Complex Analysis and Operator Theory (2023) 17:102
https://doi.org/10.1007/s11785-023-01397-x

Complex Analysis and Operator Theory

Continuous Dependence of Szegő Kernel on a Weight of Integration

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Received: 7 November 2022 / Accepted: 21 July 2023 / Published online: 20 August 2023

Abstract

The weighted Szegő kernel was investigated in a few papers (see Nehari in J d'Analyse Mathématique 2:126–149, 1952; Alenitsin in Zapiski Nauchnykh Seminarov LOMI 24:16–28, 1972; Uehara and Saitoh in Mathematica Japonica 29:887–891, 1984; Uehara in Mathematica Japonica 42:459–469, 1995). In all of these, however, only continuous weights were considered. The aim of this paper is to show that the Szegő kernel depends in a continuous way on a weight of integration in the case when the weights are not necessarily continuous. A topology on the set of admissible weights will be constructed and Pasternak's theorem (see Pasternak-Winiarski in Studia Mathematica 128:1, 1998) on the dependence of the orthogonal projector on a deformation of an inner product will be used in the proof of the main theorem.

Communicated by Filippo Bracci.

This article is part of the topical collection "Reproducing kernel spaces and applications" edited by Daniel Alpay.

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Keywords Szegő kernel · Weight of integration · Continuous dependence

Mathematics Subject Classification 46E22 · 32A25

1 Introduction

This paper is a continuation of [10]. For the reader's convenience we will recall preliminaries from that article. The reader is also referred to publications devoted to various applications of reproducing kernel Hilbert spaces ([9]). The main result of this paper is Theorem 2.2 which shows that Szegő kernel depends continuously on a weight of integration.

Let Ω be a bounded domain with the boundary of class C^2. Suppose that the function $\mu : \partial\Omega \to \mathbb{R}$ is measurable and almost everywhere greater than 0 with respect to the surface measure $\mathrm{d}S$ on $\partial\Omega$. Such a function will be called a *weight*. By $L^2(\partial\Omega, \mu)$ we will denote a set of classes of functions $f : \partial\Omega \to \mathbb{C}$, square-integrable in the sense that

$$||f||_\mu^2 := \int_{\partial\Omega} |f(w)|^2 \mu(w) \mathrm{d}S < \infty\,,$$

where the integral is understood as an integral of a scalar field. The set $L^2(\partial\Omega, \mu)$ with an inner product given by

$$\langle f|g\rangle_\mu := \int_{\partial\Omega} \overline{f(w)} g(w) \mu(w) \mathrm{d}S$$

is a Hilbert space. Now let us consider the space $A(\Omega)$ of continuous functions $f : \overline{\Omega} \to \mathbb{C}$, such that $f|_\Omega$ is holomorphic. Let us write $B(\Omega, \mu) := \{f|_{\partial\Omega} : f \in A(\Omega)\} \cap L^2(\partial\Omega, \mu)$. By $L^2H(\partial\Omega, \mu)$ we will understand the closure of $B(\Omega, \mu)$ in the $L^2(\partial\Omega, \mu)$ topology.

Of course $L^2H(\partial\Omega, \mu)$ can change as a set with a change of μ. However

Proposition 1.1 *If μ_1, μ_2 are weights and there exist $m, M > 0$, such that*

$$m\mu_1(z) \le \mu_2(z) \le M\mu_1(z) \tag{1}$$

a.e. on $\partial\Omega$, then for any $f \in L^2(\partial\Omega, \mu_j)$ we have $f \in L^2(\partial\Omega, \mu_k)$, $j, k \in \{1, 2\}$, and $m||f||_{\mu_1}^2 \le ||f||_{\mu_2}^2 \le M||f||_{\mu_1}^2$. Hence $L^2H(\partial\Omega, \mu_1) = L^2H(\partial\Omega, \mu_2)$ as a set and the norms $||\cdot||_{\mu_1}$ and $||\cdot||_{\mu_2}$ are equivalent. In particular, if $0 < m \le \mu \le M < \infty$, then $L^2H(\partial\Omega, \mu) = L^2H(\partial\Omega, 1)$ as a vector space.

If $L^2H(\partial\Omega, \mu_1) = L^2H(\partial\Omega, \mu_2)$ as vector spaces, we will write $\mu_1 \approx \mu_2$. It is easy to show that this is an equivalence relation.

Each element of $B(\Omega, 1)$ has a unique holomorphic prolongation to Ω by Poisson integration (see [3] for more details), so it is also true for any element from $B(\Omega, \mu)$, because $B(\Omega, \mu) \subset B(\Omega, 1)$ for any μ. (Remember that $B(\Omega, \mu) = \{f|_{\partial\Omega} : f \in$

$A(\Omega)\} \cap L^2(\partial\Omega, \mu)$ and $B(\Omega, 1) = \{f|_{\partial\Omega} : f \in A(\Omega)\} \cap L^2(\partial\Omega, 1) = \{f|_{\partial\Omega} : f \in A(\Omega)\}$, because Ω is a bounded domain of class C^2, therefore $\partial\Omega$ has finite Lebesgue measure). We will denote the set of all such prolongations by $\widetilde{B}(\Omega, \mu)$ (where $\widetilde{B}(\Omega, \mu) \subset A(\Omega)$).

A good question to ask is how to find a holomorphic prolongation of functions from $L^2 H(\partial\Omega, \mu)\backslash B(\Omega, \mu)$ for an arbitrary μ? We will answer this question in a moment.

We will use the same symbol for a function and its prolongation, which should not be misleading.

Let μ be a weight with the following property:

(CB) for any compact set $X \subset \Omega$ there exists $C_X > 0$, such that for any $f \in \widetilde{B}(\Omega, \mu)$ and $z \in X$

$$|f(z)| \leq C_X ||f||_\mu .$$

Then for functions from $L^2 H(\partial\Omega, \mu)\backslash B(\Omega, \mu)$ we can define their prolongation to Ω in the following way:

Let (f_n) be a sequence of functions from $\widetilde{B}(\Omega, \mu)$. Let f be the limit of this sequence in $L^2 H(\partial\Omega, \mu)$. Since by (CB) the sequence of functions $\{f_n|_\Omega\}$ is a locally uniformly Cauchy sequence on Ω, the function

$$f(z) := \lim_{n\to\infty} f_n(z), z \in \Omega$$

is well defined and holomorphic on Ω.

From now on, if μ fullfills (CB), we will interpret $L^2 H(\partial\Omega, \mu)$ as a set of functions on $\overline{\Omega}$.

Definition 1.1 Let μ be a weight satisfying (CB). A function (if it exists) $S_\mu : \Omega\times\overline{\Omega} \to \mathbb{C}$, such that for any $z \in \Omega$, $\overline{S_\mu(z, \cdot)} \in L^2 H(\partial\Omega, \mu)$ and for any $f \in L^2 H(\partial\Omega, \mu)$ (reproducing property)

$$f(z) = \langle\overline{S_\mu(z, \cdot)}|f(\cdot)\rangle_\mu,$$

will be called the *Szegő kernel* of $L^2 H(\partial\Omega, \mu)$.

It is true (as for any reproducing kernel Hilbert space) that if S_μ and S'_μ are Szegő kernels of the same space, then $S_\mu = S'_\mu$ and if the Szegő kernel exists, then it is given uniquelly by the formula

$$S_\mu(z, w) = \sum_{i\in I} \varphi_i(z)\overline{\varphi_i(w)},$$

where $\{\varphi_i\}_{i\in I}$ is an arbitrary complete orthonormal system of $L^2 H(\partial\Omega, \mu)$. Hence for any $z, w \in \Omega$ we have $S_\mu(w, z) = \overline{S_\mu(z, w)}$ and by Hartogs's theorem on separate analyticity the function $\Omega \times \Omega' \ni (z, w) \mapsto S^0(z, w) := S_\mu(z, \overline{w})$ is holomorphic, where $\Omega' = \{w \in \mathbb{C}^N : \overline{w} \in \Omega\}$. So S_μ is real analytic on $\Omega \times \Omega$, holomorphic with respect to the first N variables and antiholomorphic with respect to the last N

variables. Moreover for any $z \in \Omega$ we have $\overline{||S_\mu(z,\cdot)||_\mu^2} = ||S_\mu(\cdot, z)||_\mu^2 = S_\mu(z,z)$. It is a natural question to ask which conditions μ must satisfy in order for $L^2H(\partial\Omega,\mu)$ to be a reproducing kernel Hilbert space.

Definition 1.2 We will say that a weight μ is *Szegő admissible* (S-admissible for short) if there exists Szegő kernel of $L^2\,H(\partial\Omega,\mu)$ space.

Theorem 1.1 *The weight μ is S-admissible if and only if Condition (CB) is satisfied.*

Proof $\Rightarrow$ comes directly from Definition 1.1. In fact we can show that the smallest possible constant C_X in Condition (CB) is

$$\max_{z\in X}\sqrt{S_\mu(z,z)}\,.$$

Indeed, by the reproducing property, the Cauchy-Schwarz inequality, and again the reproducing property:

$$|f(z)| \leq |\langle\overline{S_\mu(z,\cdot)}, f\rangle_\mu| \leq ||\overline{S_\mu(z,\cdot)}||_\mu ||f||_\mu = \sqrt{S_\mu(z,z)}||f||_\mu.$$

On the other hand, for $f = \overline{S_\mu(z,\cdot)}$, the above inequality becomes equality.

$\Leftarrow$ (CB) means that the point evaluation functionals

$$\widetilde{E_z} : \widetilde{B}(\Omega,\mu) \ni f \mapsto f(z) \in \mathbb{C}$$

are continuous. Since $B(\Omega,\mu)$ is dense in $L^2\,H(\partial\Omega,\mu)$, we can prolong $\widetilde{E_z}$ to the functional $E_z \in L^2H(\partial\Omega,\mu)^*$ with the same majorizing constant C_X for any $z \in \Omega$. By the Riesz representation theorem for E_z we see that for $z \in \Omega$ there exists $\overline{e_z} \in L^2H(\partial\Omega,\mu)$ such that, for any $f \in L^2H(\partial\Omega,\mu)$,

$$f(z) = \langle\overline{e_z}, f\rangle$$

and the function

$$S_\mu(z,w) := e_z(w) \qquad (z,w) \in \Omega \times \overline{\Omega}$$

is the Szegő kernel of $L^2H(\partial\Omega,\mu)$. □

Theorem 1.2 *Let μ be a weight on $\partial\Omega$, such that*

$$\int_{\partial\Omega}\frac{1}{\mu}\mathrm{d}S < \infty.$$

Then μ is S-admissible.

For more details and the proof see [10].

2 Main Results

At the beginning, we will introduce the appropriate topology in the set $\mathrm{SAW}(\partial\Omega)$ of S-admissible weights on $\partial\Omega$. Let us recall that we denote by $\approx$ the equivalence relation on $\mathrm{SAW}(\partial\Omega)$ defined as follows: for any $\mu_1, \mu_2 \in \mathrm{SAW}(\partial\Omega)$ $\mu_1 \approx \mu_2$ if and only if $L^2H(\partial\Omega, \mu_1)$ is equal as a vector space to $L^2H(\partial\Omega, \mu_2)$. Just as in the work [4] it can be proved that in this case the norms in $L^2\, H(\partial\Omega, \mu_1)$ and $L^2\, H(\partial\Omega, \mu_2)$ are equivalent, i.e. there are positive constants c and C, such that for any $f \in L^2H(\partial\Omega, \mu_1)$ we have

$$c||f||_{\mu_2} \leq ||f||_{\mu_1} \leq C||f||_{\mu_2}.$$

See [11] for details about weighted Szegő space.

For any $\mu \in \mathrm{SAW}(\partial\Omega)$, denote by $\mathrm{SAW}(\partial\Omega, \mu)$ the equivalence class of μ with respect to the relation $\approx$. Note that $\mathrm{SAW}(\partial\Omega, \mu)$ contains infinitely many elements, because for any function $g \in L^\infty(\partial\Omega)$ such that

$$\mathrm{essinf}_{z\in\partial\Omega} g(z) > 0$$

the ordinary product $g\mu$ is an element of $\mathrm{SAW}(\partial\Omega, \mu)$ (see Proposition (1.1)) and if $g_1 \neq g_2$, then $g_1\mu \neq g_2\mu$.

On $\mathrm{SAW}(\partial\Omega, \mu)$ we consider the map:

$$\mathrm{SAW}(\partial\Omega, \mu) \ni \nu \mapsto B_\mu(\nu) := \langle -, \cdot \rangle_\nu \in \mathrm{Her}(L^2H(\partial\Omega, \mu)\,,$$

where $\mathrm{Her}(\mathcal{H})$ denotes the real Banach space of all continuous hermitian forms on a Hilbert space $\mathcal{H}$ with the standard Banach space norm:

$$||B||_{op} := \sup_{||x||=||y||=1} |B(x, y)| \qquad B \in \mathrm{Her}(\mathcal{H}). \tag{2}$$

(It is possible that for some weights zero is the only element of our space. In that case the norm of the operator is equal to zero).

We denote by τ_μ the weakest topology on $\mathrm{SAW}(\partial\Omega, \mu)$ with respect to which the map B_μ is continuous. By the Lax-Milgram Theorem each inner (hermitian) product $\langle -, \cdot \rangle_\nu$ equivalent to $\langle -, \cdot \rangle_\mu$ on $L^2H(\partial\Omega, \mu)$ uniquely determines an invertible positive definite continuous operator A_ν on $L^2H(\partial\Omega, \mu)$, such that

$$\langle f, g \rangle_\nu = \langle A_\nu f, g \rangle_\mu \qquad f, g \in L^2H(\partial\Omega, \mu).$$

Moreover, if $\mathrm{Her}_+(L^2H(\partial\Omega, \mu))$ denotes the cone in $\mathrm{Her}(L^2H(\partial\Omega, \mu))$ of all positive definite hermitian forms (the set of hermitian products) on $L^2H(\partial\Omega, \mu)$ equivalent to $\langle -, \cdot \rangle_\mu$, then the map

$$\mathrm{Her}_+(L^2H(\partial\Omega, \mu)) \ni \nu \mapsto \Psi_\mu(\nu) := A_\nu \in L(L^2H(\partial\Omega, \mu))$$

is an isometry (onto its image) with respect to the standard norms in $\mathrm{Her}(L^2H(\partial\Omega,\mu))$ and in the space $L(L^2H(\partial\Omega,\mu))$ of all bounded endomorphisms of $L^2H(\partial\Omega,\mu)$. Therefore the map Ψ_μ is a homeomorphism. Hence τ_μ is the weakest topology with respect to which the map $\Psi_\mu \circ B_\mu$ is continuous.

On the other hand, if $\mu_1 \approx \mu$ and $\nu \in \mathrm{SAW}(\partial\Omega,\mu) = \mathrm{SAW}(\partial\Omega,\mu_1)$, then

$$\langle f, g\rangle_\nu = \langle(\Psi_{\mu_1} \circ B_{\mu_1})(\nu) f, g\rangle_{\mu_1} = \langle(\Psi_\mu \circ B_\mu)(\mu_1)(\Psi_{\mu_1} \circ B_{\mu_1})(\nu) f, g\rangle_\mu.$$

We can write

$$\langle(\Psi_\mu \circ B_\mu)(\mu_1)(\Psi_{\mu_1} \circ B_{\mu_1})(\nu) f, g\rangle_\mu = \langle G \circ [(\Psi_{\mu_1} \circ B_{\mu_1})(\nu)] f, g\rangle_\mu$$

for $f, g \in L^2H(\partial\Omega,\mu)$, where $G : L(L^2\,H(\partial\Omega,\mu)) \to L(L^2\,H(\partial\Omega,\mu))$ is the map of composition with constant invertible operator

$$G(A) := [(\Psi_\mu \circ B_\mu)(\mu_1)] \circ A,\ A \in L(L^2H(\partial\Omega,\mu))\,.$$

Of course such a map G is a homeomorphism of $L(L^2\,H(\partial\Omega,\mu))$. Hence $\Psi_\mu \circ B_\mu = G \circ (\Psi_{\mu_1} \circ B_{\mu_1})$ and therefore $\tau_\mu = \tau_{\mu_1}$. This means that the topology τ_μ does not depend on the choice of an equivalence class representative from $\mathrm{SAW}(\partial\Omega,\mu)$.

Let us consider the family $\bigcup_{\mu\in\mathrm{SAW}(\partial\Omega)} \tau_\mu$ of subsets of $\mathrm{SAW}(\partial\Omega)$. It is, of course, the base of the same topology τ on $\mathrm{SAW}(\partial\Omega)$. From now on we will consider $\mathrm{SAW}(\partial\Omega)$ as a topological space endowed with this topology.

Note that for any $\mu \in \mathrm{SAW}(\partial\Omega)$ the set $\mathrm{SAW}(\partial\Omega,\mu)$ is open in $\mathrm{SAW}(\partial\Omega)$, but it is also closed, because

$$\mathrm{SAW}(\partial\Omega) \setminus \mathrm{SAW}(\partial\Omega,\mu) = \bigcup_{\nu\in\mathrm{SAW}(\partial\Omega)\setminus\mathrm{SAW}(\partial\Omega,\mu)} \mathrm{SAW}(\partial\Omega,\nu).$$

Moreover, from the definition of $\mathrm{SAW}(\partial\Omega)$ it follows almost immediately that for any $\nu_1, \nu_2 \in \mathrm{SAW}(\partial\Omega,\mu)$ and any $t \in [0,1]$ we have

$$t\nu_1 + (1-t)\nu_2 \in \mathrm{SAW}(\partial\Omega,\mu).$$

In addition, the map

$$\begin{aligned}[0,1] \ni t \mapsto (\Psi_\mu \circ B_\mu)(t\nu_1 + (1-t)\nu_2)) &= t(\Psi_\mu \circ B_\mu)(\nu_1)\\ &+ (1-t)(\Psi_\mu \circ B_\mu)(\nu_2) \in L(L^2H(\partial\Omega,\mu))\end{aligned}$$

is evidently continuous and therefore the map

$$[0,1] \ni t \mapsto t\nu_1 + (1-t)\nu_2 \in \mathrm{SAW}(\partial\Omega,\mu)$$

is also continuous. Hence $\mathrm{SAW}(\partial\Omega,\mu)$ is connected and consequently it is a connected component of $\mathrm{SAW}(\partial\Omega)$ with respect to τ.

It may happen for some $\mu \in \mathrm{SAW}(\partial\Omega)$ that B_μ is not a 1–1 map. In this case τ_μ is not a Hausdorff topology. In extreme cases it may happen that $L^2 H(\partial\Omega, \mu) = \{0\}$ and $B_\mu \equiv 0$. For example let $f : [0, 1] \to \mathbb{R}$ be a measurable and almost everywhere positive function which is not integrable on any interval. (It is well-known that such functions exist; see e.g. [2], Example 26(c), page 327). Let $\gamma : [0, 1] \to \partial\Omega$ be a parametrization of $\partial\Omega$. Then, for the weight $\mu := f \circ \gamma^{-1}$, the space $L^2 H(\partial\Omega, \mu)$ is equal to $\{0\}$ as a vector space and its reproducing kernel is a zero function of two variables. Indeed, it is clear that any non-zero function which is continuous on $\overline{\Omega}$ and holomorphic on Ω is not an element of $\widetilde{B}(\Omega, \mu)$.

On the other hand, in cases that are important for applications (for example, when μ is bounded from above and below by non-zero constants), B_μ is a 1–1 mapping and τ_μ is Hausdorff. Indeed, any weight bounded from above and below by non-zero constants is an element of $\mathrm{SAW}(\partial\Omega, 1)$, as a consequence of Proposition 1.1. For such μ all polynomials are elements of $L^2 H(\partial\Omega, \mu)$ and it is easy to see that B_1 is an injection.

Our results are true in all cases.

Now let us recall this theorem (see Theorem 5.1. in [6] for more details):

Theorem 2.1 *Let $\mathcal{H}$ be a Hilbert space and V be a closed vector subspace of $\mathcal{H}$. Let $P(\cdot)$ denote the mapping that assigns to each positive defined and invertible operator $A \in L(\mathcal{H})$ the projection of $\mathcal{H}$ onto V orthogonal with respect to the hermitian product*

$$\langle f, g\rangle_A := \langle f, Ag\rangle,\ f, g \in \mathcal{H}.$$

Then $P(\cdot)$ is analytic with respect to the natural analytic structure on an open set of all positively defined and invertible operators in $\mathcal{H}$.

We are ready to prove main result of this section.

Theorem 2.2 *For any $\mu \in \mathrm{SAW}(\partial\Omega)$, denote by S_μ the weighted Szegő kernel of $L^2 H(\partial\Omega, \mu)$ defined on $\Omega \times \overline{\Omega}$. Then, for any $\nu \in \mathrm{SAW}(\partial\Omega, \mu)$ and any $z \in \Omega$, the map*

$$\mathrm{SAW}(\partial\Omega, \mu) \ni \nu \mapsto \overline{S_\nu(z, \cdot)} \in L^2 H(\partial\Omega, \mu)$$

is continuous with respect to the topology τ_μ on $\mathrm{SAW}(\partial\Omega, \mu)$ and the Hilbert space topology on $L^2 H(\partial\Omega, \mu)$.

Proof Fix $z \in \Omega$. We know that $\overline{S_\nu(z, \cdot)}$ is a vector representing the point evaluation functional $E_z : f \mapsto f(z)$ in the sense of the Riesz Representation Theorem for the space $L^2 H(\partial\Omega, \nu)$ (i.e. in $L^2 H(\partial\Omega, \mu)$ endowed with the inner product $\langle -, \cdot\rangle_\nu$). Let P_ν denote orthogonal projection onto $\ker E_z$ in $L^2 H(\partial\Omega, \nu)$. It follows from the proof of the Riesz Theorem that $\overline{S_\nu(z, \cdot)}$ can be expressed in terms of P_ν in this way:

Fix $f \in L^2 H(\partial\Omega, \mu)$ (f does not depend on ν), such that $E_z(f) = f(z) \neq 0$. (If $\ker E_z = L^2 H(\partial\Omega, \mu)$, then $S_\nu(z, \cdot) = 0$ for any $\nu \in \mathrm{SAW}(\partial\Omega, \mu)$ and therefore our theorem is true).

Let $g_\nu := (I - P_\nu) f$, where I denotes the identity operator of $L^2 H(\partial\Omega, \mu)$. Then $g_\nu \neq 0$ for any $\nu \in \mathrm{SAW}(\partial\Omega, \mu)$. Using the fact that the subspace $(\ker E_z)^{\perp \nu}$

orthogonal to $\ker E_z$ in $L^2H(\partial\Omega,\nu)$ is one-dimensional, the following formula can be easily derived

$$\overline{S_\nu(z,\cdot)} = \frac{E_z(g_\nu)}{||g_\nu||_\nu^2} g_\nu = \frac{E_z((I-P_\nu)f)}{||(I-P_\nu)f||_\nu^2}(I-P_\nu)f.$$

In this formula E_z, I, and f do not depend on ν. On the other hand, if $h \in L^2H(\partial\Omega,\mu)$, then by definition of τ_μ the map

$$\mathrm{SAW}(\partial\Omega,\mu) \ni \nu \mapsto ||h||_\nu^2 = B_\nu(h,h) \in \mathbb{R}$$

is continuous (We used the same notation as in the definition of the topology τ_ν). Moreover, by standard arguments from calculus, we get that if a map

$$\mathrm{SAW}(\partial\Omega,\mu) \ni \nu \mapsto g_\nu \in L^2H(\partial\Omega,\mu)$$

is continuous, then the map

$$\mathrm{SAW}(\partial\Omega,\mu) \ni \nu \mapsto ||g_\nu||_\nu^2 = B_\nu(g_\nu,g_\nu) \in \mathbb{R}$$

is continuous.

To complete the proof of the theorem it is enough to show that the map

$$\mathrm{SAW}(\partial\Omega,\mu) \ni \nu \mapsto P_\nu \in L(L^2H(\partial\Omega,\mu))$$

is continuous. But this is a direct consequence of Theorem 2.1.

In our case, taking $\mathcal{H} = L^2H(\partial\Omega,\mu)$, $V = \ker E_z$, and $A = (\Psi_\mu \circ B_\mu)(\nu)$ for $\nu \in \mathrm{SAW}(\partial\Omega,\mu)$, we obtain that the map

$$\mathrm{SAW}(\partial\Omega,\mu) \ni \nu \mapsto P_\nu = P((\Psi_\mu \circ B_\mu)(\nu)) \in L(L^2H(\partial\Omega,\mu))$$

is continuous. This ends the proof of the theorem. □

We could formulate Theorem 2.2 as follows: for any $z \in \Omega$ the map

$$\mathrm{SAW}(\partial\Omega) \ni \mu \mapsto \overline{S_\mu(z,\cdot)} \in \bigcup_{\nu\in\mathrm{SAW}(\partial\Omega)} L^2H(\partial\Omega,\nu)$$

is continuous. This requires, however, introducing a topology on the set

$$\bigcup_{\nu\in\mathrm{SAW}(\partial\Omega)} L^2H(\partial\Omega,\nu)$$

that is locally compatible with the topologies of Hilbert spaces on its components. It is possible, but we shall omit it, because it complicates the considerations and is not needed in this paper.

3 Concluding Remarks

Our purpose here has been to study Szegő kernels calculated with discontinuous weights. We introduce a topology on the set of admissible weights, and prove some deformation theorems. This expands and enhances the known theory of weights.

Author Contributions (1) TŻ is involved in receiving all the results, about 35% of the total; (2) ZP-W is involved in receiving all the results, about 30% of the total; (3) JS is involved in receiving results of Section 2, about 20% of the total; (4) SGK participates in receiving results of Section 2 (Theorems), about 15% of the total. TŻ wrote the main manuscript text.

Declarations

Conflict of interests The authors declare no competing interests.

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