{-1} R_{T^*_{q,n}}(z)\left[H_n^- R_{T_{q,n}}(\alpha),\, V_{q,n}H^-_{\alpha\triangleright n-1}V^*_{q,n}H_n\right]\\
&\cdot(I_2\otimes v_{q,n})\begin{bmatrix}\phi(z)\\ \psi(z)\end{bmatrix} y = 0.
\end{aligned}\tag{4.10}$$

From (4.8), $\Xi_{n,\alpha}(z) = \widehat{\Xi}_{n,\alpha}(z)$, Lemma 3.23, and $v^*_{q,n}R_{T_{q,n}}(\alpha)v_{q,n} = I_q$, we get

$$\begin{aligned}
\widetilde{\phi}(z)y &= [I_q,\ 0_{q\times q}]\begin{bmatrix}\widetilde{\phi}(z)\\ \widetilde{\psi}(z)\end{bmatrix} y = [I_q,\ 0_{q\times q}]\Xi_{n,\alpha}(z)\begin{bmatrix}\phi(z)\\ \psi(z)\end{bmatrix} y\\
&= \big[I_q + (z-\alpha)v^*_{q,n}H_nT^*_{q,n}R_{T^*_{q,n}}(z)H_n^- R_{T_{q,n}}(\alpha)v_{q,n},\\
&\quad v^*_{q,n}H_n[R_{T^*_{q,n}}(\alpha)]^{-1}R_{T^*_{q,n}}(z)V_{q,n}H^-_{\alpha\triangleright n-1}V^*_{q,n}H_nv_{q,n}\big]\begin{bmatrix}\phi(z)\\ \psi(z)\end{bmatrix} y\\
&= \big[v^*_{q,n}R_{T_{q,n}}(\alpha)v_{q,n} + (z-\alpha)v^*_{q,n}H_nT^*_{q,n}R_{T^*_{q,n}}(z)H_n^- R_{T_{q,n}}(\alpha)v_{q,n},\\
&\quad v^*_{q,n}H_n[R_{T^*_{q,n}}(\alpha)]^{-1}R_{T^*_{q,n}}(z)V_{q,n}H^-_{\alpha\triangleright n-1}V^*_{q,n}H_nv_{q,n}\big]\begin{bmatrix}\phi(z)\\ \psi(z)\end{bmatrix} y\\
&= v^*_{q,n}\big[R_{T_{q,n}}(\alpha) + (z-\alpha)H_nT^*_{q,n}R_{T^*_{q,n}}(z)H_n^- R_{T_{q,n}}(\alpha),\\
&\quad H_n[R_{T^*_{q,n}}(\alpha)]^{-1}R_{T^*_{q,n}}(z)V_{q,n}H^-_{\alpha\triangleright n-1}V^*_{q,n}H_n\big](I_2\otimes v_{q,n})\begin{bmatrix}\phi(z)\\ \psi(z)\end{bmatrix} y.
\end{aligned}\tag{4.11}$$

Remark 3.7 yields $(z-\alpha)T^*_{q,n}R_{T^*_{q,n}}(z) = [R_{T^*_{q,n}}(\alpha)]^{-1}R_{T^*_{q,n}}(z) - I_{(n+1)q}$, which implies

$$(z-\alpha)H_nT^*_{q,n}R_{T^*_{q,n}}(z)H_n^- R_{T_{q,n}}(\alpha)$$

$$= H_n \left([R_{T_{q,n}^*}(\alpha)]^{-1} R_{T_{q,n}^*}(z) - I_{(n+1)q} \right) H_n^- R_{T_{q,n}}(\alpha)$$
$$= H_n [R_{T_{q,n}^*}(\alpha)]^{-1} R_{T_{q,n}^*}(z) H_n^- R_{T_{q,n}}(\alpha) - H_n H_n^- R_{T_{q,n}}(\alpha). \tag{4.12}$$

Taking (4.11), (4.12), and (4.10) into account, we have

$$\begin{aligned}
&\widetilde{\phi}(z)y \\
&= v_{q,n}^* \Big[R_{T_{q,n}}(\alpha) + H_n [R_{T_{q,n}^*}(\alpha)]^{-1} R_{T_{q,n}^*}(z) H_n^- R_{T_{q,n}}(\alpha) - H_n H_n^- R_{T_{q,n}}(\alpha), \\
&\quad H_n [R_{T_{q,n}^*}(\alpha)]^{-1} R_{T_{q,n}^*}(z) V_{q,n} H_{\alpha \triangleright n-1}^- V_{q,n}^* H_n \Big] \\
&\quad \cdot (I_2 \otimes v_{q,n}) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} y \\
&= v_{q,n}^* \left[R_{T_{q,n}}(\alpha) - H_n H_n^- R_{T_{q,n}}(\alpha), 0_{(n+1)q \times (n+1)q} \right] (I_2 \otimes v_{q,n}) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} y \\
&\quad + v_{q,n}^* \Big[H_n [R_{T_{q,n}^*}(\alpha)]^{-1} R_{T_{q,n}^*}(z) H_n^- R_{T_{q,n}}(\alpha), \\
&\quad H_n [R_{T_{q,n}^*}(\alpha)]^{-1} R_{T_{q,n}^*}(z) V_{q,n} H_{\alpha \triangleright n-1}^- V_{q,n}^* H_n \Big] \\
&\quad \cdot (I_2 \otimes v_{q,n}) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} y \\
&= v_{q,n}^* \left[R_{T_{q,n}}(\alpha) - H_n H_n^- R_{T_{q,n}}(\alpha), 0_{(n+1)q \times (n+1)q} \right] \\
&\quad \cdot (I_2 \otimes v_{q,n}) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} y.
\end{aligned} \tag{4.13}$$

Then, using (4.13), [21, Remark 6.13], and (4.6), we conclude

$$\begin{aligned}
\widetilde{\phi}(z)y &= \left[v_{q,n}^* R_{T_{q,n}}(\alpha) v_{q,n} - v_{q,n}^* H_n H_n^- R_{T_{q,n}}(\alpha) v_{q,n},\ 0_{q \times q} \right] \begin{bmatrix} \phi(z)y \\ \psi(z)y \end{bmatrix} \\
&= v_{q,n}^* \left(I - H_n H_n^- \right) R_{T_{q,n}}(\alpha) v_{q,n} \phi(z) y \\
&= v_{q,n}^* \left(I - H_n H_n^- \right) \left(I - H_n^+ H_n \right) R_{T_{q,n}}(\alpha) v_{q,n} \phi(z) y = 0
\end{aligned}$$

and, consequently, $y \in \mathcal{N}(\widetilde{\phi}(z))$. Thus, $\mathcal{N}(\widetilde{\psi}(z)) \subseteq \mathcal{N}(\widetilde{\phi}(z))$.
Now suppose (4.1). We consider again an arbitrary $y \in \mathcal{N}(\widetilde{\psi}(z))$. Then we already know that $y \in \mathcal{N}(\widetilde{\phi}(z))$. In view of Remark 3.29, (4.8), and due to the choice of y, we infer $\begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} y = [\Xi_{n,\alpha}(z)]^{-1} \widehat{\Xi}_{n,\alpha}(z) \begin{bmatrix} \phi(z) \\ \psi(z) \end{bmatrix} y = [\Xi_{n,\alpha}(z)]^{-1} \begin{bmatrix} \phi(z)y \\ \psi(z)y \end{bmatrix} = 0$. Because of (4.1), this implies $y = 0_{q \times 1}$, and, hence, $\det \widetilde{\psi}(z) \neq 0$. In the case $n \geq 1$, the proof is complete. If $n = 0$, then the assertion analogously can be checked using Remarks 3.24 and A.1. □

The following remark is similar to [29, Lemma 10.23] and [21, Lemma 13.2].

Remark 4.7 Suppose $\kappa \geq 2$. Let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$, let $n \in \mathbb{N}$ be such that $2n \leq \kappa$, and let $[\phi; \psi] \in \mathcal{P}_{-\tilde{J}_q,\geq}^{(q,q)}(\mathbb{C}\setminus[\alpha,\infty))$ be such that $\left(I_{(n+1)q} - H_n^+ H_n\right) R_{T_{q,n}}(\alpha) v_{q,n}\phi = 0_{(n+1)q\times q}$. Let $\Xi_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ be defined by (3.24), and let (4.4) be the $q \times q$ block partition of $\widehat{\Xi}_{n,\alpha} := \operatorname{Rstr}_{\mathbb{C}\setminus[\alpha,\infty)} \Xi_{n,\alpha}$. Similar to [21, Lemma 13.2], using Definition 4.1 and Lemma 4.6, it is readily checked that then there is a discrete subset $\mathcal{D}$ of $\mathbb{C}\setminus[\alpha,\infty)$ such that ϕ and ψ are holomorphic in $\mathbb{C}\setminus([\alpha,\infty)\cup\mathcal{D})$ and that

$$\det\left(\widehat{\Xi}_{n,\alpha}^{(2,1)}(z)\phi(z) + \widehat{\Xi}_{n,\alpha}^{(2,2)}(z)\psi(z)\right) \neq 0 \tag{4.14}$$

holds true for each $z \in \mathbb{C}\setminus(\mathbb{R}\cup\mathcal{D})$.

5 A Particular Subclass of Stieltjes Pairs

In this section, we present a special class of pairs of meromorphic matrix-valued functions in order to parametrize the solution set of Problem $\mathsf{S}[[\alpha,\infty); (s_j)_{j=0}^m, \leq]$ in the case of an even integer m, generated still using the given data. In this way, we will modify certain statements of [21].

Notation 5.1 Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. For all $n \in \mathbb{N}_0$ with $2n \leq \kappa$, let $\mathcal{P}_{-\tilde{J}_q,\geq}^{(q,q)}[\mathbb{C}\setminus[\alpha,\infty), (s_j)_{j=0}^{2n}]$ be the set of all $[\phi; \psi] \in \mathcal{P}_{-\tilde{J}_q,\geq}^{(q,q)}(\mathbb{C}\setminus[\alpha,\infty))$ such that

$$(I_{(n+1)q} - H_n^+ H_n) R_{T_{q,n}}(\alpha) v_{q,n}\phi = 0_{(n+1)q\times q} \tag{5.1}$$

as well as, in the case $n \geq 1$, moreover

$$(I_{nq} - H_{\alpha\triangleright n-1}^+ H_{\alpha\triangleright n-1}) V_{q,n}^* H_n v_{q,n}\psi = 0_{nq\times q}. \tag{5.2}$$

hold true.

Observe that, in view of Lemma 3.44, equation (5.1) holds true if and only if $P_{\mathcal{U}_{n,\alpha}}\phi = 0$ is valid and that, moreover, (5.2) is equivalent to $P_{\mathcal{W}_{n,\alpha}}\psi = 0$.

Remark 5.2 Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n}$ be a sequence of complex $q \times q$ matrices. Let $[\phi; \psi] \in \mathcal{P}_{-\tilde{J}_q,\geq}^{(q,q)}[\mathbb{C}\setminus[\alpha,\infty), (s_j)_{j=0}^{2n}]$ and let g be a $q \times q$ matrix-valued function which is meromorphic in $\mathbb{C}\setminus[\alpha,\infty)$ such that the function $\det g$ does not vanish identically. In view of Remark 4.3, then $[\phi g; \psi g]$ belongs to $\mathcal{P}_{-\tilde{J}_q,\geq}^{(q,q)}[\mathbb{C}\setminus[\alpha,\infty), (s_j)_{j=0}^{2n}]$ as well.

Lemma 5.3 *Suppose $\kappa \geq 2$. Let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$, and let $n \in \mathbb{N}$ be such that $2n \leq \kappa$. Let $\Xi_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ be defined by (3.24), let $\widehat{\Xi}_{n,\alpha} := \operatorname{Rstr}_{\mathbb{C}\setminus[\alpha,\infty)} \Xi_{n,\alpha}$, let (4.4) be the $q \times q$ block representation of $\widehat{\Xi}_{n,\alpha}$, and let $\widehat{R}_{T_{q,n}} := \operatorname{Rstr}_{\mathbb{C}\setminus[\alpha,\infty)} R_{T_{q,n}}$.*

Let $[\phi;\psi] \in \mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C}\setminus[\alpha,\infty))$ *be such that* $\det(\widehat{\Xi}^{(2,1)}_{n,\alpha}\phi + \widehat{\Xi}^{(2,2)}_{n,\alpha}\psi)$ *does not vanish identically. Let*

$$\widehat{S}_{n,\alpha} := \big(\widehat{\Xi}^{(1,1)}_{n,\alpha}\phi + \widehat{\Xi}^{(1,2)}_{n,\alpha}\psi\big)\big(\widehat{\Xi}^{(2,1)}_{n,\alpha}\phi + \widehat{\Xi}^{(2,2)}_{n,\alpha}\psi\big)^{-1}. \tag{5.3}$$

Let $\widehat{\mathcal{E}} : \mathbb{C}\setminus[\alpha,\infty) \to \mathbb{C}$ *be given by* $\widehat{\mathcal{E}}(z) := z$. *Then* $[\phi;\psi] \in \mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}[\mathbb{C}\setminus[\alpha,\infty), (s_j)_{j=0}^{2n}]$ *if and only if*

$$(I_{(n+1)q} - H_n^+ H_n)\widehat{R}_{T_{q,n}}\left[I_{(n+1)q}, T_{q,n}H_n\right](I_2 \otimes v_{q,n})\begin{bmatrix}\widehat{S}_{n,\alpha}\\ I_q\end{bmatrix} = 0_{(n+1)q\times q} \tag{5.4}$$

and

$$\begin{aligned}&(I_{nq} - H^+_{\alpha\triangleright n-1}H_{\alpha\triangleright n-1})V^*_{q,n}\widehat{R}_{T_{q,n}}\left[I_{(n+1)q},\ [R_{T_{q,n}}(\alpha)]^{-1}H_n\right]\\ &\cdot (I_2\otimes v_{q,n})\begin{bmatrix}(\widehat{\mathcal{E}}-\alpha)\widehat{S}_{n,\alpha}\\ I_q\end{bmatrix} = 0_{nq\times q}.\end{aligned} \tag{5.5}$$

Proof Our proof is subdivided into twelve steps.
(I) Since $\det\left(\widehat{\Xi}^{(2,1)}_{n,\alpha}\phi + \widehat{\Xi}^{(2,2)}_{n,\alpha}\psi\right)$ does not vanish identically, there is a discrete subset $\mathcal{D}$ of $\mathbb{C}\setminus[\alpha,\infty)$ such that the conditions (i), (ii), and (iii) of Definition 4.1 hold true and that (4.14) is fulfilled for each $z \in \mathbb{C}\setminus([\alpha,\infty)\cup\mathcal{D})$.
(II) In view of (i), (5.3), and (4.14), the function $\widehat{S}_{n,\alpha}$ admits the representation

$$\widehat{S}_{n,\alpha}(z) = \left(\widehat{\Xi}^{(1,1)}_{n,\alpha}(z)\phi(z) + \widehat{\Xi}^{(1,2)}_{n,\alpha}(z)\psi(z)\right)\left(\widehat{\Xi}^{(2,1)}_{n,\alpha}(z)\phi(z) + \widehat{\Xi}^{(2,2)}_{n,\alpha}(z)\psi(z)\right)^{-1} \tag{5.6}$$

for each $z \in \mathbb{C}\setminus([\alpha,\infty)\cup\mathcal{D})$. Because of (i), (4.14), (4.4), and (5.6), we see that

$$\begin{aligned}&\widehat{\Xi}_{n,\alpha}(z)\begin{bmatrix}\phi(z)\\ \psi(z)\end{bmatrix}\left(\widehat{\Xi}^{(2,1)}_{n,\alpha}(z)\phi(z) + \widehat{\Xi}^{(2,2)}_{n,\alpha}(z)\psi(z)\right)^{-1}\\ &= \begin{bmatrix}\left[\widehat{\Xi}^{(1,1)}_{n,\alpha}(z)\phi(z) + \widehat{\Xi}^{(1,2)}_{n,\alpha}(z)\psi(z)\right]\left[\widehat{\Xi}^{(2,1)}_{n,\alpha}(z)\phi(z) + \widehat{\Xi}^{(2,2)}_{n,\alpha}(z)\psi(z)\right]^{-1}\\ \left[\widehat{\Xi}^{(2,1)}_{n,\alpha}(z)\phi(z) + \widehat{\Xi}^{(2,2)}_{n,\alpha}(z)\psi(z)\right]\left[\widehat{\Xi}^{(2,1)}_{n,\alpha}(z)\phi(z) + \widehat{\Xi}^{(2,2)}_{n,\alpha}(z)\psi(z)\right]^{-1}\end{bmatrix}\\ &= \begin{bmatrix}\widehat{S}_{n,\alpha}(z)\\ I_q\end{bmatrix}\end{aligned} \tag{5.7}$$

is valid for each $z \in \mathbb{C}\setminus([\alpha,\infty)\cup\mathcal{D})$. Let $\widetilde{\Xi}_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ be given by (3.25). Because of (4.14), Lemma 3.26, and (5.7), for each $z \in \mathbb{C}\setminus([\alpha,\infty)\cup\mathcal{D})$, we get

$$\left(\mathrm{Rstr}_{\mathbb{C}\setminus[\alpha,\infty)}\widetilde{\Xi}_{n,\alpha}(z)\right)\begin{bmatrix}(\widehat{\mathcal{E}}(z)-\alpha)\phi(z)\\ \psi(z)\end{bmatrix}\left(\widehat{\Xi}^{(2,1)}_{n,\alpha}(z)\phi(z) + \widehat{\Xi}^{(2,2)}_{n,\alpha}(z)\psi(z)\right)^{-1}$$

$$= \operatorname{diag}\left((z-\alpha)I_q,\ I_q\right)\cdot\widehat{\Xi}_{n,\alpha}(z)\cdot\operatorname{diag}\left((z-\alpha)^{-1}I_q,\ I_q\right)$$
$$\cdot\begin{bmatrix}(z-\alpha)\phi(z)\\ \psi(z)\end{bmatrix}\left(\widehat{\Xi}_{n,\alpha}^{(2,1)}(z)\phi(z)+\widehat{\Xi}_{n,\alpha}^{(2,2)}(z)\psi(z)\right)^{-1}$$
$$= \operatorname{diag}\left((\widehat{\mathcal{E}}(z)-\alpha)I_q,\ I_q\right)\cdot\widehat{\Xi}_{n,\alpha}(z)\begin{bmatrix}\phi(z)\\ \psi(z)\end{bmatrix}\left(\widehat{\Xi}_{n,\alpha}^{(2,1)}(z)\phi(z)+\widehat{\Xi}_{n,\alpha}^{(2,2)}(z)\psi(z)\right)^{-1}$$
$$= \begin{bmatrix}(\widehat{\mathcal{E}}(z)-\alpha)\widehat{S}_{n,\alpha}(z)\\ I_q\end{bmatrix}. \tag{5.8}$$

(III) Since the functions $\widehat{\mathcal{E}}$ and $\widehat{R}_{T_{q,n}}$ are holomorphic in $\mathbb{C}\backslash[\alpha,\infty)$, we see that (5.4) and (5.5) are fulfilled if and only if the following statement holds true:

(a) There exists a discrete subset $\widetilde{\mathcal{D}}$ of $\mathbb{C}\backslash[\alpha,\infty)$ such that $\widehat{S}_{n,\alpha}$ is holomorphic in $\mathbb{C}\setminus([\alpha,\infty)\cup\widetilde{\mathcal{D}})$ and that

$$(I_{(n+1)q}-H_n^+H_n)\widehat{R}_{T_{q,n}}(z)\left[I_{(n+1)q},\ T_{q,n}H_n\right](I_2\otimes v_{q,n})\begin{bmatrix}\widehat{S}_{n,\alpha}(z)\\ I_q\end{bmatrix}=0 \tag{5.9}$$

and

$$(I_{nq}-H_{\alpha\triangleright n-1}^+H_{\alpha\triangleright n-1})V_{q,n}^*\widehat{R}_{T_{q,n}}(z)\left[I_{(n+1)q},\ [R_{T_{q,n}}(\alpha)]^{-1}H_n\right](I_2\otimes v_{q,n})$$
$$\cdot\begin{bmatrix}(\widehat{\mathcal{E}}(z)-\alpha)\widehat{S}_{n,\alpha}(z)\\ I_q\end{bmatrix}=0 \tag{5.10}$$

are valid for each $z\in\mathbb{C}\setminus([\alpha,\infty)\cup\widetilde{\mathcal{D}})$.

(IV) In this step of the proof, we suppose (a). We are going to prove the following:

(b) There is a discrete subset $\widehat{\mathcal{D}}$ of $\mathbb{C}\backslash[\alpha,\infty)$ such that ϕ and ψ are holomorphic in $\mathbb{C}\backslash([\alpha,\infty)\cup\widehat{\mathcal{D}})$ and that

$$(I_{(n+1)q}-H_n^+H_n)\widehat{R}_{T_{q,n}}(z)\left[I_{(n+1)q},\ T_{q,n}H_n\right](I_2\otimes v_{q,n})$$
$$\cdot\widehat{\Xi}_{n,\alpha}(z)\begin{bmatrix}\phi(z)\\ \psi(z)\end{bmatrix}\left(\widehat{\Xi}_{n,\alpha}^{(2,1)}(z)\phi(z)+\widehat{\Xi}_{n,\alpha}^{(2,2)}(z)\psi(z)\right)^{-1}=0 \tag{5.11}$$

and

$$(I_{nq}-H_{\alpha\triangleright n-1}^+H_{\alpha\triangleright n-1})V_{q,n}^*\widehat{R}_{T_{q,n}}(z)\left[I_{(n+1)q},[R_{T_{q,n}}(\alpha)]^{-1}H_n\right](I_2\otimes v_{q,n})$$
$$\cdot\left(\operatorname{Rstr}_{\mathbb{C}\backslash[\alpha,\infty)}\widetilde{\Theta}_{n,\alpha}(z)\right)\begin{bmatrix}(\widehat{\mathcal{E}}(z)-\alpha)\phi(z)\\ \psi(z)\end{bmatrix}\left(\widehat{\Xi}_{n,\alpha}^{(2,1)}(z)\phi(z)+\widehat{\Xi}_{n,\alpha}^{(2,2)}(z)\psi(z)\right)^{-1}$$
$$=0 \tag{5.12}$$

are fulfilled for each $z\in\mathbb{C}\backslash([\alpha,\infty)\cup\widehat{\mathcal{D}})$.

Obviously, $\mathcal{D}_\#:=\mathcal{D}\cup\widetilde{\mathcal{D}}$ is a discrete subset of $\mathbb{C}\backslash[\alpha,\infty)$. Since (5.9) and (5.10) are valid for each $z\in\mathbb{C}\backslash([\alpha,\infty)\cup\mathcal{D}_\#)$ and since (II) shows that (5.7) and (5.8) are fulfilled for each $z \in \mathbb{C}\setminus([\alpha,\infty)\cup\mathcal{D}_\#)$, we get (5.11) and (5.12) for each $z \in \mathbb{C}\setminus([\alpha,\infty)\cup\mathcal{D}_\#)$. Setting $\widehat{\mathcal{D}} = \mathcal{D}_\#$, statement (b) is proved.

(V) In this step of the proof, we suppose (b). We are going to prove that (a) holds true. Obviously, $\mathcal{D}_\square := \mathcal{D} \cup \widehat{\mathcal{D}}$ is a discrete subset of $\mathbb{C}\setminus[\alpha,\infty)$. According to (I) and (II), we get (4.14), (5.7), and (5.8) for each $z \in \mathbb{C} \setminus ([\alpha,\infty) \cup \mathcal{D}_\square)$. Using these arguments and (5.11) as well as (5.12), we see that (5.9) and (5.10) are fulfilled for each $z \in \mathbb{C} \setminus ([\alpha,\infty) \cup \mathcal{D}_\square)$. Consequently, statement (a) holds true, using $\widetilde{\mathcal{D}} = \mathcal{D}_\square$.

(VI) We now verify that statement (b) implies the following statement:

(c) There is a discrete subset $\widetilde{\mathcal{D}}_\#$ of $\mathbb{C}\setminus[\alpha,\infty)$ such that ϕ and ψ are holomorphic in $\mathbb{C} \setminus ([\alpha,\infty) \cup \widetilde{\mathcal{D}}_\#)$ and that

$$(I_{(n+1)q} - H_n^+ H_n)\widehat{R}_{T_{q,n}}(z)\left[I_{(n+1)q},\ T_{q,n}H_n\right](I_2 \otimes v_{q,n})\widehat{\Xi}_{n,\alpha}(z)\begin{bmatrix}\phi(z)\\ \psi(z)\end{bmatrix} = 0 \tag{5.13}$$

and

$$\begin{aligned}(I_{nq} - H^+_{\alpha\triangleright n-1}H_{\alpha\triangleright n-1})V^*_{q,n}\widehat{R}_{T_{q,n}}(z)\left[I_{(n+1)q},\ [R_{T_{q,n}}(\alpha)]^{-1}H_n\right](I_2 \otimes v_{q,n})\\ \cdot\left(\mathrm{Rstr}_{\mathbb{C}\setminus[\alpha,\infty)}\ \widetilde{\Xi}_{n,\alpha}(z)\right)\begin{bmatrix}(\widehat{\mathcal{E}}(z)-\alpha)\phi(z)\\ \psi(z)\end{bmatrix} = 0\end{aligned} \tag{5.14}$$

hold true for each $z \in \mathbb{C}\setminus([\alpha,\infty) \cup \widetilde{\mathcal{D}}_\#)$.

Let us assume that (b) is fulfilled. Because of (I), we know that $\widetilde{\mathcal{D}}_\square := \mathcal{D} \cup \widehat{\mathcal{D}}$ is a discrete subset of $\mathbb{C}\setminus[\alpha,\infty)$. From (I) and (b) we see that (4.14), (5.11), and (5.12) are valid for each $z \in \mathbb{C} \setminus ([\alpha,\infty) \cup \widetilde{\mathcal{D}}_\square)$, which implies (5.13) and (5.14) for each $z \in \mathbb{C}([\alpha,\infty) \cup \widetilde{\mathcal{D}}_\square)$. Consequently, (c) holds true, setting $\widetilde{\mathcal{D}}_\# = \widetilde{\mathcal{D}}_\square$.

(VII) We now show that (c) implies (b). Let (c) be fulfilled. Obviously, $\widehat{\mathcal{D}}_\# := \widetilde{\mathcal{D}}_\# \cup \mathcal{D}$ is a discrete subset of $\mathbb{C} \setminus [\alpha,\infty)$. Because of (I) and (c), we know that (4.14), (5.13), and (5.14) are valid for each z belonging to $\mathbb{C}\setminus([\alpha,\infty)\cup\widehat{\mathcal{D}}_\#)$. Thus, (5.11) and (5.12) hold true for all $z \in \mathbb{C}\setminus([\alpha,\infty) \cup \widehat{\mathcal{D}}_\#)$. Hence, (b) is fulfilled, taking $\widehat{\mathcal{D}} = \widehat{\mathcal{D}}_\#$.

(VIII) Since $\widehat{R}_{T_{q,n}}$ marks the restriction of $R_{T_{q,n}}$ onto $\mathbb{C}\setminus[\alpha,\infty)$, we see that (c) is equivalent to the following statement:

(d) There is a discrete subset $\mathcal{D}'$ of $\mathbb{C}\setminus[\alpha,\infty)$ such that ϕ and ψ are holomorphic in $\mathbb{C}\setminus([\alpha,\infty) \cup \mathcal{D}')$ and that

$$(I_{(n+1)q} - H_n^+ H_n)R_{T_{q,n}}(z)\left[I_{(n+1)q},\ T_{q,n}H_n\right](I_2 \otimes v_{q,n})\Xi_{n,\alpha}(z)\begin{bmatrix}\phi(z)\\ \psi(z)\end{bmatrix} = 0 \tag{5.15}$$

and

$$\begin{aligned}(I_{nq} - H^+_{\alpha\triangleright n-1}H_{\alpha\triangleright n-1})V^*_{q,n}R_{T_{q,n}}(z)\left[I_{(n+1)q},\ [R_{T_{q,n}}(\alpha)]^{-1}H_n\right](I_2 \otimes v_{q,n})\\ \cdot\ \widetilde{\Xi}_{n,\alpha}(z)\cdot\mathrm{diag}\left((z-\alpha)I_q,\ I_q\right)\begin{bmatrix}\phi(z)\\ \psi(z)\end{bmatrix} = 0\end{aligned} \tag{5.16}$$

hold true for each $z \in \mathbb{C}\backslash([\alpha, \infty) \cup \mathcal{D}')$.

(IX) Let $\widetilde{P}_{n,\alpha}$, $\widetilde{Q}_{n,\alpha}$, and $\widetilde{S}_{n,\alpha}$ be the matrix-valued functions defined on $\mathbb{C}$ by (3.42), (3.43), and (3.44). According to part (a) of Lemma 3.43, we see that $\mathcal{N} := \mathcal{N}_{\det \widetilde{P}_{n,\alpha}} \cup \mathcal{N}_{\det \widetilde{Q}_{n,\alpha}} \cup \mathcal{N}_{\det \widetilde{S}_{n,\alpha}}$ is a finite and, in particular, a discrete subset of $\mathbb{C}$.

(X) Now we suppose (d). By virtue of (IX), we know that $\mathcal{N}' := \mathcal{N} \cup \mathcal{D}'$ is a discrete subset of $\mathbb{C}$. From (d) and part (b) of Lemma 3.43 we see that following statement holds true:

(e) There is a discrete subset $\mathcal{D}''$ of $\mathbb{C}\backslash[\alpha, \infty)$ such that ϕ and ψ are holomorphic in $\mathbb{C}\backslash([\alpha, \infty) \cup \mathcal{D}'')$ and that the equations

$$(I_{(n+1)q} - H_n^+ H_n) R_{T_{q,n}}(\alpha) v_{q,n} \phi(z) = 0 \tag{5.17}$$

and

$$(I_{nq} - H_{\alpha \triangleright n-1}^+ H_{\alpha \triangleright n-1}) V_{q,n}^* H_n v_{q,n} \psi(z) = 0 \tag{5.18}$$

are valid for each $z \in \mathbb{C} \setminus ([\alpha, \infty) \cup \mathcal{D}'')$.

(XI) Conversely, we now suppose (e). We are going to prove (d). From (IX) we see that $\widetilde{\mathcal{N}} := \mathcal{N} \cap [\mathbb{C}\backslash[\alpha, \infty)]$ is a discrete subset of $\mathbb{C}\backslash[\alpha, \infty)$. Hence, $\mathcal{D}'_\square := \mathcal{D}'' \cup \widetilde{\mathcal{N}}$ is a discrete subset of $\mathbb{C} \setminus [\alpha, \infty)$. Because of (e), the functions ϕ and ψ are holomorphic in $\mathbb{C}\backslash([\alpha, \infty) \cup \mathcal{D}'_\square)$ and (5.17) and (5.18) are valid for each $z \in \mathbb{C}\backslash([\alpha, \infty) \cup \mathcal{D}'_\square)$. Let us consider an arbitrary $z \in \mathbb{C}\backslash([\alpha, \infty) \cup \mathcal{D}'_\square)$. From (5.17) and (5.18) we then get that $x := \phi(z)$ and $y := \psi(z)$ fulfill (3.60) and (3.61). Consequently, part (b) of Lemma 3.43 yields that (3.58) and (3.59) hold true. Thus, we see that (5.15) and (5.16) are valid. Hence, (d) is true, using $\mathcal{D}' = \mathcal{D}'_\square$.

(XII) The equivalence of (e) and $[\phi; \psi] \in \mathcal{P}^{(q,q)}_{-\tilde{J}_q, \geq}[\mathbb{C}\backslash[\alpha, \infty), (s_j)_{j=0}^{2n}]$ follows from Notation 5.1.

Combining (III)-(VIII) and (X)-(XII) completes the proof. □

We are now able to prove a parametrization of the solution set of the matricial Stieltjes-type problem $\mathsf{S}[[\alpha, \infty); (s_j)_{j=0}^{2n}, \leq]$, where, however, the set of parameters still depends on the given data.

Proposition 5.4 *Let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}^{\geq, e}_{q,\kappa,\alpha}$ and let $n \in \mathbb{N}_0$ be such that $2n \leq \kappa$. Let (4.4) be the $q \times q$ block representation of $\widehat{\Xi}_{n,\alpha} := \mathrm{Rstr}_{\mathbb{C}\backslash[\alpha,\infty)} \Xi_{n,\alpha}$, where $\Xi_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2q \times 2q}$ is given by (3.27) in the case $n = 0$ and by (3.24) if $n \geq 1$. Then:*

(a) *For each $[\phi; \psi] \in \mathcal{P}^{(q,q)}_{-\tilde{J}_q, \geq}[\mathbb{C}\backslash[\alpha, \infty), (s_j)_{j=0}^{2n}]$, the function $\det(\widehat{\Xi}^{(2,1)}_{n,\alpha} \phi + \widehat{\Xi}^{(2,2)}_{n,\alpha} \psi)$ is meromorphic in $\mathbb{C}\backslash[\alpha, \infty)$ and does not vanish identically and*

$$S := \left(\widehat{\Xi}^{(1,1)}_{n,\alpha} \phi + \widehat{\Xi}^{(1,2)}_{n,\alpha} \psi\right) \left(\widehat{\Xi}^{(2,1)}_{n,\alpha} \phi + \widehat{\Xi}^{(2,2)}_{n,\alpha} \psi\right)^{-1} \tag{5.19}$$

belongs to the class $\mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^{2n}, \leq]$.

(b) *For each* $S \in \mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^{2n}, \leq]$, *there exists a pair* $[\phi; \psi] \in \mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}[\mathbb{C}\setminus[\alpha,\infty), (s_j)_{j=0}^{2n}]$ *of* $q \times q$ *matrix-valued functions* ϕ *and* ψ *both being holomorphic in* $\mathbb{C}\setminus[\alpha,\infty)$ *such that (4.14) and*

$$S(z) = \left[\widehat{\Xi}_{n,\alpha}^{(1,1)}(z)\phi(z) + \widehat{\Xi}_{n,\alpha}^{(1,2)}(z)\psi(z)\right]\left[\widehat{\Xi}_{n,\alpha}^{(2,1)}(z)\phi(z) + \widehat{\Xi}_{n,\alpha}^{(2,2)}(z)\psi(z)\right]^{-1} \tag{5.20}$$

hold true for each $z \in \mathbb{C}\setminus[\alpha,\infty)$.

(c) *Let* $[\phi_1; \psi_1], [\phi_2; \psi_2] \in \mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}[\mathbb{C}\setminus[\alpha,\infty), (s_j)_{j=0}^{2n}]$. *Then*

$$\begin{aligned}&\left(\widehat{\Xi}_{n,\alpha}^{(1,1)}\phi_1 + \widehat{\Xi}_{n,\alpha}^{(1,2)}\psi_1\right)\left(\widehat{\Xi}_{n,\alpha}^{(2,1)}\phi_1 + \widehat{\Xi}_{n,\alpha}^{(2,2)}\psi_1\right)^{-1}\\ &\quad= \left(\widehat{\Xi}_{n,\alpha}^{(1,1)}\phi_2 + \widehat{\Xi}_{n,\alpha}^{(1,2)}\psi_2\right)\left(\widehat{\Xi}_{n,\alpha}^{(2,1)}\phi_2 + \widehat{\Xi}_{n,\alpha}^{(2,2)}\psi_2\right)^{-1}\end{aligned} \tag{5.21}$$

if and only if $\langle[\phi_1; \psi_1]\rangle = \langle[\phi_2; \psi_2]\rangle$.

Proof First we consider the case $n \geq 1$. Since $(s_j)_{j=0}^{\kappa}$ belongs to $\mathcal{K}^{\geq,e}_{q,\kappa,\alpha}$, [21, Remarks 5.7 and 5.8] yields $s_j^* = s_j$ for each $j \in \mathbb{Z}_{0,\kappa}$ as well as $H_n \in \mathbb{C}_{\geq}^{(n+1)q\times(n+1)q}$ and $H_{\alpha\triangleright n-1} \in \mathbb{C}_{\geq}^{nq\times nq}$. From [21, Remark 6.13] we conclude

$$H_n^+ H_n = H_n H_n^+ \qquad \text{and} \qquad H_{\alpha\triangleright n-1}^+ H_{\alpha\triangleright n-1} = H_{\alpha\triangleright n-1} H_{\alpha\triangleright n-1}^+. \tag{5.22}$$

Lemma 3.35 provides $\widehat{\Xi}_{n,\alpha} \in \mathfrak{W}_{\tilde{J}_q,\alpha}$. Let $\widetilde{\Xi}_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ be defined by (3.25), Lemma 3.23 shows that $\Xi_{n,\alpha}$ and $\widetilde{\Xi}_{n,\alpha}$ are holomorphic in $\mathbb{C}$. Remark 3.29 yields

$$\det \Xi_{n,\alpha}(z) \neq 0 \qquad \text{and} \qquad \det \widetilde{\Xi}_{n,\alpha}(z) \neq 0 \tag{5.23}$$

for each $z \in \mathbb{C}$ and, in particuar, $\det \widehat{\Xi}_{n,\alpha}(z) \neq 0$ for each $z \in \mathbb{C}\setminus[\alpha,\infty)$.

(a) Let $[\phi; \psi] \in \mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}[\mathbb{C}\setminus[\alpha,\infty), (s_j)_{j=0}^{2n}]$. According to Notation 5.1, equations (5.1) and (5.2) are fulfilled. Using Remark 4.7, we see that there is a discrete subset $\widetilde{\mathcal{D}}$ of $\mathbb{C}\setminus[\alpha,\infty)$ such that ϕ and ψ are holomorphic in $\mathbb{C}\setminus([\alpha,\infty)\cup\widetilde{\mathcal{D}})$ and that inequality (4.14) holds true for each $z \in \mathbb{C}\setminus(\mathbb{R}\cup\widetilde{\mathcal{D}})$. In particular, $\det\left(\widehat{\Xi}_{n,\alpha}^{(2,1)}\phi + \widehat{\Xi}_{n,\alpha}^{(2,2)}\psi\right)$ does not vanish identically. Because of $\widehat{\Xi}_{n,\alpha} \in \mathfrak{W}_{\tilde{J}_q,\alpha}$ and [21, Proposition 10.1 (c)], the following three statements are valid:

(I) There is a discrete subset $\mathcal{D}$ of $\mathbb{C}\setminus[\alpha,\infty)$ such that $\widehat{\Xi}_{n,\alpha}$, ϕ, and ψ are holomorphic in $\mathbb{C}\setminus([\alpha,\infty)\cup\mathcal{D})$.

(II) The matrix-valued function S given by (5.19) is holomorphic in $\mathbb{C}\setminus([\alpha,\infty)\cup\mathcal{D})$, and inequality (4.14) as well the representation (5.20) of S are fulfilled for each $z \in \mathbb{C}\setminus([\alpha,\infty)\cup\mathcal{D})$.

(III) For each $z \in \mathbb{C}\backslash(\mathbb{R} \cup \mathcal{D})$, the inequalities

$$\begin{bmatrix} S(z) \\ I_q \end{bmatrix}^* \widehat{\Xi}_{n,\alpha}^{-*}(z) \left(\frac{-\tilde{J}_q}{2\Im z} \right) \widehat{\Xi}_{n,\alpha}^{-1}(z) \begin{bmatrix} S(z) \\ I_q \end{bmatrix} \geq 0_{q \times q} \tag{5.24}$$

and

$$\begin{bmatrix} S(z) \\ I_q \end{bmatrix}^* \widehat{\Xi}_{n,\alpha}^{-*}(z) \left(\operatorname{diag}\left((z-\alpha)I_q,\ I_q\right)\right)^* \left(\frac{-\tilde{J}_q}{2\Im z} \right) \cdot \operatorname{diag}\left((z-\alpha)I_q,\ I_q\right) \cdot \widehat{\Xi}_{n,\alpha}^{-1}(z) \begin{bmatrix} S(z) \\ I_q \end{bmatrix} \geq 0_{q \times q} \tag{5.25}$$

hold true.

In view of (5.1) and (5.2), Lemma 5.3 provides (5.4) and (5.5) with $\widehat{R}_{T_{q,n}} := \operatorname{Rstr}_{\mathbb{C}\backslash[\alpha,\infty)} R_{T_{q,n}}$ and with $\widehat{\mathcal{E}} : \mathbb{C}\backslash[\alpha, \infty) \to \mathbb{C}$ given by $\widehat{\mathcal{E}}(z) := z$. Using (5.4) and (5.5), we obtain

$$\begin{aligned} &\widehat{R}_{T_{q,n}} \left[I_{(n+1)q}, T_{q,n} H_n\right] (I_2 \otimes v_{q,n}) \begin{bmatrix} S \\ I_q \end{bmatrix} \\ &= H_n^+ H_n \widehat{R}_{T_{q,n}} \left[I_{(n+1)q}, T_{q,n} H_n\right] (I_2 \otimes v_{q,n}) \begin{bmatrix} S \\ I_q \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} &V_{q,n}^* \widehat{R}_{T_{q,n}} \left[I_{(n+1)q}, [R_{T_{q,n}}(\alpha)]^{-1} H_n\right] (I_2 \otimes v_{q,n}) \begin{bmatrix} (\widehat{\mathcal{E}} - \alpha) S \\ I_q \end{bmatrix} \\ &= H_{\alpha \triangleright n-1}^+ H_{\alpha \triangleright n-1} V_{q,n}^* \widehat{R}_{T_{q,n}} \left[I_{(n+1)q}, [R_{T_{q,n}}(\alpha)]^{-1} H_n\right] \\ &\cdot (I_2 \otimes v_{q,n}) \begin{bmatrix} (\widehat{\mathcal{E}} - \alpha) \widehat{S}_{n,\alpha} \\ I_q \end{bmatrix}. \end{aligned} \tag{5.26}$$

Consequently, from (5.22) and (II) we then conclude

$$\mathcal{R}\left(\widehat{R}_{T_{q,n}}(z) \left[I_{(n+1)q}, T_{q,n} H_n\right] (I_2 \otimes v_{q,n}) \begin{bmatrix} S(z) \\ I_q \end{bmatrix} \right) \subseteq \mathcal{R}(H_n) \tag{5.27}$$

for each $z \in \mathbb{C}\backslash(\mathbb{R} \cup \mathcal{D})$. Obviously, Remark 3.6 yields $V_{q,n}^* \widehat{R}_{T_{q,n}}(z) = \widehat{R}_{T_{q,n-1}}(z) V_{q,n}^*$ for each $z \in \mathbb{C}\backslash(\mathbb{R} \cup \mathcal{D})$. Therefore, Lemma 3.39, (5.22), (5.26), and (II) provide

$$\begin{aligned} &\mathcal{R}\left(\widehat{R}_{T_{q,n-1}}(z) \left[I_{nq},\ [R_{T_{q,n-1}}(\alpha)]^{-1} H_{n-1}\right] (I_2 \otimes v_{q,n-1}) \begin{bmatrix} (z-\alpha) S(z) \\ I_q \end{bmatrix} \right) \\ &= \mathcal{R}\left(V_{q,n}^* \widehat{R}_{T_{q,n}}(z) \left[I_{(n+1)q},\ [R_{T_{q,n}}(\alpha)]^{-1} H_n\right] (I_2 \otimes v_{q,n}) \begin{bmatrix} (z-\alpha) S(z) \\ I_q \end{bmatrix} \right) \\ &\subseteq \mathcal{R}(H_{\alpha \triangleright n-1}) \end{aligned} \tag{5.28}$$

for each $z \in \mathbb{C}\setminus(\mathbb{R} \cup \mathcal{D})$. For all $z \in \mathbb{C}\setminus(\mathbb{R} \cup \mathcal{D})$, from Remark 3.9 (a) we get

$$\begin{aligned}
&\widehat{R}_{T_{q,n}}(z)\left[I_{(n+1)q}, T_{q,n}H_n\right](I_2 \otimes v_{q,n})\begin{bmatrix} S(z) \\ I_q \end{bmatrix} \\
&\quad= \widehat{R}_{T_{q,n}}(z)\left[v_{q,n}, T_{q,n}H_n v_{q,n}\right]\begin{bmatrix} S(z) \\ I_q \end{bmatrix} = \widehat{R}_{T_{q,n}}(z)(v_{q,n}S(z) - u_n), \\
&[R_{T_{q,n-1}}(\alpha)]^{-1}H_{n-1}v_{q,n-1} = (I_{nq} - \alpha T_{q,n-1})H_{n-1}v_{q,n-1} \\
&\quad= H_{n-1}v_{q,n-1} - \alpha T_{q,n-1}H_{n-1}v_{q,n-1} = y_{0,n-1} + \alpha u_{n-1}
\end{aligned} \tag{5.29}$$

and, hence,

$$\begin{aligned}
&\widehat{R}_{T_{q,n-1}}(z)\left[I_{nq},\ [R_{T_{q,n-1}}(\alpha)]^{-1}H_{n-1}\right](I_2 \otimes v_{q,n-1})\begin{bmatrix} (z-\alpha)S(z) \\ I_q \end{bmatrix} \\
&\quad= \widehat{R}_{T_{q,n-1}}(z)\left(v_{q,n-1}(z-\alpha)S(z) + [R_{T_{q,n-1}}(\alpha)]^{-1}H_{n-1}v_{q,n-1}\right) \\
&\quad= \widehat{R}_{T_{q,n-1}}(z)\left[v_{q,n-1}(z-\alpha)S(z) - (-\alpha u_{n-1} - y_{0,n-1})\right].
\end{aligned} \tag{5.30}$$

For all $z \in \mathbb{C}\setminus(\mathbb{R} \cup \mathcal{D})$, equations (5.29) and (5.27) imply

$$\mathcal{R}\left(R_{T_{q,n}}(z)(v_{q,n}S(z) - u_n)\right) = \mathcal{R}\left(\widehat{R}_{T_{q,n}}(z)(v_{q,n}S(z) - u_n)\right) \subseteq \mathcal{R}(H_n) \tag{5.31}$$

and, in view of (5.30) and (5.28), furthermore

$$\begin{aligned}
&\mathcal{R}\left(R_{T_{q,n-1}}(z)\left[v_{q,n-1}(z-\alpha)S(z) - (-\alpha u_{n-1} - y_{0,n-1})\right]\right) \\
&\quad= \mathcal{R}\left(\widehat{R}_{T_{q,n-1}}(z)\left[v_{q,n-1}(z-\alpha)S(z) - (-\alpha u_{n-1} - y_{0,n-1})\right]\right) \subseteq \mathcal{R}(H_{\alpha\triangleright n-1}).
\end{aligned} \tag{5.32}$$

Lemma 3.37 shows that, for each $z \in \mathbb{C}\setminus(\mathbb{R} \cup \mathcal{D})$, the matrix $\Sigma_{2n}^{[S]}(z)$ given by (3.6) admits the representation

$$\Sigma_{2n}^{[S]}(z) = \begin{bmatrix} S(z) \\ I_q \end{bmatrix}^* \widehat{\Xi}_{n,\alpha}^{-*}(z)\left(\frac{-\tilde{J}_q}{2\Im z}\right)\widehat{\Xi}_{n,\alpha}^{-1}(z)\begin{bmatrix} S(z) \\ I_q \end{bmatrix}. \tag{5.33}$$

In view of (5.23), for each $z \in \mathbb{C}\setminus[\alpha, \infty)$, we infer

$$\begin{aligned}
&\left(\operatorname{diag}\left((z-\alpha)I_q,\ I_q\right) \cdot \Xi_{n,\alpha}(z) \cdot \operatorname{diag}\left((z-\alpha)^{-1}I_q,\ I_q\right)\right)^{-1} \\
&\quad= \operatorname{diag}\left((z-\alpha)I_q,\ I_q\right) \cdot \widehat{\Xi}_{n,\alpha}^{-1}(z) \cdot \operatorname{diag}\left((z-\alpha)^{-1}I_q,\ I_q\right).
\end{aligned} \tag{5.34}$$

From Lemma 3.37 we see that, for each $z \in \mathbb{C}\setminus(\mathbb{R} \cup \mathcal{D})$, the matrix $\Sigma_{2n-1}^{[S]}(z)$ given by (3.7) can be represented by

$$\Sigma_{2n-1}^{[S]}(z) = \begin{bmatrix} (z-\alpha)S(z) \\ I_q \end{bmatrix}^* \widetilde{\Xi}_{n,\alpha}^{-*}(z) \left(\frac{-\tilde{J}_q}{2\Im z} \right) \widetilde{\Xi}_{n,\alpha}^{-1}(z) \begin{bmatrix} (z-\alpha)S(z) \\ I_q \end{bmatrix}. \tag{5.35}$$

For all $z \in \mathbb{C}\setminus[\alpha, \infty)$, the second inequality in (5.23), Lemma 3.26, and (5.34) yield

$$\widetilde{\Xi}_{n,\alpha}^{-1}(z) \begin{bmatrix} (z-\alpha)S(z) \\ I_q \end{bmatrix} = \operatorname{diag}\left((z-\alpha)I_q,\ I_q\right) \cdot \widehat{\Xi}_{n,\alpha}^{-1}(z) \begin{bmatrix} S(z) \\ I_q \end{bmatrix}, \tag{5.36}$$

which, because of (5.35), implies

$$\begin{aligned} \Sigma_{2n-1}^{[S]}(z) = \begin{bmatrix} S(z) \\ I_q \end{bmatrix}^* \widehat{\Xi}_{n,\alpha}^{-*}(z) \left(\operatorname{diag}\left((z-\alpha)I_q,\ I_q\right)\right)^* \left(\frac{-\tilde{J}_q}{2\Im z} \right) \\ \cdot \operatorname{diag}\left((z-\alpha)I_q,\ I_q\right) \cdot \widehat{\Xi}_{n,\alpha}^{-1}(z) \begin{bmatrix} S(z) \\ I_q \end{bmatrix} \end{aligned} \tag{5.37}$$

for each $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D})$. Taking into account (5.33), (5.24), (5.37), and (5.25), it follows that

$$\Sigma_{2n}^{[S]}(z) \in \mathbb{C}_{\geq}^{q\times q} \qquad \text{and} \qquad \Sigma_{2n-1}^{[S]}(z) \in \mathbb{C}_{\geq}^{q\times q} \tag{5.38}$$

hold true for each $z \in \mathbb{C}\setminus(\mathbb{R} \cup \mathcal{D})$. Thus, for all $z \in \mathbb{C}\setminus(\mathbb{R} \cup \mathcal{D})$, using Notation 3.3, $H_n \geq 0$, $H_{\alpha \triangleright n-1} \geq 0$, (5.31), (5.32), (5.38), (3.6), (3.7), and a well-known characterization of non-negative Hermitian block matrices (see, e.g., [21, Lemma 4.6 (a)]), we conclude that (3.5) is fulfilled. Obviously, $\widetilde{\mathcal{D}} := \mathcal{D} \cap \Pi_+$ is a discrete subset of Π_+, and (3.5) implies $P_{2n}^{[S]}(z) \in \mathbb{C}_{\geq}^{(n+2)q\times(n+2)q}$ and $P_{2n-1}^{[S]}(z) \in \mathbb{C}_{\geq}^{(n+1)q\times(n+1)q}$ for each $z \in \Pi_+\setminus\widetilde{\mathcal{D}}$. Thus, Theorem 3.4 provides $S \in \mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^{2n}, \leq]$.

(b) We now consider an arbitrary $S \in \mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^{2n}, \leq]$. Then S is holomorphic in $\mathbb{C}\setminus[\alpha, \infty)$, and Theorems 2.5 and 3.4 yield (3.5) for each $z \in \mathbb{C}\setminus\mathbb{R}$. Consequently, the above mentioned characterization of non-negative Hermitian block matrices shows that, for each $z \in \mathbb{C}\setminus\mathbb{R}$, the following three statements hold true:

(i) $\mathcal{R}\left(R_{T_{q,n}}(z)[v_{q,n}S(z) - u_n]\right) \subseteq \mathcal{R}(H_n)$.
(ii) $\mathcal{R}\left(R_{T_{q,n-1}}(z)[v_{q,n-1}(z-\alpha)S(z) - (-\alpha u_{n-1} - y_{0,n-1})]\right) \subseteq \mathcal{R}(H_{\alpha\triangleright n-1})$.
(iii) The matrices $\Sigma_{2n-1}^{[S]}(z)$ and $\Sigma_{2n}^{[S]}(z)$ are non-negative Hermitian.

For each $z \in \mathbb{C}\setminus[\alpha, \infty)$, from Remark 3.9 (a) we get

$$\begin{aligned} R_{T_{q,n}}(z)(v_{q,n}S(z) - u_n) &= \widehat{R}_{T_{q,n}}(z)[v_{q,n},\ -u_n]\begin{bmatrix} S(z) \\ I_q \end{bmatrix} \\ &= \widehat{R}_{T_{q,n}}(z)[I_{(n+1)q},\ T_{q,n}H_n](I_2 \otimes v_{q,n}) \begin{bmatrix} S(z) \\ I_q \end{bmatrix} \end{aligned} \tag{5.39}$$

and, because of Remark 3.9 (a), we furthermore obtain (5.30). Remark 3.29 shows that $\det \widehat{\Xi}_{n,\alpha}(z) \neq 0$ holds true for each $z \in \mathbb{C}\setminus[\alpha,\infty)$. Using (5.23), Remark 2.2, Lemma 3.37, and (iii), we see that (5.24) and

$$\begin{bmatrix}(z-\alpha)S(z)\\ I_q\end{bmatrix}^* \widetilde{\Xi}_{n,\alpha}^{-*}(z)\left(\frac{-\tilde{J}_q}{2\Im z}\right)\widetilde{\Xi}_{n,\alpha}^{-1}(z)\begin{bmatrix}(z-\alpha)S(z)\\ I_q\end{bmatrix} \geq 0 \tag{5.40}$$

hold true for each $z \in \mathbb{C}\setminus\mathbb{R}$. In view of (5.23) and Lemma 3.26, we get (5.36) for each $z \in \mathbb{C}\setminus\mathbb{R}$. Consequently, from (5.40) equation (5.25) follows for each $z \in \mathbb{C}\setminus\mathbb{R}$. Since S and $\widehat{\Xi}_{n,\alpha}$ are holomorphic in $\mathbb{C}\setminus[\alpha,\infty)$, from $\widehat{\Xi}_{n,\alpha} \in \mathfrak{W}_{\tilde{J}_q,\alpha}$, (4.4), (5.24), (5.25), and [21, Proposition 10.1 (b)] we get that there is a pair $[\phi;\psi] \in \mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C}\setminus[\alpha,\infty))$ of $q\times q$ matrix-valued functions ϕ and ψ both being holomorphic in $\mathbb{C}\setminus[\alpha,\infty)$ such that (4.14) and (5.20) hold true for each $z \in \mathbb{C}\setminus[\alpha,\infty)$. Because of (5.39) and (i), we infer (5.27) for each $z \in \mathbb{C}\setminus\mathbb{R}$. Consequently, (5.22), Remark A.3, and the Identity Theorem for holomorphic functions yield

$$(I_{(n+1)q} - H_n^+ H_n)\widehat{R}_{T_{q,n}}[I_{(n+1)q},\ T_{q,n}H_n](I_2 \otimes v_{q,n})\begin{bmatrix}S\\ I_q\end{bmatrix} = 0. \tag{5.41}$$

Because of (5.27) and (ii), we obtain

$$\mathcal{R}\left(\widehat{R}_{T_{q,n-1}}(z)\left[I_{nq},\ [R_{T_{q,n-1}}(\alpha)]^{-1}H_{n-1}\right]\cdot(I_2\otimes v_{q,n-1})\begin{bmatrix}(z-\alpha)S(z)\\ I_q\end{bmatrix}\right) \subseteq \mathcal{R}(H_{\alpha\triangleright n-1})$$

for each $z \in \mathbb{C}\setminus\mathbb{R}$. Let $\widehat{\mathcal{E}} : \mathbb{C}\setminus[\alpha,\infty) \to \mathbb{C}$ be defined by $\widehat{\mathcal{E}}(z) := z$. Thus, we see that

$$\begin{aligned} f &:= (I_{nq} - H^+_{\alpha\triangleright n-1}H_{\alpha\triangleright n-1})\widehat{R}_{T_{q,n-1}}\left[I_{nq}, [R_{T_{q,n-1}}(\alpha)]^{-1}H_{n-1}\right] \\ &\quad\cdot(I_2\otimes v_{q,n-1})\begin{bmatrix}(\widehat{\mathcal{E}}-\alpha)S\\ I_q\end{bmatrix}\end{aligned}$$

fulfills $f(z) = 0$ for all $z \in \mathbb{C}\setminus\mathbb{R}$. Applying the Identity Theorem for holomorphic functions once more, it follows that $f(z) = 0$. Consequently, using additionally (4.4), (4.14), (5.20), and (5.41), we infer, due to Lemma 5.3, that $[\phi;\psi]$ belongs to $\mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}[\mathbb{C}\setminus[\alpha,\infty), (s_j)_{j=0}^{2n}]$.

(c) In view of part (a), we know that, for each $k \in \mathbb{Z}_{1,2}$, the function $\det(\widehat{\Xi}^{(2,1)}_{n,\alpha}\phi_k + \widehat{\Xi}^{(2,2)}_{n,\alpha}\psi_k)$ does not vanish identically in $\mathbb{C}\setminus[\alpha,\infty)$ and that $S_k := \left(\widehat{\Xi}^{(1,1)}_{n,\alpha}\phi_k + \widehat{\Xi}^{(1,2)}_{n,\alpha}\psi_k\right)\left(\widehat{\Xi}^{(2,1)}_{n,\alpha}\phi_k + \widehat{\Xi}^{(2,2)}_{n,\alpha}\psi_k\right)^{-1}$ is a well-defined $q\times q$ matrix-valued function being meromorphic in $\mathbb{C}\setminus[\alpha,\infty)$. Because of $\widehat{\Xi}_{n,\alpha} \in \mathfrak{W}_{\tilde{J}_q,\alpha}$, the application of [21, Proposition 10.1 (d)] provides the asserted equivalence.

In the case $n \geq 1$, the proof is complete. Using additionally Remark 3.24 and [18, Lemma 4.2] as well as Remarks 3.15, 3.2, 3.24 and Notation 5.1 the case of $n = 0$ can be treated analogously. We omit the details. □

6 Parametrization of the Solution Set of the Truncated Matricial Stieltjes Moment Probem in the Non-degenerate and Degenerate Cases

In this section, we give a parametrization of the solution set of problem $\mathsf{S}[[\alpha, \infty); (s_j)_{j=0}^{2n}, \leq]$. In view of Theorem 2.5, this will be a description of the solution set of the corresponding Stieltjes-type moment problem $\mathsf{M}[[\alpha, \infty); (s_j)_{j=0}^{2n}, \leq]$.

According to Lemma 3.44 and a well-known result of linear algebra, we infer that

$$k{:=}\operatorname{rank}\left[\left(I_{(n+1)q} - H_n^+ H_n\right) R_{T_{q,n}}(\alpha) v_{q,n}\right] \tag{6.1}$$

and

$$l{:=}\begin{cases}\operatorname{rank}\left[\left(I_{nq} - H_{\alpha\triangleright n-1}^+ H_{\alpha\triangleright n-1}\right) V_{q,n}^* H_n v_{q,n}\right], & \text{if } n \geq 1\\ 0, & \text{if } n = 0\end{cases} \tag{6.2}$$

are non-negative integers such that $k+l \leq q$ and fulfill $\dim \mathcal{U}_{n,\alpha} = k$ and $\dim \mathcal{W}_{n,\alpha} = l$. We separately consider the following three cases:

(I) The non-degenerate case that $k = 0$ and $l = 0$ hold true.
(II) The degenerate, but not completely degenerate case $1 \leq k + l \leq q - 1$.
(III) The completely degenerate case $k + l = q$.

For technical reasons, it will be an advantage to subdivide the cases (II) and (III) into sub-cases:

(II-1) $k \geq 1, l \geq 1$, and $k + l \leq q - 1$,
(II-2) $k = 0$ and $1 \leq l \leq q - 1$,
(II-3) $1 \leq k \leq q - 1$ and $l = 0$,

and

(III-1) $k + l = q, k \geq 1$, and $l \geq 1$,
(III-2) $k = 0$ and $l = q$,
(III-3) $k = q$ and $l = 0$.

We observe that the inequality $l \geq 1$ necessarily implies $n \geq 1$ and, consequently, $2 \leq 2n \leq \kappa$. Hence, regarding cases (II-1), (II-2), (III-1), and (III-2), we can assume that $\kappa \geq 2$ and that n is a positive integer such that $2n \leq \kappa$.

Throughout this chapter, let k and l be given by (6.1) and (6.2). Furthermore, let $\Xi_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ be defined by (3.24) and (3.27), let $\widehat{\Xi}_{n,\alpha}{:=}\operatorname{Rstr}_{\mathbb{C}\setminus[\alpha,\infty)} \Xi_{n,\alpha}$, and let (4.4) be the $q \times q$ block partition of $\widehat{\Xi}_{n,\alpha}$.

First we study the so-called non-degenerate case (I), i.e., we consider the situation that $\left(I_q - H_n^+ H_n\right) R_{T_{q,n}}(\alpha) v_{q,n} = 0$ holds true and that, in case $n \geq 1$, furthermore the equation $\left(I_q - H^{+}_{\alpha\triangleright n-1} H_{\alpha\triangleright n-1}\right) V^{*}_{q,n} H_n v_{q,n} = 0$ is also fulfilled. We start by stating a result similar to [21, Remark 13.4].

Remark 6.1 Let $\left(s_j\right)_{j=0}^{\kappa} \in \mathcal{K}^{\geq,e}_{q,\kappa,\alpha}$. Let $n \in \mathbb{N}$ be such that $2n \leq \kappa$ and let k and l be given by (6.1) and (6.2), respectively. Let $\mathcal{U}_{n,\alpha}$ and $\mathcal{W}_{n,\alpha}$ be the subspaces of $\mathbb{C}^q$ given by (3.64) and (3.65), respectively. If $k = 0$ and $l = 0$, then one can easily see from Lemma 3.44 and Notation 5.1 that $\mathcal{U}_{n,\alpha} = \{0_{q\times 1}\}$, $\mathcal{W}_{n,\alpha} = \{0_{q\times 1}\}$, and $\mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}[\mathbb{C} \setminus [\alpha,\infty), \left(s_j\right)_{j=0}^{2n}] = \mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C}\setminus[\alpha,\infty))$.

Theorem 6.2 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$, let $\left(s_j\right)_{j=0}^{\kappa} \in \mathcal{K}^{\geq,e}_{q,\kappa,\alpha}$, and let $n \in \mathbb{N}_0$ be such that $2n \leq \kappa$. Suppose that k and l given by (6.1) and (6.2) fulfill $k = 0$ and $l = 0$. Let (4.4) be the $q \times q$ block partition of $\widehat{\Xi}_{n,\alpha} := \operatorname{Rstr}_{\mathbb{C}\setminus[\alpha,\infty)} \Xi_{n,\alpha}$, where $\Xi_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ is given by (3.24) if $n \geq 1$ and by (3.27) if $n = 0$. Then statements (a), (b), and (c) of Proposition 5.4 hold true with the class $\mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C}\setminus[\alpha,\infty))$ instead of the class $\mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}[\mathbb{C}\setminus[\alpha,\infty), \left(s_j\right)_{j=0}^{2n}]$.*

Proof Apply Proposition 5.4 and Lemma 6.1. □

Now we are going to study the degenerate, but not completely degenerate case.

Lemma 6.3 *Suppose $\kappa \geq 2$. Let $\left(s_j\right)_{j=0}^{\kappa} \in \mathcal{K}^{\geq,e}_{q,\kappa,\alpha}$ and let $n \in \mathbb{N}$ be such that $2n \leq \kappa$. Let $\mathcal{U}_{n,\alpha}$ and $\mathcal{W}_{n,\alpha}$ be the subspaces of $\mathbb{C}^q$ given by (3.64) and (3.65), respectively, let $P_{\mathcal{U}_{n,\alpha}}$ (resp. $P_{\mathcal{W}_{n,\alpha}}$) be the orthogonal projection matrix onto $\mathcal{U}_{n,\alpha}$ (resp. $\mathcal{W}_{n,\alpha}$), and let the integers k and l be defined by (6.1) and (6.2). Suppose that $r := q - (k + l)$ fulfills $1 \leq r \leq q - 1$. Then:*

(a) There exists a unitary complex $q \times q$ matrix U such that

$$U^* P_{\mathcal{U}_{n,\alpha}} U = \triangle_{k,l} \quad \text{and} \quad U^* P_{\mathcal{W}_{n,\alpha}} U = \nabla_{k,l} \tag{6.3}$$

hold true where

$$\triangle_{k,l} := \begin{cases} \operatorname{diag}\left(0_{r\times r},\ I_k,\ 0_{l\times l}\right), & \text{if } k \geq 1 \text{ and } l \geq 1 \\ 0_{q\times q}, & \text{if } k = 0 \text{ and } l \geq 1 \\ \operatorname{diag}\left(0_{(q-k)\times(q-k)},\ I_k\right), & \text{if } k \geq 1 \text{ and } l = 0 \end{cases} \tag{6.4}$$

and

$$\nabla_{k,l} := \begin{cases} \operatorname{diag}\left(0_{r\times r},\ 0_{k\times k},\ I_l\right), & \text{if } k \geq 1 \text{ and } l \geq 1 \\ \operatorname{diag}\left(0_{(q-l)\times(q-l)},\ I_l\right), & \text{if } k = 0 \text{ and } l \geq 1\ . \\ 0_{q\times q}, & \text{if } k \geq 1 \text{ and } l = 0 \end{cases} \tag{6.5}$$

(b) Let U be a unitary complex $q \times q$ matrix such that (6.3) is valid.

(b1) *If* $\left[\widetilde{\phi}; \widetilde{\psi}\right] \in \mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C}\backslash[\alpha,\infty))$ *is such that*

$$P_{\mathcal{U}_{n,\alpha}}\widetilde{\phi} = 0 \quad \text{and} \quad P_{\mathcal{W}_{n,\alpha}}\widetilde{\psi} = 0, \tag{6.6}$$

then there exists a pair $[\phi; \psi] \in \mathcal{P}^{(r,r)}_{-\tilde{J}_r,\geq}(\mathbb{C}\backslash[\alpha,\infty))$ *such that* ϕ *and* ψ *as well as*

$$\phi^{\square} := \begin{cases} U \cdot \operatorname{diag}(\phi,\ 0_{k\times k},\ I_l), & \text{if } k \geq 1 \text{ and } l \geq 1 \\ U \cdot \operatorname{diag}(\phi,\ I_l), & \text{if } k = 0 \text{ and } l \geq 1 \\ U \cdot \operatorname{diag}(\phi,\ 0_{k\times k}), & \text{if } k \geq 1 \text{ and } l = 0 \end{cases} \tag{6.7}$$

and

$$\psi^{\square} := \begin{cases} U \cdot \operatorname{diag}(\psi,\ I_k,\ 0_{l\times l}), & \text{if } k \geq 1 \text{ and } l \geq 1 \\ U \cdot \operatorname{diag}(\psi,\ 0_{l\times l}), & \text{if } k = 0 \text{ and } l \geq 1 \\ U \cdot \operatorname{diag}(\psi,\ I_k), & \text{if } k \geq 1 \text{ and } l = 0 \end{cases} \tag{6.8}$$

are holomorphic in Π_+ *and that* $\left[\phi^{\square}; \psi^{\square}\right]$ *is a pair belonging to* $\mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C}\backslash[\alpha,\infty))$ *and fulfilling* $\left\langle\left[\phi^{\square}; \psi^{\square}\right]\right\rangle = \langle[\widetilde{\phi}; \widetilde{\psi}]\rangle$.

(b2) *For each pair* $[\phi; \psi] \in \mathcal{P}^{(r,r)}_{-\tilde{J}_r,\geq}(\mathbb{C}\backslash[\alpha,\infty))$*, the pair* $\left[\phi^{\square}; \psi^{\square}\right]$ *given by (6.7) and (6.8) belongs to* $\mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C}\backslash[\alpha,\infty))$ *and each pair* $\left[\widetilde{\phi}; \widetilde{\psi}\right] \in \mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C}\backslash[\alpha,\infty))$ *such that* $\langle[\widetilde{\phi}; \widetilde{\psi}]\rangle = \left\langle\left[\phi^{\square}; \psi^{\square}\right]\right\rangle$ *fulfills both equations in (6.6).*

Proof Combine Lemma 3.44 and [21, Lemma 15.3]. □

Theorem 6.4 *Let* $\alpha \in \mathbb{R}$*, let* $\kappa \in \mathbb{N} \cup \{\infty\}$ *be such that* $\kappa \geq 2$*, let* $(s_j)_{j=0}^{\kappa} \in \mathcal{K}^{\geq,e}_{q,\kappa,\alpha}$*, and let* $n \in \mathbb{N}$ *be such that* $2n \leq \kappa$*. Let* $\mathcal{U}_{n,\alpha}$ *and* $\mathcal{W}_{n,\alpha}$ *be the subspaces of* $\mathbb{C}^q$ *defined by (3.64) and (3.65), respectively. Let* k *and* l *be given by (6.1) and (6.2). Suppose that* $r := q-(k+l)$ *fulfills* $1 \leq r \leq q-1$*. Let* U *be a unitary complex* $q \times q$ *matrix such that (6.3) holds true and let (4.4) be the* $q \times q$ *block partition of* $\widehat{\Xi}_{n,\alpha} := \operatorname{Rstr}_{\mathbb{C}\backslash[\alpha,\infty)} \Xi_{n,\alpha}$ *where* $\Xi_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ *is given by (3.24). Then:*

(a) *For each* $[\phi; \psi] \in \mathcal{P}^{(r,r)}_{-\tilde{J}_r,\geq}(\mathbb{C}\backslash[\alpha,\infty))$*, the pair* $\left[\phi^{\square}; \psi^{\square}\right]$ *given by (6.7) and (6.8) belongs to* $\mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C}\backslash[\alpha,\infty))$*, the function* $\det\left(\widehat{\Xi}^{(2,1)}_{n,\alpha}\phi^{\square} + \widehat{\Xi}^{(2,2)}_{n,\alpha}\psi^{\square}\right)$ *does not vanish identically, and* S *defined by*

$$S = \left(\widehat{\Xi}^{(1,1)}_{n,\alpha}\phi^{\square} + \widehat{\Xi}^{(1,2)}_{n,\alpha}\psi^{\square}\right)\left(\widehat{\Xi}^{(2,1)}_{n,\alpha}\phi^{\square} + \widehat{\Xi}^{(2,2)}_{n,\alpha}\phi^{\square}\right)^{-1} \tag{6.9}$$

belongs to $\mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^{2n}, \leq]$.

(b) *For each* $S \in \mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^{2n}, \leq]$, *there exists a pair* $[\phi; \psi]$ *belonging to* $\mathcal{P}^{(r,r)}_{-\tilde{J}_r,\geq}(\mathbb{C}\setminus[\alpha,\infty))$ *such that the function* $\det\left(\widehat{\Xi}^{(2,1)}_{n,\alpha}\phi^{\square} + \widehat{\Xi}^{(2,2)}_{n,\alpha}\psi^{\square}\right)$ *does not vanish identically and that S can be represented via (6.9), where* $\phi^{\square}$ *and* $\psi^{\square}$ *are given by (6.7) and (6.8).*

(c) *Let* $[\phi_1; \psi_1], [\phi_2; \psi_2] \in \mathcal{P}^{(r,r)}_{-\tilde{J}_r,\geq}(\mathbb{C} \setminus [\alpha,\infty))$. *With obvious notation, then* $\langle[\phi_1; \psi_1]\rangle = \langle[\phi_2; \psi_2]\rangle$ *if and only if*

$$\left(\widehat{\Xi}^{(1,1)}_{n,\alpha}\phi_1^{\square} + \widehat{\Xi}^{(1,2)}_{n,\alpha}\psi_1^{\square}\right) \cdot \left(\widehat{\Xi}^{(2,1)}_{n,\alpha}\phi_1^{\square} + \widehat{\Xi}^{(2,2)}_{n,\alpha}\psi_1^{\square}\right)^{-1}$$
$$= \left(\widehat{\Xi}^{(1,1)}_{n,\alpha}\phi_2^{\square} + \widehat{\Xi}^{(1,2)}_{n,\alpha}\psi_2^{\square}\right) \cdot \left(\widehat{\Xi}^{(2,1)}_{n,\alpha}\phi_2^{\square} + \widehat{\Xi}^{(2,2)}_{n,\alpha}\psi_2^{\square}\right)^{-1}.$$

Proof (a) Let $[\phi; \psi] \in \mathcal{P}^{(r,r)}_{-\tilde{J}_r,\geq}(\mathbb{C}\setminus[\alpha,\infty))$. From part (b2) of Lemma 6.3 we conclude that $\left[\phi^{\square}; \psi^{\square}\right]$ belongs to $\mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}(\mathbb{C}\setminus[\alpha,\infty))$ and fulfills (6.6). In view of Notation 5.1 and Lemma 3.44, then we see that $\left[\phi^{\square}; \psi^{\square}\right]$ belongs to $\mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}[\mathbb{C}\setminus[\alpha,\infty), (s_j)_{j=0}^{2n}]$. Applying part (a) of Proposition 5.4 completes the proof of part (a).
(b) Let $S \in \mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^{2n}, \leq]$. According to part (b) of Proposition 5.4, then there is a pair $[\widetilde{\phi}; \widetilde{\psi}] \in \mathcal{P}^{(q,q)}_{-\tilde{J}_q,\geq}[\mathbb{C}\setminus[\alpha,\infty), (s_j)_{j=0}^{2n}]$ such that $\widetilde{\phi}$ and $\widetilde{\psi}$ are holomorphic in $\mathbb{C}\setminus[\alpha,\infty)$ and fulfill that $\det\left(\widehat{\Xi}^{(2,1)}_{n,\alpha}(z)\widetilde{\phi}(z) + \widehat{\Xi}^{(2,2)}_{n,\alpha}(z)\widetilde{\psi}(z)\right) \neq 0$ and

$$S(z) = \left[\widehat{\Xi}^{(1,1)}_{n,\alpha}(z)\widetilde{\phi}(z) + \widehat{\Xi}^{(1,2)}_{n,\alpha}(z)\widetilde{\psi}(z)\right]\left[\widehat{\Xi}^{(2,1)}_{n,\alpha}(z)\widetilde{\phi}(z) + \widehat{\Xi}^{(2,2)}_{n,\alpha}(z)\widetilde{\psi}(z)\right]^{-1} \tag{6.10}$$

hold true for all $z \in \mathbb{C}\setminus[\alpha,\infty)$. Moreover, then Notation 5.1 and Lemma 3.44 show that (6.6) is valid and, using part (b1) of Lemma 6.3 and the notation given there, we see that there exists a pair $[\phi; \psi] \in \mathcal{P}^{(r,r)}_{-\tilde{J}_r,\geq}(\mathbb{C}\setminus[\alpha,\infty))$ fulfilling $\left\langle\left[\phi^{\square}; \psi^{\square}\right]\right\rangle = \langle[\widetilde{\phi}; \widetilde{\psi}]\rangle$. Consequently, there is a discrete subset $\mathcal{D}$ of $\mathbb{C}\setminus[\alpha,\infty)$ and a $q \times q$ matrix-valued function g meromorphic in $\mathbb{C}\setminus[\alpha,\infty)$ such that $\widetilde{\phi}, \widetilde{\psi}, \phi, \psi$, and g are holomorphic in $\mathbb{C}\setminus([\alpha,\infty) \cup \mathcal{D})$ and that $\det g(z) \neq 0$ as well as $\widetilde{\phi}(z) = \phi^{\square}(z)g(z)$ and $\widetilde{\psi}(z) = \psi^{\square}(z)g(z)$ hold true for all $z \in \mathbb{C}\setminus([\alpha,\infty) \cup \mathcal{D})$. Thus, the inequality $\det\left(\widehat{\Xi}^{(2,1)}_{n,\alpha}(z)\widetilde{\phi}(z) + \widehat{\Xi}^{(2,2)}_{n,\alpha}(z)\widetilde{\psi}(z)\right) \neq 0$ implies $0 \neq \det\left(\widehat{\Xi}^{(2,1)}_{n,\alpha}(z)\phi^{\square}(z) + \widehat{\Xi}^{(2,2)}_{n,\alpha}(z)\psi^{\square}(z)\right) \cdot \det g(z)$, whereas (6.10), $\widetilde{\phi}(z) = \phi^{\square}(z)g(z)$, and $\widetilde{\psi}(z) = \psi^{\square}(z)g(z)$ yield then

$$S(z) = \left[\widehat{\Xi}^{(1,1)}_{n,\alpha}(z)\phi^{\square}(z) + \widehat{\Xi}^{(1,2)}_{n,\alpha}(z)\psi^{\square}(z)\right]\left[\widehat{\Xi}^{(2,1)}_{n,\alpha}(z)\phi^{\square}(z) + \widehat{\Xi}^{(2,2)}_{n,\alpha}(z)\psi^{\square}(z)\right]^{-1}$$

for all $z \in \mathbb{C}\setminus([\alpha,\infty) \cup \mathcal{D})$. In particular, (6.9) is valid.

(c) In view of [21, Lemma 15.2, Remark 15.1], part (c) can be proved analogous to the proof of [21, Theorem 15.5, part (c)]. We omit the details. □

Now we turn our attention to the completely degenerate case that the integers k and l given by (6.1) and (6.2) fulfill $k + l = q$.

Lemma 6.5 *Suppose $\kappa \geq 2$. Let $\left(s_j\right)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$ and let $n \in \mathbb{N}$ be such that $2n \leq \kappa$. Suppose that the integers k and l given by (6.1) and (6.2) fulfill $k+l = q$. Let $\mathcal{U}_{n,\alpha}$ and $\mathcal{W}_{n,\alpha}$ be the subspaces of $\mathbb{C}^q$ given by (3.64) and (3.65). Then there exists a unitary complex $q \times q$ matrix U such that*

$$U^* P_{\mathcal{U}_{n,\alpha}} U = \Delta_k \quad \text{and} \quad U^* P_{\mathcal{W}_{n,\alpha}} U = I_q - \Delta_k \tag{6.11}$$

where

$$\Delta_k := \begin{cases} \operatorname{diag}\left(I_k,\ 0_{l\times l}\right), & \text{if } 1 \leq k \leq q-1 \\ 0_{q\times q}, & \text{if } k = 0 \\ I_q, & \text{if } k = q \end{cases}. \tag{6.12}$$

If U is a unitary complex $q \times q$ matrix such that (6.11) is valid, then $\phi_\sharp, \psi_\sharp : \mathbb{C}\setminus[\alpha,\infty) \to \mathbb{C}^{q\times q}$ defined by $\phi_\sharp(z) := U\left(I_q - \Delta_k\right)$ and $\psi_\sharp(z) := U\Delta_k$ form a pair $\left[\phi_\sharp; \psi_\sharp\right]$ belonging to $\mathcal{P}_{-\tilde{J}_q,\geq}^{(q,q)}(\mathbb{C}\setminus[\alpha,\infty))$ and fulfilling $\mathcal{P}_{-\tilde{J}_q,\geq}^{(q,q)}[\mathbb{C}\setminus[\alpha,\infty), \left(s_j\right)_{j=0}^{2n}] = \left\{[\phi;\psi] : \langle[\phi;\psi]\rangle = \left\langle\left[\phi_\sharp;\psi_\sharp\right]\right\rangle\right\}$.

Proof In the case $1 \leq k \leq q-1$, the assertion follows immediately using Lemma 3.44, Notation 5.1, and [21, Lemma 16.1]. If $k = 0$, then $\mathcal{U}_{n,\alpha} = \left\{0_{q\times 1}\right\}$ and $\mathcal{W}_{n,\alpha} = \mathbb{C}^q$, i.e, $P_{\mathcal{U}_{n,\alpha}} = 0_{q\times q}$ and $P_{\mathcal{W}_{n,\alpha}} = I_q$, which, in view of [21, Remark 16.3], Notation 5.1, and Lemma 3.44, yields the assertion. Similarly, the case $k = q$ can be treated by the same arguments. □

Theorem 6.6 *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ be such that $\kappa \geq 2$, let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$, and let $n \in \mathbb{N}_0$ be such that $2n \leq \kappa$. Let $\Xi_{n,\alpha} : \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ be given by (3.24) and (3.27), let $\widehat{\Xi}_{n,\alpha} := \operatorname{Rstr}_{\mathbb{C}\setminus[\alpha,\infty)} \Xi_{n,\alpha}$, and let (4.4) be the $q \times q$ block representation of $\widehat{\Xi}_{n,\alpha}$. Suppose that the integers k and l given by (6.1) and (6.2) fulfill $k + l = q$. Then $\mathcal{S}_{0,q,[\alpha,\infty)}[\left(s_j\right)_{j=0}^{2n}, \leq]$ consists of exactly one function S and the following statements hold true:*

(a) If $k = 0$, then $\det \widehat{\Xi}_{n,\alpha}^{(2,1)}$ does not vanish identically and $S = \widehat{\Xi}_{n,\alpha}^{(1,1)} \left(\widehat{\Xi}_{n,\alpha}^{(2,1)}\right)^{-1}$.

(b) If $l = 0$, then $\det \widehat{\Xi}_{n,\alpha}^{(2,2)}$ does not vanish identically and $S = \widehat{\Xi}_{n,\alpha}^{(1,2)} \left(\widehat{\Xi}_{n,\alpha}^{(2,2)}\right)^{-1}$.

(c) Suppose $k \geq 1$ and $l \geq 1$. Let $\mathcal{U}_{n,\alpha}$ and $\mathcal{W}_{n,\alpha}$ be the subspaces of $\mathbb{C}^q$ given by (3.64) and (3.65). Furthermore, let U be a unitary complex $q \times q$ matrix such that (6.11) holds true where Δ_k is defined by (6.12). Then $\det\left(\widehat{\Xi}_{n,\alpha}^{(2,1)} U\left(I_q - \Delta_k\right) + \widehat{\Xi}_{n,\alpha}^{(2,2)} U\Delta_k\right)$ does not vanish identically and

$$S = \left(\widehat{\Xi}_{n,\alpha}^{(1,1)} U\left(I_q - \Delta_k\right) + \widehat{\Xi}_{n,\alpha}^{(1,2)} U\Delta_k\right)\left(\widehat{\Xi}_{n,\alpha}^{(2,1)} U\left(I_q - \Delta_k\right) + \widehat{\Xi}_{n,\alpha}^{(2,2)} U\Delta_k\right)^{-1}.$$

Proof Combine Proposition 5.4 and Lemma 6.5. □

It remains to discuss the completely degenerate case if $n = 0$ holds true. Then $k = q$ and $l = 0$. Because of (6.1), we have $q = k = \operatorname{rank}(I - s_0^+ s_0)$, which implies $\mathcal{N}(I - s_0^+ s_0) = \{0_{q\times 1}\}$ and, in view of $(I - s_0^+ s_0)s_0^+ = 0$, consequently, $s_0 = (s_0^+)^+ = 0_{q\times q}^+ = 0_{q\times q}$. Thus, we have:

Remark 6.7 (*cf.* [9, Remark 6.7]) Let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$. Let $n = 0$ and let k given by (6.1) fulfill $k = q$. Then $s_j = 0_{q\times q}$ for all $j \in \mathbb{Z}_{0,\kappa}$ and $\mathcal{S}_{0,q,[\alpha,\infty)}[(s_j)_{j=0}^{0}, \leq] = \{0_{q\times q}\}$ as well as $\mathcal{M}_{\geq}^{q}[[\alpha,\infty); (s_j)_{j=0}^{0}, \leq] = \{\mathcal{O}_q\}$ where $\mathcal{O}_q : \mathcal{B}_{[\alpha,\infty)} \to \mathbb{C}^{q\times q}$ is given by $\mathcal{O}_q(B) := 0_{q\times q}$.

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Declarations

Conflict of interest The authors have no conflicts of interest to declare that are relevant to the content of this article.

Appendix A. Appendix

Remark A.1 Let $A \in \mathbb{C}^{q\times q}$. Then $A = A^*$ if and only if $(A^+)^* = A^+$. Furthermore, if $\mathcal{R}(A) = \mathcal{R}(A^*)$, then $AA^+ = A^+A$.

Remark A.2 Let $A \in \mathbb{C}^{q\times q}$. Then $A \in \mathbb{C}_{\geq}^{q\times q}$ if and only if $A^+ \in \mathbb{C}_{\geq}^{q\times q}$. If $A^+ \in \mathbb{C}_{\geq}^{q\times q}$, then $\sqrt{A}^+ = \sqrt{A^+}$ and $AA^+ = A^+A = \sqrt{A}\sqrt{A}^+ = \sqrt{A}^+\sqrt{A}$.

Remark A.3 Let $A \in \mathbb{C}^{p\times q}$ and let $B \in \mathbb{C}^{p\times r}$. Then $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ if and only if $BB^+A = A$.

Remark A.4 Let $A \in \mathbb{C}^{p\times q}$ and let $X \in \mathbb{C}^{q\times p}$ be such that $AXA = A$ and $XAX = X$ hold true. Then $P{:=}AX$ and $Q{:=}XA$ are both idempotent. Thus, [1, Ch. 2, Thm. 8] yields $\mathcal{R}(P) \dotplus \mathcal{N}(P) = \mathbb{C}^p$ and $\mathcal{R}(Q) \dotplus \mathcal{N}(Q) = \mathbb{C}^q$. In view of $\mathcal{R}(P) = \mathcal{R}(A)$, $\mathcal{N}(P) = \mathcal{N}(X)$, $\mathcal{R}(Q) = \mathcal{R}(X)$, and $\mathcal{N}(Q) = \mathcal{N}(A)$, then $\mathcal{R}(A) \dotplus \mathcal{N}(X) = \mathbb{C}^p$ and $\mathcal{N}(A) \dotplus \mathcal{R}(X) = \mathbb{C}^q$ are valid.

Lemma A.5 *Let $A \in \mathbb{C}_{\mathrm{H}}^{q\times q}$, let $U \in \mathbb{C}^{q\times p}$, and let $Y \in \mathbb{C}^{p\times p}$ be such that $U^*AUYU^*AU = U^*AU$. Let $X{:=}UYU^*$. Then the following statements are equivalent:*

(i) $\mathcal{N}(A) \dotplus \mathcal{R}(U) = \mathbb{C}^q$ *and* $X = A^{-}_{\mathcal{R}(U)}$.
(ii) $\operatorname{rank}(U^*AU) = \operatorname{rank} U$ *and* $\operatorname{rank} U = \operatorname{rank} A$.

Proof We state a modified proof given by T. Makarevich in an unpublished extended version of [29].

"(i) ⇒ (ii)": Because of $A^* = A$ and (i), we also have $\mathcal{R}(A) \dotplus \mathcal{R}(U)^\perp = \mathbb{C}^q$ and $X = A^{-}_{\mathcal{R}(U)} = A^{(1,2)}_{\mathcal{R}(U),\mathcal{R}(U)^\perp} = A^{(1,2)}_{\mathcal{R}(U),\mathcal{N}(U^*)}$. Applying [1, Ch. 2, Thm. 13(d)], we get then (ii).
"(ii)⇒(i)": By virtue of (ii) and [1, Ch. 2, Thm. 13(a), (b)], we have $AXA = A$ and $XAX = X$ as well as $\mathcal{R}(X) = \mathcal{R}(U)$. Consequently, Remark A.4 yields $\mathbb{C}^q = \mathcal{N}(A) \dotplus \mathcal{R}(X) = \mathcal{N}(A) \dotplus \mathcal{R}(U)$, whereas (ii) and [1, Ch. 2, Thm. 13(d)] provide $X = A^{(1,2)}_{\mathcal{R}(U),\mathcal{N}(U^*)} = A^{(1,2)}_{\mathcal{R}(U),\mathcal{R}(U)^\perp} = A^{-}_{\mathcal{R}(U)}$.

□

Lemma A.6 *Let $A \in \mathbb{C}_{\geq}^{q\times q} \setminus \{0_{q\times q}\}$ and let $Q \in \mathbb{C}^{p\times q}$. Then the following statements are equivalent:*

(i) $QAQ^* \in \mathbb{C}_{>}^{p\times p}$ *and* $\operatorname{rank}(QAQ^*) = \operatorname{rank} A$.
(ii) $\operatorname{rank} A = p$ *and* $\operatorname{rank}(QAQ^*) = p$.
(iii) $\mathcal{N}(A) \dotplus \mathcal{R}(Q^*) = \mathbb{C}^q$ *and* $\operatorname{rank} Q = p$.

If (i) is fulfilled, then $A^{-}_{\mathcal{R}(Q^*)} = Q^*(QAQ^*)^{-1}Q$.

Proof We state a modified proof given by T. Makarevich in an unpublished extended version of [29]. The equivalence of (i) and (ii) is obvious.

Now suppose (i). Then $B{:=}QAQ^*$ is a non-singular Hermitian matrix. Consequently, $BB^{-1}B = B$ and $p = \operatorname{rank} B = \operatorname{rank} A$ as well as $p \geq \operatorname{rank} Q^* \geq \operatorname{rank} B = p$. Thus, $\operatorname{rank} Q^* = \operatorname{rank} B = \operatorname{rank} A$ and $\operatorname{rank} Q = p$. Applying Lemma A.5 provides $\mathcal{N}(A) \dotplus \mathcal{R}(Q^*) = \mathbb{C}^q$ and $A^{-}_{\mathcal{R}(Q^*)} = Q^*B^{-1}Q = Q^*(QAQ^*)^{-1}Q$. In particular, (iii) is valid.

Conversely, now suppose (iii). Consider an arbitrary $x \in \mathbb{C}^p \setminus \{0_{p\times 1}\}$. From (iii) we obtain $Q^*x \in \mathcal{R}(Q^*) \setminus \{0_{q\times 1}\}$ and, using (iii) again, then $Q^*x \in \mathcal{R}(Q^*) \setminus \mathcal{N}(A) = \mathcal{R}(Q^*) \setminus \mathcal{N}(\sqrt{A})$. This implies $x^*QAQ^*x = (\sqrt{A}Q^*x)^*(\sqrt{A}Q^*x) \neq 0$. Therefore, $QAQ^* \in \mathbb{C}_{>}^{p\times p}$. In particular, $p = \operatorname{rank}(QAQ^*) \leq \operatorname{rank} Q \leq p$, which implies $\operatorname{rank}(QAQ^*) = p$ and $p = \operatorname{rank} Q^* = \dim \mathcal{R}(Q^*)$. Because of (iii), then $p = \dim \mathbb{C}^q - \dim \mathcal{N}(A) = \dim \mathcal{R}(A) = \operatorname{rank} A$ follows. Thus, (i) holds true. □

Lemma A.7 *Let $n \in \mathbb{N}$, let $A \in \mathbb{C}_H^{p\times p}$, and let $B \in \mathbb{C}_H^{q\times q}$. Furthermore, let $T \in \mathbb{C}^{p\times p}$, let $V \in \mathbb{C}^{p\times q}$, and let $\mathfrak{V} \in \mathbb{C}^{p\times q}$ be such that $TV = \mathfrak{V}$. Suppose that $\mathcal{U}$ is a subspace of $\mathbb{C}^p$ and $\mathcal{V}$ is a subspace of $\mathbb{C}^q$ such that the four conditions $\mathcal{N}(A) \dotplus \mathcal{U} = \mathbb{C}^p$, $\mathcal{N}(B) \dotplus \mathcal{V} = \mathbb{C}^q$, $T^*(\mathcal{U}) \subseteq \mathcal{U}$, and $\mathfrak{V}^*(\mathcal{U}) \subseteq \mathcal{V}$ are fulfilled. Then $A_{\mathcal{U}}^- T^l V(I_q - BB_{\mathcal{V}}^-) = 0_{p\times q}$ for each $l \in \mathbb{N}$.*

Proof Obviously, $V^*T^* = \mathfrak{V}^*$ and $(T^*)^l u \in \mathcal{U}$ for each $l \in \mathbb{N}$ and each $u \in \mathcal{U}$. Let $l \in \mathbb{N}$ and let $u \in \mathcal{U}$. Due to the assumption $T^*(\mathcal{U}) \subseteq \mathcal{U}$, then $\hat{u} := (T^*)^{l-1}u$ belongs to $\mathcal{U}$. Hence, $\mathfrak{V}^*(\mathcal{U}) \subseteq \mathcal{V}$ delivers $V^*(T^*)^l u = (V^*T^*)(T^*)^{l-1}u = \mathfrak{V}^*\hat{u} \in \mathcal{V}$. Thus, for each $\tilde{v} \in \mathcal{V}^\perp$, we get

$$\langle u, T^l V\tilde{v}\rangle_E = \left(T^l V\tilde{v}\right)^* u = \tilde{v}^* V^* (T^l)^* u = \tilde{v}^* V^* (T^*)^l u = \langle V^*(T^*)^l u, \tilde{v}\rangle_E = 0.$$

Therefore, $T^l V(\mathcal{V}^\perp) \subseteq \mathcal{U}^\perp$ for each $l \in \mathbb{N}$. Since [21, Lemma 17.3] provides $\mathcal{N}(A_{\mathcal{U}}^-) = \mathcal{U}^\perp$ and since Lemma [21, Lemma 17.5] yields $\mathcal{R}\left(I_q - BB_{\mathcal{V}}^-\right) = \mathcal{V}^\perp$, we obtain finally $T^l V\left(\mathcal{R}\left(I_q - BB_{\mathcal{V}}^-\right)\right) = T^l V\left(\mathcal{V}^\perp\right) \subseteq \mathcal{U}^\perp = \mathcal{N}(A_{\mathcal{U}}^-)$ for each $l \in \mathbb{N}$. □

Lemma A.8 *Let $(\Omega, \mathfrak{A})$ be a measurable space, let $\mu \in \mathcal{M}_{\geq}^q(\Omega, \mathfrak{A})$, and let $f : \Omega \to \mathbb{C}$ be an $\mathfrak{A} - \mathfrak{B}_{\mathbb{C}}$-measurable mapping. Then the following statements are equivalent:*

(i) $f \in \mathcal{L}^1(\Omega, \mathfrak{A}, \mu; \mathbb{C})$.
(ii) $f \in \mathcal{L}^1(\Omega, \mathfrak{A}, u^*\mu u; \mathbb{C})$ *for each* $u \in \mathbb{C}^q$.
(iii) $f \in \mathcal{L}^1(\Omega, \mathfrak{A}, \tau; \mathbb{C})$ *where* τ *is the trace measure of* μ.

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