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A Peak Set with a Maximal Hausdorff Dimension on Each Slice

Piotr Kot[1]

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Abstract
We consider a bounded balanced strictly convex domain $\Omega \subset \mathbb{C}^d$ with C^2 boundary. Then there exists a peak set E with Hausdorff dimension equal to 1 on each slice. In particular E has maximal possible Hausdorff dimension equal to $2d - 1$.

Keywords Peak set · Maximum modulus set · Inner function · Hausdorff dimension

Mathematics Subject Classification 32E30 · 32E35

1 Introduction

Let Ω be a bounded, convex and balanced domain with C^2 boundary. Each $\eta \in \partial\Omega$ sets out the slice $\eta\mathbb{D}$ where $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ is the unit disc.

1.1 Historical Background

If we have a compact set $K \subset \partial\Omega$ and $f \in A(\Omega)$ such that $|f| < 1$ on $\overline{\Omega}\backslash K$ and $f = 1$ on K we say that K is a peak set for[1] $A(\Omega)$ and f is a peak function for K. It is possible to generalize this concept to a peak interpolation set:

If for a given continuous function g on K there exists $f \in A(\Omega)$ such that $f = g$ on K and $\|f\|_\infty \leq \|g\|_\infty$ then we say that K is a peak interpolation set.

Crucial sources of information about peak sets can be found in the following works: [4, 9, 11, 13].

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✉ Piotr Kot
Piotr.Kot@agh.edu.pl

1 Faculty of Applied Mathematics, AGH University of Science and Technology, Aleja Mickiewicza 30, 30-059 Kraków, Poland

[1] As usually, by $A(\Omega)$ we denote a space of all functions which are holomorphic in Ω and continuous to the boundary.

Peak sets on the unit disc $\mathbb{D}$ are extensively studied. The most important result in this area is the Fatou-Rudin-Carleson theorem (see [1, 2, 8, 15] and [17, p 205]) which states that the classes of peak and peak interpolations sets for $A(\mathbb{D})$ coincide and are precisely the subsets of Lebesgue measure zero in $\partial\mathbb{D}$.

However, as one can expect the situation in $\mathbb{C}^d$ is not so obvious. There exist non trivial examples of strange behavior of peak sets in several complex variables.

The regular case[2] Rudin described in [14].

Topologically, peak sets are small in strictly pseudoconvex domains. The real topological dimension of a peak set is not bigger than $d - 1$ (see [16]). In particular, a peak set must have an empty interior. But from the measure-theoretic point of view peak sets no longer have to be so small. Tumanov [18] constructed a peak set of Hausdorff dimension 2.5 in the unit sphere $\mathbb{B}^3 \subset \mathbb{C}^3$. Stensönes Henriksen proved [5] that every strictly pseudoconvex domain with C^∞ boundary in $\mathbb{C}^d$ has a peak set with a Hausdorff dimension $2d - 1$. Moreover, If Ω is a circular, bounded, strictly convex domain with C^2 boundary it is possible to construct a peak set $K \subset \partial\Omega$ which intersects all the circles in $\partial\Omega$ with the center at zero (see [8]). These examples indicate that the question of complete characterization of all peak sets for $A(\Omega)$ for a strictly pseudoconvex domain is far from trivial.

It is known nowadays that every peak set K is also a peak interpolation set, which implies that any compact $T \subset K$ is also a peak set (see [13, p. 206]). Moreover, any subset of euclidean space of Hausdorff dimension m contains a compact subset of Hausdorff dimension β for each $0 \leq \beta \leq m$. (see [3, Theorem 2.10.47]). Therefore the peak sets mentioned in [5, 8, 18] contain peak sets for any lower Hausdorff dimension. However, if we choose a compact set $K \subset \partial\Omega$ then it is usually impossible to construct a peak set as in [5, 8, 18] inside K.

1.2 Main Result

In this paper we give an example (see Theorem 3.1) of peak set $E \subset \partial\Omega$ such that $\eta\partial\mathbb{D} \cap E$ has Hausdorff dimension equal to 1 for all $\eta \in \partial\Omega$. In fact our peak set has maximal possible Hausdorff dimension on $\partial\Omega$ (see Remark 3.2).

Our inspiration is a Henriksen's result [5]. Henriksen's method is based on the $\overline{\partial}$ problem and requires C^∞ boundary of a considered domain. Our methods do not require the use of a theory related to the $\overline{\partial}$ problem. We consider domains with only C^2 boundary. Note that our peak set crosses precisely all circles with the center at zero and has maximal possible Hausdorff dimension on each slice. Even in the case of the unit ball our result is new. Some generalizations are possible (see Remark 3.3).

1.3 Applications

Assume that $K \subset \partial\Omega$ is a peak set for $A(\Omega)$, then a set K has the following properties (see [13, 19]):

(1) there exists $f \in A(\Omega)$ such that $K = f^{-1}(0)$ (K is a zero set).

[2] The paper [14] is in fact generalization Fatou-Rudin-Carleson theorem to a higher dimension for C^1 peak sets.

(2) if $T \subset K$ is compact then T is a peak set.
(3) $A(\Omega)|_K = C(K)$.
(4) if g is a nonzero, continuous function on K then there exists $f \in A(\Omega)$ such that $f = g$ on K and $|f(z)| < \|g\|_K$ for $z \in \overline{\Omega} \backslash K$ (K is a peak interpolation set).
(5) $|\mu|(K) = 0$ for all $\mu \in A^{\perp}(\partial\Omega)$.
(6) if $G \in A(\Omega)$ is a peak function for K, then $F = \exp(i \log(1 - G))$ is a bounded holomorphic function on Ω with no limit along any curve in Ω that ends at a point of K.

1.4 Organization of the Paper

We start our paper by describing some property of compact set $E \subset [0, 1]$ that guarantees Hausdorff dimension equal to 1 (see Lemma 2.2). In fact it is enough to divide recursively into n_i equal intervals and choose inside smaller intervals with length controlled by θ_i parameter. Then we will show that combination of special homogeneous polynomials retain this property (see Lemma 2.5). Next we amplify the big ($\Re Q_j \geq \alpha$) real part values of constructed polynomials Q_j and we suppress values that are too small ($\Re Q_j \leq \frac{\alpha}{2}$) (see Theorem 3.1).

1.5 Notations

We use the following notation:

For $\alpha > 0$ and $\delta > 0$ and countable family $\mathfrak{U}$ of open sets $U \in \mathfrak{U}$ with diameter $d(U)$ we define

$$h^{\alpha}(\mathfrak{U}) := \sum_{U \in \mathfrak{U}} d(U)^{\alpha}$$

Now we have Hausdorff measure:

$$H^{\alpha}_{\delta}(E) = \inf_{\mathfrak{U}} \left\{ h^{\alpha}(\mathfrak{U}) : E \subset \bigcup_{U \in \mathfrak{U}, d(U) < \delta} U \right\},$$

where the infimum is taken over all countable covers $\mathfrak{U}$ of E by open sets. We can define Hausdorff dimension

$$\dim_H(E) := \sup\{\alpha \geq 0 : H^{\alpha}(E) > 0\} = \inf\{\alpha \geq 0 : H^{\alpha}(E) = 0\},$$

where $H^{\alpha}(E) = \lim_{\delta \to 0} H^{\alpha}_{\delta}(E)$.

For a given $z \in \partial\Omega$ and $0 \leq a \leq b \leq 2\pi$ let us denote

$$z_{[a,b]} = z \exp(2\pi i [a, b])$$
$$z_{[a]} = z \exp(2\pi i a)$$

2 Preliminary Estimates

2.1 Hausdorff Dimension

We use the following Tumanov's Lemma to calculate Hausdorff dimension:

Lemma 2.1 [18, Lemma 3] *Let* $1 > \alpha > 0$ *and* $c > 0$. *Let* $\{r_j\}_{j=0}^{\infty}$ *be a sequence of positive numbers decreasing to zero. For a given* $j \in \mathbb{N}$ *let* ω_j *be a subset of* $[0, 1]$. *Let us define a sum of intervals* $E_j = \bigcup_{x\in\omega_j}[x, x + r_j]$.

Suppose that:

(1) *In each interval* $[x, x + r_{j-1}]$ *of* E_{j-1} *there are at least* $\left(\frac{r_{j-1}}{r_j}\right)^{\alpha}$ *intervals of* E_j.
(2) *Distance between intervals* $[x, x + r_j]$ *and* $[y, y + r_j]$ *is at least*

$$\rho_j := c r_{j-1}\left(\frac{r_j}{r_{j-1}}\right)^{\alpha}$$

for $x \neq y \in \omega_j$.

Then $H^{\alpha}(E) > 0$ *where* $E = \bigcap_{j\in\mathbb{N}} E_j$.

In fact our crucial property of E gives us a so large set that it is impossible to use directly Tumanov's Lemma. Fortunately, we can choose a subset of E for which we can easily use this method.

Lemma 2.2 *Let* $\{n_j\}$ *be a sequence of natural numbers and* $\{\theta_j\}$ *be a sequence of positive numbers such that* $n_j > 4$, $\theta_j > 1$ *for* $j \geq 2$ *and* $\lim_{j\to\infty}\frac{\theta_j}{n_j^{\alpha}} = 0$ *for* $\alpha \in (0, 1)$. *Let* $E_1 = [0, 1]$, $r_1 = 1$ *and* $r_j = \frac{r_{j-1}}{\theta_j n_j}$. *Suppose that a decreasing sequence of compact sets* $\{E_j\}$ *has the following property:*

If $[z, z + r_{j-1}] \subset E_{j-1}$ *and* $k \in \{0, \dots, n_j - 1\}$ *then there exists* $x_{z,k}$ *such that*

$$[x_{z,k}, x_{z,k} + r_j] \subset \left[z + \frac{k r_{j-1}}{n_j}, z + \frac{(k+1) r_{j-1}}{n_j}\right] \cap E_j.$$

Then $\dim_H(E) = 1$ *where* $E = \bigcap_{j\in\mathbb{N}} E_j$.

Proof Let $\alpha \in (0, 1)$ and $\tilde{\omega}_1 = \{0\}$. Let $\tilde{\omega}_j$ be a maximal possible subset of

$$\bigcup_{z\in\tilde{\omega}_{j-1}} \bigcup_{k=0}^{n_j-1} \{x_{z,k}\}$$

such that distance between intervals $[x, x + r_j]$ and $[y, y + r_j]$ is at least

$$\rho_j := \frac{1}{4} r_{j-1}\left(\frac{r_j}{r_{j-1}}\right)^{\alpha}$$

for $x \neq y \in \tilde{\omega}_j$. Let us observe that if $z_1 \neq z_2 \in \tilde{\omega}_{j-1}$ then distance between $[z_1, z_1 + r_{j-1}]$ and $[z_2, z_2 + r_{j-1}]$ is at least $\rho_{j-1} > \rho_j$, so without lost of the generality we can assume that $x_{z,0} \in \tilde{\omega}_j$ for all $z \in \tilde{\omega}_{j-1}$.

For a given $\tilde{\omega}_{j-1}$ let us denote $\tilde{E}_{j-1} = \bigcup_{z\in\tilde{\omega}_{j-1}}[z, z + r_{j-1}]$.

Let $z \in \tilde{\omega}_{j-1}$ and

$$\tilde{N}_j = \#\left\{x_{z,k} \in \tilde{\omega}_j\right\}.$$

Let $[a, b] \subset [z, z + r_{j-1}] \subset \tilde{E}_{j-1}$ with $b - a \geq 2\rho_j + 4\frac{r_{j-1}}{n_j}$. Since length of

$$\left[a + \rho_j + r_j, b - \rho_j - 2r_j\right]$$

is larger than $\frac{r_{j-1}}{n_j}$ there exists $k_0 \in \{0, \dots, n_j - 1\}$ such that

$$\left[z + \frac{k_0 r_{j-1}}{n_j}, z + \frac{(k_0 + 1)r_{j-1}}{n_j}\right] \subset \left(a + \rho_j + r_j, b - \rho_j - 2r_j\right),$$

so there exists x_{z,k_0} with $\left[x_{z,k_0}, x_{z,k_0} + r_j\right] \subset \left(a + \rho_j + r_j, b - \rho_j - 2r_j\right)$. But $\tilde{\omega}_j$ is maximal possible subset of

$$\bigcup_{z\in\tilde{\omega}_{j-1}} \bigcup_{k=0}^{n_j-1} \{x_{z,k}\}$$

so there exists $x_{z,k_1} \in \tilde{\omega}_j$ with $\left|x_{z,k_0} - x_{z,k_1}\right| \leq \rho_j + r_j$, which implies that

$$\left[x_{z,k_1}, x_{z,k_1} + r_j\right] \subset (a, b),$$

so (a, b) contains at least one element of $\tilde{\omega}_j$.

Since $x_{z,0} \in \tilde{\omega}_j$ we have:

$$\begin{aligned}
\tilde{N}_j &\geq \frac{r_{j-1}\left(1 - \frac{1}{n_j}\right)}{2\rho_j + 4\frac{r_{j-1}}{n_j}} \geq \frac{\frac{3}{4}r_{j-1}}{\frac{1}{2}r_{j-1}\left(\frac{r_j}{r_{j-1}}\right)^{\alpha} + \frac{4r_{j-1}}{n_j}} = \frac{3}{2\left(\theta_j n_j\right)^{-\alpha} + 16\left(n_j\right)^{-1}} \\
&\geq \frac{3\left(\theta_j n_j\right)^{\alpha}}{2 + 16\theta_j^{\alpha}\left(n_j\right)^{\alpha-1}} \geq \left(\theta_j n_j\right)^{\alpha} = \left(\frac{r_{j-1}}{r_j}\right)^{\alpha}
\end{aligned}$$

iff $16\theta_j^{\alpha}\left(n_j\right)^{\alpha-1} \leq 1$. Since $0 \leq \lim_{j\to\infty}\theta_j^{\alpha}\left(n_j\right)^{\alpha-1} \leq \lim_{j\to\infty}\frac{\theta_j}{n_j^{1-\alpha}} = 0$ we can use Lemma 2.1 and conclude that $H^{\alpha}(E) \geq H^{\alpha}(\tilde{E}) > 0$, which finishes the proof. $\square$

2.2 Homogeneous Polynomials

We need polynomials from [7] but with Lipschitz constants:

Lemma 2.3 *There exists $K \in \mathbb{N}$ such that for each $N \in \mathbb{N}$ sufficiently large and $N < m_1 < m_2 < \cdots < m_K \leq 2N$, we can choose a sequence p_m of homogeneous polynomials of degree m which satisfy*

(1) $\left|p_{m_j}(z)\right| \leq 2$ *for all* $z \in \partial\Omega$,
(2) $\max_{k\in\{1,\ldots,K\}} \left|p_{m_k}(z)\right| \geq \frac{1}{2}$ *for all* $z \in \partial\Omega$,
(3) $\left|p_{m_j}(z) - p_{m_j}(w)\right| \leq 3m_j \, \|z - w\|$ *for* $z, w \in \partial\Omega$

Proof Let $0 < c_1 < c_2$ be from [7, Lemma 2.1]. For $a = \frac{1}{4}$ we can choose C from [7, Lemma 2.5]. Let $K = K(\alpha, \beta)$ be from [7, Lemma 2.3] for $\alpha = \frac{1}{4\sqrt{c_2}}$ and $\beta = \frac{C}{\sqrt{c_1}}$. For $N \in \mathbb{N}$ fix a maximal $1/(4\sqrt{c_2 N})$-separated subset $A \subset \partial\Omega$. Using [7, Lemma 2.3] we can divide A into at most K disjoint $C/\sqrt{c_1 N}$-separated subsets $A_0, A_1, \ldots, A_{K-1}$. We define the same way as in [7, Theorem 2.6]:

$$p_{m_j}(z) := \sum_{\xi \in A_j} \left\langle z, \nu_\xi \right\rangle^{m_j}$$

for $j = 0, 1, \ldots, K - 1$.

Using the same arguments as in [7, Theorem 2.6] we conclude properties (1)–(2). Since $m_j - 1 \geq N$ we can use [7, Lemma 2.5 (4)] and observe that

$$\sum_{\xi \in A_j} \left|\left\langle z, \nu_\xi \right\rangle\right|^{m_j - 1} \leq 1 + a = \frac{5}{4}$$

for $z \in \partial\Omega$. Now we can estimate:

$$\begin{aligned}
\left|p_{m_j}(z) - p_{m_j}(w)\right| &\leq \sum_{\xi \in A_j} \left|\left\langle z, \nu_\xi \right\rangle^{m_j} - \left\langle w, \nu_\xi \right\rangle^{m_j}\right| \\
&\leq \sum_{\xi \in A_j} \left|\left\langle z - w, \nu_\xi \right\rangle\right| \sum_{k=0}^{m_j - 1} \left|\left\langle z, \nu_\xi \right\rangle\right|^k \left|\left\langle w, \nu_\xi \right\rangle\right|^{m_j - k - 1} \\
&\leq \|z - w\| \sum_{k=0}^{m_j - 1} \sum_{\xi \in A_j} \left(\left|\left\langle z, \nu_\xi \right\rangle\right|^{m_j - 1} + \left|\left\langle w, \nu_\xi \right\rangle\right|^{m_j - 1}\right) \\
&\leq \|z - w\| \sum_{k=0}^{m_j - 1} \frac{10}{4} \leq 3m_j \, \|z - w\|
\end{aligned}$$

for $z, w \in \partial\Omega$. ∎

Since we use combinations homogeneous polynomials and Lipschitz functions we will need the following tool for efficient small interval integration.

Lemma 2.4 *Let f be a Lipschitz function on* $[0, 1]$ *then*

$$\int_0^{\frac{1}{n}} \exp(2\pi int)\, f(t)dt = O\left(\frac{1}{n^2}\right)$$

Proof If $|f(t_1) - f(t_2)| \leq C|t_1 - t_2|$ then we can estimate

$$\begin{aligned}\left|\int_0^{\frac{1}{n}} \exp(2\pi int)\, f(t)dt\right| &= \left|\int_0^{\frac{1}{n}} \exp(2\pi int)\,(f(t) - f(0))\, dt\right| \\ &\leq \int_0^{\frac{1}{n}} Ctdt \leq \frac{C}{n^2}\end{aligned}$$

□

Our crucial one-dimensional property of E (Lemma 2.2) is preserved by polynomials:

Lemma 2.5 *There exists a constant $\alpha \in (0, 1)$ such that for a given Lipschitz positive function h on $\partial\Omega$ there exists a constant $\theta > 1$ such that for all sufficiently large $m \in \mathbb{N}$ we can choose polynomial P_m with:*

(1) $|P_m| < h$ *on* $\partial\Omega$,
(2) $|P_m(r\eta)| \leq \|h\|\, r^m$ *for* $\eta \in \partial\Omega$, $r \in (0, 1)$,
(3) *for all* $\eta \in \partial\Omega$ *there exists* $x \in \left[0, \frac{1}{m}\right]$ *such that:*

$$\Re P_m \geq \alpha h$$

on $\eta_{\left[x, x+\frac{1}{\theta m}\right]}$.

Proof There exists $N = N(\Omega) \in \mathbb{N}$ such that for a given continuous positive function h on $\partial\Omega$ we can choose holomorphic functions $f_1, \ldots, f_N$ on $\mathbb{C}^d$ such that (see [7]):

$$|f_i| < h < 2 \max_{i=1,\ldots,N} |f_i|$$

on $\partial\Omega$. There exists $C > 0$ such that

$$|f_i(\eta) - f_i(\xi)| \leq C\, \|\eta - \xi\|$$

for $\eta, \xi \in \overline{\Omega}$. Let $K \in \mathbb{N}$ be as in Lemma 2.3. For a given $m, n \in \mathbb{N}$ let $m_{i,j} = m(n + K(i-1) + j - 1)$. Now for sufficiently large n_0 and all positive m we have: $mn \leq m_{1,1} < \cdots < m_{N,K} < 2mn$ and there exist homogeneous polynomials $p_{m_{i,j}}$ of degree $m_{i,j}$ such that:

(1) $\left|p_{m_{i,j}}\right| \leq 2$ on $\partial\Omega$,

(2) $\max_{j\in\{1,\dots,K\}} \left|p_{m_{i,j}}\right| \geq \frac{1}{2}$ on $\partial\Omega$,
(3) $\left|p_{m_{i,j}}(\eta) - p_{m_{i,j}}(\xi)\right| \leq 3m_{i,j}\left\|\eta - \xi\right\| \leq 6mn_0\left\|\eta - \xi\right\|$ for $z, w \in \partial\Omega$.

We show that it is enough to define $\beta = 1 - \frac{1}{2^{11}K^2N^2}$, $\alpha = \frac{1}{2}(1-\beta^2)$,

$$\theta = \frac{24\pi\left\|h\right\| n_0}{\alpha \min_{\xi\in\partial\Omega} h(\xi)}$$

and

$$P_m := \frac{1}{2KN}\sum_{i=1}^{N} f_i \sum_{j=1}^{K} p_{m_{i,j}}$$

for m large enough.

We have on $\partial\Omega$:

$$|P_m| \leq \frac{1}{2KN}\sum_{i=1}^{N} h \sum_{j=1}^{K} 2 \leq h$$

and

$$|P_m(r\eta)| \leq \frac{1}{2KN}\sum_{i=1}^{N} \|h\| \sum_{j=1}^{K} 2r^m \leq \|h\|\, r^m$$

for $\eta \in \partial\Omega$ and $r \in (0,1)$. Moreover for $\eta, \xi \in \partial\Omega$ and m sufficiently large:

$$\begin{aligned} |P_m(\eta) - P_m(\xi)| \leq \frac{1}{2KN}\sum_{i,j} |f_i(\eta)| \left|p_{m_{i,j}}(\eta) - p_{m_{i,j}}(\xi)\right| + \\ + \frac{1}{2KN}\sum_{i,j}\left|p_{m_{i,j}}(\xi)\right| |f_i(\eta) - f_i(\xi)| \\ \leq \frac{1}{2KN}\sum_{i,j} h(z) 6mn_0 \left\|\eta - \xi\right\| + \frac{1}{2KN}\sum_{i,j} 2C\left\|\eta - \xi\right\| \\ \leq 6\left\|h\right\| mn_0 \left\|\eta - \xi\right\|. \end{aligned}$$

Let us observe that $\sqrt{1+x} \leq 1 + \frac{1}{2}x - \frac{1}{16}x^2$ holds for $x \in (-1, 1)$.
Now we can use Lemma 2.4 and estimate for m large enough, $\eta \in \partial\Omega$:

$$\begin{aligned} \int_0^{\frac{1}{m}} \sqrt{h - \Re P_m}(\eta_t)dt &\leq \int_0^{\frac{1}{m}} \sqrt{h}\left(1 - \frac{P_m + \overline{P_m}}{4h} - \frac{1}{16}\left(\frac{P_m + \overline{P_m}}{2h}\right)^2\right)(\eta_{[t]})dt \\ &\leq \int_0^{\frac{1}{m}} \sqrt{h}\left(1 - \frac{1}{2^6h^2K^2N^2}\sum_{i,j}|f_i|^2\left|p_{m_{i,j}}\right|^2\right)(\eta_{[t]})dt + O\left(\frac{1}{m^2}\right) \end{aligned}$$

$$\leq \int_0^{\frac{1}{m}} \sqrt{h}\left(1-\frac{1}{2^6 h^2 K^2 N^2}\sum_{i,j}|f_i|^2\frac{1}{4}+O\left(\frac{1}{m}\right)\right)(\eta_{[t]})dt$$
$$\leq \int_0^{\frac{1}{m}} \sqrt{h}\left(1-\frac{1}{2^{10}K^2N^2}+O\left(\frac{1}{m}\right)\right)(\eta_{[t]})dt$$
$$\leq \int_0^{\frac{1}{m}} \sqrt{h}\left(1-\frac{1}{2^{11}K^2N^2}\right)(\eta_{[t]})dt = \int_0^{\frac{1}{m}} \beta\sqrt{h}(\eta_{[t]})dt.$$

In particular there exists $x \in \left[0, \frac{1}{m}\right]$ such that $(h - \Re P_m)(\eta_{[x]}) \leq \beta^2 h(\eta_{[x]})$ so we have:

$$2\alpha h(\eta_{[x]}) = (1-\beta^2)h(\eta_{[x]}) \leq (\Re P_m)(\eta_{[x]}).$$

Without lost of the generality we can assume that m is so large that for $t \in \left[x, x+\frac{1}{\theta m}\right]$ we have

$$\frac{3}{4}h(\eta_{[t]}) \leq h(\eta_{[x]}).$$

Since

$$\left|(\Re P_m)(\eta_{[x]}) - (\Re P_m)(\eta_{[t]})\right| \leq 6\|h\| m n_0 2\pi(t-x) \leq \frac{12\pi\|h\| n_0}{\theta} = \frac{\alpha \min_{\xi\in\partial\Omega} h(\xi)}{2},$$

we can estimate:

$$\frac{3}{2}\alpha h(\eta_{[t]}) \leq 2\alpha h(\eta_{[x]}) \leq (\Re P_m)(\eta_{[x]}) \leq (\Re P_m)(\eta_{[x]}) + (\Re P_m)(\eta_{[t]}) - (\Re P_m)(\eta_{[t]}) \leq (\Re P_m)(\eta_{[t]}) + \frac{1}{2}\alpha h(\eta_{[t]}),$$

which implies

$$\alpha h(\eta_{[t]}) \leq (\Re P_m)(\eta_{[t]})$$

and finishes the proof. □

3 Peak Set

Theorem 3.1 *There exists a peak set $E \subset \partial\Omega$ such that $\eta\partial\mathbb{D} \cap E$ has Hausdorff dimension equal to* 1 *for all $\eta \in \partial\Omega$.*

Proof Let $\alpha \in (0, 1)$ be from Lemma 2.5.

Before constructing the desired peak set and the function F exhibiting the definition of the peak set we present an inductive construction of several sequences (of $Q_j, E_j m_j, \theta_j, r_j$) satisfying the following properties:

(1) $Q_1 = 1$, $E_1 = \partial\Omega$, $m_1 = 1$, $\theta_1 = 1$, $r_1 = 1$, Q_j is a polynomial, E_j–compact subset of $\partial\Omega$, $m_j, \theta_j, \frac{1}{r_j}$–natural numbers.

(2) $\frac{\theta_j}{\sqrt[j]{m_j}} \leq 2^{-j}$, $r_j = \frac{r_{j-1}}{\theta_j m_j}$,

(3) $E_j = \left\{\eta \in E_{j-1} : \Re Q_j(\eta) \geq \alpha\right\}$,

(4) $\left|Q_j\right| \leq 1$ and $\left|Q_j\right| \leq 2^{-j}$ on[3]

$$D_{j-1} := \left(\overline{\Omega} \backslash K(E_{j-1}, 2^{-j})\right) \cup \left(\bigcup_{k<j} \left\{\Re Q_k \leq \frac{\alpha}{2}\right\} \cap \overline{\Omega}\right),$$

(5) If $\eta_{\left[0, \frac{r_{j-1}}{m_j}\right]} \subset E_{j-1}$ then there exists $x \in \left[0, \frac{r_{j-1}}{2m_j}\right]$ such that

$$\eta_{[x, x+r_j]} \subset E_j \cap \eta_{\left[0, \frac{r_{j-1}}{m_j}\right]}.$$

Let us observe that $Q_1, E_1, m_1, \theta_1, r_1$ are just defined by property (1), so suppose that $Q_{j-1}, E_{j-1} m_{j-1}, \theta_{j-1}, r_{j-1}$ have properties (1)–(5).

Since $D_{j-1} \cap E_{j-1} = \emptyset$ therefore $2\varepsilon = \inf_{(\xi,\eta)\in E_{j-1}\times D_{j-1}} \|\xi - \eta\| > 0$. For a given $\eta \in \partial\Omega \cap D_{j-1}$ there exists an open neighbourhood V_η of η with the following properties:

- $V_\eta \cap K(E_{j-1}, \varepsilon) = \emptyset$
- If f is holomorphic on $\overline{V_\eta}$ then:

$$\sup_{\xi \in \overline{V_\eta} \cap \Omega} |f(\xi)| \leq \sup_{\xi \in \partial\Omega \cap V_\eta} |f(\xi)|.$$

To guarantee the last inequality, it suffices to choose the neighborhood U_η of η in $\partial\Omega$ so that $U_\eta \cap \overline{K(E_{j-1}, \varepsilon)} = \emptyset$. Now it is enough to define $V_\eta = \bigcup_{\omega:\partial\omega\subset U_\eta} \omega$ where ω is any one-dimensional complex disc disjoint with $\overline{K(E_{j-1}, \varepsilon)}$.

Let $V = \bigcup_{\eta\in D_{j-1}} V_\eta$. Since D_{j-1} is compact there exists $s \in (0, 1)$ such that $D_{j-1} \subset s\Omega \cup V$. We can choose a positive Lipschitz function h on $\partial\Omega$ such that

- $h \leq 1$ on $\partial\Omega$,
- $h \equiv 1$ on E_{j-1},
- $h \leq 2^{-j}$ on $\overline{V} \cap \partial\Omega$.

Now for a given h we can use Lemma 2.5 and choose $\theta > 1$ such that for all sufficiently large $m \in \mathbb{N}$ we can choose polynomial P_m with:

[3] If S is a set then $K(S, r) = \left\{\eta \in \mathbb{C}^d : \inf_{\xi\in S} \|\xi - \eta\| < r\right\}$

- $|P_m| < h$ on $\partial\Omega$,
- $|P_m(r\eta)| \le r^m$ for $\eta \in \partial\Omega$, $r \in (0,1)$,
- for all $\eta \in \partial\Omega$ there exists $x \in \left[0, \frac{1}{m}\right]$ such that:

$$\Re P_m \ge \alpha h \tag{3.1}$$

  on $\eta_{\left[x, x+\frac{1}{\theta m}\right]}$.

We can define $\theta_j = 2\lceil\theta\rceil$. Since $\frac{1}{r_{j-1}} \in \mathbb{N}$ we can also assume that $m = \frac{2m_j}{r_{j-1}}$ for $m_j \in \mathbb{N}$ sufficiently large. In particular m_j can be so large that $|P_m| \le 2^{-j}$ on $s\overline{\Omega}$ and $\frac{\theta_j}{\sqrt[j]{m_j}} \le 2^{-j}$. Now we can define $r_j = \frac{r_{j-1}}{\theta_j m_j}$, $Q_j = P_m$ and $E_j = \left\{\eta \in E_{j-1} : \Re Q_j(\eta) \ge \alpha\right\}$. If $\eta \in D_{j-1}$ then $\eta \in s\overline{\Omega}$ or $\eta \in V$. In both cases $|Q_j(\eta)| \le 2^{-j}$ so we have properties (1)–(4). Suppose that $\eta_{\left[0, \frac{r_{j-1}}{m_j}\right]} = \eta_{\left[0, \frac{2}{m}\right]} \subset E_{j-1}$. Now for a given η there exists $x \in \left[0, \frac{1}{m}\right] = \left[0, \frac{r_{j-1}}{2m_j}\right]$ with (3.1). But $\eta_{\left[x, x+\frac{1}{\theta m}\right]} \subset \eta_{\left[0, \frac{2}{m}\right]} \subset E_{j-1}$ so $h = 1$ on $\eta_{\left[x, x+\frac{1}{\theta m}\right]}$ and

$$\eta_{[x, x+r_j]} = \eta_{\left[x, x+\frac{r_{j-1}}{\theta_j m_j}\right]} = \eta_{\left[x, x+\frac{1}{\lceil\theta\rceil m}\right]} \subset \eta_{\left[x, x+\frac{1}{\theta m}\right]} \subset E_j.$$

We just proved that our sequences fulfills properties (1)-(5).

Let us choose sequences $Q_j E_j, m_j, \theta_j, r_j$. The set $E = \bigcap_j E_j$ is a compact subset of $\partial\Omega$. Let us consider $Q = \sum_j Q_j$. If $j_0 \in \mathbb{N}$ then we can observe $\left|Q_j\right| \le 2^{-j}$ on $\overline{\Omega} \setminus K(E_{j_0-1}, 2^{-j_0})$ for $j \ge j_0$. In particular Q is holomorphic function on Ω and continuous on $\overline{\Omega} \setminus E$.

For a given $\eta \in \overline{\Omega} \setminus (D_1 \cup E)$ let index $j_\eta \ge 2$ be such that $\eta \in D_{j_\eta} \setminus D_{j_\eta - 1}$. We have $\left|Q_j(\eta)\right| \le 2^{-j}$ for $j \ge j_\eta + 1$ and $\Re Q_j(\eta) \ge \frac{\alpha}{2}$ for $j \le j_\eta - 1$, so we can estimate:

$$\begin{aligned}\Re Q(\eta) &\ge \sum_{k \le j_\eta - 1} \Re Q_k(\eta) + \Re Q_{j_\eta}(\eta) - \sum_{k \ge j_\eta + 1} |Q_k(\eta)| \ge j_\eta \frac{\alpha}{2} - 1 - \sum_{k \ge j_\eta + 1} 2^{-k} \\ &> \frac{j_\eta}{2} - 2.\end{aligned}$$

If $\overline{\Omega} \setminus (D_1 \cup E) \ni \eta_n \to \xi \in E$ then $j_{\eta_n} \to \infty$. In particular

$$\lim_{\overline{\Omega} \setminus E \ni \eta_n \to \xi \in E} \Re Q(\xi) \ge \lim_{n \to \infty} \frac{1}{2} j_{\eta_n} - 2 = \infty$$

and $\Re Q(\eta) > -1$ for $\eta \in \overline{\Omega} \setminus (D_1 \cup E)$.

If $\eta \in D_1$ then $\left|Q_j(\eta)\right| \le 2^{-j}$ for $j \ge 2$, so we have $\Re Q(\eta) \ge -1 - \sum_{j \ge 2} 2^{-j} > -2$. In particular

$$\Re Q > -2$$

on $\overline{\Omega}\backslash E$, which implies that E is a peak set for the function $F = \exp\left(\frac{-1}{Q+2}\right)$.

Now we calculate Hausdorff dimension of E on each slice. Let $\alpha \in (0,1)$ we can observe for $\frac{1}{j} \le \alpha$:

$$0 \le \frac{\theta_j}{m_j^{\alpha}} \le \frac{\theta_j}{m_j^{\frac{1}{j}}} \to 0.$$

For a given $\xi \in \partial\Omega$ we have $\xi_{[0,1]} \subset E_1$. Moreover if $\xi_{[z,z+r_{j-1}]} \subset E_{j-1}$ for some $z \in [0,1]$ and $k \in \{0,\dots,m_j - 1\}$ then we can set $\eta = \xi_{z+s} \in E_{j-1}$ for $s = \frac{kr_{j-1}}{m_j}$. In particular

$$\eta_{\left[0,\frac{r_{j-1}}{m_j}\right]} = \xi_{\left[z+\frac{kr_{j-1}}{m_j},z+\frac{(k+1)r_{j-1}}{m_j}\right]} \subset \xi_{[z,z+r_{j-1}]} \subset E_{j-1},$$

so there exists $x \in \left[0, \frac{r_{j-1}}{2m_j}\right]$ such that

$$\eta_{[x,x+r_j]} \subset E_j \cap \eta_{\left[0,\frac{r_{j-1}}{m_j}\right]}.$$

Now we have:

$$\xi_{\left[z+\frac{kr_{j-1}}{m_j}+x,z+\frac{kr_{j-1}}{m_j}+x+r_j\right]} = \eta_{[x,x+r_j]} \subset E_j \cap \eta_{\left[0,\frac{r_{j-1}}{m_j}\right]} = E_j \cap \xi_{\left[z+\frac{kr_{j-1}}{m_j},z+\frac{(k+1)r_{j-1}}{m_j}\right]}$$

so we can use Lemma 2.2 and conclude that $\xi_{[0,1]} \cap E$ has Hausdorff dimension equal 1. □

Remark 3.2 Peak set E with Hausdorff dimension equals to 1 on each slice has maximal possible Hausdorff dimension equal to $2d - 1$.

Proof It follows from Fubini theorem for Hausdorff measure see ([18, Proposition 2] and [20]). □

Remark 3.3 Let $\varepsilon, r \in (0,1)$ and $S_\varepsilon = \{\eta \in \partial\Omega : \eta_1 \ge \varepsilon\}$, $\Lambda_{[0,r]} = \exp(2\pi i[0,r])$. Then there exists $E \subset S_\varepsilon\Lambda_r$ peak set such that $\eta\Lambda_r \cap E$ has Hausdorff dimension equal to 1 for $\eta \in S_\varepsilon$.

Proof Let r_i be as in the proof of Theorem 3.1. It is enough to observe that $S_\varepsilon\Lambda_{[0,r_i]}$ contains a peak set with Hausdorff dimension equal to 1 on each slice crossing S_ε. □

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Declarations

Conflict of interest The authors declare no competing interests.

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