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Complex Symmetry of Linear Combinations of Composition Operators on the McCarthy–Bergman Space of Dirichlet Series

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Abstract

The complex symmetric linear combinations of composition operators on the McCarthy–Bergman spaces of Dirichlet series are completely characterized. The normality and self-adjointness of complex symmetric linear combinations of composition operators on such spaces are also characterized. Some images are given in order to find some interesting phenomena of $\mathcal{J}$-symmetric such combinations.

Keywords Complex symmetry · Linear combinations · Composition operators · McCarthy–Bergman spaces · Dirichlet series

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1 Introduction

As usual, let $\mathbb{N}$ be the set of natural numbers, H a separable complex Hilbert space and $\mathcal{B}(H)$ the set of all bounded linear operators on H. For an operator $T \in \mathcal{B}(H)$, let T^* denote the adjoint operator of T.

In this section, we need to introduce some definitions. One of the definitions is the complex symmetric operators. It is widely recognized that numerous analytical problems necessitate extensive research on non-Hermitian operators. Among these problems, complex symmetric operators have emerged as particularly crucial in both theoretic and application aspects (see [16]).

Definition 1.1 A mapping $T : H \to H$ is said to be anti-linear (also conjugate-linear), if it satisfies

$$T(\alpha x + \beta y) = \bar{\alpha}T(x) + \bar{\beta}T(y),$$

for all $\alpha, \beta \in \mathbb{C}$ and $x, y \in H$.

Definition 1.2 An anti-linear mapping $C : H \to H$ is said to be a conjugation if it satisfies the following conditions:

(a) involutive: $C^2 = I_d$, where I_d is an identity operator;
(b) isometric: $\|C(x)\| = \|x\|$, for all $x \in H$.

Following [17, Lemma 1], we see that for any conjugation C, there exists an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ for H satisfying $Ce_n = e_n$ for all $n \in \mathbb{N}$. Actually, there are many conjugations on some holomorphic function spaces. For example, the common conjugation of complex numbers $(\mathcal{J} f)(z) = \overline{f(\bar{z})}$ and more general conjugation $\mathcal{J}_{\mu,\sigma}$, which will be defined on McCarthy–Bergman spaces of Dirichlet series later.

Based on Definition 1.2, we give the following definition.

Definition 1.3 Let C be a conjugation on H. An operator $T \in \mathcal{B}(H)$ is said to be complex symmetric with C if $T = CT^*C$.

Significantly, if an operator $T \in \mathcal{B}(H)$ is complex symmetric, then it can be represented as a symmetric matrix relative to some orthonormal basis of H (see [17, Proposition 2]). For this reason, the complex symmetric operators can be considered as an extension of symmetric matrices. As expected, with the continuous people's studies, the class of complex symmetric operators has become increasingly diverse. The class includes all normal operators, Hankel operators (matrices), operators that are algebraic of order two, finite Toeplitz matrices, all (truncated or compressed) Toeplitz operators, and some Volterra integration operators. The investigations of this operator were carried out by Garcia, Putinar, and Wogen in [17–20]. Many studies for the operator have been conducted on holomorphic function spaces (see [13, 14, 23, 25–27, 32]).

In the next time, we would like to provide the research motivations of this paper. With the basic questions such as boundedness and compactness settled, more attention has been paid to the study of the topological structure of the (weighted) composition operators in the operator norm topology. In this research background, Shapiro and Sundberg in [30] posed a question on whether two composition operators belong to the same connected component, when their difference is compact. Motivated by this question, people started to investigate compact differences, or more generally linear combinations of composition operators; see for example [5, 7–9, 22]. In the study of the compactness of linear combinations of composition operators, people indeed found some interesting phenomena. For example, the compactness of linear combinations $\sum_{j=1}^{N} \lambda_j C_{\varphi_j}$, for finitely many distinct linear fractional maps φ_j and nonzero complex numbers λ_j, implies that each composition operator C_{φ_j} is compact on the Hardy space $H^2(\mathbb{B}_n)$ over the unit ball (see [10]). Most recently, Xu et al. in [31] characterized complex symmetry of linear combinations of composition operators on the Fock space and proved that the bounded operator $\sum_{j=1}^{N} \lambda_j C_{\varphi_j}$ is $\mathcal{J}$-symmetric if and only if each C_{φ_j} is $\mathcal{J}$-symmetric.

Motivated by the above-mentioned interesting studies, a very natural thing is to study complex symmetry of linear combinations of composition operators on some other holomorphic function spaces. Here, we shall extend such problem from classical spaces to the McCarthy–Bergman spaces of Dirichlet series. Actually, in this work, we give a complete characterization of complex symmetry for linear combinations of composition operators on the McCarthy-Bergman spaces of Dirichlet series. We also characterize the normal and self-adjoint complex symmetric linear combinations of composition operators on such spaces. At the same time, some images are given in order to find some interesting phenomena of $\mathcal{J}$-symmetric such combinations. These results well demonstrate the innovation of the work. Our work can be regarded as a good continuous study of the composition operators on the McCarthy–Bergman spaces of Dirichlet series.

2 Preliminaries

Let $\mathbb{C}_\theta$ denote the half-plane of complex numbers $s = \sigma + it$ with $\sigma > \theta$, that is, $\mathbb{C}_\theta = \{s \in \mathbb{C} : \operatorname{Re} s > \theta\}$. For $a \le 0$, the McCarthy–Bergman space $\mathcal{A}_a$ of Dirichlet series is defined by (see [21])

$$\mathcal{A}_a = \left\{ f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} : \|f\|_{\mathcal{A}_a}^2 = |a_1|^2 + \sum_{n=2}^{\infty} |a_n|^2 (\log n)^a < \infty \right\}.$$

$\mathcal{A}_a$ is a Hilbert space with the inner product

$$\langle f, g \rangle_{\mathcal{A}_a} = a_1 \bar{b}_1 + \sum_{n=2}^{\infty} a_n \bar{b}_n (\log n)^a,$$

where $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ and $g(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s} \in \mathcal{A}_a$. If $a = 0$, $\mathcal{A}_a$ is reduced to the Hardy space $\mathcal{H}^2$ of Dirichlet series with square summable coefficients. One can see [6] for more information on the space. The reproducing kernel $K_{w,a}$ of $\mathcal{A}_a$ at a point

$w \in \mathbb{C}_{1/2}$ is given by

$$K_{w,a}(s) = 1 + \sum_{n=2}^{\infty} \frac{1}{(\log n)^a} \frac{1}{n^{\overline{w}+s}}, \quad s \in \mathbb{C}_{1/2}.$$

By the Cauchy-Schwarz inequality, $\mathcal{A}_a$ is a space of analytic functions in $\mathbb{C}_{1/2}$.

Let φ be an analytic self-map of the half-plane $\mathbb{C}_{1/2}$. The composition operator induced by φ on $\mathcal{A}_a$ is defined as

$$C_\varphi f = f \circ \varphi, \ \ f \in \mathcal{A}_a.$$

It is clear that $f \circ \varphi$ is an analytic function in $\mathbb{C}_{1/2}$ for any $f \in \mathcal{A}_a$. Composition operators on $\mathcal{A}_a$ (or other spaces of Dirichlet series) have been extensively studied in recent years (see [1–4, 12, 15, 24, 28]). Among these studies, the following result obtained in [4, 24] characterizes the bounded composition operators on $\mathcal{A}_a$. For the convenience, if φ satisfies Theorem A, then it is called a c_0-symbol.

Theorem A *Let $a \leq 0$ and φ be an analytic self-map of $\mathbb{C}_{1/2}$. Then the operator C_φ is bounded on $\mathcal{A}_a$ if and only if*

$$\varphi(s) = c_0 s + \sum_{n=1}^{\infty} c_n n^{-s} =: c_0 s + \psi(s)$$

where c_0 is a nonnegative integer and ψ is a Dirichlet series that converges in $\mathbb{C}_\theta$ for some $\theta > 0$ and has the following mapping properties:

(a) If $c_0 = 0$, then $\psi(\mathbb{C}_0) \subset \mathbb{C}_{1/2}$.
(b) If $c_0 \geq 1$, then either $\psi \equiv 0$ or $\psi(\mathbb{C}_0) \subset \mathbb{C}_0$.

3 Complex Symmetry of Linear Combinations of Composition Operators

In this section, we characterize the linear combinations of composition operators on $\mathcal{A}_a$ which are $\mathcal{J}$-symmetric with respect to the conjugation

$$(\mathcal{J} f)(s) = \overline{f(\overline{s})}, \quad f \in \mathcal{A}_a \text{ and } s \in \mathbb{C}_{1/2}.$$

By the similar method of Lemma 3.1 in [29]. We have the folowing result on $\mathcal{A}_a$.

Lemma 3.1 *Let $\varphi(s) = c_0 s + \sum_{k=1}^{\infty} c_k k^{-s}$ be a c_0-symbol. Then the following statements hold:*

(i) If $c_0 = 0$, then $C_\varphi^* 1 = 1 + \sum_{n=2}^{\infty} n^{-\bar{c}_1} n^{-s} (\log n)^{-a}$.
(ii) If $c_0 \geq 1$, then $C_\varphi^* 1 = 1$.

Let

$$\varphi_1(s) = c_0^{(1)}s + \sum_{k=1}^{\infty} c_k^{(1)}k^{-s} \text{ and } \varphi_2(s) = c_0^{(2)}s + \sum_{k=1}^{\infty} c_k^{(2)}k^{-s}$$

be two c_0-symbols and $\varphi_1 \neq \varphi_2$. Let λ_1 and λ_2 be two nonzero complex numbers. The linear combination of composition operators on $\mathcal{A}_a$ is defined as

$$\mathfrak{S}_2 = \lambda_1 C_{\varphi_1} + \lambda_2 C_{\varphi_2}.$$

Note that if one of λ_1 and λ_2 is equal to zero, then $\mathfrak{S}_2$ returns to the case of a single composition operator, which has been studied on $\mathcal{A}_0$ (see [32]) and the results are also applicable to the general spaces $\mathcal{A}_a$. For this reason, we assume that λ_1 and λ_2 are nonzero.

One of the aims of this section is to characterize $\mathcal{J}$-symmetric operator $\mathfrak{S}_2$ on $\mathcal{A}_a$. For this problem we obtain the following result.

Theorem 3.1 *Let $a \leq 0$. Then the operator $\mathfrak{S}_2$ is $\mathcal{J}$-symmetric on $\mathcal{A}_a$ if and only if $\varphi_1(s) = s + c_1^{(1)}$ and $\varphi_2(s) = s + c_1^{(2)}$ with $\operatorname{Re} c_1^{(1)} \geq 0$ and $\operatorname{Re} c_1^{(2)} \geq 0$.*

Proof Since φ_1 and φ_2 are c_0-symbols, both C_{φ_1} and C_{φ_2} are bounded on $\mathcal{A}_a$, which means that $\mathfrak{S}_2$ is bounded on $\mathcal{A}_a$. Now, suppose that $\mathfrak{S}_2$ is $\mathcal{J}$-symmetric on $\mathcal{A}_a$. Since φ_1 and φ_2 are c_0-symbols, there exist the following four possible cases:

(i) $c_0^{(1)} = 0$ and $c_0^{(2)} = 0$;
(ii) $c_0^{(1)} = 0$ and $c_0^{(2)} \geq 1$;
(iii) $c_0^{(1)} \geq 1$ and $c_0^{(2)} = 0$;
(iv) $c_0^{(1)} \geq 1$ and $c_0^{(2)} \geq 1$.

It is easy to see that above four cases can be reduced to the following two cases:

(a) $c_0^{(1)} = 0$ or $c_0^{(2)} = 0$;
(b) $c_0^{(1)} \geq 1$ and $c_0^{(2)} \geq 1$.

Case (a). Assume that $c_0^{(1)} = 0$ or $c_0^{(2)} = 0$. From Definition 1.3, it follows that $\mathfrak{S}_2$ is $\mathcal{J}$-symmetric on $\mathcal{A}_a$ if and only if

$$\lambda_1 C_{\varphi_1} + \lambda_2 C_{\varphi_2} = \mathcal{J}(\lambda_1 C_{\varphi_1} + \lambda_2 C_{\varphi_2})^* \mathcal{J} = \mathcal{J}(\overline{\lambda}_1 C_{\varphi_1}^* + \overline{\lambda}_2 C_{\varphi_2}^*)\mathcal{J}. \tag{3.1}$$

From (3.1) and $1 \in \mathcal{A}_a$, we obtain

$$(\overline{\lambda}_1 C_{\varphi_1}^* + \overline{\lambda}_2 C_{\varphi_2}^*)\mathcal{J}1 = \mathcal{J}(\lambda_1 C_{\varphi_1} + \lambda_2 C_{\varphi_2})1. \tag{3.2}$$

From the fact $\mathcal{J}1 = 1$ and (3.2), we have

$$\overline{\lambda}_1 C_{\varphi_1}^* 1 + \overline{\lambda}_2 C_{\varphi_2}^* 1 = \overline{\lambda}_1 + \overline{\lambda}_2. \tag{3.3}$$

By Lemma 3.1 (i) and since $c_0^{(1)} = 0$ or $c_0^{(2)} = 0$, we see that the left side of (3.3) is a nonconstant function but the right side of (3.3) is a constant. This is a contradiction, which shows that this case does not happen.

Case (b). Assume that $c_0^{(1)} \geq 1$ and $c_0^{(2)} \geq 1$. First, from an elementary calculation we see that for $f, g \in \mathcal{A}_a$, it follows that

$$\langle \mathcal{J} f, \mathcal{J} g \rangle_{\mathcal{A}_a} = \langle g, f \rangle_{\mathcal{A}_a}. \tag{3.4}$$

Since $\mathcal{J}\left(n^{-s}\right) = n^{-s}$ for all $n \in \mathbb{N}$, from (3.1) and (3.4) we have

$$\left\langle \mathfrak{S}_2\left(m^{-s}\right), n^{-s}\right\rangle_{\mathcal{A}_a} = \left\langle \mathfrak{S}_2\left(n^{-s}\right), m^{-s}\right\rangle_{\mathcal{A}_a} \tag{3.5}$$

for each $m, n \in \mathbb{N}$, which shows that the matrix of the operator $\mathfrak{S}_2$ in the base $\{n^{-s}\}_{n=1}^{\infty}$ is symmetric. We also know that in the base $\{n^{-s}\}_{n=1}^{\infty}$, $\{\mathfrak{S}_2 n^{-s}\}_{n=1}^{\infty}$ can be expressed as

$$\left(\mathfrak{S}_2 1, \mathfrak{S}_2 2^{-s}, \mathfrak{S}_2 3^{-s}, \ldots\right) = \left(1, 2^{-s}, 3^{-s}, \ldots\right) \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}. \tag{3.6}$$

Since $\mathfrak{S}_2 1 = \lambda_1 + \lambda_2$, from some calculations we see that the matrix in (3.6) is equal to

$$\begin{pmatrix} \lambda_1 + \lambda_2 & 0 & 0 & \cdots \\ 0 & a_{2,2} & a_{2,3} & \cdots \\ 0 & a_{3,2} & a_{3,3} & \cdots \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}. \tag{3.7}$$

So far, it has shown that the matrix in (3.7) is symmetric. On the other hand, we have

$$\begin{aligned} \mathfrak{S}_2 n^{-s} &= (\lambda_1 C_{\varphi_1} + \lambda_2 C_{\varphi_2}) n^{-s} = \lambda_1 n^{-\varphi_1(s)} + \lambda_2 n^{-\varphi_2(s)} \\ &= \lambda_1 n^{-c_0^{(1)} s - c_1^{(1)}} n^{-\sum_{k=2}^{\infty} c_k^{(1)} k^{-s}} + \lambda_2 n^{-c_0^{(2)} s - c_1^{(2)}} n^{-\sum_{k=2}^{\infty} c_k^{(2)} k^{-s}} \\ &= \lambda_1 n^{-c_0^{(1)} s - c_1^{(1)}} \prod_{k=2}^{\infty} \left(1 + \sum_{j=1}^{\infty} \frac{\left(-c_k^{(1)} \log n\right)^j}{j!} k^{-js} \right) \\ &\quad + \lambda_2 n^{-c_0^{(2)} s - c_1^{(2)}} \prod_{k=2}^{\infty} \left(1 + \sum_{j=1}^{\infty} \frac{\left(-c_k^{(2)} \log n\right)^j}{j!} k^{-js} \right). \end{aligned} \tag{3.8}$$

Interestingly, Sect. 3 in [15] shows that the Dirichlet series of $(\lambda_1 C_{\varphi_1} + \lambda_2 C_{\varphi_2}) n^{-s}$ can be obtained by expanding the brackets in (3.8). From this and (3.7), we obtain

$a_{m,n} = 0$ for all $n > m \geq 2$. Since the matrix in (3.7) is symmetric, we obtain $a_{m,n} = 0$ for all $2 \leq n < m$. This implies that the matrix is a diagonal matrix

$$\begin{pmatrix} \lambda_1 + \lambda_2 & 0 & 0 & \cdots \\ 0 & a_{2,2} & 0 & \cdots \\ 0 & 0 & a_{3,3} & \cdots \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}. \tag{3.9}$$

Since λ_1 and λ_2 are two nonzero complex numbers, from (3.9) we obtain $c_k^{(1)} = c_k^{(2)} = 0$ for all $k \geq 2$. From this and (3.8), it follows that

$$\mathfrak{S}_2 n^{-s} = \lambda_1 n^{-c_0^{(1)} s - c_1^{(1)}} + \lambda_2 n^{-c_0^{(2)} s - c_1^{(2)}} \tag{3.10}$$

for all $n \geq 2$. Since $c_0^{(1)} \geq 1$ and $c_0^{(2)} \geq 1$, we will divide into the following subcases to discuss.

Subcase (1). If $c_0^{(1)} = 1$ and $c_0^{(2)} = 1$, then from (3.10) we have

$$\mathfrak{S}_2 n^{-s} = \lambda_1 n^{-s - c_1^{(1)}} + \lambda_2 n^{-s - c_1^{(2)}} = (\lambda_1 n^{-c_1^{(1)}} + \lambda_2 n^{-c_1^{(2)}}) n^{-s} \tag{3.11}$$

for all $n \geq 2$. From (3.9) and (3.11), it follows that

$$a_{n,n} = \lambda_1 n^{-c_1^{(1)}} + \lambda_2 n^{-c_1^{(2)}}$$

for all $n \geq 2$. Thus, we obtain $\varphi_1(s) = s + c_1^{(1)}$ and $\varphi_2(s) = s + c_1^{(2)}$. Moreover, from Theorem A, we have $\operatorname{Re} c_1^{(1)} \geq 0$ and $\operatorname{Re} c_1^{(2)} \geq 0$.

Subcase (2). If $c_0^{(1)} = 1$ and $c_0^{(2)} > 1$, or $c_0^{(1)} > 1$ and $c_0^{(2)} = 1$, then from (3.10) we have

$$\mathfrak{S}_2 n^{-s} = \lambda_1 n^{-s - c_1^{(1)}} + \lambda_2 n^{-c_0^{(2)} s - c_1^{(2)}} \tag{3.12}$$

or

$$\mathfrak{S}_2 n^{-s} = \lambda_1 n^{-c_0^{(1)} s - c_1^{(1)}} + \lambda_2 n^{-s - c_1^{(2)}} \tag{3.13}$$

for all $n \geq 2$. Since λ_1 and λ_2 are nonzero complex numbers, $\lambda_1 n^{-c_1^{(1)}} \neq 0$ and $\lambda_2 n^{-c_1^{(2)}} \neq 0$ for all $n \geq 2$. From these facts, we see that $\mathfrak{S}_2 n^{-s} \neq a_{n,n} n^{-s}$ for all $n \geq 2$, which shows that this subcase does not happen.

Subcase (3). If $c_0^{(1)} > 1$ and $c_0^{(2)} > 1$, then from (3.10) we have

$$\mathfrak{S}_2 n^{-s} = \lambda_1 n^{-c_0^{(1)} s - c_1^{(1)}} + \lambda_2 n^{-c_0^{(2)} s - c_1^{(2)}} \tag{3.14}$$

for all $n \geq 2$. Moreover, we see that

$$n^{-c_0^{(1)}s} \neq n^{-s} \quad \text{and} \quad n^{-c_0^{(2)}s} \neq n^{-s}.$$

Also, we see that if $\mathfrak{S}_2 n^{-s} = a_{n,n} n^{-s}$ for all $n \geq 2$, then the coefficient $a_{n,n}$ should be equal to zero. Thus, the coefficients of $n^{-c_0^{(1)}s}$ and $n^{-c_0^{(2)}s}$ in (3.14) must be one of the following two cases:

$$\lambda_1 n^{-c_1^{(1)}} = 0 \quad \text{and} \quad \lambda_2 n^{-c_1^{(2)}} = 0 \tag{3.15}$$

or

$$c_0^{(1)} = c_0^{(2)} > 1 \quad \text{and} \quad \lambda_1 n^{-c_1^{(1)}} + \lambda_2 n^{-c_1^{(2)}} = 0 \tag{3.16}$$

for all $n \geq 2$. But, since λ_1 and λ_2 are nonzero, (3.15) is clearly not true. So, (3.16) holds for all $n \geq 2$. However, if (3.16) holds for all $n \geq 2$, then from (3.14) it follows that $\lambda_1 + \lambda_2 = 0$ and $c_1^{(1)} = c_1^{(2)}$. This shows that $\varphi_1 = \varphi_2$, which is a contradiction since $\varphi_1 \neq \varphi_2$.

Combining these cases, we have proven that if the operator $\mathfrak{S}_2$ is $\mathcal{J}$-symmetric on $\mathcal{A}_a$, then $\varphi_1(s) = s + c_1^{(1)}$ and $\varphi_2(s) = s + c_1^{(2)}$ with $\operatorname{Re} c_1^{(1)} \geq 0$ and $\operatorname{Re} c_1^{(2)} \geq 0$.

Conversely, assume that $\varphi_1(s) = s + c_1^{(1)}$ and $\varphi_2(s) = s + c_1^{(2)}$ with $\operatorname{Re} c_1^{(1)} \geq 0$ and $\operatorname{Re} c_1^{(2)} \geq 0$. Since $\operatorname{span}\{K_{w,a} : w \in \mathbb{C}_{1/2}\}$ is dense in $\mathcal{A}_a$ and it is obvious that $\mathcal{J} K_{w,a} = K_{\overline{w},a}$, we obtain

$$\mathfrak{S}_2 \mathcal{J} K_{w,a}(s) = \sum_{j=1}^{2} \lambda_j K_{\overline{w},a}(\varphi_j(s)) \tag{3.17}$$

for all $w, s \in \mathbb{C}_{1/2}$. Using the fact $C_\varphi^* K_{w,a} = K_{\varphi(w),a}$, we have

$$\mathcal{J} \mathfrak{S}_2^* K_{w,a}(s) = \sum_{j=1}^{2} \lambda_j K_{\overline{\varphi_j(w)},a}(s) \tag{3.18}$$

for all $w, s \in \mathbb{C}_{1/2}$. Hence, from (3.17) and (3.18) we see that $\mathfrak{S}_2$ is $\mathcal{J}$-symmetric on $\mathcal{A}_a$ if and only if

$$\sum_{j=1}^{2} \lambda_j K_{\overline{w},a}(\varphi_j(s)) = \sum_{j=1}^{2} \lambda_j K_{\overline{\varphi_j(w)},a}(s). \tag{3.19}$$

From the assumptions, we see that (3.19) holds by using a tedious computation. This shows that the operator $\mathfrak{S}_2$ is $\mathcal{J}$-symmetric on $\mathcal{A}_a$. From this, the desired conclusion follows. □

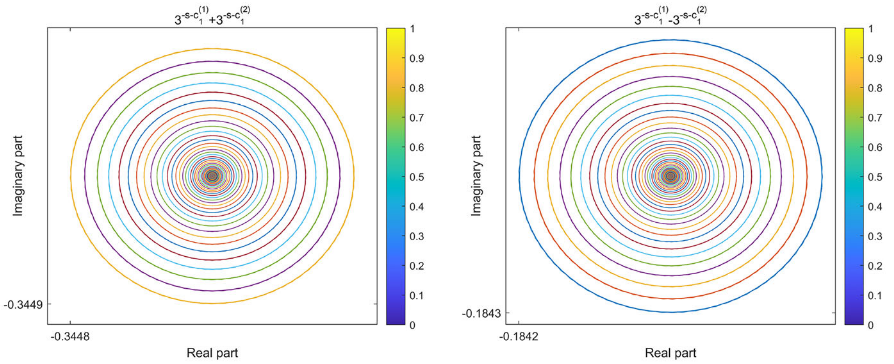

Fig. 1 The images of ranges for functions $\mathfrak{S}_2^{(1)}3^{-s}$ and $\mathfrak{S}_2^{(2)}3^{-s}$

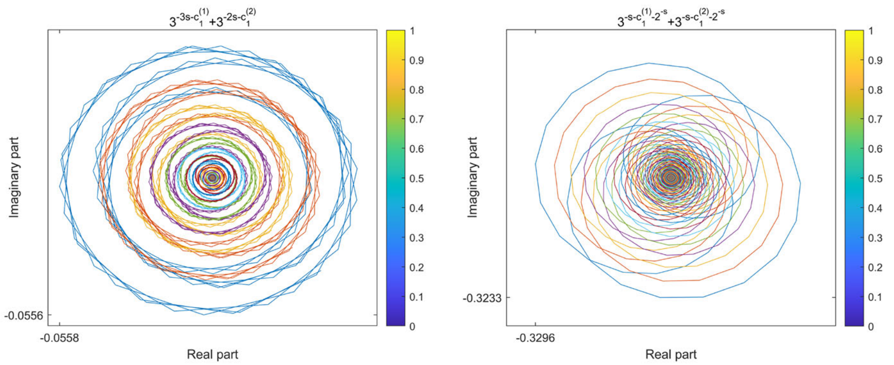

Fig. 2 The images of ranges for functions $\mathfrak{S}_2^{(3)}$ and $\mathfrak{S}_2^{(4)}$ acting on 3^{-s}

Next, we give some examples.

Example 3.1 (a) Let $c_1^{(1)} = 1+2i$ and $c_1^{(2)} = 3+4i$. Define $\mathfrak{S}_2^{(1)} = C_{s+c_1^{(1)}} + C_{s+c_1^{(2)}}$ and $\mathfrak{S}_2^{(2)} = C_{s+c_1^{(1)}} - C_{s+c_1^{(2)}}$. By Theorem 3.1, $\mathfrak{S}_2^{(1)}$ and $\mathfrak{S}_2^{(2)}$ are $\mathcal{J}$-symmetric on $\mathcal{A}_a$.

(b) Let $c_1^{(1)} = 1+2i$ and $c_1^{(2)} = 3+4i$. Define $\mathfrak{S}_2^{(3)} = C_{3s+c_1^{(1)}} + C_{2s+c_1^{(2)}}$ and $\mathfrak{S}_2^{(4)} = C_{s+c_1^{(1)}+2^{-s}} + C_{s+c_1^{(2)}+2^{-s}}$. Also, from Theorem 3.1 we see that $\mathfrak{S}_2^{(3)}$ and $\mathfrak{S}_2^{(4)}$ are not $\mathcal{J}$-symmetric on $\mathcal{A}_a$.

In order to find some interesting phenomena of $\mathcal{J}$-symmetric operators, we give the images of ranges for functions $\mathfrak{S}_2^{(1)}3^{-s}$, $\mathfrak{S}_2^{(2)}3^{-s}$, $\mathfrak{S}_2^{(3)}3^{-s}$ and $\mathfrak{S}_2^{(4)}3^{-s}$, respectively (Figs. 1, 2).

From these images, one can easily find that there indeed exist some distinct differences between $\mathcal{J}$-symmetric and non $\mathcal{J}$-symmetric operators $\mathfrak{S}_2$ on $\mathcal{A}_a$.

Theoretically speaking, there may be many conjugations on $\mathcal{A}_a$. However, contrary to expectations, we have the following result, which shows that the complex symmetry of the composition operator on $\mathcal{A}_a$ is independent of the conjugations. By reading Theorem 2.5 in [32] and Theorem 3.1, we can easily give the proof and so we omit it.

Lemma 3.2 *Let $a \le 0$ and $\varphi(s) = c_0 s + \sum_{k=1}^{\infty} c_k k^{-s}$ be a non-constant c_0-symbol. Then the following statements are equivalent:*

(a) C_φ is complex symmetric on $\mathcal{A}_a$.
(b) C_φ is $\mathcal{J}$-symmetric on $\mathcal{A}_a$.
(c) $\varphi(s) = s + c_1$ with $\Re c_1 \ge 0$.

From Theorem 3.1 and Lemma 3.2, we have the next result.

Corollary 3.1 *Let $a \le 0$ and φ_j be a non-constant c_0-symbol. Then the following statements are equivalent:*

(a) $\mathfrak{S}_2$ is $\mathcal{J}$-symmetric on $\mathcal{A}_a$.
(b) C_{φ_j} is complex symmetric on $\mathcal{A}_a$ for each $j \in \{1, 2\}$.
(c) C_{φ_j} is $\mathcal{J}$-symmetric on $\mathcal{A}_a$ for each $j \in \{1, 2\}$.
(d) $\varphi_1(s) = s + c_1^{(1)}$ and $\varphi_2(s) = s + c_1^{(2)}$ with $\operatorname{Re} c_1^{(1)} \ge 0$ and $\operatorname{Re} c_1^{(2)} \ge 0$.

Remark 3.1 Let $|\mu| = 1$ and $\{\sigma_n\}$ be a sequence of real numbers. From [33], we obtain the following conjugation on $\mathcal{A}_a$

$$(\mathcal{J}_{\mu,\sigma} f)(s) = \mu \overline{\sum_{n=1}^{\infty} a_n n^{-\bar{s}-i\sigma_n}},$$

for any $f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{A}_a$. From Corollary 3.1, we obtain that $\mathfrak{S}_2$ is $\mathcal{J}_{\mu,\sigma}$-symmetric on $\mathcal{A}_a$ if and only if $\varphi_1(s) = s + c_1^{(1)}$ and $\varphi_2(s) = s + c_1^{(2)}$ with $\operatorname{Re} c_1^{(1)} \ge 0$ and $\operatorname{Re} c_1^{(2)} \ge 0$. This corollary is very theoretical significance.

Now, we extend $\mathfrak{S}_2$ to the more complicated case. To this end, let

$$\begin{aligned}\varphi_1(s) &= c_0^{(1)} s + \sum_{k=1}^{\infty} c_k^{(1)} k^{-s}, \varphi_2(s) = c_0^{(2)} s + \sum_{k=1}^{\infty} c_k^{(2)} k^{-s}, \ldots, \varphi_N(s) = c_0^{(N)} s \\ &\quad + \sum_{k=1}^{\infty} c_k^{(N)} k^{-s}\end{aligned}$$

be c_0-symbols and $\varphi_i \ne \varphi_k$ whenever $i \ne k$ for $i, k \in \{1, 2, \ldots, N\}$. Define

$$\mathfrak{S}_N = \sum_{j=1}^{N} \lambda_j C_{\varphi_j}$$

as the linear combination of the operators C_{φ_j} on $\mathcal{A}_a$, where $\lambda_1, \ldots, \lambda_N \in \mathbb{C}\backslash\{0\}$. Since each operator C_{φ_j} is bounded on $\mathcal{A}_a$, the operator $\mathfrak{S}_N$ is also bounded on $\mathcal{A}_a$.

Theorem 3.2 *Let $a \le 0$. Then the operator $\mathfrak{S}_N$ is $\mathcal{J}$-symmetric on $\mathcal{A}_a$ if and only if $\varphi_j(s) = s + c_1^{(j)}$ with $\operatorname{Re} c_1^{(j)} \ge 0$ for each $j \in \{1, 2, \ldots, N\}$.*

Proof Suppose that the operator $\mathfrak{S}_N$ is $\mathcal{J}$-symmetric on $\mathcal{A}_a$. From this and the fact $1 \in \mathcal{A}_a$, we have

$$(\overline{\lambda}_1 C_{\varphi_1}^* + \overline{\lambda}_2 C_{\varphi_2}^* + \cdots + \overline{\lambda}_N C_{\varphi_N}^*)\mathcal{J}1 = \mathcal{J}(\lambda_1 C_{\varphi_1} + \lambda_2 C_{\varphi_2} + \cdots + \lambda_N C_{\varphi_N})1. \tag{3.20}$$

Similarly, we divide into two cases to consider:

(i) There is at least a zero in $c_0^{(1)}, c_0^{(2)}, \ldots, c_0^{(N)}$;
(ii) $c_0^{(1)} \ge 1, c_0^{(2)} \ge 1, \ldots,$ and $c_0^{(N)} \ge 1$.

Case (i). Assume that there is at least a zero in $c_0^{(1)}, c_0^{(2)}, \ldots, c_0^{(N)}$. From the fact $\mathcal{J}1 = 1$ and (3.20), we have

$$\overline{\lambda}_1 C_{\varphi_1}^* 1 + \overline{\lambda}_2 C_{\varphi_2}^* 1 + \cdots + \overline{\lambda}_N C_{\varphi_N}^* 1 = \overline{\lambda}_1 + \overline{\lambda}_2 + \cdots + \overline{\lambda}_N. \tag{3.21}$$

From Lemma 3.1, we see that this case is obviously incorrect.

Case (ii). Assume that $c_0^{(1)} \ge 1, c_0^{(2)} \ge 1, \ldots,$ and $c_0^{(N)} \ge 1$. Since $\mathcal{J}\left(n^{-s}\right) = n^{-s}$ for all $n \ge 1$, from (3.20) and (3.4) we have

$$\left\langle \mathfrak{S}_N\left(m^{-s}\right), n^{-s}\right\rangle_{\mathcal{A}_a} = \left\langle \mathfrak{S}_N\left(n^{-s}\right), m^{-s}\right\rangle_{\mathcal{A}_a} \tag{3.22}$$

for each $m, n \ge 1$. Then, there is a symmetric matrix of the operator $\mathfrak{S}_N$ with respect to the base $\{n^{-s}\}_{n=1}^{\infty}$. On the other hand, since $\mathfrak{S}_N 1 = \lambda_1 + \lambda_2 + \ldots + \lambda_N$, the matrix of the operator $\mathfrak{S}_N$ in this base is

$$\begin{pmatrix} \lambda_1 + \lambda_2 + \ldots + \lambda_N & 0 & 0 & \cdots \\ 0 & a_{2,2} & a_{2,3} & \cdots \\ 0 & a_{3,2} & a_{3,3} & \cdots \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}. \tag{3.23}$$

But, we also have

$$\begin{aligned} \mathfrak{S}_N n^{-s} &= \lambda_1 n^{-\varphi_1(s)} + \lambda_2 n^{-\varphi_2(s)} + \cdots + \lambda_N n^{-\varphi_N(s)} \\ &= \lambda_1 n^{-c_0^{(1)} s - c_1^{(1)}} n^{-\sum_{k=2}^{\infty} c_k^{(1)} k^{-s}} + \lambda_2 n^{-c_0^{(2)} s - c_1^{(2)}} n^{-\sum_{k=2}^{\infty} c_k^{(2)} k^{-s}} \\ &\quad + \cdots + \lambda_N n^{-c_0^{(N)} s - c_1^{(N)}} n^{-\sum_{k=2}^{\infty} c_k^{(N)} k^{-s}} \\ &= \lambda_1 n^{-c_0^{(1)} s - c_1^{(1)}} \prod_{k=2}^{\infty} \left(1 + \sum_{j=1}^{+\infty} \frac{\left(-c_k^{(1)} \log n\right)^j}{j!} k^{-js} \right) \end{aligned}$$

$$+\lambda_2 n^{-c_0^{(2)}s-c_1^{(2)}}\prod_{k=2}^{\infty}\left(1+\sum_{j=1}^{+\infty}\frac{\left(-c_k^{(2)}\log n\right)^j}{j!}k^{-js}\right)$$

$$+\cdots+\lambda_N n^{-c_0^{(N)}s-c_1^{(N)}}\prod_{k=2}^{\infty}\left(1+\sum_{j=1}^{+\infty}\frac{\left(-c_k^{(N)}\log n\right)^j}{j!}k^{-js}\right). \tag{3.24}$$

By expanding the brackets in (3.24), we obtain $a_{m,n}=0$ for all $n>m\geq 2$. Since the matrix (3.23) is symmetric by the previous argument, we obtain $a_{m,n}=0$ for all $2\leq n<m$. This implies that the matrix (3.23) is a diagonal matrix

$$\begin{pmatrix}\lambda_1+\lambda_2+\ldots+\lambda_N & 0 & 0 & \cdots\\ 0 & a_{2,2} & 0 & \cdots\\ 0 & 0 & a_{3,3} & \cdots\\ \vdots & \vdots & \ddots & \vdots\end{pmatrix}. \tag{3.25}$$

Then, we have $c_k^{(1)}=c_k^{(2)}=\ldots=c_k^{(N)}=0$ for all $k\geq 2$. By this and (3.24), we have

$$\mathfrak{S}_N n^{-s}=\lambda_1 n^{-c_0^{(1)}s-c_1^{(1)}}+\lambda_2 n^{-c_0^{(2)}s-c_1^{(2)}}+\ldots+\lambda_N n^{-c_0^{(N)}s-c_1^{(N)}} \tag{3.26}$$

for all $n\geq 2$. Similar to the proof of Theorem 3.1, here we also will divide into the following subcases:

Subcase (1). Assume that $c_0^{(1)}=c_0^{(2)}=\cdots=c_0^{(N)}=1$. Then from (3.26) we have

$$\mathfrak{S}_N n^{-s}=(\lambda_1 n^{-c_1^{(1)}}+\lambda_2 n^{-c_1^{(2)}}+\cdots+\lambda_N n^{c_1^{(N)}})n^{-s} \tag{3.27}$$

for all $n\geq 2$. From (3.25) and (3.27), it follows that $a_{n,n}=\lambda_1 n^{-c_1^{(1)}}+\lambda_2 n^{-c_1^{(2)}}+\cdots+\lambda_N n^{c_1^{(N)}}$ for all $n\geq 2$. Thus, we obtain that $\mathfrak{S}_N$ is $\mathcal{J}$-symmetric such that $\varphi_1(s)=s+c_1^{(1)}, \varphi_2(s)=s+c_1^{(2)},\ldots,\varphi_N(s)=s+c_1^{(N)}$. Moreover, from Theorem A, we have $\operatorname{Re} c_1^{(1)}\geq 0, \operatorname{Re} c_1^{(2)}\geq 0,\ldots,\operatorname{Re} c_1^{(N)}\geq 0$.

Subcase (2). Assume that there is only one $c_0^{(j)}>1$ and $c_0^{(i)}=1$, where $j\in\{1,2,\ldots,N\}$ and $i\in\{1,2,\ldots,N\}\setminus\{j\}$. From (3.26), we have

$$\mathfrak{S}_N n^{-s}=\lambda_1 n^{-s-c_1^{(1)}}+\cdots+\lambda_j n^{-c_0^{(j)}s-c_1^{(j)}}+\cdots+\lambda_N n^{-s-c_1^{(N)}} \tag{3.28}$$

for all $n\geq 2$. Since $\lambda_1,\lambda_2,\ldots,\lambda_N\in\mathbb{C}\backslash\{0\}$, and $\lambda_j n^{-c_1^{(j)}}\neq 0$ for all $n\geq 2$ and $j\in\{1,2,\ldots,N\}$, then we have that $\mathfrak{S}_N n^{-s}\neq a_{n,n}n^{-s}$ for all $n\geq 2$. Clearly, this case does not happen.

Subcase (3). Assume that $c_0^{(j)} > 1$ and $j \in \Lambda$, where Λ contains at least two or more elements of $\{1, 2, \ldots, N\}$. Then from (3.26) we have

$$\mathfrak{S}_N n^{-s} = \sum_{j\in\Lambda} \lambda_j n^{-c_0^{(j)}s-c_1^{(j)}} + \sum_{i\in\{1,2,\ldots,N\}\setminus\Lambda} \lambda_i n^{-s-c_1^{(i)}} \tag{3.29}$$

for all $n \geq 2$. Moreover, we know that

$$n^{-c_0^{(j)}s} \neq n^{-s}$$

for every $j \in \Lambda$. We see that if $\mathfrak{S}_N n^{-s} = a_{n,n} n^{-s}$ for all $n \geq 2$, then $a_{n,n} = 0$. But, because $\lambda_1, \lambda_2, \ldots, \lambda_N \in \mathbb{C}\setminus\{0\}$ and $\lambda_j n^{-c_1^{(j)}} \neq 0$ for all $n \geq 2$ and $j \in \{1, 2, \ldots, N\}$, from (3.29) we have that $c_0^{(i)} = c_0^{(j)} > 1$ for each $i, j \in \Lambda$ and

$$\sum_{j\in\Lambda} \lambda_j n^{-c_1^{(j)}} = 0$$

for all $n \geq 2$. This shows that

$$\sum_{j\in\Lambda} \lambda_j = 0$$

and $c_1^{(i)} = c_1^{(j)}$ for each $i, j \in \Lambda$. From this, it follows that $\varphi_i = \varphi_j$ for each $i, j \in \{1, 2, \ldots, N\}$, which is a contradiction.

Combining the above discussions, we prove that if the operator $\mathfrak{S}_N$ is $\mathcal{J}$-symmetric on $\mathcal{A}_a$, then $\varphi_j(s) = s + c_1^{(j)}$ with $\operatorname{Re} c_1^{(j)} \geq 0$ for each $j \in \{1, 2, \ldots, N\}$.

Conversely, assume that $\varphi_j(s) = s + c_1^{(j)}$ with $\operatorname{Re} c_1^{(j)} \geq 0$, where $j \in \{1, 2, \ldots, N\}$. By Lemma 3.2, each C_{φ_j}, $j \in \{1, 2, \ldots, N\}$ is $\mathcal{J}$-symmetric on $\mathcal{A}_a$. This shows that the operator $\mathfrak{S}_N$ is $\mathcal{J}$-symmetric on $\mathcal{A}_a$, and we complete the proof. □

By Lemma 3.2 and Theorem 3.2, the following result is true.

Corollary 3.2 *Let $a \leq 0$ and φ_j be a non-constant c_0-symbol. Then the following statements are equivalent:*

(a) $\mathfrak{S}_N$ is $\mathcal{J}$-symmetric on $\mathcal{A}_a$.
(b) C_{φ_j} is complex symmetric on $\mathcal{A}_a$ for all $j \in \{1, 2, \ldots, N\}$.
(c) C_{φ_j} is $\mathcal{J}$-symmetric on $\mathcal{A}_a$ for all $j \in \{1, 2, \ldots, N\}$.
(d) $\varphi_j(s) = s + c_1^{(j)}$ with $\operatorname{Re} c_1^{(j)} \geq 0$, for all $j \in \{1, 2, \ldots, N\}$.

Finally, we present some applications of $\mathcal{J}$-symmetric linear combinations of composition operators. Recall that each complex symmetric composition operator is normal on $\mathcal{A}_a$ (see Theorem B in [32]) and this result is also applicable to $\mathcal{A}_a$.

Lemma 3.3 *Let $a \leq 0$ and φ be a non-constant c_0-symbol. Then the operator C_φ is complex symmetric on $\mathcal{A}_a$ if and only if C_φ is normal on $\mathcal{A}_a$.*

Theorem 3.3 *Let $a \leq 0$ and φ_j be a non-constant c_0-symbol. Then the operator $\mathfrak{S}_N$ is complex symmetric on $\mathcal{A}_a$ if and only if each C_{φ_j} is normal on $\mathcal{A}_a$ for each $j \in \{1, 2, \dots, N\}$. Furthermore, $\mathfrak{S}_N$ is also normal on $\mathcal{A}_a$.*

Proof The proof is easily obtained from Corollary 3.2 and Lemma 3.3, so we omit it. □

Next, we characterize the self-adjoint composition operators.

Lemma 3.4 *Let $a \leq 0$. Then the operator C_φ is self-adjoint on $\mathcal{A}_a$ if and only if $\varphi(s) = s + c$, where c is a nonnegative real number.*

Proof Assume that the operator C_φ is self-adjoint on $\mathcal{A}_a$. Since C_φ is self-adjoint on $\mathcal{A}_a$, it is normal. From Lemma 3.2, we have that $\varphi(s) = s + c_1$, where $\Re c_1 \geq 0$. Then, on $\mathcal{A}_a$ it follows that $C_\varphi^* = C_{\widetilde{\varphi}}$, where $\widetilde{\varphi}(s) = s + \bar{c}_1$ and $\bar{c}_1$ is the complex conjugation of c_1. Since $n^{-s} \in \mathcal{A}_a$ for each $n \in \mathbb{N}$, we have

$$C_\varphi n^{-s} = C_\varphi^* n^{-s} = C_{\widetilde{\varphi}} n^{-s},$$

which implies that $n^{-c_1} = n^{-\bar{c}_1}$. From this and Lemma 4.2 in [11], we obtain $c_1 = \bar{c}_1$, that is, c_1 is a nonnegative real number. The converse is trivial. The proof is complete. □

Let $\varphi_1 \neq \varphi_2$. The next result is about the difference $\mathfrak{D}_2 = C_{\varphi_1} - C_{\varphi_2}$ on $\mathcal{A}_a$.

Theorem 3.4 *Let $a \leq 0$. Then $\mathfrak{D}_2$ is self-adjoint on $\mathcal{A}_a$ if and only if both C_{φ_1} and C_{φ_2} are self-adjoint on $\mathcal{A}_a$.*

Proof Assume that $\mathfrak{D}_2$ is self-adjoint on $\mathcal{A}_a$. Obviously, $\mathfrak{D}_2$ is normal on $\mathcal{A}_a$. By Theorem 3.3, $\varphi_1(s) = s + c_1^{(1)}$ and $\varphi_2(s) = s + c_1^{(2)}$. From the proof of Lemma 3.4, we have

$$(C_{\varphi_1} - C_{\varphi_2})n^{-s} = (C_{\widetilde{\varphi}_1} - C_{\widetilde{\varphi}_2})n^{-s},$$

where $\widetilde{\varphi}_1(s) = s + \bar{c}_1^{(1)}$ and $\widetilde{\varphi}_2(s) = s + \bar{c}_1^{(2)}$. That is,

$$n^{-c_1^{(1)}} - n^{-c_1^{(2)}} = n^{-\bar{c}_1^{(1)}} - n^{-\bar{c}_1^{(2)}}. \tag{3.30}$$

Let $c_1^{(1)} = \sigma_1 + it_1$ and $c_1^{(2)} = \sigma_2 + it_2$. Replacing them by the expression of (3.30), we obtain

$$n^{-\sigma_1 - it_1} - n^{-\sigma_2 - it_2} = n^{-\sigma_1 + it_1} - n^{-\sigma_2 + it_2}. \tag{3.31}$$

From (3.31), we obtain

$$n^{-\sigma_2}(n^{it_2} - n^{-it_2}) = n^{-\sigma_1}(n^{it_1} - n^{-it_1}). \tag{3.32}$$

By taking the natural logarithm in (3.32), we have

$$e^{-\sigma_2 \log n}(e^{it_2 \log n} - e^{-it_2 \log n}) = e^{-\sigma_1 \log n}(e^{it_1 \log n} - e^{-it_1 \log n}). \tag{3.33}$$

From (3.33), we have

$$2e^{-\sigma_2 \log n} \sin(t_2 \log n)i = 2e^{-\sigma_1 \log n} \sin(t_1 \log n)i. \tag{3.34}$$

Assume that $\sigma_1 \neq \sigma_2$. Then, from (3.34) we have

$$t_1 \log n = k_1\pi, \quad t_2 \log n = k_2\pi, \quad k_1, k_2 \in \mathbb{Z}. \tag{3.35}$$

If t_1 or t_2 does not equal zero in (3.35), then $\log 3/\log 2 \in \mathbb{Q}$. However, this is not true. So, both t_1 and t_2 must be equal to zero. As a result of this, from Lemma 3.4 it follows that both C_{φ_1} and C_{φ_2} are self-adjoint on $\mathcal{A}_a$. The reverse is obvious. The proof is finished. □

Remark 3.2 If we define the operator $\mathcal{S}_2 = aC_{\varphi_1} + bC_{\varphi_2}$, then the operator $\mathfrak{D}_2$ is obtained by choosing $a = 1$ and $b = -1$. At this moment, a and b are two special real numbers. Similarly, we can prove that if a and b are general real numbers, then the operator $\mathcal{S}_2$ is self-adjoint on $\mathcal{A}_a$ if and only if both C_{φ_1} and C_{φ_2} are self-adjoint on $\mathcal{A}_a$.

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Declarations

Conflict of interest The authors declare no competing interests.

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