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Stepanov and Weyl Classes of *c*-Almost Periodic Type Functions

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Abstract

As an extension of some classes of generalized almost periodic functions, in this paper we develop the notion of c-almost periodicity in the sense of Stepanov and Weyl approaches. In fact, we extend some basic results of this theory which were already demonstrated for the standard cases. In particular, we prove that every c-almost periodic function in the sense of Stepanov approach (in the sense of equi-Weyl or Weyl approaches, respectively) is also c^m-almost periodic in the sense of Stepanov approach (in the sense of equi-Weyl or Weyl approaches, respectively) for each non-zero integer number m. This study is performed for both representative cases of functions defined on the real axis and with values in a Banach space and the complex functions defined on vertical strips in the complex plane.

Keywords Almost periodic functions · c-almost periodic functions · Stepanov almost periodicity · Weyl almost periodicity · equi-Weyl almost periodicity · Almost anti-periodicity · Functions of a complex variable · Banach spaces

Mathematics Subject Classification 42A75 · 43A60 · 30Axx · 30D20

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1 Introduction

The theory of almost periodic functions, mainly created during the 1920s by the Danish mathematician H. Bohr (1887–1951), is a powerful tool to study a wide class of trigonometric series of the general type and even exponential series (in this context, we can cite among others the papers [4, 5, 7, 9, 24]). If $(X, \|\cdot\|)$ is an arbitrary Banach space and $f : \mathbb{R} \to X$ is a function of an unrestricted real variable x, the notion of almost periodicity in the sense of Bohr involves the fact that $f(x)$ must be continuous, and for every $\varepsilon > 0$ there corresponds a number $l = l(\varepsilon) > 0$ such that any interval of length l contains at least a number τ satisfying $\|f(x+\tau) - f(x)\| \leq \varepsilon$ for all $x \in \mathbb{R}$. We will denote as $AP(\mathbb{R}, X)$ the space of almost periodic functions in the sense of this definition (Bohr's condition). Shortly after its development, this theory acquired numerous applications to various areas of mathematics, from harmonic analysis to differential equations.

In the course of time, outstanding mathematicians were developing several variants and extensions of Bohr's concept (see for example [2–5, 8, 10, 15, 16, 22, 23]). In particular, the first generalizations of the notion of almost periodicity in Bohr's sense were given by W. Stepanov (1889–1950) [25], who succeeded in removing the continuity restrictions and characterize this new class in terms of mean values over integrals of fixed length. In this sense, given $1 \leq p < \infty$, it is not difficult to prove that

$$\|f\|_{S^p} := \sup_{x \in \mathbb{R}} \left(\int_x^{x+1} \|f(t)\|^p \, dt \right)^{1/p}$$

defines a norm on the space of locally p-integrable maps from $\mathbb{R}$ into X. This norm leads to the spaces $S^p(\mathbb{R}, X)$, $1 \leq p < \infty$, containing the primary space $AP(\mathbb{R}, X)$, which can also be characterized through a Bohr-type definition in the sense that a locally integrable map $f : \mathbb{R} \to X$ is in $S^p(\mathbb{R}, X)$ if and only if for every $\varepsilon > 0$ there corresponds a relatively dense set of real numbers $\{\tau\}$ satisfying $\|f(t+\tau) - f(t)\|_{S^p} \leq \varepsilon$ (see [4, pp. 79,88]).

A generalization of these functions was given by H. Weyl (1885–1955) through the spaces which we will denote as $e\text{-}W^p(\mathbb{R}, X) \supset S^p(\mathbb{R}, X)$, $1 \leq p < \infty$. Specifically, the functions in $e\text{-}W^p(\mathbb{R}, X)$ are obtained through locally integrable maps f from $\mathbb{R}$ into X such that for every $\varepsilon > 0$ there corresponds a relatively dense set $\{\tau\}$ of real numbers and a number $L_0 > 0$ satisfying $\|f(t+\tau) - f(t)\|_{S^p_{L_0}} \leq \varepsilon \ \forall \tau \in \{\tau\}$, where

$$\|f\|_{S^p_{L_0}} := \sup_{x \in \mathbb{R}} \left(\frac{1}{L_0} \int_x^{x+L_0} \|f(t)\|^p \, dt \right)^{1/p}.$$

(See for example [4, pp. 82,88] or [2, Definition 4.1 and p. 140] where the functions in this space are called equi-almost periodic in the sense of Weyl).

On the other hand, given $c \in \mathbb{C} \setminus \{0\}$, it is said that a continuous function $f : \mathbb{R} \to X$ is c-periodic if there exists $w > 0$ such that $f(x+w) = cf(x)$ for all $x \in \mathbb{R}$. This concept, which was proposed in [1], extends the more known notions of periodicity (with $c = 1$), anti-periodicity (with $c = -1$) and Bloch periodicity (with c depending on w in the form $c = e^{ikw}$, $k \in \mathbb{R}$), and it has practical relevance for engineering science (especially condensed matter physics).

In connection with the notions of almost periodicity and c-periodicty, Khalladi et al. [12] have recently considered the following notion, which is called c-almost periodicity: a continuous function $f : \mathbb{R} \to X$ is said to be c-almost periodic if for every $\varepsilon > 0$ there corresponds a number $l = l(\varepsilon) > 0$ such that every open interval of length l contains at least a number τ satisfying $\|f(x+\tau) - cf(x)\| \leq \varepsilon$ for all x. We will denote as $AP_c(\mathbb{R}, X)$ the space of c-almost periodic functions in the sense of this definition. Note that the case $c = 1$ leads to the space $AP(\mathbb{R}, X)$. Also, the case $c = -1$ leads to the space of almost anti-periodic functions. Note also that a c-periodic function is not necessarily c-almost periodic ($f(x) = 2^{-x}$ is an example of a function which is $\frac{1}{2}$-periodic but not $\frac{1}{2}$-almost periodic). See also [11, 13, 17, 18] for more information on this type of spaces of functions defined on the real line.

Furthermore, the concept of c-almost periodicity (and hence almost periodicity) can also be extended to the important case of complex functions defined on arbitrary vertical strips of the complex plane (see [20, Definition 1]). Let $f : U \to \mathbb{C}$ be a continuous function in a strip of the form $U = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$, with $-\infty \leq \alpha < \beta \leq \infty$. Then f is called c-almost periodic in U if, for every $\varepsilon > 0$ and reduced strip $U_1 = \{z \in \mathbb{C} : \alpha_1 \leq \operatorname{Re} z \leq \beta_1\} \subset U$ (with $\alpha < \alpha_1 < \beta_1 < \beta$), there corresponds a number $l = l(\varepsilon) > 0$ such that any open interval of length l contains at least a point τ satisfying $|f(z + i\tau) - cf(z)| \leq \varepsilon$ for all $z \in U_1$. We will denote as $AP_c(U, \mathbb{C})$ the space of c-almost periodic functions in the sense of this definition. In particular, the case $c = 1$ corresponds with the set $AP(U, \mathbb{C})$ of almost periodic functions defined on U, which was theorized in [6] and has been widely studied in the literature as an extension of the real case (see for example Chapter 3 of the books [4, 9] and the references [7, 21]).

In this context, Sects. 2 and 4 of this paper are focused on the notions of c-almost periodicity in the sense of Stepanov and Weyl approaches, which provide the sets of functions $S_c^p(\mathbb{R}, X)$, e-$W_c^p(\mathbb{R}, X)$, $W_c^p(\mathbb{R}, X)$, $S_c^p(U, \mathbb{C})$, e-$W_c^p(U, \mathbb{C})$ and $W_c^p(U, \mathbb{C})$, and the corresponding concepts of boundedness, uniform continuity and uniform convergence which will be used later. The main definitions for the real case are based on [19, Section 2.9] and [14]. In comparison with the primary work previously made for the spaces $AP_c(\mathbb{R}, X)$ and $AP_c(U, \mathbb{C})$, Sects. 3 and 5 develop the main properties of the sets of S_c^p, e-W_c^p and W_c^p-almost periodic functions. In particular, Propositions 6 and 15 prove that every c-almost periodic function in the sense of Stepanov approach (in the sense of equi-Weyl or Weyl approaches, resp.) is also c^m-almost periodic in the sense of Stepanov approach (in the sense of equi-Weyl or Weyl approaches, resp.) for each $m \in \mathbb{Z} \setminus \{0\}$. Furthermore, we show some conditions under which the sets of c-almost periodic functions in the sense of Stepanov, equi-Weyl or Weyl approaches are included in the respective spaces of almost periodic functions in the sense of Stepanov, equi-Weyl or Weyl approaches (see Propositions 8, 9, 17 and 18).

This study is performed for both representative cases of functions of the type $f : \mathbb{R} \to X$ and $f : U \to \mathbb{C}$. Although the demonstrations of the properties for the case of the complex functions defined on vertical strips are similar to the case of the functions defined on the real line and with values in a Banach space, we include them for the sake of completeness.

2 Main Definitions for the Case of Functions from the Real Line to a Banach Space

We will devote this section to introduce the spaces of c-almost periodic functions in the sense of Stepanov and Weyl approaches for the case of mappings defined on $\mathbb{R}$ with values in a generic Banach space X, whose norm is indicated by $\|\cdot\|$. These sets, which were introduced by Khalladi et al. (see [14]), are natural generalizations of the space of c-almost periodic functions $AP_c(\mathbb{R}, X)$ which was described in the introduction.

Given $1 \leq p < \infty$, we will also denote as $L^p_{\text{loc}}(\mathbb{R}, X)$ the set of locally p-integrable functions, i.e. Lebesgue measurable functions $f : \mathbb{R} \to X$ such that $\int_K \|f(x)\|^p \, dx < \infty$ for all compact subsets K of $\mathbb{R}$. If $f \in L^p_{\text{loc}}(\mathbb{R}, X)$, it is clear that the functions $(cf)(x) := cf(x)$ (with $c \in \mathbb{C}$), $x \in \mathbb{R}$, and $f^\alpha(x) := f(x+\alpha)$ (with $\alpha \in \mathbb{R}$), $x \in \mathbb{R}$, are also in $L^p_{\text{loc}}(\mathbb{R}, X)$. Furthermore, by Minkowski inequality, the sum of functions in $L^p_{\text{loc}}(\mathbb{R}, X)$ is also in $L^p_{\text{loc}}(\mathbb{R}, X)$. We will also use the symbols $\overline{\lim}$ and $\underline{\lim}$ to denote the upper limit and the lower limit, respectively.

Definition 1 (*c-almost periodicity in the sense of Stepanov and Weyl approaches*) Let $1 \leq p < \infty$, $c \in \mathbb{C} \setminus \{0\}$ and $f \in L^p_{\text{loc}}(\mathbb{R}, X)$.

(a) We will say that f is Stepanov-(p, c)-almost periodic (we will also say that it is a S^p_c-almost periodic function) if for every $\varepsilon > 0$ there corresponds a relatively dense set $\{\tau\}$ of real numbers (i.e. there exists $l > 0$ such that any interval of length l contains at least a point τ) whose elements satisfy

$$\sup_{x \in \mathbb{R}} \left(\int_x^{x+1} \|f(t+\tau) - cf(t)\|^p \, dt \right)^{\frac{1}{p}} \leq \varepsilon.$$

Fixed p, c and ε, the elements of the set $\{\tau\}$ satisfying the above condition are called S^p_c-translation numbers belonging to ε (or simply (ε, c)-Stepanov translation numbers of $f(x)$). We will denote as $S^p_c(\mathbb{R}, X)$ the set of c-almost periodic functions in the sense of Stepanov approach.

(b) We will say that f is equi-Weyl-(p, c)-almost periodic (we will also say that it is an e-W^p_c-almost periodic function) if for every $\varepsilon > 0$ we can find a real number $L_0 = L_0(\varepsilon)$ and a relatively dense set $\{\tau\}$ of real numbers (i.e. there exists $l > 0$ such that any interval of length l contains at least a point τ) satisfying

$$\sup_{x \in \mathbb{R}} \left(L_0^{-1} \int_x^{x+L_0} \|f(t+\tau) - cf(t)\|^p \, dt \right)^{\frac{1}{p}} \leq \varepsilon.$$

Fixed p, c and ε, the elements of the set $\{\tau\}$ satisfying the above condition are called e-W^p_c-translation numbers belonging to ε and associated with the value L_0

(or simply (ε, c)-equi-Weyl translation numbers of $f(x)$ associated with the value L_0). We will denote as $e\text{-}W_c^p(\mathbb{R}, X)$ the set of c-almost periodic functions in the sense of equi-Weyl.

(c) We will say that f is Weyl-(p, c)-almost periodic (we will also say that it is a W_c^p-almost periodic function) if for every $\varepsilon > 0$ we can find a relatively dense set $\{\tau\}$ of real numbers satisfying

$$\overline{\lim_{L\to\infty}} \sup_{x\in\mathbb{R}} \left(L^{-1} \int_x^{x+L} \|f(t+\tau) - cf(t)\|^p \, dt \right)^{\frac{1}{p}} \leq \varepsilon.$$

Fixed p, c and ε, the elements of the set $\{\tau\}$ satisfying the above condition are called W_c^p-translation numbers belonging to ε (or simply (ε, c)-Weyl translation numbers of $f(x)$). We will denote as $W_c^p(\mathbb{R}, X)$ the set of c-almost periodic functions in the sense of Weyl approach.

For the case $c = 1$, the set of S_1^p, $e\text{-}W_1^p$ and W_1^p-almost periodic functions (with $1 \leq p < \infty$) will be also denoted respectively as $S^p(\mathbb{R}, X)$, $e\text{-}W^p(\mathbb{R}, X)$ and $W^p(\mathbb{R}, X)$.

It is easy to see that the sets $S_c^p(\mathbb{R}, X)$, $e\text{-}W_c^p(\mathbb{R}, X)$ and $W_c^p(\mathbb{R}, X)$ (with $1 \leq p < \infty$) are generalizations of the class of c-almost periodic functions in the sense of Bohr (see also [2, Table 2] for the case $c = 1$).

Remark 1 (*Extensions of the primary notion of c-almost periodicity*) If $1 \leq p < \infty$ and $c \in \mathbb{C} \setminus \{0\}$, then

$$AP_c(\mathbb{R}, X) \subset S_c^p(\mathbb{R}, X) \subset e\text{-}W_c^p(\mathbb{R}, X) \subset W_c^p(\mathbb{R}, X).$$

Indeed, let $f \in AP_c(\mathbb{R}, X)$ and fix $\varepsilon > 0$. This means that there corresponds a number $l = l(\varepsilon) > 0$ such that any open interval of length l contains at least a real number τ satisfying

$$\|f(x+\tau) - cf(x)\| \leq \varepsilon \text{ for all } x \in \mathbb{R}.$$

Therefore, it is also accomplished that

$$\sup_{x\in\mathbb{R}} \left(\int_x^{x+1} \|f(t+\tau) - cf(t)\|^p \, dt \right)^{\frac{1}{p}} \leq \sup_{x\in\mathbb{R}} \left(\int_x^{x+1} \varepsilon^p \, dt \right)^{\frac{1}{p}} \leq \varepsilon,$$

which yields that f is in $S_c^p(\mathbb{R}, X)$. Furthermore, the inclusion $S_c^p(\mathbb{R}, X) \subset e\text{-}W_c^p(\mathbb{R}, X)$ is direct by taking $L_0 = 1$ in the definition of equi-Weyl-(p, c)-almost periodicity. Finally, it is clear that any equi-Weyl-(p, c)-almost periodic function is also Weyl-(p, c)-almost periodic.

Example 1 (*Stepanov-(p, c)-almost periodicity does not yield c-almost periodicity*) Fix $1 \leq p < \infty$ and $c \in \mathbb{C} \setminus \{0\}$. Consider the function

$$f(x) = \sin\left(\frac{1}{2 + \cos x + \cos(\sqrt{2}x)} \right), \ x \in \mathbb{R},$$

which is clearly in $L^p_{\mathrm{loc}}(\mathbb{R},\mathbb{R})$. It is straightforward that the function $\varphi(x) := 2 + \cos x + \cos(\sqrt{2}x)$ is greater than 0 for all $x \in \mathbb{R}$ (otherwise, $\varphi(x_0) = 0$ would yield $\cos x_0 = -1$ and $\cos(\sqrt{2}x_0) = -1$, which is impossible because the numbers 1 and $\sqrt{2}$ are incommensurable). In fact, by Kronecker's theorem, for all $\delta > 0$ there exists $t_0 \in \mathbb{R}$ satisfying the inequalities $|t_0 - \pi| < \delta$ and $|t_0\sqrt{2} - \pi| < \delta$ (mod. 2π). Therefore $\inf_{x\in\mathbb{R}} \varphi(x) = 0$ and the function $\frac{1}{\varphi(x)}$ is unbounded. Also by continuity (the range of φ is $(0, 4]$) and Kronecker's theorem, fixed $n \in \mathbb{N}$, there exists t_n such that $\frac{1}{\varphi(t_n)} = n\pi$ and t'_n such that $\frac{1}{\varphi(t'_n)} = \left(n + \frac{1}{2}\right)\pi$, and we can choose them so that $|t_n - t'_n| \to 0$ when n goes to ∞. Since $|f(t_n) - f(t'_n)| = 1$, we get that f is not uniformly continuous, which yields that f is not a c-almost periodic function. However, it is Stepanov-(p, c)-almost periodic for $c = 1$. Indeed, consider the function

$$f_n(x) := \sin\left(\frac{1}{2 + \max\{\cos x, -1 + \frac{1}{n}\} + \cos(\sqrt{2}x)}\right), \; x \in \mathbb{R}.$$

Since $\varphi_n(x) := 2 + \max\{\cos x, -1 + \frac{1}{n}\} + \cos(\sqrt{2}x)$, $x \in \mathbb{R}$, is bounded below by a positive constant, we have that f_n is almost periodic for each $n \in \mathbb{N}$ (see particularly [9, Theorem 1.7]). Hence f_n is also Stepanov-$(p, 1)$-almost periodic. Moreover, $\varphi_n(x) = \varphi(x)$ when $\max\{\cos x, -1 + \frac{1}{n}\} = \cos x$, which yields that

$$\int_0^{2\pi} |f_n(x+t) - f(x+t)|^p \, dt \le 2^p \mu(E_{n,x}),$$

where μ is the Lebesgue measure on $\mathbb{R}$ and $E_{n,x}$ is the set $\{\tau \in [x, x + 2\pi] : \max\{\cos\tau, -1 + \frac{1}{n}\} = -1 + \frac{1}{n}\}$. Thus $\mu(E_{n,x}) = \mu(E_{n,0}) = \mu([\pi - \delta_n, \pi + \delta_n])$ with $\cos\delta_n = 1 - \frac{1}{n}$ and $\delta_n \to 0$ when n goes to ∞. It follows that

$$\lim_{n\to\infty} \left(\int_x^{x+1} |f_n(t) - f(t)|^p\right) dt = 0$$

uniformly with respect to $x \in \mathbb{R}$. This means that for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that

$$\sup_{x\in\mathbb{R}} \left(\int_x^{x+1} |f_n(t) - f(t)|^p \, dt\right)^{\frac{1}{p}} \le \varepsilon \text{ for each } n \ge n_0.$$

Hence $\{f_n(x)\}_{n\ge1}$ is S^p-uniformly convergent to f. We deduce from Proposition 7, point v), that f is in $S^p_1(\mathbb{R},\mathbb{R})$.

Example 2 (*Equi-Weyl-(p, c)-almost periodicity does not yield Stepanov-(p, c)-almost periodicity*) Fix $1 \le p < \infty$ and $c \in \mathbb{C} \setminus \{0\}$. Consider the function

$$f(x) = \begin{cases} \frac{1}{c} & \text{if } x \in (0, 1) \\ 0 & \text{otherwise} \end{cases},$$

which is clearly in $L^p_{loc}(\mathbb{R}, \mathbb{R})$. Given $\varepsilon > 0$, suppose the existence of a relatively dense set $\{\tau\}$ of real numbers whose elements satisfy

$$\sup_{x\in\mathbb{R}} \left(\int_x^{x+1} |f(t+\tau) - cf(t)|^p \, dt \right)^{\frac{1}{p}} \leq \varepsilon. \tag{1}$$

However, by taking $0 < \varepsilon < 1$, $x_0 = 0$ and $\tau > 1$, we have

$$\int_{x_0}^{x_0+1} |f(t+\tau) - cf(t)|^p \, dt = \int_{x_0}^{x_0+1} \left| 0 - c \cdot \frac{1}{c} \right|^p dt = 1 > \varepsilon,$$

which means that (1) is not true and hence f is not Stepanov-(p, c)-almost periodic. Furthermore, it can be proved that f is equi-Weyl-(p, c)-almost periodic. Indeed, for every $\tau, x \in \mathbb{R}$ and $L_0 > 1$, we have

$$\begin{aligned} &\frac{1}{L_0} \int_x^{x+L_0} |f(t+\tau) - cf(t)|^p \, dt \\ &\quad \leq \frac{1}{L_0} \left(\int_x^{x+L_0} |f(t+\tau)|^p \, dt + |c| \int_x^{x+L_0} |f(t)|^p \, dt \right) \\ &\quad = \frac{1}{L_0} \left(\int_{x+\tau}^{x+\tau+L_0} |f(t)|^p \, dt + |c| \int_x^{x+L_0} |f(t)|^p \, dt \right) \\ &\quad \leq \frac{1}{L_0} \left(\frac{1}{|c|^p} + |c| \frac{1}{|c|^p} \right) = \frac{1}{L_0} \frac{1}{|c|^p} (1 + |c|). \end{aligned}$$

Then

$$\sup_{x\in\mathbb{R}} \left(L_0^{-1} \int_x^{x+L_0} |f(t+\tau) - cf(t)|^p \, dt \right)^{\frac{1}{p}} \leq \frac{1}{L_0^{\frac{1}{p}}} \frac{1}{|c|} (1 + |c|)^{\frac{1}{p}}.$$

Consequently, for every $\varepsilon > 0$ there exists $L_0 > 1$ such that

$$\sup_{x\in\mathbb{R}} \left(L_0^{-1} \int_x^{x+L_0} |f(t+\tau) - cf(t)|^p \, dt \right)^{\frac{1}{p}} \leq \varepsilon$$

for every $\tau \in \mathbb{R}$.

Example 3 (*Weyl-(p, c)-almost periodicity does not yield equi-Weyl-(p, c)-almost periodicity*) Fix $1 \leq p < \infty$ and $c \in \mathbb{C} \setminus \{0\}$. Consider the function

$$f(x) = \begin{cases} \frac{1}{c} & \text{if } x \leq 0 \\ 0 & \text{if } x > 0 \end{cases},$$

which is clearly in $L^p_{\mathrm{loc}}(\mathbb{R},\mathbb{R})$. Given $\varepsilon > 0$, suppose the existence of $L_0 = L_0(\varepsilon)$ and a relatively dense set $\{\tau\}$ of real numbers (which requires arbitrarily large values of τ's) satisfying

$$\sup_{x\in\mathbb{R}} \left(L_0^{-1} \int_x^{x+L_0} |f(t+\tau) - cf(t)|^p \, dt \right)^{\frac{1}{p}} \leq \varepsilon. \tag{2}$$

However, take a negative real number x_0 so that $x_0 + L_0 < 0$ and note that for large values of $\tau > 0$ it is accomplished that

$$\left(L_0^{-1} \int_{x_0}^{x_0+L_0} |f(t+\tau) - cf(t)|^p \, dt \right)^{\frac{1}{p}} = \left(L_0^{-1} \int_{x_0}^{x_0+L_0} \left| 0 - c\frac{1}{c} \right|^p \, dt \right)^{\frac{1}{p}} = 1.$$

Hence, if $0 < \varepsilon < 1$, it is clear that (2) is not true, which means that f is not equi-Weyl-(p, c)-almost periodic. Furthermore, it can be proved that f is Weyl-$(p, 1)$-almost periodic (see the reasoning for the Heaviside function in [19, Example 2.5.34]). Indeed, for every $\tau \in \mathbb{R}$, we get

$$\sup_{x\in\mathbb{R}} \left(\int_x^{x+L} |f(t+\tau) - f(t)|^p \, dt \right)^{\frac{1}{p}} = |\tau|^{\frac{1}{p}}$$

for every $L > |\tau|$ in virtue of the fact that t and $t+\tau$, $t \in \mathbb{R}$, can be of distinct sign in an interval of at most length $|\tau|$. Consequently, for every $\varepsilon > 0$ we can find a relatively dense set $\{\tau\}$ of real numbers satisfying

$$\overline{\lim_{L\to\infty}} \sup_{x\in\mathbb{R}} \left(L^{-1} \int_x^{x+L} |f(t+\tau) - f(t)|^p \, dt \right)^{\frac{1}{p}} = \overline{\lim_{L\to\infty}} \frac{1}{L^{1/p}} |\tau|^{\frac{1}{p}} \leq \varepsilon.$$

Remark 2 (*On the concept of c-almost periodicity in the sense of Stepanov approach*) Actually, given $1 \leq p < \infty$ and $c \in \mathbb{C} \setminus \{0\}$, the concept of c-almost periodicity in the sense of Stepanov approach could have defined in an analogous manner by including a positive constant L for which

$$\sup_{x\in\mathbb{R}} \left(\frac{1}{L} \int_x^{x+L} \|f(t+\tau) - cf(t)\|^p \, dt \right)^{\frac{1}{p}} \leq \varepsilon.$$

However, we write $L = 1$ in our Definition 1.a) because it is easy to prove that all the L-Stepanov norms are equivalent, i.e. for every $L_1, L_2 > 0$ there exist $k_1, k_2 > 0$ such that

$$k_1 \|f\|_{S^p_{L_1}} \leq \|f\|_{S^p_{L_2}} \leq k_2 \|f\|_{S^p_{L_1}},$$

where $\|f\|_{S^p_L} := \sup\limits_{x\in\mathbb{R}} \left(\frac{1}{L} \int_x^{x+L} \|f(t)\|^p \, dt \right)^{\frac{1}{p}}$ (see also [2, p. 132]).

Remark 3 (*On the notions of c-almost periodicity in the sense of Weyl approach*) We note that the difference between S_c^p and e-W_c^p-almost periodic functions is that in the latter case the value L_0 varies with ε.

Note also that Definition 1.b) is analogous to that of [4, p. 77] or [5, p. 226] for the case $c = 1$. Equivalently, we can state that $f \in e\text{-}W_c^p(\mathbb{R}, X)$ if for every $\varepsilon > 0$ we can find a real number $L_0 = L_0(\varepsilon)$ and a relatively dense set $\{\tau\}$ of real numbers satisfying

$$\sup_{x\in\mathbb{R}} \left(L^{-1} \int_x^{x+L} \|f(t+\tau) - cf(t)\|^p \, dt \right)^{\frac{1}{p}} \leq \varepsilon \ \forall L \geq L_0.$$

This is the version which is analogous to that of [2, Definition 4.1] for the case $c = 1$. The equivalence between these two definitions is justified by the fact that for every $L_0, L_1 > 0$ with $L_0 < L_1$ we have that

$$\|f\|_{S^p_{L_1}} \leq \left(1 + \frac{L_0}{L_1}\right)^{\frac{1}{p}} \|f\|_{S^p_{L_0}}. \tag{3}$$

Indeed, if $L_1 > L_0 > 0$ take $m \in \mathbb{N}$ such that $(m-1)L_0 < L_1 < mL_0$. Then

$$\begin{aligned}
\|f\|_{S^p_{L_1}} &= \sup_{x\in\mathbb{R}} \left(\frac{1}{L_1} \int_x^{x+L_1} \|f(t)\|^p \, dt \right)^{\frac{1}{p}} \\
&\leq \sup_{x\in\mathbb{R}} \left(\frac{mL_0}{L_1} \frac{1}{mL_0} \int_x^{x+mL_0} \|f(t)\|^p \, dt \right)^{\frac{1}{p}} \\
&= \left(\frac{mL_0}{L_1} \right)^{\frac{1}{p}} \|f\|_{S^p_{mL_0}} = \left(\frac{(m-1)L_0 + L_0}{L_1} \right)^{\frac{1}{p}} \|f\|_{S^p_{mL_0}} \\
&< \left(\frac{L_1 + L_0}{L_1} \right)^{\frac{1}{p}} \|f\|_{S^p_{mL_0}} \leq \left(1 + \frac{L_0}{L_1}\right)^{\frac{1}{p}} \|f\|_{S^p_{L_0}},
\end{aligned}$$

where the last inequality is given by the fact that

$$\begin{aligned}
&\frac{1}{mL_0} \int_x^{x+mL_0} \|f(t)\|^p \, dt \\
&= \frac{1}{mL_0} \left(\int_x^{x+L_0} \|f(t)\|^p \, dt + \int_{x+L_0}^{x+2L_0} \|f(t)\|^p \, dt + \ldots + \int_{x+(m-1)L_0}^{x+mL_0} \|f(t)\|^p \, dt \right) \\
&\leq \frac{m}{mL_0} \sup_{x\in\mathbb{R}} \left(\int_x^{x+L_0} \|f(t)\|^p \, dt \right) = \|f\|^p_{S^p_{L_0}},
\end{aligned}$$

which yields that $\|f\|^p_{S^p_{mL_0}} \leq \|f\|^p_{S^p_{L_0}}$.

Note that inequality (3) can also be used to prove the existence of the limit $\lim_{L\to\infty} \|f\|_{S^p_L}$ because, fixed an arbitrary L_0, it is deduced from there that

$\overline{\lim}_{L\to\infty}\|f\|_{S^p_L} \leq \|f\|_{S^p_{L_0}}$, which yields $\overline{\lim}_{L\to\infty}\|f\|_{S^p_L} \leq \underline{\lim}_{L_0\to\infty}\|f\|_{S^p_{L_0}} = \underline{\lim}_{L\to\infty}\|f\|_{S^p_L}$ and, therefore, the existence of the limit.
In this way, fixed $\varepsilon > 0$, if f satisfies our Definition 1.b), then there exists $L_0 = L_0(\frac{\varepsilon}{2})$ and a relatively dense set $\{\tau\}$ of real numbers such that $\|f^\tau - cf\|_{S^p_{L_0}} \leq \frac{\varepsilon}{2}$, where $f^\tau(x) := f(x+\tau)$ for all $x \in \mathbb{R}$. This yields, by virtue of (3), that $\|f^\tau - cf\|_{S^p_{L_1}} \leq \left(1 + \frac{L_0}{L_1}\right)\frac{\varepsilon}{2}$ for any $L_1 > L_0$. Therefore, if we take L_1 arbitrarily large ($L_1 \to \infty$) it is clear that $\|f^\tau - cf\|_{S^p_L} \leq \varepsilon$ for every $L \geq L_1'$ for a certain L_1' sufficiently large, which means that f satisfies this alternative definition. The converse is trivial.

With respect to the Stepanov, equi-Weyl or Weyl metrics (denoted as S^p, e-W^p and W^p, respectively), we next define the notions of boundedness, uniform continuity and uniform convergence which will be used in this paper. The interconnection among these metrics (for every notion) is given by the relations $S^p \Rightarrow e\text{-}W^p \Rightarrow W^p$.

Definition 2 (*S^p, e-W^p and W^p-boundedness*) Given $1 \leq p < \infty$, let $f \in L^p_{\mathrm{loc}}(\mathbb{R}, X)$.

(a) We will say that f is S^p-bounded (or Stepanov-p-bounded) if there exists $M > 0$ such that

$$\sup_{x\in\mathbb{R}}\left(\int_x^{x+1}\|f(t)\|^p\,dt\right)^{\frac{1}{p}} \leq M.$$

(b) We will say that f is e-W^p-bounded (or equi-Weyl-p-bounded) if there exist $L_0 > 0$ and $M > 0$ such that

$$\sup_{x\in\mathbb{R}}\left(L_0^{-1}\int_x^{x+L_0}\|f(t)\|^p\,dt\right)^{\frac{1}{p}} \leq M.$$

By Remark 3 (see (3)), it is equivalent to stating the existence of $L_0 > 0$ and $M > 0$ such that

$$\sup_{x\in\mathbb{R}}\left(L^{-1}\int_x^{x+L}\|f(t)\|^p\,dt\right)^{\frac{1}{p}} \leq M \;\forall L \geq L_0.$$

(c) We will say that f is W^p-bounded (or Weyl-p-bounded) if there exists $M > 0$ such that

$$\overline{\lim_{L\to\infty}}\sup_{x\in\mathbb{R}}\left(L^{-1}\int_x^{x+L}\|f(t)\|^p\,dt\right)^{\frac{1}{p}} \leq M.$$

Definition 3 (*S^p, e-W^p and W^p-uniform continuity*) Given $1 \leq p < \infty$, let $f \in L^p_{\mathrm{loc}}(\mathbb{R}, X)$.

(a) We will say that f is S^p-uniformly continuous if for every $\varepsilon > 0$ there is a positive number $\delta = \delta(\varepsilon)$ such that any $|h| < \delta$ satisfies

$$\sup_{x\in\mathbb{R}} \left(\int_x^{x+1} \|f(t+h) - f(t)\|^p \, dt \right)^{\frac{1}{p}} \le \varepsilon.$$

(b) We will say that f is e-W^p-uniformly continuous if for every $\varepsilon > 0$ there exist two numbers $L_0 = L_0(\varepsilon)$ and $\delta = \delta(\varepsilon)$ such that any $|h| < \delta$ satisfies

$$\sup_{x\in\mathbb{R}} \left(L_0^{-1} \int_x^{x+L_0} \|f(t+h) - f(t)\|^p \, dt \right)^{\frac{1}{p}} \le \varepsilon.$$

Again by Remark 3 (see (3)), it is equivalent to stating the existence of $L_0 = L_0(\varepsilon)$ and $\delta = \delta(\varepsilon)$ such that any $|h| < \delta$ satisfies

$$\sup_{x\in\mathbb{R}} \left(L^{-1} \int_x^{x+L} \|f(t+h) - f(t)\|^p \, dt \right)^{\frac{1}{p}} \le \varepsilon \; \forall L \ge L_0(\varepsilon).$$

(c) We will say that f is W^p-uniformly continuous if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that any $|h| < \delta$ satisfies

$$\overline{\lim_{L\to\infty}} \sup_{x\in\mathbb{R}} \left(L^{-1} \int_x^{x+L} \|f(t+h) - f(t)\|^p \, dt \right)^{\frac{1}{p}} \le \varepsilon.$$

Definition 4 (S^p, *e*-W^p *and* W^p*-uniform convergence*) Let $1 \le p < \infty$.

(a) We will say that a sequence $\{f_n\}_{n\ge 1}$ of S^p-bounded functions is S^p-uniformly convergent to a S^p-bounded function $f : \mathbb{R} \to X$ if for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that

$$\sup_{x\in\mathbb{R}} \left(\int_x^{x+1} \|f_n(t) - f(t)\|^p \, dt \right)^{\frac{1}{p}} \le \varepsilon \text{ for each } n \ge n_0.$$

(b) We will say that a sequence $\{f_n\}_{n\ge 1}$ of e-W^p-bounded functions is e-W^p-uniformly convergent to an e-W^p-bounded function $f : \mathbb{R} \to X$ if for every $\varepsilon > 0$ there exist $L_0 = L_0(\varepsilon)$ and $n_0 \in \mathbb{N}$ satisfying

$$\sup_{x\in\mathbb{R}} \left(L_0^{-1} \int_x^{x+L_0} \|f_n(t) - f(t)\|^p \, dt \right)^{\frac{1}{p}} \le \varepsilon \text{ for each } n \ge n_0.$$

Equivalently, by Remark 3, it is equivalent to stating the existence of $L_0 = L_0(\varepsilon)$ and $n_0 \in \mathbb{N}$ satisfying

$$\sup_{x\in\mathbb{R}} \left(L^{-1} \int_x^{x+L} \|f_n(t) - f(t)\|^p \, dt \right)^{\frac{1}{p}} \le \varepsilon \; \forall L \ge L_0(\varepsilon) \text{ and } n \ge n_0.$$

(c) We will say that a sequence $\{f_n\}_{n\geq 1}$ of W^p-bounded functions is W^p-uniformly convergent to a W^p-bounded function $f : \mathbb{R} \to X$ if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ satisfying

$$\overline{\lim_{L\to\infty}} \sup_{x\in\mathbb{R}} \left(L^{-1} \int_x^{x+L} \| f_n(t) - f(t)\|^p \, dt \right)^{\frac{1}{p}} \leq \varepsilon \text{ for each } n \geq n_0.$$

3 Main Properties of the Spaces $S_c^p(\mathbb{R}, X)$, e-$W_c^p(\mathbb{R}, X)$ and $W_c^p(\mathbb{R}, X)$

We next expose the most important properties of the classes of functions $f : \mathbb{R} \to X$ which are Stepanov-(p, c)-almost periodic, equi-Weyl-(p, c)-almost periodic or Weyl-(p, c)-almost periodic. The most of the following properties were already obtained for some particular cases (see [14, 20]).

It is worth already noting that the reasoning or the proofs which we will show for the case of the c-almost periodic functions in the sense of Stepanov approach are similar or analogous to that of equi-Weyl-(p, c)-almost periodicity.

Proposition 1 *(S^p and e-W^p-boundedness of the functions in $S_c^p(\mathbb{R}, X)$ and e-$W_c^p(\mathbb{R}, X)$) Let $c \in \mathbb{C}\setminus\{0\}$ and $1 \leq p < \infty$.*

(i) *If $f \in S_c^p(\mathbb{R}, X)$, then f is S^p-bounded.*
(ii) *If $f \in$ e-$W_c^p(\mathbb{R}, X)$, then f is e-W^p-bounded.*

Proof Let $1 \leq p < \infty$, $c \in \mathbb{C}\setminus\{0\}$ and $f \in L^p_{\text{loc}}(\mathbb{R}, X)$. Given $L > 0$ and $x_0 \in \mathbb{R}$, by the Minkowski inequality we have

$$\left(L^{-1} \int_{x_0}^{x_0+L} \|cf(x)\|^p \, dx \right)^{\frac{1}{p}} \leq \left(L^{-1} \int_{x_0}^{x_0+L} \|cf(x) - f(x+\tau)\|^p \, dx \right)^{\frac{1}{p}} + \left(L^{-1} \int_{x_0}^{x_0+L} \|f(x+\tau)\|^p dx \right)^{\frac{1}{p}}. \tag{4}$$

Now, take $\varepsilon = 1$. If $f \in S_c^p(\mathbb{R}, X)$ or $f \in e\text{-}W_c^p(\mathbb{R}, X)$, there exists $l > 0$ such that any interval of length l contains at least a number in the respective sets of the (ε, c)-Stepanov or (ε, c)-equi-Weyl translation numbers of $f(x)$. In particular, if x_0 is an arbitrary real number, there corresponds a value τ belonging to these respective sets such that $x_0 + \tau$ is in the interval $[0, l]$. Also, take $L_0 = L_0(1)$ the number corresponding to $\varepsilon = 1$ in the definition of equi-Weyl-(p, c)-almost periodicity. In this way, we deduce from (4) for the value $L = L_0$ (or $L = 1$ for the case of $f \in S_c^p(\mathbb{R}, X)$) that

$$\left(L_0^{-1} \int_{x_0}^{x_0+L_0} \|cf(x)\|^p \, dx \right)^{\frac{1}{p}} \leq 1 + \left(L_0^{-1} \int_{x_0+\tau}^{x_0+\tau+L_0} \|f(x)\|^p \, dx \right)^{\frac{1}{p}} \leq 1 + \left(L_0^{-1} \int_0^{l+L_0} \|f(x)\|^p \, dx \right)^{\frac{1}{p}},$$

which yields that

$$\sup_{x\in\mathbb{R}}\left(L_0^{-1}\int_x^{x+L_0}\|f(t)\|^p\,dt\right)^{\frac{1}{p}}\leq M:=\frac{1}{|c|}\left(1+\left(L_0^{-1}\int_0^{l+L_0}\|f(x)\|^p\,dx\right)^{\frac{1}{p}}\right).$$

This proves (i) and (ii). □

Proposition 2 *(S^p and e-W^p-uniform continuity of the functions in $S_c^p(\mathbb{R},X)$ and e-$W_c^p(\mathbb{R},X)$) Let $c\in\mathbb{C}\setminus\{0\}$ and $1\leq p<\infty$.*

(i) *If $f\in S_c^p(\mathbb{R},X)$, then f is S^p-uniformly continuous.*
(ii) *If $f\in$ e-$W_c^p(\mathbb{R},X)$, then f is e-W^p-uniformly continuous.*

Proof Fix $\varepsilon>0$. As the set of the $(\frac{|c|\varepsilon}{3},c)$-Stepanov or $(\frac{|c|\varepsilon}{3},c)$-equi-Weyl translation numbers of $f(x)$ is relatively dense, there exists $l>0$ such that every interval of length l contains at least one number of this set. In particular, fixed an arbitrary real number x_0 there corresponds such a translation number τ such that $x_0+\tau$ belongs to the interval $[0,l]$. Also, take L_0 the positive number corresponding to the case $\frac{|c|\varepsilon}{3}$ in the definition of equi-Weyl-(p,c)-almost periodicity (we can take $L_0=1$ for the case $S_c^p(\mathbb{R},X)$). Then for any $\delta>0$, by the Minkowski inequality, it is accomplished that

$$\begin{aligned}
&\left(L_0^{-1}\int_{x_0}^{x_0+L_0}\|cf(x+\delta)-cf(x)\|^p\,dx\right)^{\frac{1}{p}}\\
&\leq\left(L_0^{-1}\int_{x_0}^{x_0+L_0}\|cf(x+\delta)-f(x+\delta+\tau)\|^p\,dx\right)^{\frac{1}{p}}\\
&\quad+\left(L_0^{-1}\int_{x_0}^{x_0+L_0}\|f(x+\delta+\tau)-f(x+\tau)\|^p\,dx\right)^{\frac{1}{p}}\\
&\quad+\left(L_0^{-1}\int_{x_0}^{x_0+L_0}\|f(x+\tau)-cf(x)\|^p\,dx\right)^{\frac{1}{p}}\\
&\leq\frac{2}{3}|c|\,\varepsilon+\left(L_0^{-1}\int_{x_0+\tau}^{x_0+\tau+L_0}\|f(x+\delta)-f(x)\|^p\,dx\right)^{\frac{1}{p}}\\
&\leq\frac{2}{3}|c|\,\varepsilon+\left(L_0^{-1}\int_0^{L_0+l}\|f(x+\delta)-f(x)\|^p\,dx\right)^{\frac{1}{p}}.
\end{aligned}$$

Now, define the functions $f_n(x):=f(x+\frac{1}{n})\in L_{\mathrm{loc}}^p(\mathbb{R},X)$. It is clear that the sequence $\{f_n\}_{n\geq 1}$ converges pointwise to f and, by the dominated convergence theorem in L^p-spaces (or as a consequence of the Brezis-Lieb theorem), also converges to f in the sense of L^p (see also [4, p. 84] or [5, pp. 233–234]). Hence there exists $\delta_0>0$ such that for any $|\delta| < \delta_0$ it is accomplished that

$$\left(L_0^{-1}\int_0^{L_0+l}\|f(x+\delta)-f(x)\|^p\,dx\right)^{\frac{1}{p}} \le \frac{|c|\,\varepsilon}{3},$$

which yields that

$$\sup_{x\in\mathbb{R}}\left(L_0^{-1}\int_x^{x+L_0}\|f(t+\delta)-f(t)\|^p\,dt\right)^{\frac{1}{p}} \le \varepsilon,\ \forall\,|\delta|\le\delta_0.$$

This proves i) and ii). □

Remark 4 It is not true that any function f in $W_c^p(\mathbb{R}, X)$ is also W^p-bounded or W^p-uniformly continuous. A counterexample for the case $c = 1$ can be seen in [2, Example 4.28]. However, we next show an analogous result (based on [3, Lemma 5]) for the W^p-uniform continuity under the following hypothesis, with $f \in L^p_{\text{loc}}(\mathbb{R}, X)$ and $1 \le p < \infty$:

For every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left(\frac{1}{L}\int_0^L\|f(t+h)-f(t)\|^p\,dt\right)^{\frac{1}{p}} \le \varepsilon \text{ for all } |h| < \delta \tag{5}$$

uniformly with respect to $L \in (0, \infty)$.

Proposition 3 *(W^p-uniform continuity of the functions in $W_c^p(\mathbb{R}, X)$ under hypothesis (5)) Let $c \in \mathbb{C}\setminus\{0\}$ and $1 \le p < \infty$. If $f \in W_c^p(\mathbb{R}, X)$ and it satisfies (5), then f is W^p-uniformly continuous.*

Proof Fix $\varepsilon > 0$. As the set of the $(\frac{|c|\varepsilon}{3}, c)$-Weyl translation numbers of $f(x)$ is relatively dense, there exists $l > 0$ such that every interval of length l contains at least one number of this set. In particular, fixed an arbitrary real number x there corresponds such a translation number τ such that $x + \tau$ belongs to the interval $[0, l]$. Since $\|cf(x+\delta) - cf(x)\|$ is less than or equal to

$$\|cf(x+\delta)-f(x+\delta+\tau)\| + \|f(x+\delta+\tau)-f(x+\tau)\| + \|f(x+\tau)-cf(x)\|$$

and

$$\begin{aligned}\int_x^{x+L}\|f(t+\delta+\tau)-f(t+\tau)\|^p\,dt &= \int_{x+\tau}^{x+\tau+L}\|f(t+\delta)-f(t)\|^p\,dt\\ &\le \int_0^{L+l}\|f(t+\delta)-f(t)\|^p\,dt,\end{aligned}$$

for any $\delta > 0$, by the Minkowski inequality it is accomplished that

$$\overline{\lim}_{L\to\infty} \sup_{x\in\mathbb{R}} \left(L^{-1} \int_x^{x+L} \|cf(t+\delta) - cf(t)\|^p \, dt \right)^{\frac{1}{p}}$$
$$\leq \frac{2}{3} |c| \, \varepsilon + \overline{\lim}_{L\to\infty} \left(L^{-1} \int_0^{L+l} \|f(t+\delta) - f(t)\|^p \, dt \right)^{\frac{1}{p}}.$$

Now, by (5), there exists $\delta_0 > 0$ such that for any $|\delta| < \delta_0$ it is accomplished that

$$\left((L+l)^{-1} \int_0^{L+l} \|f(t+\delta) - f(t)\|^p \, dt \right)^{\frac{1}{p}} \leq \frac{|c| \, \varepsilon}{3},$$

uniformly with respect to $L > 0$, which yields that

$$\left(L^{-1} \int_0^{L+l} \|f(t+\delta) - f(t)\|^p \, dt \right)^{\frac{1}{p}} \leq \left(\frac{L+l}{L} \right)^{\frac{1}{p}} \frac{|c| \, \varepsilon}{3},$$

for any $|\delta| < \delta_0$ uniformly with respect to $L > 0$. Consequently, we get

$$\overline{\lim}_{L\to\infty} \sup_{x\in\mathbb{R}} \left(L^{-1} \int_x^{x+L} \|f(t+\delta) - f(t)\|^p \, dt \right)^{\frac{1}{p}} \leq \varepsilon, \ \forall |\delta| < \delta_0,$$

which means that f is W^p-uniformly continuous. □

Proposition 4 *($S^p_{|c|}$, e-$W^p_{|c|}$ and $W^p_{|c|}$-almost periodicity of the norm) Let $1 \leq p < \infty$ and $c \in \mathbb{C}\setminus \{0\}$. Given $f : \mathbb{R} \to X$, consider the function $\|f\| : \mathbb{R} \to [0, \infty)$ defined as $\|f\|(x) = \|f(x)\|$ for all $x \in \mathbb{R}$.*

(i) *If $f \in S^p_c(\mathbb{R}, X)$, then $\|f\| \in S^p_{|c|}(\mathbb{R}, [0, \infty))$.*
(ii) *If $f \in e\text{-}W^p_c(\mathbb{R}, X)$, then $\|f\| \in e\text{-}W^p_{|c|}(\mathbb{R}, [0, \infty))$.*
(iii) *If $f \in W^p_c(\mathbb{R}, X)$, then $\|f\| \in W^p_{|c|}(\mathbb{R}, [0, \infty))$.*

Proof Let $f \in e\text{-}W^p_c(\mathbb{R}, X)$ (or $f \in S^p_c(\mathbb{R}, X)$, respectively). For every $\varepsilon > 0$ we can find $l > 0$ and $L_0 = L_0(\varepsilon) > 0$ (take $L_0 = 1$ for the case of $f \in S^p_c(\mathbb{R}, X)$) such that any interval of length l contains at least a point τ satisfying

$$\sup_{x\in\mathbb{R}} \left(L_0^{-1} \int_x^{x+L_0} \|f(t+\tau) - cf(t)\|^p \, dt \right)^{\frac{1}{p}} \leq \varepsilon.$$

Now, if $c \in \mathbb{C} \setminus \{0\}$ and $x \in \mathbb{R}$, note that

$$|\|f(x+\tau)\| - |c|\|f(x)\|| = |\|f(x+\tau)\| - \|cf(x)\|| \leq \|f(x+\tau) - cf(x)\|,$$

which yields that

$$\sup_{x\in\mathbb{R}}\left(L_0^{-1}\int_x^{x+L_0}|\|f(t+\tau)\|-|c|\|f(t)\||^p\,dt\right)^{\frac{1}{p}}$$
$$\leq \sup_{x\in\mathbb{R}}\left(L_0^{-1}\int_x^{x+L_0}\|f(t+\tau)-cf(t)\|^p\,dt\right)^{\frac{1}{p}}\leq\varepsilon.$$

This proves (i) and (ii). Property (iii) can also be proved analogously by the above inequality. □

Proposition 5 *(S_c^p, e-W_c^p and W_c^p-almost periodicity of the function $\check{f}(x)=f(-x)$) Let $c\in\mathbb{C}\setminus\{0\}$ and $1\leq p<\infty$. Given $f:\mathbb{R}\to X$, consider the function $\check{f}:\mathbb{R}\to X$ defined as $\check{f}(x)=f(-x)$ for all $x\in\mathbb{R}$.*

(i) *$f\in S_c^p(\mathbb{R},X)$ if and only if $\check{f}\in S_{1/c}^p(\mathbb{R},X)$.*
(ii) *$f\in e$-$W_c^p(\mathbb{R},X)$ if and only if $\check{f}\in e$-$W_{1/c}^p(\mathbb{R},X)$.*
(iii) *$f\in W_c^p(\mathbb{R},X)$ if and only if $\check{f}\in W_{1/c}^p(\mathbb{R},X)$.*

Proof Given $1\leq p<\infty$, $f\in L_{\text{loc}}^p(\mathbb{R},X)$, $c\in\mathbb{C}\setminus\{0\}$, $L>0$ and $\tau\in\mathbb{R}$, we have that

$$\sup_{x\in\mathbb{R}}\left(L^{-1}\int_x^{x+L}\|\check{f}(t+\tau)-\frac{1}{c}\check{f}(t)\|^p\,dt\right)^{\frac{1}{p}}$$
$$=\sup_{x\in\mathbb{R}}\left(\frac{1}{|c|^pL}\int_x^{x+L}\|cf(-t-\tau)-f(-t)\|^p\,dt\right)^{\frac{1}{p}}$$
$$=\sup_{x\in\mathbb{R}}\left(\frac{1}{|c|^pL}\int_{-(x+\tau)-L}^{-(x+\tau)}\|f(t+\tau)-cf(t)\|^p\,dt\right)^{\frac{1}{p}}$$
$$=\frac{1}{|c|}\sup_{y\in\mathbb{R}}\left(\frac{1}{L}\int_y^{y+L}\|f(t+\tau)-cf(t)\|^p\,dt\right)^{\frac{1}{p}}.$$

This means that τ is a $(|c|\varepsilon,c)$-Stepanov, equi-Weyl or Weyl translation number of $f(x)$ if and only if τ is an $(\varepsilon,\frac{1}{c})$-Stepanov, equi-Weyl (associated with the same L) or Weyl translation number of $\check{f}(x)$, respectively. Hence the proposition holds. □

Lemma 1 (S_c^p, *e*-W_c^p and W_c^p-almost periodicity iff $S_{1/c}^p$, *e*-$W_{1/c}^p$ and $W_{1/c}^p$-almost periodicity) *Let $c\in\mathbb{C}\setminus\{0\}$ and $1\leq p<\infty$. Then $S_c^p(\mathbb{R},X)=S_{1/c}^p(\mathbb{R},X)$, e-$W_c^p(\mathbb{R},X)=e$-$W_{1/c}^p(\mathbb{R},X)$ and $W_c^p(\mathbb{R},X)=W_{1/c}^p(\mathbb{R},X)$.*

Proof Suppose that $f\in e$-$W_c^p(\mathbb{R},X)$ (or $f\in S_c^p(\mathbb{R},X)$, respectively). Given $\varepsilon>0$, take L_0 the positive number corresponding to the case $|c|\,\varepsilon$ in the definition of equi-Weyl-(p,c)-almost periodicity (we can take $L_0=1$ for the case $S_c^p(\mathbb{R},X)$). Note that every τ in the set of the $(|c|\,\varepsilon, c)$-equi-Weyl translation numbers of $f(x)$ associated with L_0 (or $(|c|\,\varepsilon, c)$-Stepanov translation numbers, with $L_0 = 1$) satisfies

$$\begin{aligned} &\sup_{x\in\mathbb{R}} \left(L_0^{-1} \int_x^{x+L_0} \| f(t-\tau) - \frac{1}{c} f(t)\|^p \, dt \right)^{\frac{1}{p}} \\ &= \sup_{x\in\mathbb{R}} \left(\frac{1}{|c|^p\, L_0} \int_x^{x+L_0} \| cf(t-\tau) - f(t)\|^p \, dt \right)^{\frac{1}{p}} \\ &= \sup_{x\in\mathbb{R}} \left(\frac{1}{|c|^p\, L_0} \int_{x-\tau}^{x-\tau+L_0} \| f(y+\tau) - cf(y)\|^p \, dy \right)^{\frac{1}{p}} \le \frac{|c|\,\varepsilon}{|c|} = \varepsilon. \end{aligned}$$

This shows that the set of the $(\varepsilon, 1/c)$-equi-Weyl translation numbers of f (associated with the same value L_0) is relatively dense (the same as the set of the $(\varepsilon, 1/c)$-Stepanov translation numbers), which means that $f \in e\text{-}W^p_{1/c}(\mathbb{R}, X)$ (or $f \in S^p_{1/c}(\mathbb{R}, X)$, respectively). The converse is analogous. The proof for the case $W^p_c(\mathbb{R}, X) = W^p_{1/c}(\mathbb{R}, X)$ is similar to the above reasoning. □

The following result generalizes that of [14, Proposition 2.3] which was stated for the case of (equi-)Weyl-almost periodicity and $m \in \mathbb{N}$.

Proposition 6 ($S^p_c, e\text{-}W^p_c, W^p_c$-almost periodicity yields $S^p_{c^m}, e\text{-}W^p_{c^m}, W^p_{c^m}$-almost periodicity) *Let $c \in \mathbb{C} \setminus \{0\}$, $1 \le p < \infty$ and $m \in \mathbb{Z} \setminus \{0\}$. Then $S^p_c(\mathbb{R}, X) \subset S^p_{c^m}(\mathbb{R}, X)$, $e\text{-}W^p_c(\mathbb{R}, X) \subset e\text{-}W^p_{c^m}(\mathbb{R}, X)$ and $W^p_c(\mathbb{R}, X) \subset W^p_{c^m}(\mathbb{R}, X)$.*

Proof Let $f \in e\text{-}W^p_c(\mathbb{R}, X)$ (or $f \in S^p_c(\mathbb{R}, X)$, respectively), and fix $m \in \mathbb{N}$. For every $\varepsilon > 0$ we can find $l > 0$ and $L_0 = L_0(\varepsilon) > 0$ (take $L_0 = 1$ for the case of $f \in S^p_c(\mathbb{R}, X)$) such that any interval of length l contains at least a point τ satisfying

$$\sup_{x\in\mathbb{R}} \left(L_0^{-1} \int_x^{x+L_0} \| f(t+\tau) - cf(t)\|^p \, dt \right)^{\frac{1}{p}} \le \frac{\varepsilon}{\left(\sum_{j=0}^{m-1} |c|^{jp} \right)^{1/p}}.$$

Also, fixed $m \in \mathbb{N}$, we have that

$$f(x+m\tau) - c^m f(x) = \sum_{j=0}^{m-1} c^j \left(f(x+(m-j)\tau) - cf(x+(m-j-1)\tau) \right).$$

Therefore, for any $x \in \mathbb{R}$ we have

$$\begin{aligned} &L_0^{-1} \int_x^{x+L_0} \| f(t+m\tau) - c^m f(t)\|^p \, dt \\ &\le \sum_{j=0}^{m-1} \frac{|c|^{jp}}{L_0} \int_x^{x+L_0} \| f(t+(m-j)\tau) - cf(t+(m-j-1)\tau)\|^p \, dt \end{aligned}$$

$$= \sum_{j=0}^{m-1} |c|^{jp} \frac{1}{L_0} \int_{x+(m-j-1)\tau}^{x+L_0+(m-j-1)\tau} \|f(t+\tau) - cf(t)\|^p \, dt$$
$$\le \sum_{j=0}^{m-1} |c|^{jp} \left(\frac{\varepsilon^p}{\sum_{j=0}^{m-1} |c|^{jp}} \right) = \varepsilon^p$$

and

$$\sup_{x \in \mathbb{R}} \left(L_0^{-1} \int_x^{x+L_0} \|f(t+m\tau) - c^m f(t)\|^p \, dt \right)^{1/p} \le \varepsilon.$$

This means that $m\tau$ is an (ε, c^m)-equi-Weyl translation number (associated with the value L_0) of $f(x)$ (or an (ε, c^m)-Stepanov translation number of $f(x)$), which means that $f \in e\text{-}W^p_{c^m}(\mathbb{R}, X)$ (or $f \in S^p_{c^m}(\mathbb{R}, X)$, respectively). Finally, by Lemma 1, f is also included in $e\text{-}W^p_{1/c^m}(\mathbb{R}, X)$ (or in $S^p_{1/c^m}(\mathbb{R}, X)$). The proof for the case $W^p_c(\mathbb{R}, X) \subset W^p_{c^m}(\mathbb{R}, X)$ is analogous. Thus the result holds. □

Proposition 7 (Some extra properties) *Let $c \in \mathbb{C} \setminus \{0\}$ and $1 \le p < \infty$. Suppose that $f \in S^p_c(\mathbb{R}, X)$ ($f \in e\text{-}W^p_c(\mathbb{R}, X)$ or $f \in W^p_c(\mathbb{R}, X)$, respectively). Then it is satisfied that:*

(i) *$\lambda f \in S^p_{c^m}(\mathbb{R}, X)$ ($\lambda f \in e\text{-}W^p_{c^m}(\mathbb{R}, X)$ or $\lambda f \in W^p_{c^m}(\mathbb{R}, X)$, respectively) for any $\lambda \in \mathbb{C}$ and for each $m \in \mathbb{Z} \setminus \{0\}$.*

(ii) *If $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R} \setminus \{0\}$, then the functions $f^\alpha(x) := f(x+\alpha)$ and $f_\beta(x) := f(\beta x)$, $x \in \mathbb{R}$, are in $S^p_{c^m}(\mathbb{R}, X)$ (in $e\text{-}W^p_{c^m}(\mathbb{R}, X)$ or $W^p_{c^m}(\mathbb{R}, X)$, respectively) for each $m \in \mathbb{Z} \setminus \{0\}$.*

(iii) *The condition of boundedness of $f(x)$ (i.e. the existence of $M > 0$ such that $\|f(x)\| \le M$, $\forall x \in \mathbb{R}$) implies that the function $f^2(x) := (f(x))^2$ for all $x \in \mathbb{R}$ is in $S^p_{c^{2k}}(\mathbb{R}, X)$ ($f^2 \in e\text{-}W^p_{c^{2k}}(\mathbb{R}, X)$ or $f^2 \in W^p_{c^{2k}}(\mathbb{R}, X)$, respectively) for each $k \in \mathbb{Z} \setminus \{0\}$.*

(iv) *If there exists $m_1 > 0$ such that $\|f(x)\| \ge m_1$ $\forall x \in \mathbb{R}$, then the function $\frac{1}{f}(x) := \frac{1}{f(x)}$, $x \in \mathbb{R}$, is in $S^p_{c^m}(\mathbb{R}, X)$ ($\frac{1}{f} \in e\text{-}W^p_{c^m}(\mathbb{R}, X)$ or $\frac{1}{f} \in W^p_{c^m}(\mathbb{R}, X)$, respectively) for each $m \in \mathbb{Z} \setminus \{0\}$.*

(v) *Let $\{f_n(x)\}_{n \ge 1}$ be a sequence of functions in $S^p_c(\mathbb{R}, X)$ (in $e\text{-}W^p_c(\mathbb{R}, X)$ or $W^p_c(\mathbb{R}, X)$, respectively). If $\{f_n(x)\}_{n \ge 1}$ is S^p-uniformly convergent ($e\text{-}W^p$-uniformly convergent or W^p-uniformly convergent, respectively) to a function $f : \mathbb{R} \to X$ (in $L^p_{\mathrm{loc}}(\mathbb{R}, X)$), then f is in $S^p_c(\mathbb{R}, X)$ (in $e\text{-}W^p_c(\mathbb{R}, X)$ or $W^p_c(\mathbb{R}, X)$, respectively).*

Proof Let $f \in e\text{-}W^p_c(\mathbb{R}, X)$ (or $f \in S^p_c(\mathbb{R}, X)$, respectively). For every $\varepsilon > 0$ we can find $l > 0$ and $L_0 = L_0(\varepsilon) > 0$ (take $L_0 = 1$ for the case of $f \in S^p_c(\mathbb{R}, X)$) such that any interval of length l contains at least a point τ (which is called an (ε, c)-equi-Weyl or Stepanov translation number of $f(x)$) satisfying

$$\sup_{x \in \mathbb{R}} \left(L_0^{-1} \int_x^{x+L_0} \|f(t+\tau) - cf(t)\|^p \, dt \right)^{\frac{1}{p}} \le \varepsilon.$$

(Although the value $\varepsilon > 0$ could be changed, we will also denote as L_0 the positive real number associated with the translation number).

If $f \in W_c^p(\mathbb{R}, X)$, for every $\varepsilon > 0$ we can find $l > 0$ such that any interval of length l contains at least a point τ (which is called an (ε, c)-Weyl translation number of $f(x)$) satisfying

$$\overline{\lim_{L\to\infty}} \sup_{x\in\mathbb{R}} \left(L^{-1} \int_x^{x+L} \| f(t+\tau) - cf(t)\|^p \, dt \right)^{\frac{1}{p}} \le \varepsilon.$$

(i) Fixed $\lambda \in \mathbb{C}\setminus\{0\}$ (the case $\lambda = 0$ is trivial), note that every τ in the set of the $(\frac{\varepsilon}{|\lambda|}, c)$-equi-Weyl translation numbers (or the set of the $(\frac{\varepsilon}{|\lambda|}, c)$-Stepanov translation numbers, respectively) satisfies

$$\begin{aligned}
&\sup_{x\in\mathbb{R}} \left(L_0^{-1} \int_x^{x+L_0} \| \lambda f(t+\tau) - \lambda c f(t)\|^p \, dt \right)^{\frac{1}{p}} \\
&= \sup_{x\in\mathbb{R}} \left(|\lambda|^p \, L_0^{-1} \int_x^{x+L_0} \| f(t+\tau) - cf(t)\|^p \, dt \right)^{\frac{1}{p}} \le |\lambda| \, \frac{\varepsilon}{|\lambda|} = \varepsilon,
\end{aligned}$$

which yields that $\lambda f \in e\text{-}W_c^p(\mathbb{R}, X)$ (or $\lambda f \in S_c^p(\mathbb{R}, X)$, respectively). Finally, by Proposition 6, we also have that $\lambda f \in e\text{-}W_{c^m}^p(\mathbb{R}, X)$ (or $\lambda f \in S_{c^m}^p(\mathbb{R}, X)$, respectively) for each $m \in \mathbb{Z} \setminus \{0\}$. The case $\lambda f \in W_{c^m}^p(\mathbb{R}, X)$ is analogous.

(ii) Note that every τ in the set of the (ε, c)-equi-Weyl translation numbers of $f(x)$ (or in the set of the (ε, c)-Stepanov translation numbers of $f(x)$, respectively) satisfies

$$\begin{aligned}
&\sup_{x\in\mathbb{R}} \left(L_0^{-1} \int_x^{x+L_0} \| f^\alpha(t+\tau) - cf^\alpha(t)\|^p \, dt \right)^{\frac{1}{p}} \\
&= \sup_{x\in\mathbb{R}} \left(L_0^{-1} \int_x^{x+L_0} \| f(t+\tau+\alpha) - cf(t+\alpha)\|^p \, dt \right)^{\frac{1}{p}} \\
&= \sup_{x\in\mathbb{R}} \left(L_0^{-1} \int_{x+\alpha}^{x+\alpha+L_0} \| f(t+\tau) - cf(t)\|^p \, dt \right)^{\frac{1}{p}} \le \varepsilon,
\end{aligned}$$

which yields that τ is also an (ε, c)-equi-Weyl translation number of $f^\alpha(x)$ (or an (ε, c)-Stepanov translation number of $f^\alpha(x)$, respectively).

On the other hand, if $\beta > 0$ and τ is in the set of the (ε, c)-equi-Weyl translation numbers of $f(x)$ associated with the value βL_0 (or in the set of the (ε, c)-Stepanov translation numbers of $f(x)$ associated with β, respectively), it is also accomplished that

$$\begin{aligned}
&\sup_{x\in\mathbb{R}}\left(L_0^{-1}\int_x^{x+L_0}\left\|f_\beta\left(t+\frac{\tau}{\beta}\right)-cf_\beta(t)\right\|^p dt\right)^{\frac{1}{p}}\\
&\quad=\sup_{x\in\mathbb{R}}\left(L_0^{-1}\int_x^{x+L_0}\left\|f\left(\beta\left(t+\frac{\tau}{\beta}\right)\right)-cf(\beta t)\right\|^p dt\right)^{\frac{1}{p}}\\
&\quad=\sup_{x\in\mathbb{R}}\left((\beta L_0)^{-1}\int_{\beta x}^{\beta x+\beta L_0}\|f(t+\tau)-cf(t)\|^p dt\right)^{\frac{1}{p}}\le\varepsilon,
\end{aligned}$$

which yields that $\frac{\tau}{\beta}$ is an (ε, c)-equi-Weyl translation number of $f_\beta(x)$ associated with the value L_0 (or an (ε, c)-Stepanov translation number of $f_\beta(x)$, respectively). The case $\beta < 0$ is solved by taking τ in the set of the (ε, c)-equi-Weyl translation numbers of $f(x)$ associated with the value $-\beta L_0$ (or in the set of the (ε, c)-Stepanov translation numbers of $f(x)$ associated with $-\beta$, respectively), which leads to the fact that $\frac{\tau}{\beta}$ is an (ε, c)-equi-Weyl translation number of $f_\beta(x)$ associated with the value L_0 (or an (ε, c)-Stepanov translation number, respectively).

Hence the functions $f^\alpha(x)$ and $f_\beta(x)$ are in e-$W_c^p(\mathbb{R}, X)$ (or $S_c^p(\mathbb{R}, X)$, respectively) and, by Proposition 6, $f^\alpha(x)$ and $f_\beta(x)$ are in e-$W_{c^m}^p(\mathbb{R}, X)$ (or $S_{c^m}^p(\mathbb{R}, X)$, respectively) for each $m \in \mathbb{Z}\setminus\{0\}$. The case $W_{c^m}^p(\mathbb{R}, X)$ is analogous.

(iii) Suppose the existence of $M > 0$ such that $\|f(x)\| \le M,\ \forall x \in \mathbb{R}$. Note that every τ in the set of the $(\frac{\varepsilon}{M(1+|c|)}, c)$-equi-Weyl translation numbers (or the set of the $(\frac{\varepsilon}{M(1+|c|)}, c)$-Stepanov translation numbers, respectively) satisfies

$$\begin{aligned}
&\sup_{x\in\mathbb{R}}\left(L_0^{-1}\int_x^{x+L_0}\|f^2(t+\tau)-c^2f^2(t)\|^p dt\right)^{\frac{1}{p}}\\
&\quad=\sup_{x\in\mathbb{R}}\left(L_0^{-1}\int_x^{x+L_0}\|(f(t+\tau)-cf(t))\,(f(t+\tau)+cf(t))\|^p dt\right)^{\frac{1}{p}}\\
&\quad\le M(1+|c|)\sup_{x\in\mathbb{R}}\left(L_0^{-1}\int_x^{x+L_0}\|f(t+\tau)-cf(t)\|^p dt\right)^{\frac{1}{p}}\\
&\quad\le M(1+|c|)\,\frac{\varepsilon}{M(1+|c|)}=\varepsilon.
\end{aligned}$$

This shows that $f^2 \in e$-$W_{c^2}^p(\mathbb{R}, X)$ (or $f^2 \in S_{c^2}^p(\mathbb{R}, X)$, respectively). Finally, we deduce from Proposition 6 that $f^2 \in e$-$W_{c^{2k}}^p(\mathbb{R}, X)$ (or $f^2 \in S_{c^{2k}}^p(\mathbb{R}, X)$, respectively) for each $k \in \mathbb{Z}\setminus\{0\}$. The case $f^2 \in W_{c^{2k}}^p(\mathbb{R}, X)$ is analogous.

(iv) Suppose the existence of $m_1 > 0$ such that $\|f(x)\| \ge m_1 > 0\ \forall x \in \mathbb{R}$. Note that every τ in the set of the $(\varepsilon\,|c|\,m_1^2, c)$-equi-Weyl translation numbers (or the set of the $(\varepsilon\,|c|\,m_1^2, c)$-Stepanov translation numbers, respectively) satisfies

$$\sup_{x\in\mathbb{R}}\left(L_0^{-1}\int_x^{x+L_0}\left\|\frac{1}{f(t+\tau)}-\frac{1}{c}\frac{1}{f(t)}\right\|^p dt\right)^{\frac{1}{p}}$$
$$=\sup_{x\in\mathbb{R}}\left(L_0^{-1}\int_x^{x+L_0}\left\|\frac{f(t+\tau)-cf(t)}{cf(t)f(t+\tau)}\right\|^p dt\right)^{\frac{1}{p}}$$
$$\leq\frac{\varepsilon\,|c|\,m_1^2}{|c|\,m_1^2}=\varepsilon,$$

which proves that the multiplicative inverse, or reciprocal, of $f(x)$ is in e-$W^p_{1/c}(\mathbb{R},X)$ (or in $S^p_{1/c}(\mathbb{R},X)$, respectively). Hence, by Proposition 6, we conclude that $\frac{1}{f}\in e\text{-}W^p_{c^m}(\mathbb{R},X)$ (or $\frac{1}{f}\in S^p_{c^m}(\mathbb{R},X)$, respectively) for each $m\in\mathbb{Z}\setminus\{0\}$. The case $\frac{1}{f}\in W^p_{c^m}(\mathbb{R},X)$ is analogous.

(v) By e-W^p-uniformly convergence (or S^p-uniformly convergence), given $\varepsilon>0$, there exist $L_0=L_0(\varepsilon)$ (take $L_0=1$ for the case of S^p-uniformly convergence) and $n_0\in\mathbb{N}$ such that

$$\sup_{x\in\mathbb{R}}\left(L_0^{-1}\int_x^{x+L_0}\|f_n(t)-f(t)\|^p dt\right)^{\frac{1}{p}}\leq\min\left\{\frac{\varepsilon}{3},\frac{\varepsilon}{3\,|c|}\right\}\ \forall n\geq n_0.$$

Then every τ in the set of the $(\frac{\varepsilon}{3},c)$-equi-Weyl translation numbers of $f_{n_0}(x)$ (or in the set of the $(\frac{\varepsilon}{3},c)$-Stepanov translation numbers, respectively) satisfies

$$\sup_{x\in\mathbb{R}}\left(L_0^{-1}\int_x^{x+L_0}\|f(t+\tau)-cf(t)\|^p dt\right)^{\frac{1}{p}}$$
$$\leq\sup_{x\in\mathbb{R}}\left(L_0^{-1}\int_x^{x+L_0}\|f(t+\tau)-f_{n_0}(t+\tau)\|^p dt\right)^{\frac{1}{p}}$$
$$+\sup_{x\in\mathbb{R}}\left(L_0^{-1}\int_x^{x+L_0}\|f_{n_0}(t+\tau)-cf_{n_0}(t)\|^p dt\right)^{\frac{1}{p}}+$$
$$+\sup_{x\in\mathbb{R}}\left(L_0^{-1}\int_x^{x+L_0}\|cf_{n_0}(t)-cf(t)\|^p dt\right)^{\frac{1}{p}}$$
$$\leq\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+|c|\,\frac{\varepsilon}{3\,|c|}=\varepsilon,$$

which yields that $f\in e\text{-}W^p_c(\mathbb{R},X)$ (or $f\in S^p_c(\mathbb{R},X)$, respectively). The case of the W^p-uniformly convergence is analogous. $\square$

The following two results show some conditions under which the sets $S^p_c(\mathbb{R},X)$, e-$W^p_c(\mathbb{R},X)$ and $W^p_c(\mathbb{R},X)$ are respectively included in $S^p(\mathbb{R},X)$, e-$W^p(\mathbb{R},X)$ and $W^p(\mathbb{R},X)$.

Proposition 8 (Connection with S^p, e-W^p and W^p-almost periodicity) *Consider* $1 \leq p < \infty$ *and* $c \in \mathbb{C} \setminus \{0\}$ *such that* $\frac{\arg c}{2\pi} \in \mathbb{Q}$. *Then*

$$S_c^p(\mathbb{R}, X) \subset S_{|c|^q}^p(\mathbb{R}, X), \ e\text{-}W_c^p(\mathbb{R}, X) \subset e\text{-}W_{|c|^q}^p(\mathbb{R}, X) \text{ and } W_c^p(\mathbb{R}, X) \subset W_{|c|^q}^p(\mathbb{R}, X),$$

where $q \in \mathbb{N}$ *is so that* $\frac{\arg c}{2\pi} = \frac{r}{q}$ *for a certain* $r \in \mathbb{Z}$ *such that* $\gcd(r, q) = 1$. *In particular, under the same condition, the case* $|c| = 1$ *yields the inclusions* $S_c^p(\mathbb{R}, X) \subset S^p(\mathbb{R}, X)$, e-$W_c^p(\mathbb{R}, X) \subset e$-$W^p(\mathbb{R}, X)$ *and* $W_c^p(\mathbb{R}, X) \subset W^p(\mathbb{R}, X)$.

Proof Put $\arg c = \frac{2\pi r}{q}$ with $r \in \mathbb{Z}$ and $q \in \mathbb{N}$ so that $\gcd(r, q) = 1$. Then

$$c^q = |c|^q e^{qi \arg c} = |c|^q e^{2r\pi i} = |c|^q.$$

Now, if $f \in S_c^p(\mathbb{R}, X)$, $f \in e$-$W_c^p(\mathbb{R}, X)$ or $f \in W_c^p(\mathbb{R}, X)$, Proposition 6 assures respectively that $f \in S_{c^q}^p(\mathbb{R}, X) = S_{|c|^q}^p(\mathbb{R}, X)$, $f \in e$-$W_{c^q}^p(\mathbb{R}, X) = e$-$W_{|c|^q}^p(\mathbb{R}, X)$ or $f \in W_{c^q}^p(\mathbb{R}, X) = W_{|c|^q}^p(\mathbb{R}, X)$. Hence the result holds. □

Proposition 9 (Connection with S^p, e-W^p and W^p-almost periodicity) *Let* $1 \leq p < \infty$ *and* $c \in \mathbb{C} \setminus \{0\}$ *such that* $\frac{\arg c}{\pi} \notin \mathbb{Q}$ *and* $|c| = 1$. *Then* $S_c^p(\mathbb{R}, X) \subset S^p(\mathbb{R}, X)$ *and* e-$W_c^p(\mathbb{R}, X) \subset e$-$W^p(\mathbb{R}, X)$. *Furthermore, every* e-W^p*-bounded function in* $W_c^p(\mathbb{R}, X)$ *is also in* $W^p(\mathbb{R}, X)$.

Proof Let $f \in e$-$W_c^p(\mathbb{R}, X)$ (or $f \in S_c^p(\mathbb{R}, X)$, respectively). By Proposition 1 (see also Definition 2), there exist $M > 0$ and $L_0 > 0$ (take $L_0 = 1$, respectively) such that

$$\sup_{x \in \mathbb{R}} \left(L^{-1} \int_x^{x+L} \|f(t)\|^p \, dt \right)^{\frac{1}{p}} \leq M \ \forall L \geq L_0. \tag{6}$$

Suppose that $\arg c$ is not a rational multiple of π, which yields that $e^{ni \arg c} \neq 1$ for all $n \in \mathbb{N}$. Given $\varepsilon > 0$, choose $n_1, n_2 \in \mathbb{N}$ (with $n_1 \neq n_2$) such that $\left| e^{n_2 i \arg c} - e^{n_1 i \arg c} \right| < \frac{\varepsilon}{2M}$ (note that the existence of n_1 and n_2 is assured by virtue of the fact that $\left\{ e^{ni \arg c} : n \in \mathbb{N} \right\} \subset \{c \in \mathbb{C} : |c| = 1\}$ and that the length of the unit circumference is finite). Hence

$$\left| c^{n_2 - n_1} - 1 \right| = \left| e^{(n_2 - n_1)i \arg c} - 1 \right| = \left| e^{n_2 i \arg c} - e^{n_1 i \arg c} \right| < \frac{\varepsilon}{2M}.$$

Moreover, by Proposition 6, it is accomplished that $f \in e$-$W_{c^{n_2-n_1}}^p(\mathbb{R}, X)$ (or $f \in S_{c^{n_2-n_1}}^p(\mathbb{R}, X)$, respectively). Consequently, for $\varepsilon > 0$ there exists L_1 (without loss of generality, we will suppose that $L_1 \geq L_0$) such that every τ in the set of $(\frac{\varepsilon}{2}, c^{n_2-n_1})$-equi-Weyl translation numbers of $f(x)$ associated with L_1 (or $(\frac{\varepsilon}{2}, c^{n_2-n_1})$-Stepanov translation numbers associated with $L_1 = 1$) satisfies

$$\begin{aligned}
&\sup_{x\in\mathbb{R}}\left(L_1^{-1}\int_x^{x+L_1}\|f(t+\tau)-f(t)\|^p\,dt\right)^{\frac{1}{p}}\\
&\quad\leq\sup_{x\in\mathbb{R}}\left(L_1^{-1}\int_x^{x+L_1}\|f(t+\tau)-c^{n_2-n_1}f(t)\|^p\,dt\right)^{\frac{1}{p}}\\
&\qquad+\sup_{x\in\mathbb{R}}\left(L_1^{-1}\int_x^{x+L_1}\|c^{n_2-n_1}f(t)-f(t)\|^p\,dt\right)^{\frac{1}{p}}\\
&\quad\leq\frac{\varepsilon}{2}+|c^{n_2-n_1}-1|\sup_{x\in\mathbb{R}}\left(L_1^{-1}\int_x^{x+L_1}\|f(t)\|^p\,dt\right)^{\frac{1}{p}}\leq\varepsilon,
\end{aligned}$$

which proves that $S_c^p(\mathbb{R}, X) \subset S^p(\mathbb{R}, X)$ and $e\text{-}W_c^p(\mathbb{R}, X) \subset e\text{-}W^p(\mathbb{R}, X)$. Finally, the condition of $e\text{-}W^p$-boundedness of a function $f \in L^p_{\mathrm{loc}}(\mathbb{R}, X)$ yields (6) and the last statement of the result is proved in an analogous manner. □

The following property reminds the relative compactness of the family of translates of a function with respect to an hypothetical S_c^p or W_c^p-metric, which constitutes an extension of some concrete results proved by Andres et al. for the case $c = 1$ (see [2, Theorems 3.5, 4.12, 4.23]).

Proposition 10 (On the family of translates of a S_c^p, $e\text{-}W_c^p$ or W_c^p-almost periodic function) *Let $c \in \mathbb{C}\setminus\{0\}$ and $1 \leq p < \infty$. Given a function $f \in L^p_{\mathrm{loc}}(\mathbb{R}, X)$, consider the family of translates $\mathcal{F}_f = \{f^h(x) := f(x+h) : h \in \mathbb{R}\} \subset L^p_{\mathrm{loc}}(\mathbb{R}, X)$.*

(i) *If $f \in S_c^p(\mathbb{R}, X)$, then there exists a finite amount of values $h_j \in \mathbb{R}$, $j = 1, \ldots, n$, satisfying the following property: for every $\varepsilon > 0$ and $f^h \in \mathcal{F}_f$, there exists $j \in \{1, \ldots, n\}$ such that*

$$\sup_{x\in\mathbb{R}}\left(\int_x^{x+1}\|cf^h(t)-f^{h_j}(t)\|^p\,dt\right)^{\frac{1}{p}}\leq\varepsilon$$

or

$$\sup_{x\in\mathbb{R}}\left(\int_x^{x+1}\|cf^{-h_j}(t)-f^h(t)\|^p\,dt\right)^{\frac{1}{p}}\leq\varepsilon.$$

(ii) *If $f \in e\text{-}W_c^p(\mathbb{R}, X)$, then there exists a finite amount of values $h_j \in \mathbb{R}$, $j = 1, \ldots, n$, and $L_0 > 0$ satisfying the following property: for every $\varepsilon > 0$ and $f^h \in \mathcal{F}_f$, there exists $j \in \{1, \ldots, n\}$ such that*

$$\sup_{x\in\mathbb{R}}\left(L^{-1}\int_x^{x+L}\|cf^h(t)-f^{h_j}(t)\|^p\,dt\right)^{\frac{1}{p}}\leq\varepsilon\ \textit{for all}\ L\geq L_0$$

or

$$\sup_{x\in\mathbb{R}}\left(L^{-1}\int_x^{x+L}\|cf^{-h_j}(t)-f^h(t)\|^p\,dt\right)^{\frac{1}{p}}\leq\varepsilon \text{ for all } L\geq L_0.$$

(iii) *If $f\in W_c^p(\mathbb{R},X)$ and it satisfies hypothesis (5), then there exists a finite amount of values $h_j\in\mathbb{R}$, $j=1,\ldots,n$, satisfying the following property: for every $\varepsilon>0$ and $f^h\in\mathcal{F}_f$, there exists $j\in\{1,\ldots,n\}$ such that*

$$\overline{\lim_{L\to\infty}}\sup_{x\in\mathbb{R}}\left(L^{-1}\int_x^{x+L}\|cf^h(t)-f^{h_j}(t)\|^p\,dt\right)^{\frac{1}{p}}\leq\varepsilon$$

or

$$\overline{\lim_{L\to\infty}}\sup_{x\in\mathbb{R}}\left(L^{-1}\int_x^{x+L}\|cf^{-h_j}(t)-f^h(t)\|^p\,dt\right)^{\frac{1}{p}}\leq\varepsilon.$$

Proof (i) and (ii) We will expose the proof for the case $f\in e\text{-}W_c^p(\mathbb{R},X)$ (the case $f\in S_c^p(\mathbb{R},X)$ is analogous by taking $L_0=1$). Fix $\varepsilon>0$. By Proposition 2, we know that f is $e\text{-}W^p$-uniformly continuous, which means that there exist two positive numbers $L_0=L_0(\varepsilon)$ and $\delta=\delta(\varepsilon)$ such that any $|h|<\delta$ satisfies

$$\sup_{x\in\mathbb{R}}\left(L_0^{-1}\int_x^{x+L_0}\|f(t+h)-f(t)\|^p\,dt\right)^{\frac{1}{p}}\leq\frac{\varepsilon}{2}. \tag{7}$$

Moreover, we can find $l>0$ and $L_1=L_1(\varepsilon)>0$ such that any interval of length l contains at least a point τ satisfying

$$\sup_{x\in\mathbb{R}}\left(L_1^{-1}\int_x^{x+L_1}\|f(t+\tau)-cf(t)\|^p\,dt\right)^{\frac{1}{p}}\leq\frac{\varepsilon}{2}. \tag{8}$$

Without loss of generality, suppose $l>\delta$. Now, fix $h\in\mathbb{R}$ and note that in the interval $[-h,-h+l]$ there exists τ satisfying (8) (note that $h+\tau\in[0,l]$). Furthermore, it is assured the existence of $n\in\mathbb{N}$ such that $n\delta\leq l<(n+1)\delta$ and choose $j\in\{1,\ldots,n\}$ such that

$$(j-1)\delta<h+\tau<(j+1)\delta$$

or, equivalently,

$$|h+\tau-j\delta|<\delta. \tag{9}$$

Put $h_j = j\delta$. Now, if $L_2 \geq \max\{L_0, L_1\}$ (see Remark 3), by (7), (8) and (9), we have that

$$\sup_{x\in\mathbb{R}}\left(L_2^{-1}\int_x^{x+L_2}\|cf(t+h)-f(t+h_j)\|^p\,dt\right)^{\frac{1}{p}}$$
$$\leq \sup_{x\in\mathbb{R}}\left(L_2^{-1}\int_x^{x+L_2}\|cf(t+h)-f(t+h+\tau)\|^p\,dt\right)^{\frac{1}{p}}$$
$$+\sup_{x\in\mathbb{R}}\left(L_2^{-1}\int_x^{x+L_2}\|f(t+h+\tau)-f(t+h_j)\|^p\,dt\right)^{\frac{1}{p}} \leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Analogously, fixed $h \in \mathbb{R}$, in the interval $[h, h+l]$ there exists τ_1 satisfying (8) (note that $-h+\tau_1 \in [0,l]$). Furthermore, it is assured the existence of $n \in \mathbb{N}$ such that $n\delta \leq l < (n+1)\delta$ and choose $j \in \{1,\ldots,n\}$ such that

$$(j-1)\delta < -h+\tau_1 < (j+1)\delta$$

or, equivalently,

$$|-h+\tau_1-j\delta| < \delta.$$

Hence

$$\sup_{x\in\mathbb{R}}\left(L_2^{-1}\int_x^{x+L_2}\|cf(t-h_j)-f(t+h)\|^p\,dt\right)^{\frac{1}{p}}$$
$$\leq \sup_{x\in\mathbb{R}}\left(L_2^{-1}\int_x^{x+L_2}\|cf(t-h_j)-f(t-h_j+\tau_1)\|^p\,dt\right)^{\frac{1}{p}}$$
$$+\sup_{x\in\mathbb{R}}\left(L_2^{-1}\int_x^{x+L_2}\|f(t-h_j+\tau_1)-f(t+h)\|^p\,dt\right)^{\frac{1}{p}} \leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

(iii) Fix $\varepsilon > 0$. By Proposition 3, we know that f is W^p-uniformly continuous, which means that there exists $\delta = \delta(\varepsilon)$ such that any $|h| < \delta$ satisfies

$$\overline{\lim_{L\to\infty}}\sup_{x\in\mathbb{R}}\left(L^{-1}\int_x^{x+L}\|f(t+h)-f(t)\|^p\,dt\right)^{\frac{1}{p}} \leq \frac{\varepsilon}{2}. \tag{10}$$

Moreover, we can find $l > 0$ such that any interval of length l contains at least a point τ satisfying

$$\overline{\lim_{L\to\infty}}\sup_{x\in\mathbb{R}}\left(L^{-1}\int_x^{x+L}\|f(t+\tau)-cf(t)\|^p\,dt\right)^{\frac{1}{p}} \leq \frac{\varepsilon}{2}. \tag{11}$$

Without loss of generality, suppose $l > \delta$. Now, fix $h \in \mathbb{R}$ and note that in the interval $[-h, -h+l]$ there exists τ satisfying (8) (note that $h+\tau \in [0, l]$). Furthermore, it is assured the existence of $n \in \mathbb{N}$ such that $n\delta \leq l < (n+1)\delta$ and choose $j \in \{1, \dots, n\}$ such that

$$(j-1)\delta < h+\tau < (j+1)\delta$$

or, equivalently,

$$|h+\tau - j\delta| < \delta. \tag{12}$$

Put $h_j = j\delta$. By (10), (11) and (12), we have that

$$\begin{aligned}
&\overline{\lim_{L\to\infty}} \sup_{x\in\mathbb{R}} \left(L^{-1}\int_x^{x+L} \|cf(t+h) - f(t+h_j)\|^p\, dt\right)^{\frac{1}{p}} \\
&\leq \overline{\lim_{L\to\infty}} \sup_{x\in\mathbb{R}} \left(L^{-1}\int_x^{x+L} \|cf(t+h) - f(t+h+\tau)\|^p\, dt\right)^{\frac{1}{p}} \\
&\quad + \overline{\lim_{L\to\infty}} \sup_{x\in\mathbb{R}} \left(L^{-1}\int_x^{x+L} \|f(t+h+\tau) - f(t+h_j)\|^p\, dt\right)^{\frac{1}{p}} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Analogously, fixed $h \in \mathbb{R}$, in the interval $[h, h+l]$ there exists τ_1 satisfying (11) (note that $-h+\tau_1 \in [0, l]$). Furthermore, it is assured the existence of $n \in \mathbb{N}$ such that $n\delta \leq l < (n+1)\delta$ and choose $j \in \{1, \dots, n\}$ such that

$$(j-1)\delta < -h+\tau_1 < (j+1)\delta$$

or, equivalently,

$$|-h+\tau_1 - j\delta| < \delta.$$

Hence

$$\begin{aligned}
&\overline{\lim_{L\to\infty}} \sup_{x\in\mathbb{R}} \left(L^{-1}\int_x^{x+L} \|cf(t-h_j) - f(t+h)\|^p\, dt\right)^{\frac{1}{p}} \\
&\leq \overline{\lim_{L\to\infty}} \sup_{x\in\mathbb{R}} \left(L^{-1}\int_x^{x+L} \|cf(t-h_j) - f(t-h_j+\tau_1)\|^p\, dt\right)^{\frac{1}{p}} \\
&\quad + \overline{\lim_{L\to\infty}} \sup_{x\in\mathbb{R}} \left(L^{-1}\int_x^{x+L} \|f(t-h_j+\tau_1) - f(t+h)\|^p\, dt\right)^{\frac{1}{p}} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

□

4 Main Definitions for the Case of Complex Functions Defined on Vertical Strips

We will devote this section to introduce the spaces of c-almost periodic functions in the sense of Stepanov and Weyl approaches for the case of complex functions defined on vertical strips in the complex plane. These sets are natural generalizations of the space of c-almost periodic functions $AP_c(U, \mathbb{C})$ which was described in the introduction, where U is of the form $\{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$, with $-\infty \leq \alpha < \beta \leq \infty$.

First of all, we introduce the following family of functions which is connected with the set of p-locally integrable functions.

Definition 5 (*Functions in the set* $L^p_{li}(U, \mathbb{C})$) Let $1 \leq p < \infty$ and $f : U \to \mathbb{C}$ a complex function defined in a vertical strip $U = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$, with $-\infty \leq \alpha < \beta \leq \infty$. We will say that f is in the set $L^p_{li}(U, \mathbb{C})$ if for every rectangle in U, say $\{z \in \mathbb{C} : \alpha_1 \leq \operatorname{Re} z \leq \beta_1,\ l_1 \leq \operatorname{Im} z \leq l_2\}$ (with $\alpha < \alpha_1 < \beta_1 < \beta$ and $-\infty < l_1 < l_2 < \infty$), we have

$$\sup_{x \in [\alpha_1, \beta_1]} \left(\int_{l_1}^{l_2} |f(x + it)|^p \, dt \right)^{\frac{1}{p}} < \infty.$$

Definition 6 (*c-almost periodicity in the sense of Stepanov or Weyl approaches for vertical strips*) Let $c \in \mathbb{C} \setminus \{0\}$ and $1 \leq p < \infty$. Consider a function $f \in L^p_{li}(U, \mathbb{C})$ defined in a strip $U = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$, with $-\infty \leq \alpha < \beta \leq \infty$.

(a) We will say that f is Stepanov-(p, c)-almost periodic in U, and we will write $f \in S^p_c(U, \mathbb{C})$, if for every $\varepsilon > 0$ and every reduced strip $U_1 = \{z \in \mathbb{C} : \alpha_1 \leq \operatorname{Re} z \leq \beta_1\} \subset U$ (with $\alpha < \alpha_1 < \beta_1 < \beta$) there corresponds a relatively dense set $\{\tau\} \subset \mathbb{R}$ (i.e. there exists $l > 0$ such that any interval of length l contains at least a point τ) whose elements satisfy

$$\sup_{x+iy \in U_1} \left(\int_{y}^{y+1} |f(x + i(t + \tau)) - cf(x + it)|^p \, dt \right)^{\frac{1}{p}} \leq \varepsilon. \tag{13}$$

The elements of the set $\{\tau\}$ satisfying the above condition are called S^p_c-translation numbers belonging to ε associated with U_1 (or simply (ε, c)-Stepanov translation numbers of f).

(b) We will say that f is equi-Weyl-(p, c)-almost periodic in U, and we will write $f \in e\text{-}W^p_c(U, \mathbb{C})$, if for every $\varepsilon > 0$ and every reduced strip $U_1 = \{z \in \mathbb{C} : \alpha_1 \leq \operatorname{Re} z \leq \beta_1\} \subset U$ (with $\alpha < \alpha_1 < \beta_1 < \beta$) we can find a real number $L_0 = L_0(\varepsilon)$ and a relatively dense set $\{\tau\}$ of real numbers (i.e. there exists $l > 0$ such that any interval of length l contains at least a point τ) satisfying

$$\sup_{x+iy \in U_1} \left(L_0^{-1} \int_{y}^{y+L_0} |f(x + i(t + \tau)) - cf(x + it)|^p \, dt \right)^{\frac{1}{p}} \leq \varepsilon. \tag{14}$$

The elements of the set $\{\tau\}$ satisfying the above condition are called e-W_c^p-translation numbers belonging to ε associated with L_0 and U_1 (or simply (ε, c)-equi-Weyl translation numbers of f associated with L_0).

(c) We will say that f is Weyl-(p, c)-almost periodic in U, and we will write $f \in W_c^p(U, \mathbb{C})$, if for every $\varepsilon > 0$ and every reduced strip of the form $U_1 = \{z \in \mathbb{C} : \alpha_1 \leq \operatorname{Re} z \leq \beta_1\} \subset U$ (with $\alpha < \alpha_1 < \beta_1 < \beta$) we can find a relatively dense set $\{\tau\}$ of real numbers (i.e. there exists $l > 0$ such that any interval of length l contains at least a point τ) satisfying

$$\overline{\lim_{L\to\infty}} \sup_{x+iy\in U_1} \left(L^{-1} \int_y^{y+L} |f(x + i(t + \tau)) - cf(x + it)|^p \, dt \right)^{\frac{1}{p}} \leq \varepsilon.$$

The elements of the set $\{\tau\}$ satisfying the above condition are called W_c^p-translation numbers belonging to ε associated with U_1 (or simply (ε, c)-Weyl translation numbers of f).

As it was said above, the sets $S_c^p(U, \mathbb{C})$, e-$W_c^p(U, \mathbb{C})$ and $W_c^p(U, \mathbb{C})$ are generalizations of the class of functions $AP_c(U, \mathbb{C})$ (see [20, Definition 1]). In fact, if $1 \leq p < \infty$ and $c \in \mathbb{C} \setminus \{0\}$, it is easy to see that

$$AP_c(U, \mathbb{C}) \subset S_c^p(U, \mathbb{C}) \subset e\text{-}W_c^p(U, \mathbb{C}) \subset W_c^p(U, \mathbb{C}).$$

With respect to the Stepanov, equi-Weyl or Weyl metrics, we next define the properties of boundedness, uniform continuity and uniform convergence which we will use in the next section. They are adaptations of the Definitions 2, 3 and 4 for the case of complex functions defined on vertical strips.

Definition 7 (*S^p, e-W^p or W^p-boundedness, for vertical strips*) Let $1 \leq p < \infty$ and consider a function $f \in L_{li}^p(U, \mathbb{C})$, where U is a vertical strip of the type $\{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$ with $-\infty \leq \alpha < \beta \leq \infty$. Also, take a vertical substrip $U_1 = \{z \in \mathbb{C} : \alpha_1 \leq \operatorname{Re} z \leq \beta_1\} \subset U$.

(i) f is said to be S^p-bounded in U_1 if there exists $M > 0$ such that

$$\sup_{x+iy\in U_1} \left(\int_y^{y+1} |f(x + it)|^p \, dt \right)^{\frac{1}{p}} \leq M.$$

(ii) f is said to be e-W^p-bounded in U_1 if there exist two positive numbers L_0 and M such that

$$\sup_{x+iy\in U_1} \left(\frac{1}{L_0} \int_y^{y+L_0} |f(x + it)|^p \, dt \right)^{\frac{1}{p}} \leq M.$$

(iii) f is said to be W^p-bounded in U_1 if there exists $M > 0$ such that

$$\overline{\lim_{L\to\infty}} \sup_{x+iy\in U_1} \left(\frac{1}{L} \int_y^{y+L} |f(x+it)|^p \, dt \right)^{\frac{1}{p}} \leq M.$$

Definition 8 (S^p, *e-W^p or W^p-uniform continuity, for vertical strips*) Let $1 \leq p < \infty$ and consider a function $f \in L^p_{li}(U, \mathbb{C})$, where U is a vertical strip of the type $\{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$ with $-\infty \leq \alpha < \beta \leq \infty$. Also, consider an arbitrary vertical line $r_x = \{z \in \mathbb{C} : \operatorname{Re} z = x\} \subset U$, where $x = \operatorname{Re} w$ for some $w \in U$.

(i) f is said to be S^p-uniformly continuous in the vertical line r_x if for any $\varepsilon > 0$ there is a positive number $\delta = \delta(\varepsilon)$ such that any $|h| < \delta$ satisfies

$$\sup_{y\in\mathbb{R}} \left(\int_y^{y+1} |f(x+i(t+h)) - f(x+it)|^p \, dt \right)^{\frac{1}{p}} \leq \varepsilon.$$

(ii) f is said to be e-W^p-uniformly continuous in the vertical line r_x if for any $\varepsilon > 0$ there exist two positive numbers L_0 and $\delta = \delta(\varepsilon)$ such that any $|h| < \delta$ satisfies

$$\sup_{y\in\mathbb{R}} \left(\frac{1}{L_0} \int_y^{y+L_0} |f(x+i(t+h)) - f(x+it)|^p \, dt \right)^{\frac{1}{p}} \leq \varepsilon.$$

iii) f is said to be W^p-uniformly continuous in the vertical line r_x if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that any $|h| < \delta$ satisfies

$$\overline{\lim_{L\to\infty}} \sup_{y\in\mathbb{R}} \left(\frac{1}{L} \int_y^{y+L} |f(x+i(t+h)) - f(x+it)|^p \, dt \right)^{\frac{1}{p}} \leq \varepsilon.$$

Definition 9 (S^p, *e-W^p or W^p-uniform convergence, for vertical strips*) Let $1 \leq p < \infty$ and $\{f_n\}_{n\geq 1}$ a sequence of functions in $L^p_{li}(U, \mathbb{C})$, where $U = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$, with $-\infty \leq \alpha < \beta \leq \infty$.

(i) If $\{f_n\}_{n\geq 1}$ is a sequence of S^p-bounded functions in every vertical substrip $U_1 \subset U$, we will say that $\{f_n\}_{n\geq 1}$ is S^p-uniformly convergent to a function $f : U \to \mathbb{C}$, which is also S^p-bounded in every $U_1 \subset U$, if for every vertical substrip $U_1 \subset U$ and $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{x+iy\in U_1} \left(\int_y^{y+1} |f_n(x+it) - f(x+it)|^p \, dt \right)^{\frac{1}{p}} \leq \varepsilon \text{ for each } n \geq n_0.$$

(ii) If $\{f_n\}_{n\geq 1}$ is a sequence of e-W^p-bounded functions in every vertical substrip $U_1 \subset U$, we will say that $\{f_n\}_{n\geq 1}$ is e-W^p-uniformly convergent to a function

$f : U \to \mathbb{C}$, which is also e-W^p-bounded in every $U_1 \subset U$, if for every vertical substrip $U_1 \subset U$ and $\varepsilon > 0$ there exist $L_0 = L_0(\varepsilon)$ and $n_0 \in \mathbb{N}$ satisfying

$$\sup_{x+iy\in U_1} \left(L_0^{-1} \int_y^{y+L_0} |f_n(x+it) - f(x+it)|^p \, dt \right)^{\frac{1}{p}} \leq \varepsilon \text{ for each } n \geq n_0.$$

(iii) If $\{f_n\}_{n\geq 1}$ is a sequence of W^p-bounded functions in every vertical substrip $U_1 \subset U$, we will say that $\{f_n\}_{n\geq 1}$ is W^p-uniformly convergent to a function $f : U \to \mathbb{C}$, which is also W^p-bounded in every $U_1 \subset U$, if for every vertical substrip $U_1 \subset U$ and $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ satisfying

$$\overline{\lim_{L\to\infty}} \sup_{x+iy\in U_1} \left(L^{-1} \int_y^{y+L} |f_n(x+it) - f(x+it)|^p \, dt \right)^{\frac{1}{p}} \leq \varepsilon \text{ for each } n \geq n_0.$$

Remark 5 (*On the notions of c-almost periodicity in the sense of Weyl and equi-Weyl*) As in Remark 3, we note that the difference between Stepanov-(p, c)-almost periodicity and equi-Weyl-(p, c)-almost periodicity is that in the latter case the value L_0 varies with ε and every reduced strip in U. Furthermore, by comparison with Definition 6.b), we note that $f \in e\text{-}W_c^p(U, \mathbb{C})$ if and only if for every $\varepsilon > 0$ and every reduced strip $U_1 \subset U$ we can find a real number $L_0 = L_0(\varepsilon)$ and a relatively dense set $\{\tau\}$ of real numbers satisfying

$$\sup_{x+iy\in U_1} \left(L^{-1} \int_y^{y+L} |f(x+i(t+\tau)) - cf(x+it)|^p \, dt \right)^{\frac{1}{p}} \leq \varepsilon \; \forall L \geq L_0.$$

Indeed, for every $L_0, L_1 > 0$ with $L_0 < L_1$ we have that

$$\|f\|_{S^p_{L_1,U_1}} \leq \left(1 + \frac{L_0}{L_1}\right)^{\frac{1}{p}} \|f\|_{S^p_{L_0,U_1}}, \tag{15}$$

where $\|f\|_{S^p_{L,U_1}} := \sup_{x+iy\in U_1} \left(L^{-1} \int_y^{y+L} |f(x+it)|^p \, dt \right)^{\frac{1}{p}}$ (for every $L > 0$). In fact, if we take $m \in \mathbb{N}$ such that $(m-1)L_0 < L_1 < mL_0$ then

$$\begin{aligned}
\|f\|_{S^p_{L_1,U_1}} &= \sup_{x+iy\in U_1} \left(L_1^{-1} \int_y^{y+L_1} |f(x+it)|^p \, dt \right)^{\frac{1}{p}} \\
&\leq \sup_{x+iy\in U_1} \left(\frac{mL_0}{L_1} \frac{1}{mL_0} \int_y^{y+mL_0} |f(x+it)|^p \, dt \right)^{\frac{1}{p}} \\
&= \left(\frac{mL_0}{L_1} \right)^{\frac{1}{p}} \|f\|_{S^p_{mL_0,U_1}}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{(m-1)L_0 + L_0}{L_1}\right)^{\frac{1}{p}} \|f\|_{S^p_{mL_0,U_1}} \\
&< \left(\frac{L_1 + L_0}{L_1}\right)^{\frac{1}{p}} \|f\|_{S^p_{mL_0,U_1}} \\
&\le \left(1 + \frac{L_0}{L_1}\right)^{\frac{1}{p}} \|f\|_{S^p_{L_0,U_1}},
\end{aligned}
$$

where the last inequality is given by the fact that any x, y with $x + iy \in U_1$ satisfies

$$
\begin{aligned}
&\frac{1}{mL_0} \int_y^{y+mL_0} |f(x+it)|^p \, dt \\
&= \frac{1}{mL_0} \left(\int_y^{y+L_0} |f(x+it)|^p \, dt + \int_{y+L_0}^{y+2L_0} |f(x+it)|^p \, dt + \ldots + \right. \\
&\quad \left. + \ldots + \int_{y+(m-1)L_0}^{y+mL_0} |f(x+it)|^p \, dt \right) \\
&\le \frac{m}{mL_0} \sup_{x+iy \in U_1} \left(\int_y^{y+L_0} |f(x+it)|^p \, dt \right) = \|f\|^p_{S^p_{L_0,U_1}},
\end{aligned}
$$

which yields that $\|f\|^p_{S^p_{mL_0,U_1}} \le \|f\|^p_{S^p_{L_0,U_1}}$. In this way, fixed $\varepsilon > 0$ and a reduced strip $U_1 \subset U$, if f satisfies our Definition 6.b), then there exists $L_0 = L_0(\frac{\varepsilon}{2})$ and a relatively dense set $\{\tau\}$ of real numbers such that it is accomplished $\|f^{i\tau} - cf\|_{S^p_{L_0,U_1}} \le \frac{\varepsilon}{2}$, where $f^{i\tau}(z) := f(z + i\tau)$ for all $z \in U$. This yields by (15) that $\|f^{i\tau} - cf\|_{S^p_{L_1,U_1}} \le \left(1 + \frac{L_0}{L_1}\right) \frac{\varepsilon}{2}$ for any $L_1 > L_0$. Therefore, if we take L_1 arbitrarily large ($L_1 \to \infty$) it is clear that $\|f^{i\tau} - cf\|_{S^p_{L,U_1}} \le \varepsilon$ for every $L \ge L_1'$ for a certain L_1' sufficiently large, which means that f satisfies this alternative definition. The converse is trivial.

5 Main Properties of the Spaces $S^p_c(U, \mathbb{C})$, e-$W^p_c(U, \mathbb{C})$ and $W^p_c(U, \mathbb{C})$

In this section we will show the basic properties of the functions $f : U \to \mathbb{C}$ which are Stepanov-(p, c)-almost periodic, equi-Weyl-(p, c)-almost periodic or Weyl-(p, c)-almost periodic and are defined on vertical strips U of the complex plane.

The demonstrations of the most of following properties are similar to the case of the functions in $S^p_c(\mathbb{R}, X)$, e-$W^p_c(\mathbb{R}, X)$ and $W^p_c(\mathbb{R}, X)$. However, we include them for the sake of completeness.

Proposition 11 (S^p, e-W^p-boundedness and uniform continuity) *Let $c \in \mathbb{C} \setminus \{0\}$ and $1 \le p < \infty$. Consider a vertical strip of the form $U = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$, with $-\infty \le \alpha < \beta \le \infty$, and a substrip $U_1 = \{z \in \mathbb{C} : \alpha_1 \le \operatorname{Re} z \le \beta_1\} \subset U$ with $\alpha < \alpha_1 < \beta_1 < \beta$.*

(i) *If* $f \in S^p_c(U,\mathbb{C})$, *then* f *is* S^p*-bounded in* U_1 *and* S^p*-uniformly continuous in every vertical line in* U.
(ii) *If* $f \in e\text{-}W^p_c(U,\mathbb{C})$, *then* f *is* $e\text{-}W^p$*-bounded in* U_1 *and* $e\text{-}W^p$*-uniformly continuous in every vertical line in* U.

Proof Take $U_1 = \{z \in \mathbb{C} : \alpha_1 \leq \operatorname{Re} z \leq \beta_1\} \subset U$ with $\alpha < \alpha_1 < \beta_1 < \beta$, and $\varepsilon = 1$. By $e\text{-}W^p_c$-almost periodicity (or S^p_c-almost periodicity, respectively), there exist $L_0 > 0$ (take $L_0 = 1$, resp.) and $l > 0$ such that every interval in $\mathbb{R}$ of length l contains at least an (ε, c)-equi-Weyl (or Stepanov, resp.) translation number of f i.e. it satisfies (14) (or (13), resp.). In particular, if $z = x + iy$ is an arbitrary complex number in U_1, we can assure the existence of a value τ satisfying (14) (or (13), resp.) and such that $y + \tau \in [0, l]$. Moreover, since $f \in L^p_{li}(U,\mathbb{C})$, it is clear that there exists $M > 0$ such that $\sup_{x\in[\alpha_1,\beta_1]} \left(\int_0^{l+L_0} |f(x+it)|^p\, dt\right)^{\frac{1}{p}} \leq M$ (take $L_0 = 1$, resp.). Consequently, for an arbitrary $x + iy \in U_1$ we have

$$
\begin{aligned}
&\left(L_0^{-1}\int_y^{y+L_0} |cf(x+it)|^p\, dt\right)^{\frac{1}{p}}\\
&\leq \left(L_0^{-1}\int_y^{y+L_0} |cf(x+it) - f(x+i(t+\tau))|^p\, dt\right)^{\frac{1}{p}}\\
&\quad + \left(L_0^{-1}\int_y^{y+L_0} |f(x+i(t+\tau))|^p\, dt\right)^{\frac{1}{p}}\\
&\leq 1 + \left(L_0^{-1}\int_{y+\tau}^{y+\tau+L_0} |f(x+it)|^p\, dt\right)^{\frac{1}{p}}\\
&\leq 1 + \left(L_0^{-1}\int_0^{l+L_0} |f(x+it)|^p\, dt\right)^{\frac{1}{p}}\\
&\leq 1 + L_0^{-\frac{1}{p}} M,
\end{aligned}
$$

which yields that

$$
\sup_{x+iy\in U_1}\left(L_0^{-1}\int_y^{y+L_0} |f(x+it)|^p\, dt\right)^{\frac{1}{p}} \leq \frac{1 + L_0^{-\frac{1}{p}} M}{|c|}.
$$

This shows that f satisfies Definition 7 concerning $e\text{-}W^p$ or S^p-boundedness.
On the other hand, fix $\varepsilon > 0$ and $x+iy \in U_1$. Denote as l, L_0 and τ (with $y+\tau \in [0, l]$) the corresponding numbers above associated with the c-almost periodicity in the sense of Stepanov or equi-Weyl-(p, c)-almost periodicity for the value $\frac{|c|\varepsilon}{3}$. In this way, for every $\delta > 0$ we have

$$\begin{aligned}
&\left(L_0^{-1}\int_y^{y+L_0}|cf(x+i(t+\delta))-cf(x+it)|^p\,dt\right)^{\frac{1}{p}}\\
&\le\left(L_0^{-1}\int_y^{y+L_0}|cf(x+i(t+\delta))-f(x+i(t+\delta+\tau))|^p\,dt\right)^{\frac{1}{p}}\\
&\quad+\left(L_0^{-1}\int_y^{y+L_0}|f(x+i(t+\delta+\tau))-f(x+i(t+\tau))|^p\,dt\right)^{\frac{1}{p}}\\
&\quad+\left(L_0^{-1}\int_y^{y+L_0}|f(x+i(t+\tau))-cf(x+it)|^p\,dt\right)^{\frac{1}{p}}\\
&=\left(L_0^{-1}\int_{y+\delta}^{y+\delta+L_0}|f(x+i(s+\tau))-cf(x+is)|^p\,ds\right)^{\frac{1}{p}}\\
&\quad+\left(L_0^{-1}\int_{y+\tau}^{y+\tau+L_0}|f(x+i(t+\delta))-f(x+it)|^p\,dt\right)^{\frac{1}{p}}\\
&\quad+\left(L_0^{-1}\int_y^{y+L_0}|f(x+i(t+\tau))-cf(x+it)|^p\,dt\right)^{\frac{1}{p}}\\
&\le\frac{2\,|c|\,\varepsilon}{3}+\left(L_0^{-1}\int_0^{l+L_0}|f(x+i(t+\delta))-f(x+it)|^p\,dt\right)^{\frac{1}{p}}.
\end{aligned}$$

Now, for every $x=\operatorname{Re} z$ with $z\in U$, define the function $f_{n,x}(t):=f(x+i(t+\frac{1}{n}))\in L^p_{\mathrm{loc}}(\mathbb{R},\mathbb{C})$. It is clear that the sequence $\{f_{n,x}(t)\}_{n\ge1}$ converges pointwise to $f_x(t):=f(x+it)$ and, by the dominated convergence theorem in L^p-spaces (or as a consequence of the Brezis-Lieb theorem), also converges to f_x in the sense of L^p. Hence there exists $\delta_x>0$ such that for any $|\delta|<\delta_x$ it is accomplished that

$$\left(L_0^{-1}\int_0^{L_0+l}|f(x+i(t+\delta))-f(x+it)|^p\,dt\right)^{\frac{1}{p}}\le\frac{|c|\,\varepsilon}{3},$$

which yields that

$$\sup_{y\in\mathbb{R}}\left(L_0^{-1}\int_y^{y+L_0}|f(x+i(t+\delta))-f(x+it)|^p\,dt\right)^{\frac{1}{p}}\le\varepsilon,\ \forall\,|\delta|\le\delta_x.$$

This proves the result. □

We next show an analogous result for the W^p-uniform continuity under the following hypothesis, which is based on (5), with $f\in L^p_{li}(U,\mathbb{C})$ and $1\le p<\infty$:

For every $x=\operatorname{Re} z$, with $z\in U$, and $\varepsilon>0$ there exists $\delta_x>0$ such that

$$\left(\frac{1}{L}\int_0^L|f(x+i(t+h))-f(x+it)|^p\,dt\right)^{\frac{1}{p}}\le\varepsilon\text{ for all }|h|<\delta_x,\qquad(16)$$

uniformly with respect to $L \in (0, \infty)$.

Proposition 12 (W^p-uniform continuity of functions in $W_c^p(U, \mathbb{C})$ under hypothesis (16)) *Let $c \in \mathbb{C} \setminus \{0\}$ and $1 \leq p < \infty$. Consider a vertical strip of the form $U = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$, with $-\infty \leq \alpha < \beta \leq \infty$. If $f \in W_c^p(U, \mathbb{C})$ and it satisfies (16), then f is W^p-uniformly continuous in every vertical line in U.*

Proof Fix $\varepsilon > 0$ and take an arbitrary $U_1 = \{z \in \mathbb{C} : \alpha_1 \leq \operatorname{Re} z \leq \beta_1\} \subset U$ with $\alpha < \alpha_1 < \beta_1 < \beta$. As the set of the W_c^p-translation numbers belonging to $\frac{|c|\varepsilon}{3}$ associated with U_1 is relatively dense, there exists $l > 0$ such that every interval of length l contains at least one number of this set. In particular, to each fixed arbitrary complex number $x + iy \in U_1$ there corresponds such a translation number τ such that $y + \tau$ belongs to the interval $[0, l]$. Now, note that for any $\delta > 0$ and $t \in \mathbb{R}$, we have

$$\begin{aligned}
&|cf(x + i(t + \delta)) - cf(x + it)| \\
&\quad \leq |cf(x + i(t + \delta)) - f(x + i(t + \delta + \tau))| \\
&\quad\quad + |f(x + i(t + \delta + \tau)) - f(x + i(t + \tau))| \\
&\quad\quad + |f(x + i(t + \tau)) - cf(x + it)|
\end{aligned}$$

and

$$\begin{aligned}
&\int_y^{y+L} |f(x + i(t + \delta + \tau)) - f(x + i(t + \tau))|^p \, dt \\
&\quad = \int_{y+\tau}^{y+\tau+L} |f(x + i(t + \delta)) - f(x + it)|^p \, dt \\
&\quad \leq \int_0^{L+l} |f(x + i(t + \delta)) - f(x + it)|^p \, dt.
\end{aligned}$$

Thus, by the Minkowski inequality, it is accomplished that

$$\begin{aligned}
&\varlimsup_{L \to \infty} \sup_{y \in \mathbb{R}} \left(L^{-1} \int_y^{y+L} |cf(x + i(t + \delta)) - cf(x + it)|^p \, dt \right)^{\frac{1}{p}} \\
&\leq \frac{2}{3} |c| \varepsilon + \varlimsup_{L \to \infty} \left(L^{-1} \int_0^{L+l} |f(x + i(t + \delta)) - f(x + it)|^p \, dt \right)^{\frac{1}{p}}.
\end{aligned}$$

Now, by (16), there exists $\delta_x > 0$ such that for any $|\delta| < \delta_x$ it is accomplished that

$$\left((L + l)^{-1} \int_0^{L+l} |f(x + i(t + \delta)) - f(x + it)|^p \, dt \right)^{\frac{1}{p}} \leq \frac{|c| \varepsilon}{3},$$

uniformly with respect to $L > 0$, which yields that

$$\left(L^{-1} \int_0^{L+l} |f(x + i(t + \delta)) - f(x + it)|^p \, dt \right)^{\frac{1}{p}} \leq \left(\frac{L + l}{L} \right)^{\frac{1}{p}} \frac{|c| \varepsilon}{3},$$

for any $|\delta| < \delta_x$ uniformly with respect to $L > 0$. Consequently, we get

$$\overline{\lim_{L\to\infty}} \sup_{y\in\mathbb{R}} \left(L^{-1} \int_y^{y+L} |f(x+i(t+\delta)) - f(x+it)|^p \, dt \right)^{\frac{1}{p}} \le \varepsilon, \ \forall |\delta| < \delta_x,$$

which means, by Definition 8, that f is W^p-uniformly continuous in the vertical line $r_x = \{z \in \mathbb{C} : \operatorname{Re} z = x\} \subset U$. $\square$

Proposition 13 ($S^p_{|c|}$, e-$W^p_{|c|}$ and $W^p_{|c|}$-almost periodicity of the modulus function) *Let $1 \le p < \infty$ and $c \in \mathbb{C}\setminus\{0\}$. Consider a vertical strip of the form $U = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$, with $-\infty \le \alpha < \beta \le \infty$. Given $f : U \to \mathbb{C}$, take the notation $|f| : U \to [0,\infty)$ for the function defined as $|f|(z) := |f(z)|$ for all $z \in U$.*

(i) *If $f \in S^p_c(U,\mathbb{C})$, then $|f| \in S^p_{|c|}(U,[0,\infty))$.*
(ii) *If $f \in e$-$W^p_c(U,\mathbb{C})$, then $|f| \in e$-$W^p_{|c|}(U,[0,\infty))$.*
(iii) *If $f \in W^p_c(U,\mathbb{C})$, then $|f| \in W^p_{|c|}(U,[0,\infty))$.*

Proof Let $f \in e$-$W^p_c(U,\mathbb{C})$ (or $f \in S^p_c(U,\mathbb{C})$, respectively), and take $U_1 = \{z \in \mathbb{C} : \alpha_1 \le \operatorname{Re} z \le \beta_1\} \subset U$ with $\alpha < \alpha_1 < \beta_1 < \beta$. For every $\varepsilon > 0$ we can find $l > 0$ and $L_0 = L_0(\varepsilon) > 0$ (take $L_0 = 1$ for the case of $f \in S^p_c(U,\mathbb{C})$) such that any interval of length l contains at least a point τ satisfying

$$\sup_{x+iy\in U_1} \left(L_0^{-1} \int_y^{y+L_0} |f(x+i(t+\tau)) - cf(x+it)|^p \, dt \right)^{\frac{1}{p}} \le \varepsilon.$$

Now, if $c \in \mathbb{C}\setminus\{0\}$ and $x+iy \in U_1$, note that

$$\begin{aligned} &||f(x+i(y+\tau))| - |c||f(x+iy)|| \\ &\quad = ||f(x+i(y+\tau))| - |cf(x+iy)|| \le |f(x+i(y+\tau)) - cf(x+iy)|, \end{aligned}$$

which yields that

$$\begin{aligned} &\sup_{x+iy\in U_1} \left(L_0^{-1} \int_y^{y+L_0} ||f(x+i(t+\tau))| - |c||f(x+it)||^p \, dt \right)^{\frac{1}{p}} \\ &\quad \le \sup_{x+iy\in U_1} \left(L_0^{-1} \int_y^{y+L_0} |f(x+i(t+\tau)) - cf(x+it)|^p \, dt \right)^{\frac{1}{p}} \le \varepsilon. \end{aligned}$$

This proves (i) and (ii). Property (iii) is also proved analogously from the above inequality. $\square$

Proposition 14 (S^p_c, e-W^p_c and W^p_c-almost periodicity of the function $\check{f}(z) = f(\overline{z})$) *Let $c \in \mathbb{C}\setminus\{0\}$ and $1 \le p < \infty$, and consider a vertical strip of the form $U = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$, with $-\infty \le \alpha < \beta \le \infty$. Given $f : U \to \mathbb{C}$, consider the function $\check{f} : U \to \mathbb{C}$ defined as $\check{f}(z) := f(\overline{z})$ for all $z \in U$.*

(i) *$f \in S_c^p(U,\mathbb{C})$ if and only if $\check{f} \in S_{1/c}^p(U,\mathbb{C})$.*
(ii) *$f \in e\text{-}W_c^p(U,\mathbb{C})$ if and only if $\check{f} \in e\text{-}W_{1/c}^p(U,\mathbb{C})$.*
(iii) *$f \in W_c^p(U,\mathbb{C})$ if and only if $\check{f} \in W_{1/c}^p(U,\mathbb{C})$.*

Proof Let $f \in e\text{-}W_c^p(U,\mathbb{C})$ (or $f \in S_c^p(U,\mathbb{C})$, respectively), and take $U_1 = \{z \in \mathbb{C} : \alpha_1 \leq \operatorname{Re} z \leq \beta_1\} \subset U$ with $\alpha < \alpha_1 < \beta_1 < \beta$. For every $\varepsilon > 0$ we can find $l > 0$ and $L_0 = L_0(\varepsilon) > 0$ (take $L_0 = 1$ for the case of $f \in S_c^p(U,\mathbb{C})$) such that any interval of length l contains at least a point τ satisfying

$$\sup_{x+iy\in U_1}\left(L_0^{-1}\int_y^{y+L_0}|f(x+i(t+\tau))-cf(x+it)|^p\,dt\right)^{\frac{1}{p}} \leq |c|\varepsilon.$$

Then

$$\begin{aligned}
&\sup_{x+iy\in U_1}\left(L_0^{-1}\int_y^{y+L_0}|\check{f}(x+i(t+\tau))-\frac{1}{c}\check{f}(x+it)|^p\,dt\right)^{\frac{1}{p}}\\
&=\sup_{x+iy\in U_1}\left(\frac{1}{|c|^p\,L_0}\int_y^{y+L_0}|cf(x-i(t+\tau))-f(x-it)|^p\,dt\right)^{\frac{1}{p}}\\
&=\sup_{x+iy\in U_1}\left(\frac{1}{|c|^p\,L_0}\int_{-(y+\tau+L_0)}^{-(y+\tau)}|f(x+i(t+\tau))-cf(x+it)|^p\,dt\right)^{\frac{1}{p}}\\
&\leq\frac{|c|\,\varepsilon}{|c|}=\varepsilon,
\end{aligned}$$

which yields that every τ is in the set of the $(\varepsilon, \frac{1}{c})$-equi-Weyl translation numbers of $\check{f}$ associated with L_0 (or $(\varepsilon, \frac{1}{c})$-Stepanov translation numbers, with $L_0 = 1$), i.e. $\check{f} \in e\text{-}W_{1/c}^p(U,\mathbb{C})$ (or $\check{f} \in S_{1/c}^p(U,\mathbb{C})$, respectively). The converse is analogous. Furthermore, the case $f \in W_c^p(U,\mathbb{C})$ is also analogous. □

Lemma 2 (S_c^p, $e\text{-}W_c^p$ and W_c^p-almost periodicity iff $S_{1/c}^p$, $e\text{-}W_{1/c}^p$ and $W_{1/c}^p$-almost periodicity) *Let $c \in \mathbb{C}\setminus\{0\}$ and $1 \leq p < \infty$. Consider a vertical strip of the form $U = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$, with $-\infty \leq \alpha < \beta \leq \infty$. Then $S_c^p(U,\mathbb{C}) = S_{1/c}^p(U,\mathbb{C})$, $e\text{-}W_c^p(U,\mathbb{C}) = e\text{-}W_{1/c}^p(U,\mathbb{C})$ and $W_c^p(U,\mathbb{C}) = W_{1/c}^p(U,\mathbb{C})$.*

Proof Suppose that $f \in e\text{-}W_c^p(U,\mathbb{C})$ (or $f \in S_c^p(U,\mathbb{C})$, respectively), and take the vertical substrip $U_1 = \{z \in \mathbb{C} : \alpha_1 \leq \operatorname{Re} z \leq \beta_1\} \subset U$ with $\alpha < \alpha_1 < \beta_1 < \beta$. Given $\varepsilon > 0$, take L_0 the positive number corresponding to the case $|c|\,\varepsilon$ in the definition of equi-Weyl-(p, c)-almost periodicity (we can take $L_0 = 1$ for the case $S_c^p(U,\mathbb{C})$). Note that every τ in the set of the $(|c|\,\varepsilon, c)$-equi-Weyl translation numbers of f associated with L_0 and U_1 (or $(|c|\,\varepsilon, c)$-Stepanov translation numbers, with $L_0 = 1$) satisfies

$$
\begin{aligned}
&\sup_{x+iy\in U_1}\left(L_0^{-1}\int_y^{y+L_0}\left|f(x+i(t-\tau))-\frac{1}{c}f(x+it))\right|^p\,dt\right)^{\frac{1}{p}}\\
&=\sup_{x+iy\in U_1}\left(\frac{1}{|c|^p\,L_0}\int_y^{y+L_0}|cf(x+i(t-\tau))-f(x+it))|^p\,dt\right)^{\frac{1}{p}}\\
&\leq\frac{|c|\,\varepsilon}{|c|}=\varepsilon.
\end{aligned}
$$

This shows that $-\tau$ is in the set of the $(\varepsilon, \frac{1}{c})$-equi-Weyl translation numbers of f associated with L_0 and U_1 (or $(\varepsilon, \frac{1}{c})$-Stepanov translation numbers, resp.). Hence $f\in e\text{-}W^p_{1/c}(U,\mathbb{C})$ (or $f\in S^p_{1/c}(U,\mathbb{C})$, resp.). The converse is analogous.
The proof for the case $W^p_c(U,\mathbb{C})=W^p_{1/c}(U,\mathbb{C})$ is similar to the above reasoning. □

Proposition 15 (S^p_c, $e\text{-}W^p_c$, W^p_c-almost periodicity yields $S^p_{c^m}$, $e\text{-}W^p_{c^m}$, $W^p_{c^m}$-almost periodicity) *Let $c\in\mathbb{C}\setminus\{0\}$, $1\leq p<\infty$ and $m\in\mathbb{Z}\setminus\{0\}$. Consider a vertical strip of the form $U=\{z\in\mathbb{C}:\alpha<\operatorname{Re}z<\beta\}$, with $-\infty\leq\alpha<\beta\leq\infty$. Then $S^p_c(U,\mathbb{C})\subset S^p_{c^m}(U,\mathbb{C})$, $e\text{-}W^p_c(U,\mathbb{C})\subset e\text{-}W^p_{c^m}(U,\mathbb{C})$ and $W^p_c(U,\mathbb{C})\subset W^p_{c^m}(U,\mathbb{C})$.*

Proof Let $f\in e\text{-}W^p_c(U,\mathbb{C})$ (or $f\in S^p_c(U,\mathbb{C})$, respectively), and fix $m\in\mathbb{N}$ and $U_1=\{z\in\mathbb{C}:\alpha_1\leq\operatorname{Re}z\leq\beta_1\}\subset U$ with $\alpha<\alpha_1<\beta_1<\beta$. For every $\varepsilon>0$ we can find $l>0$ and $L_0=L_0(\varepsilon)>0$ (take $L_0=1$ for the case of $f\in S^p_c(U,\mathbb{C})$) such that any interval of length l contains at least a point τ satisfying

$$
\sup_{x+iy\in U_1}\left(L_0^{-1}\int_y^{y+L_0}|f(x+i(t+\tau))-cf(x+it)|^p\,dt\right)^{\frac{1}{p}}\leq\frac{\varepsilon}{\left(\sum_{j=0}^{m-1}|c|^{jp}\right)^{1/p}}.
$$

Also, for fixed $m\in\mathbb{N}$ and $x+iy\in U$, we have that

$$
\begin{aligned}
&f(x+i(y+m\tau))-c^mf(x+iy)\\
&=\sum_{j=0}^{m-1}c^j\left(f(x+i(y+(m-j)\tau))-cf(x+i(y+(m-j-1)\tau))\right).
\end{aligned}
$$

Therefore, for any $x+iy\in U_1$ we have

$$
\begin{aligned}
&L_0^{-1}\int_y^{y+L_0}|f(x+i(t+m\tau))-c^mf(x+it)|^p\,dt\\
&\leq\sum_{j=0}^{m-1}\frac{|c|^{jp}}{L_0}\int_y^{y+L_0}|f(x+i(t+(m-j)\tau))-cf(x+i(t+(m-j-1)\tau))|^p\,dt\\
&=\sum_{j=0}^{m-1}|c|^{jp}\,\frac{1}{L_0}\int_{y+(m-j-1)\tau}^{y+L_0+(m-j-1)\tau}|f(x+i(t+\tau))-cf(x+it)|^p\,dt\\
&\leq\sum_{j=0}^{m-1}|c|^{jp}\left(\frac{\varepsilon^p}{\sum_{j=0}^{m-1}|c|^{jp}}\right)=\varepsilon^p
\end{aligned}
$$

and

$$\sup_{x+iy\in U_1}\left(L_0^{-1}\int_y^{y+L_0}|f(x+i(t+m\tau))-c^m f(x+it)|^p\,dt\right)^{1/p}\le\varepsilon.$$

This means that $m\tau$ is an (ε, c^m)-equi-Weyl translation number (associated with the value L_0) of f (or an (ε, c^m)-Stepanov translation number of f), which means that $f\in e\text{-}W^p_{c^m}(U,\mathbb{C})$ (or $f\in S^p_{c^m}(U,\mathbb{C})$, respectively). Finally, by Lemma 2, the function f is also included in $e\text{-}W^p_{1/c^m}(U,\mathbb{C})$ (or in $S^p_{1/c^m}(U,\mathbb{C})$). The proof for the case $W^p_c(U,\mathbb{C})\subset W^p_{c^m}(U,\mathbb{C})$ is analogous. This proves the result. □

Proposition 16 (Some extra properties) *Let $c\in\mathbb{C}\setminus\{0\}$ and $1\le p<\infty$. Consider a vertical strip of the form $U=\{z\in\mathbb{C}:\alpha<\operatorname{Re} z<\beta\}$, with $-\infty\le\alpha<\beta\le\infty$. Suppose that $f\in S^p_c(U,\mathbb{C})$ ($f\in e\text{-}W^p_c(U,\mathbb{C})$ or $f\in W^p_c(U,\mathbb{C})$, respectively). Then it is satisfied*

- (i) *If $c\in\mathbb{R}$, then $\operatorname{Re} f$ and $\operatorname{Im} f$ are in $S^p_{c^m}(U,\mathbb{R})$ (in $e\text{-}W^p_{c^m}(U,\mathbb{R})$ or $W^p_{c^m}(U,\mathbb{R})$, respectively) for each $m\in\mathbb{Z}\setminus\{0\}$.*
- (ii) *$\lambda f\in S^p_{c^m}(U,\mathbb{C})$ ($\lambda f\in e\text{-}W^p_{c^m}(U,\mathbb{C})$ or $\lambda f\in W^p_{c^m}(U,\mathbb{C})$, respectively) for any $\lambda\in\mathbb{C}$ and for each $m\in\mathbb{Z}\setminus\{0\}$.*
- (iii) *If $\alpha\in\mathbb{R}$ and $\beta\in\mathbb{R}\setminus\{0\}$, then the functions $f^{i\alpha}(z):=f(z+i\alpha)$ and $f_{i\beta}(z):=f(x+i\beta y)$, with $z=x+iy\in U$, are in $S^p_{c^m}(U,\mathbb{C})$ (in $e\text{-}W^p_{c^m}(U,\mathbb{C})$ or $W^p_{c^m}(U,\mathbb{C})$, respectively) for each $m\in\mathbb{Z}\setminus\{0\}$.*
- (iv) *$\overline{f}\in S^p_{\overline{c}^m}(U,\mathbb{C})$ ($\overline{f}\in e\text{-}W^p_{\overline{c}^m}(U,\mathbb{C})$ or $\overline{f}\in W^p_{\overline{c}^m}(U,\mathbb{C})$, respectively) for each $m\in\mathbb{Z}\setminus\{0\}$.*
- (v) *The condition of boundedness of f (i.e. the existence of $M>0$ such that $|f(z)|\le M$, $\forall z\in U$) implies that the function $f^2(z):=(f(z))^2$ for all $z\in U$ is in $S^p_{c^{2k}}(U,\mathbb{C})$ ($f^2\in e\text{-}W^p_{c^{2k}}(U,\mathbb{C})$ or $f^2\in W^p_{c^{2k}}(U,\mathbb{C})$, respectively) for each $k\in\mathbb{Z}\setminus\{0\}$.*
- (vi) *If there exists $m_1>0$ such that $|f(z)|\ge m_1$ $\forall z\in U$, then the function $\frac{1}{f}(z):=\frac{1}{f(z)}$, $z\in U$, is in $S^p_{c^m}(U,\mathbb{C})$ ($\frac{1}{f}\in e\text{-}W^p_{c^m}(U,\mathbb{C})$ or $\frac{1}{f}\in W^p_{c^m}(U,\mathbb{C})$, respectively) for each $m\in\mathbb{Z}\setminus\{0\}$.*
- (vii) *Let $\{f_n(z)\}_{n\ge1}$ be a sequence of functions in $S^p_c(U,\mathbb{C})$ (in $e\text{-}W^p_c(U,\mathbb{C})$ or $W^p_c(U,\mathbb{C})$, respectively). If $\{f_n(z)\}_{n\ge1}$ is S^p-uniformly convergent ($e\text{-}W^p$-uniformly convergent or W^p-uniformly convergent, respectively) to a function $f:U\to\mathbb{C}$ (in $L^p_{\text{li}}(U,\mathbb{C})$), then f is in $S^p_c(U,\mathbb{C})$ (in $e\text{-}W^p_c(U,\mathbb{C})$ or $W^p_c(U,\mathbb{C})$, respectively).*

Proof Let $f\in e\text{-}W^p_c(U,\mathbb{C})$ (or $f\in S^p_c(U,\mathbb{C})$, respectively), and take $U_1=\{z\in\mathbb{C}:\alpha_1\le\operatorname{Re} z\le\beta_1\}\subset U$ with $\alpha<\alpha_1<\beta_1<\beta$. For every $\varepsilon>0$ we can find $l>0$ and $L_0=L_0(\varepsilon)>0$ (take $L_0=1$ for the case of $f\in S^p_c(U,\mathbb{C})$) such that any interval of length l contains at least a point τ (which is called an (ε, c)-equi-Weyl or Stepanov translation number of f) satisfying

$$\sup_{x+iy\in U_1}\left(L_0^{-1}\int_y^{y+L_0}|f(x+i(t+\tau))-cf(x+it)|^p\,dt\right)^{\frac{1}{p}}\le\varepsilon.$$

(Although the value $\varepsilon > 0$ could be changed in the different paragraphs, without loss of generality we will also denote as L_0 the positive real number associated with the translation number).
If $f \in W_c^p(U, \mathbb{C})$, for every $\varepsilon > 0$ and every reduced strip of the form $U_1 = \{z \in \mathbb{C} : \alpha_1 \leq \operatorname{Re} z \leq \beta_1\} \subset U$ (with $\alpha < \alpha_1 < \beta_1 < \beta$) we can find $l > 0$ such that any interval of length l contains at least a point τ satisfying

$$\overline{\lim_{L\to\infty}} \sup_{x+iy\in U_1} \left(L^{-1} \int_y^{y+L} |f(x+i(t+\tau)) - cf(x+it)|^p \, dt \right)^{\frac{1}{p}} \leq \varepsilon.$$

(i) As $c \in \mathbb{R}$ and $|\operatorname{Re} w| \leq |w|$ for all $w \in \mathbb{C}$, by above every τ in the set of the (ε, c)-equi-Weyl translation numbers of f (or in the set of the (ε, c)-Stepanov translation numbers of f, respectively) satisfies

$$\begin{aligned} &\sup_{x+iy\in U_1} \left(L_0^{-1} \int_y^{y+L_0} |\operatorname{Re} f(x+i(t+\tau)) - c \operatorname{Re} f(x+it)|^p \, dt \right)^{\frac{1}{p}} \\ &= \sup_{x+iy\in U_1} \left(L_0^{-1} \int_y^{y+L_0} |\operatorname{Re} (f(x+i(t+\tau)) - cf(x+it))|^p \, dt \right)^{\frac{1}{p}} \\ &\leq \sup_{x+iy\in U_1} \left(L_0^{-1} \int_y^{y+L_0} |f(x+i(t+\tau)) - cf(x+it)|^p \, dt \right)^{\frac{1}{p}} \leq \varepsilon, \end{aligned}$$

which yields that $\operatorname{Re} f \in e\text{-}W_c^p(U, \mathbb{C})$ (or $\operatorname{Re} f \in S_c^p(U, \mathbb{C})$, respectively). The case of the W_c^p-almost periodicity is analogous, as also is the case $\operatorname{Im} f$. Finally, Proposition 15 assures our result.

(ii) If $\lambda \in \mathbb{C}\setminus\{0\}$ (the case $\lambda = 0$ is trivial), note that every τ in the set of the $(\frac{\varepsilon}{|\lambda|}, c)$-equi-Weyl translation numbers (or the set of the $(\frac{\varepsilon}{|\lambda|}, c)$-Stepanov translation numbers, respectively) satisfies

$$\begin{aligned} &\sup_{x+iy\in U_1} \left(L_0^{-1} \int_y^{y+L_0} |\lambda f(x+i(t+\tau)) - \lambda cf(x+it)|^p \, dt \right)^{\frac{1}{p}} \\ &= \sup_{x+iy\in U_1} \left(|\lambda|^p L_0^{-1} \int_y^{y+L_0} |f(x+i(t+\tau)) - cf(x+it)|^p \, dt \right)^{\frac{1}{p}} \leq |\lambda| \frac{\varepsilon}{|\lambda|} = \varepsilon, \end{aligned}$$

which yields that $\lambda f \in e\text{-}W_c^p(U, \mathbb{C})$ (or $\lambda f \in S_c^p(U, \mathbb{C})$, respectively). Finally, by Proposition 15, we also have that $\lambda f \in e\text{-}W_{c^m}^p(U, \mathbb{C})$ (or $\lambda f \in S_{c^m}^p(U, \mathbb{C})$, respectively) for each $m \in \mathbb{Z} \setminus \{0\}$. The case $\lambda f \in W_{c^m}^p(U, \mathbb{C})$ is analogous.

(iii) Note that every τ in the set of the (ε, c)-equi-Weyl translation numbers of f (or in the set of the (ε, c)-Stepanov translation numbers of f, respectively) satisfies

$$\sup_{x+iy\in U_1}\left(L_0^{-1}\int_y^{y+L_0}|f^{i\alpha}(x+i(t+\tau))-cf^{i\alpha}(x+it)|^p\,dt\right)^{\frac{1}{p}}$$
$$=\sup_{x+iy\in U_1}\left(L_0^{-1}\int_y^{y+L_0}|f(x+i(t+\tau+\alpha))-cf(x+i(t+\alpha))|^p\,dt\right)^{\frac{1}{p}}$$
$$=\sup_{x+iy\in U_1}\left(L_0^{-1}\int_{y+\alpha}^{y+\alpha+L_0}|f(x+i(t+\tau))-cf(x+it)|^p\,dt\right)^{\frac{1}{p}}\le\varepsilon,$$

which yields that τ is also an (ε, c)-equi-Weyl translation number of $f^{i\alpha}(z)$ (or an (ε, c)-Stepanov translation number of $f^{i\alpha}(z)$, respectively).
On the other hand, if $\beta > 0$ and τ is in the set of the (ε, c)-equi-Weyl translation numbers of f associated with the value βL_0 (or in the set of the (ε, c)-Stepanov translation numbers of f associated with β, respectively), it is also accomplished that

$$\sup_{x+iy\in U_1}\left(L_0^{-1}\int_y^{y+L_0}\left|f_{i\beta}\left(x+i(t+\frac{\tau}{\beta})\right)-cf_{i\beta}(x+it)\right|^p\,dt\right)^{\frac{1}{p}}$$
$$=\sup_{x+iy\in U_1}\left(L_0^{-1}\int_y^{y+L_0}\left|f\left(x+i\beta(t+\frac{\tau}{\beta})\right)-cf(x+i\beta t)\right|^p\,dt\right)^{\frac{1}{p}}$$
$$=\sup_{x+iy\in U_1}\left((\beta L_0)^{-1}\int_{\beta y}^{\beta y+\beta L_0}|f(x+i(t+\tau))-cf(x+it)|^p\,dt\right)^{\frac{1}{p}}\le\varepsilon,$$

which yields that $\frac{\tau}{\beta}$ is an (ε, c)-equi-Weyl translation number of $f_{i\beta}(z)$ associated with L_0 (or an (ε, c)-Stepanov translation number, respectively). The case $\beta < 0$ is analogous to that of the proof of Proposition 7, point ii).
Hence the functions $f^{i\alpha}(z)$ and $f_{i\beta}(z)$ are in e-$W_c^p(U, \mathbb{C})$ (or $S_c^p(U, \mathbb{C})$, respectively) and, by Proposition 15, $f^{i\alpha}(z)$ and $f_{i\beta}(z)$ are in e-$W_{c^m}^p(U, \mathbb{C})$ (or $S_{c^m}^p(U, \mathbb{C})$, respectively) for each $m \in \mathbb{Z}\setminus\{0\}$. The case $W_{c^m}^p(U, \mathbb{C})$ is analogous.

(iv) The result is immediately deduced from the fact that any $\tau \in \mathbb{R}$ satisfies

$$\left|\overline{f}(z+i\tau)-\overline{c}\overline{f}(z)\right|^p=|f(z+i\tau)-cf(z)|^p\ \ \forall z\in U.$$

Proposition 15 completes the proof.

(v) Suppose the existence of $M > 0$ such that $|f(z)| \le M,\ \forall z \in U$. Note that every τ in the set of the $(\frac{\varepsilon}{M(1+|c|)}, c)$-equi-Weyl translation numbers of f (or the set of the $(\frac{\varepsilon}{M(1+|c|)}, c)$-Stepanov translation numbers of f, respectively) satisfies

$$
\begin{aligned}
&\sup_{x+iy\in U_1}\left(L_0^{-1}\int_y^{y+L_0}|f^2(x+i(t+\tau))-c^2f^2(x+it)|^p\,dt\right)^{\frac{1}{p}}\\
&=\sup_{x+iy\in U_1}\left(L_0^{-1}\int_y^{y+L_0}|(f(x+i(t+\tau))-cf(x+it))\right.\\
&\quad\left.\times\ (f(x+i(t+\tau))+cf(x+it))|^p\,dt\right)^{\frac{1}{p}}\\
&\leq M(1+|c|)\sup_{x+iy\in U_1}\left(L_0^{-1}\int_y^{y+L_0}|f(x+i(t+\tau))-cf(x+it)|^p\,dt\right)^{\frac{1}{p}}\\
&\leq M(1+|c|)\,\frac{\varepsilon}{M(1+|c|)}=\varepsilon.
\end{aligned}
$$

This shows that $f^2\in e\text{-}W^p_{c^2}(U,\mathbb{C})$ (or $f^2\in S^p_{c^2}(U,\mathbb{C})$, respectively). Finally, we deduce from Proposition 15 that $f^2\in e\text{-}W^p_{c^{2k}}(U,\mathbb{C})$ (or $f^2\in S^p_{c^{2k}}(U,\mathbb{C})$, respectively) for each $k\in\mathbb{Z}\backslash\{0\}$. The case $f^2\in W^p_{c^{2k}}(U,\mathbb{C})$ is analogous.

(vi) Suppose the existence of $m_1>0$ such that $|f(z)|\geq m_1>0\ \forall z\in U$. Note that every τ in the set of the $(\varepsilon\,|c|\,m_1^2,c)$-equi-Weyl translation numbers of f (or in the set of the $(\varepsilon\,|c|\,m_1^2,c)$-Stepanov translation numbers of f, respectively) satisfies

$$
\begin{aligned}
&\sup_{x+iy\in U_1}\left(L_0^{-1}\int_y^{y+L_0}\left|\frac{1}{f(x+i(t+\tau))}-\frac{1}{c}\,\frac{1}{f(x+it)}\right|^p\,dt\right)^{\frac{1}{p}}\\
&=\sup_{x+iy\in U_1}\left(L_0^{-1}\int_y^{y+L_0}\left|\frac{f(x+i(t+\tau))-cf(x+it)}{cf(x+it)f(x+i(t+\tau))}\right|^p\,dt\right)^{\frac{1}{p}}\\
&\leq\frac{\varepsilon\,|c|\,m_1^2}{|c|\,m_1^2}=\varepsilon,
\end{aligned}
$$

which proves that the multiplicative inverse, or reciprocal, of f is in $e\text{-}W^p_{1/c}(U,\mathbb{C})$ (or in $S^p_{1/c}(U,\mathbb{C})$, respectively). Hence, by Proposition 15, we conclude that $\frac{1}{f}\in e\text{-}W^p_{c^m}(U,\mathbb{C})$ (or $\frac{1}{f}\in S^p_{c^m}(U,\mathbb{C})$, respectively) for each $m\in\mathbb{Z}\setminus\{0\}$. The case $\frac{1}{f}\in W^p_{c^m}(U,\mathbb{C})$ is analogous.

(vii) By $e\text{-}W^p$-uniformly convergence (or S^p-uniformly convergence), we know that given $\varepsilon>0$, there exist $L_0=L_0(\varepsilon)$ (take $L_0=1$ for the case of S^p-uniformly convergence) and $n_0\in\mathbb{N}$ such that

$$
\sup_{x+iy\in U_1}\left(L_0^{-1}\int_y^{y+L_0}|f_n(x+it)-f(x+it)|^p\,dt\right)^{\frac{1}{p}}\leq\min\left\{\frac{\varepsilon}{3},\frac{\varepsilon}{3\,|c|}\right\}\ \forall n\geq n_0.
$$

Then every τ in the set of the $(\frac{\varepsilon}{3},c)$-equi-Weyl translation numbers of $f_{n_0}(z)$ (or in the set of the $(\frac{\varepsilon}{3},c)$-Stepanov translation numbers of $f_{n_0}(z)$, respectively) satisfies

$$\begin{aligned}
&\sup_{x+iy\in U_1}\left(L_0^{-1}\int_y^{y+L_0}|f(x+i(t+\tau))-cf(x+it)|^p\,dt\right)^{\frac{1}{p}}\\
&\le \sup_{x+iy\in U_1}\left(L_0^{-1}\int_y^{y+L_0}|f(x+i(t+\tau))-f_{n_0}(x+i(t+\tau))|^p\,dt\right)^{\frac{1}{p}}+\\
&\quad+\sup_{x+iy\in U_1}\left(L_0^{-1}\int_y^{y+L_0}|f_{n_0}(x+i(t+\tau))-cf_{n_0}(x+it)|^p\,dt\right)^{\frac{1}{p}}+\\
&\quad+\sup_{x+iy\in U_1}\left(L_0^{-1}\int_y^{y+L_0}|cf_{n_0}(x+it)-cf(x+it)|^p\,dt\right)^{\frac{1}{p}}\\
&\le \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+|c|\,\frac{\varepsilon}{3\,|c|}=\varepsilon,
\end{aligned}$$

which yields that $f\in e\text{-}W_c^p(U,\mathbb{C})$ (or $f\in S_c^p(U,\mathbb{C})$, respectively). The case of the W^p-uniformly convergence is analogous. □

The following two results show some conditions under which the sets $S_c^p(U,\mathbb{C})$, $e\text{-}W_c^p(U,\mathbb{C})$ and $W_c^p(U,\mathbb{C})$ are included in $S^p(U,\mathbb{C})$, $e\text{-}W^p(U,\mathbb{C})$ and $W^p(U,\mathbb{C})$, respectively.

Proposition 17 (Connection with S^p, $e\text{-}W^p$ and W^p almost periodicity) *Let $1\le p<\infty$ and $c\in\mathbb{C}\setminus\{0\}$ such that $\frac{\arg c}{2\pi}\in\mathbb{Q}$. Consider a vertical strip of the form $U=\{z\in\mathbb{C}:\alpha<\operatorname{Re} z<\beta\}$, with $-\infty\le\alpha<\beta\le\infty$. Then*

$$S_c^p(U,\mathbb{C})\subset S_{|c|^q}^p(U,\mathbb{C}),\ \ e\text{-}W_c^p(U,\mathbb{C})\subset e\text{-}W_{|c|^q}^p(U,\mathbb{C})\ \textit{and}\ W_c^p(U,\mathbb{C})\subset W_{|c|^q}^p(U,\mathbb{C}),$$

where $q\in\mathbb{N}$ is so that $\frac{\arg c}{2\pi}=\frac{r}{q}$ for a certain $r\in\mathbb{Z}$ such that $\gcd(r,q)=1$. In particular, under the same condition, the case $|c|=1$ yields the inclusions $S_c^p(U,\mathbb{C})\subset S^p(U,\mathbb{C})$, $e\text{-}W_c^p(U,\mathbb{C})\subset e\text{-}W^p(U,\mathbb{C})$ and $W_c^p(U,\mathbb{C})\subset W^p(U,\mathbb{C})$.

Proof Put $\arg c=\frac{2\pi r}{q}$ with $r\in\mathbb{Z}$ and $q\in\mathbb{N}$ so that $\gcd(r,q)=1$. Then

$$c^q=|c|^q e^{qi\arg c}=|c|^q e^{2r\pi i}=|c|^q.$$

Now, if $f\in S_c^p(U,\mathbb{C})$, $f\in e\text{-}W_c^p(U,\mathbb{C})$ or $f\in W_c^p(U,\mathbb{C})$, Proposition 15 assures that $f\in S_{c^q}^p(U,\mathbb{C})=S_{|c|^q}^p(U,\mathbb{C})$, $f\in e\text{-}W_{c^q}^p(U,\mathbb{C})=e\text{-}W_{|c|^q}^p(U,\mathbb{C})$ or $f\in W_{c^q}^p(U,\mathbb{C})=W_{|c|^q}^p(U,\mathbb{C})$, and the result holds. □

Proposition 18 (Connection with S^p, $e\text{-}W^p$ and W^p-almost periodicity) *Let $1\le p<\infty$ and $c\in\mathbb{C}\setminus\{0\}$ such that $\frac{\arg c}{\pi}\notin\mathbb{Q}$ and $|c|=1$. Consider a vertical strip of the form $U=\{z\in\mathbb{C}:\alpha<\operatorname{Re} z<\beta\}$, with $-\infty\le\alpha<\beta\le\infty$. Then $S_c^p(U,\mathbb{C})\subset S^p(U,\mathbb{C})$ and $e\text{-}W_c^p(U,\mathbb{C})\subset e\text{-}W^p(U,\mathbb{C})$. Furthermore, if f is a function in $W_c^p(U,\mathbb{C})$ which is $e\text{-}W^p$-bounded in every vertical substrip in U, then f is also in $W^p(U,\mathbb{C})$.*

Proof Let $f \in S_c^p(U, \mathbb{C})$ and take $U_1 = \{z \in \mathbb{C} : \alpha_1 \leq \operatorname{Re} z \leq \beta_1\} \subset U$ with $\alpha < \alpha_1 < \beta_1 < \beta$. By Proposition 11 (see also Definition 7), there exists $M > 0$ such that

$$\sup_{x+iy \in U_1} \left(\int_y^{y+1} |f(x+it)|^p \, dt \right)^{\frac{1}{p}} \leq M.$$

Suppose that $\arg c$ is not a rational multiple of π, which yields that $e^{ni \arg c} \neq 1$ for all $n \in \mathbb{N}$. Given $\varepsilon > 0$, choose $n_1, n_2 \in \mathbb{N}$ (with $n_1 \neq n_2$) such that $\left| e^{n_2 i \arg c} - e^{n_1 i \arg c} \right| < \frac{\varepsilon}{2M}$ (note that the existence of n_1 and n_2 is assured by virtue of the fact that $\left\{ e^{ni \arg c} : n \in \mathbb{N} \right\} \subset \{c \in \mathbb{C} : |c| = 1\}$ and that the length of the unit circumference is finite). Hence

$$\left| c^{n_2 - n_1} - 1 \right| = \left| e^{(n_2 - n_1) i \arg c} - 1 \right| = \left| e^{n_2 i \arg c} - e^{n_1 i \arg c} \right| < \frac{\varepsilon}{2M}.$$

Moreover, by Proposition 15, it is accomplished that $f \in S^p_{c^{n_2 - n_1}}(U, \mathbb{C})$. Consequently, given $\varepsilon > 0$, every τ in the set of $(\frac{\varepsilon}{2}, c^{n_2 - n_1})$-Stepanov translation numbers of f satisfies

$$\begin{aligned}
&\sup_{x+iy \in U_1} \left(\int_y^{y+1} |f(x+i(t+\tau)) - f(x+it)|^p \, dt \right)^{\frac{1}{p}} \\
&\leq \sup_{x+iy \in U_1} \left(\int_y^{y+1} |f(x+i(t+\tau)) - c^{n_2 - n_1} f(x+it)|^p \, dt \right)^{\frac{1}{p}} \\
&\quad + \sup_{x+iy \in U_1} \left(\int_y^{y+1} |c^{n_2 - n_1} f(x+it) - f(x+it)|^p \, dt \right)^{\frac{1}{p}} \\
&\leq \frac{\varepsilon}{2} + |c^{n_2 - n_1} - 1| \sup_{x+iy \in U_1} \left(\int_y^{y+1} |f(x+it)|^p \, dt \right)^{\frac{1}{p}} \leq \varepsilon,
\end{aligned}$$

which proves the inclusion $S_c^p(U, \mathbb{C}) \subset S^p(U, \mathbb{C})$. The case $e\text{-}W_c^p(U, \mathbb{C}) \subset e\text{-}W^p(U, \mathbb{C})$ is similar. To prove the last statement, take U_1 an arbitrary vertical substrip in U and note that, by Definition 7, the condition of $e\text{-}W^p$-boundedness in U_1 of a function $f \in L^p_{\mathrm{li}}(U, \mathbb{C})$ yields the existence of two positive numbers L_0 and M such that $\sup\limits_{x+iy \in U_1} \left(\frac{1}{L_0} \int_y^{y+L_0} |f(x+it)|^p \, dt \right)^{\frac{1}{p}} \leq M$, which leads to $f \in W^p(U, \mathbb{C})$ in a manner analogous to the above reasoning. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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