The $\mathcal{F}$-Resolvent Equation and Riesz Projectors for the $\mathcal{F}$-Functional Calculus

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Abstract

The Fueter-Sce-Qian mapping theorem gives a constructive way to extend holomorphic functions of one complex variable to slice hyperholomorphic functions. By means of the Cauchy formula for slice hyperholomorphic functions it is possible to have a Fueter-Sce-Qian mapping theorem in integral form for n odd. On this theorem it is based the $\mathcal{F}$-functional calculus for n-tuples of commuting operators. It is a functional calculus based on the commutative version of the S spectrum. Furthermore, it is a monogenic functional calculus in the spirit of McIntosh and collaborators. In this paper, inspired by the quaternionic case and some particular Clifford algebras cases, we show a general resolvent equation for the $\mathcal{F}$-functional calculus in the Clifford algebra setting. Moreover, we prove that the $\mathcal{F}$-resolvent equation is the suitable equation to study the Riesz projectors.

Keywords Spectral theory on the S-spectrum · $\mathcal{F}$-Resolvent operators · $\mathcal{F}$-Resolvent equation · Riesz projectors for the $\mathcal{F}$-functional calculus

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1 Introduction

One of the deepest and important result in hypercomplex analysis is the Fueter-Sce-Qian mapping theorem. This gives a two step procedure to construct a monogenic function, i.e., null solutions of the generalized Cauchy Riemann operator in $\mathbb{R}^{n+1}$, starting from a holomorphic function of one complex variable. In the first step the class of holomorphic functions is extended to the one of slice hyperholomorphic functions. In the second step the class of monogenic functions is generated by applying the so-called Fueter-Sce-Qian map, namely $\Delta^{\frac{n-1}{2}}$ (where Δ is the Laplace operator), to the class of slice hyperholomorphic functions. For more details see [27].

Nowadays both slice hyperholomorphic and monogenic functions are widely studied. See, for example, the books, [25, 26, 33, 34], and the references therein, for further information about slice hyperholomorphic functions. Recently, this theory of functions has generated the following research directions: quaternionic integral transforms see [30, 31], quaternionic perturbation theory and invariant subspaces [8], the characteristic operator functions and applications to linear system theory [5], Schur analysis [4], fractional powers of quaternionic linear operators [14]. Based on these there are new classes of fractional diffusion problems [9, 10, 20].

For further information about the theory of monogenic functions see, for instance, [7, 24, 28, 35] and references therein. Some applications of this function theory are related to the functional calculus, see [36], peculiar integral transforms [29], singular integrals [38].

A Cauchy formula holds for slice hyperholomorphic functions and it is the heart of the S-functional calculus. In order to give further information about this we need some preliminary material.

Let $\mathbb{R}_n$ be the real Clifford algebra over n imaginary units $e_1, \ldots, e_n$ satisfying the relations $e_\ell e_m + e_m e_\ell = 0$, $\ell \neq m$, $e_\ell^2 = -1$. An element in the Clifford algebra will be denoted by $\sum_A e_A x_A$ where $A = \{\ell_1 \ldots \ell_r\} \in \mathcal{P}\{1, 2, \ldots, n\}$, $\ell_1 < \ldots < \ell_r$ is a multi-index and $e_A = e_{\ell_1} e_{\ell_2} \ldots e_{\ell_r}$, $e_\emptyset = 1$. A point $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ will be identified with the element $x = x_0 + \underline{x} = x_0 + \sum_{j=1}^n x_j e_j \in \mathbb{R}_n$ called paravector and the real part x_0 of x will also be denoted by $\text{Re}(x)$. The vector part of x is defined by $\underline{x} = x_1 e_1 + \cdots + x_n e_n$. The conjugate of x is denoted by $\overline{x} = x_0 - \underline{x}$ and the Euclidean modulus of x is given by $|x|^2 = x_0^2 + \cdots + x_n^2$.

In this paper we work in a Clifford algebra setting, and in the sequel we denote by $\mathcal{B}(V_n)$ the Banach space of all bounded right linear operators acting on a two sided Clifford Banach module $V_n = V \otimes \mathbb{R}_n$, where V is a real Banach space.

Let $T : V_n \to V_n$ be a bounded right linear operator. The formulation of a quaternionic quantum mechanic given by G.Birkhoff and J.Von Neumann in the paper [6] suggests the existence of an appropriate quaternionic spectrum. The notion of S-spectrum was discovered in 2006 by F.Colombo and I.Sabadini, see [15, 26].

The S-spectrum is defined in an unconventional way because the square of the linear operator T is involved, it is given by:

$$\sigma_S(T) := \{s \in \mathbb{R}^{n+1} \,:\, T^2 - 2\text{Re}(s)T + |s|^2 \mathcal{I} \text{ is not invertible}\},$$

and the resolvent set is defined as its complementary set, namely:

$$\rho_S(T) = \mathbb{R}^{n+1} \setminus \sigma_S(T).$$

The left and the right S-resolvent operators are defined as

$$S_L^{-1}(s, T) := -(T^2 - 2\mathrm{Re}(s)T + |s|^2 \mathcal{I})^{-1}(T - \bar{s}\mathcal{I}), \quad s \in \rho_S(T),$$

and

$$S_R^{-1}(s, T) := -(T - \bar{s}\mathcal{I})(T^2 - 2\mathrm{Re}(s)T + |s|^2 \mathcal{I})^{-1}, \quad s \in \rho_S(T),$$

respectively. The left S-resolvent operator satisfies the equation

$$S_L^{-1}(s, T)s - T S_L^{-1}(s, T) = \mathcal{I}, \quad s \in \rho_S(T), \tag{1.1}$$

and the right S-resolvent operator satisfies

$$s S_R^{-1}(s, T) - S_R^{-1}(s, T)T = \mathcal{I}, \quad s \in \rho_S(T). \tag{1.2}$$

By combining in a suitable way formulas (1.1) and (1.2) it is possible to get the so-called S-resolvent equation, see [1]:

$$S_R^{-1}(s, T)S_L^{-1}(p, T) = \left[[S_R^{-1}(s, T) - S_L^{-1}(p, T)]p - \bar{s}[S_R^{-1}(s, T) - S_L^{-1}(p, T)] \right] \\ (p^2 - 2s_0 p + |s|^2)^{-1}, \tag{1.3}$$

for $s, p \in \rho_S(T)$. The peculiarity of this equation is that both the left and the right resolvent operators are involved. Moreover it preserves the right slice hyperholomorphicity in s and the left slice hyperholomorphicity in p.

Remark 1.1 In the S-resolvent equation the product $S_L^{-1}(p, T)S_R^{-1}(s, T)$ cannot be used, because it destroys the slice hyperholomorphicity.

We observe that in this setting the spectral mapping theorem plays a very important role, see the quaternionic case [2], the particular cases [3] and the extension to fully Clifford algebras, see [17]. For recent contributions on the S-functional calculus, see for example [16, 18].

In [22] a commutative version of the S-functional calculus is studied. To introduce this we need some preliminary notations.

In the sequel, we will consider bounded paravector operators T, with commuting components $T_\ell \in \mathcal{B}(V)$ for $\ell = 0, 1, \ldots, n$, n odd. By $\mathcal{BC}(V_n)$ we will denote the subset of $\mathcal{B}(V_n)$ consisting of Clifford operators with commuting components, i.e., operators of the type $\sum_A e_A T_A$ where $A = \{\ell_1 \ldots \ell_r\} \in \mathcal{P}\{1, 2, \ldots, n\}$, $\ell_1 < \cdots < \ell_r$ is a multi-index, $T_\emptyset = T_0$, and the operators T_A commute among themselves.

Let us consider the paravector operator $T = T_0 + e_1 T_1 + \cdots + e_n T_n$ in $\mathcal{BC}(V_n)$, the $\mathcal{F}$-spectrum of T is defined as

$$\sigma_{\mathcal{F}}(T) = \{s \in \mathbb{R}^{n+1} : s^2 \mathcal{I} - s(T + \overline{T}) + T\overline{T} \text{ is not invertible in } \mathcal{B}(V_n)\},$$

where we have set $\overline{T} := T_0 - e_1 T_1 - \ldots - e_n T_n$, and the $\mathcal{F}$-resolvent set

$$\rho_{\mathcal{F}}(T) := \mathbb{R}^{n+1} \setminus \sigma_{\mathcal{F}}(T).$$

In [23] it is showed that the $\mathcal{F}$-spectrum is the commutative version of the S-spectrum, i.e., we have

$$\sigma_{\mathcal{F}}(T) = \sigma_S(T), \quad \text{for } T \in \mathcal{BC}(V_n).$$

The definition of $\mathcal{F}$-spectrum comes from the shape of the commutative S-resolvent operators. Let us consider an operator $T \in \mathcal{BC}(V_n)$, the commutative version of the left S-resolvent operator is defined as

$$S_L^{-1}(s, T) := (s\mathcal{I} - \overline{T})(s^2 \mathcal{I} - s(T + \overline{T}) + T\overline{T})^{-1}, \quad s \in \rho_{\mathcal{F}}(T), \qquad (1.4)$$

and the commutative version of the right S-resolvent operator is

$$S_R^{-1}(s, T) := (s^2 \mathcal{I} - s(T + \overline{T}) + T\overline{T})^{-1}(s\mathcal{I} - \overline{T}), \quad s \in \rho_{\mathcal{F}}(T). \qquad (1.5)$$

For the sake of simplicity we have still denoted the commutative version of the S-resolvent operators with the same symbols as for the noncommutative ones. The operator

$$\mathcal{Q}_s(T) := (s^2 \mathcal{I} - s(T + \overline{T}) + T\overline{T})^{-1}, \quad s \in \rho_{\mathcal{F}}(T),$$

is called the commutative pseudo S-resolvent operator (for short, it is called pseudo resolvent operator).

In the sequel, when we mention the S-resolvent operators we intend their commutative versions.

By applying the Fueter-Sce map, namely $\Delta^{\frac{n-1}{2}}$ with n odd, to the slice hyperholomorphic Cauchy formulas it is possible to get a Fueter-Sce mapping theorem in integral form, see [23]. This result is crucial to define the $\mathcal{F}$-functional calculus, which is a monogenic functional calculus in the spirit of A.McIntosh and collaborators, see [36]. For more information about the $\mathcal{F}$-functional see Sect. 2 of this paper and the papers [11, 13, 21], here we are only interested to recall the definitions of $\mathcal{F}$-resolvent operators. Let us consider $T \in \mathcal{BC}(X)$ and $s \in \rho_{\mathcal{F}}(T)$, for n being an odd number we define the left $\mathcal{F}$-resolvent operator as

$$\mathcal{F}_n^L(s, T) := \gamma_n (s\mathcal{I} - \overline{T})(s^2 \mathcal{I} - s(T + \overline{T}) + T\overline{T})^{-\frac{n+1}{2}}, \qquad (1.6)$$

and the right $\mathcal{F}$-resolvent operator as

$$\mathcal{F}_n^R(s, T) := \gamma_n (s^2 \mathcal{I} - s(T + \overline{T}) + T\overline{T})^{-\frac{n+1}{2}} (s\mathcal{I} - \overline{T}), \tag{1.7}$$

where the constant γ_n is defined by

$$\gamma_n := (-1)^{\frac{n-1}{2}} 2^{n-1} \left[\left(\frac{n-1}{2} \right)! \right]^2. \tag{1.8}$$

In [13] a resolvent equation for the $\mathcal{F}$-functional calculus in the quaternionic case (which coincides with the case $n = 3$) was obtained, and it is given by

$$
\begin{aligned}
&\mathcal{F}_3^R(s, T)S_L^{-1}(p, T) + S_R^{-1}(s, T)\mathcal{F}_3^L(p, T) \\
&\quad - \frac{1}{4}\Big(s\mathcal{F}_3^R(s, T)\mathcal{F}_3^L(p, T)p - s\mathcal{F}_3^R(s, T)T\mathcal{F}_3^L(p, T) - \mathcal{F}_3^R(s, T)T\mathcal{F}_3^L(p, T)p \\
&\quad + \mathcal{F}_3^R(s, T)T^2\mathcal{F}_3^L(p, T)\Big) \\
&= \Big[\Big(\mathcal{F}_3^R(s, T) - \mathcal{F}_3^L(p, T)\Big) p - \bar{s}\Big(\mathcal{F}_3^R(s, T) - \mathcal{F}_3^L(p, T)\Big)\Big](p^2 - 2s_0 p + |s|^2)^{-1}.
\end{aligned}
\tag{1.9}
$$

for $T \in \mathcal{BC}(V_3)$ and for any $p, s \in \rho_{\mathcal{F}}(T)$, with $s \notin [p]$.

The $\mathcal{F}$-functional calculus depends crucially on the powers of the $\Delta^{\frac{n-1}{2}}$. This means that when we increase the dimension of the algebra we have to increase the power of the Laplacian and this generates more involved computations.

The $\mathcal{F}$-resolvent equation when $n = 5$ is written in terms of the S-resolvent operators and of the commutative pseudo S-resolvent operators, it is given by

$$
\begin{aligned}
&\mathcal{F}_5^R(s, T)S_L^{-1}(p, T) + S_R^{-1}(s, T)\mathcal{F}_5^L(p, T) + \gamma_5 \mathcal{Q}_s(T)S_R^{-1}(s, T)S_L^{-1}(p, T)\mathcal{Q}_p(T) \\
&\qquad\qquad + \gamma_5[\mathcal{Q}_s^2(T)\mathcal{Q}_p(T) + \mathcal{Q}_s(T)\mathcal{Q}_p^2(T)] \\
&= \{[\mathcal{F}_5^R(s, T) - \mathcal{F}_5^L(p, T)]p - \bar{s}[\mathcal{F}_5^R(s, T) - \mathcal{F}_5^L(p, T)]\}(p^2 - 2s_0 p + |s|^2)^{-1}. \quad (1.10)
\end{aligned}
$$

for $p, s \in \rho_{\mathcal{F}}(T)$ and where γ_5 is given by (1.8) for $n = 5$. In [11] is showed that this equation can be still written in terms of $\mathcal{F}$-resolvent operators.

For the case $n = 7$ we have the following resolvent equation

$$
\begin{aligned}
&\mathcal{F}_7^R(s, T)S_L^{-1}(p, T) + S_R^{-1}(s, T)\mathcal{F}_7^L(p, T) + \gamma_7[\mathcal{Q}_s(T)S_R^{-1}(s, T)S_L^{-1}(p, T)\mathcal{Q}_p^2(T) \\
&+ \mathcal{Q}_s^2(T)S_R^{-1}(s, T)S_L^{-1}(p, T)\mathcal{Q}_p(T) + \mathcal{Q}_s(T)\mathcal{Q}_p^3(T) + \mathcal{Q}_s^3(T)\mathcal{Q}_p(T) + \mathcal{Q}_s^2(T)\mathcal{Q}_p^2(T)] \\
&= \{[\mathcal{F}_7^R(s, T) - \mathcal{F}_7^L(p, T)]p - \bar{s}[\mathcal{F}_7^R(s, T) - \mathcal{F}_7^L(p, T)]\}(p^2 - 2s_0 p + |s|^2)^{-1}. \quad (1.11)
\end{aligned}
$$

Nevertheless, in this case it is too much complicated to write the equation (1.11) in terms of the $\mathcal{F}$-resolvent operators.

This last case shows that the general case of n-tuples of operators can be treated only in terms of S-resolvent operators and commutative pseudo S-resolvent operators. In this paper we show, with all the details, the proof of the following formula

$$\mathcal{F}_n^R(s, T)S_L^{-1}(p, T) + S_R^{-1}(s, T)\mathcal{F}_n^L(p, T)$$

$$+ \gamma_n \left[\sum_{i=0}^{h-2} \mathcal{Q}_s^{h-i-1}(T)S_R^{-1}(s, T)S_L^{-1}(p, T)\mathcal{Q}_p^{i+1}(T) + \sum_{i=0}^{h-1} \mathcal{Q}_s^{h-i}(T)\mathcal{Q}_p^{i+1}(T) \right]$$

$$= \left\{ \left[\mathcal{F}_n^R(s, T) - \mathcal{F}_n^L(p, T) \right]p - \bar{s}\left[\mathcal{F}_n^R(s, T) - \mathcal{F}_n^L(p, T) \right] \right\}(p^2 - 2s_0 p + |s|^2)^{-1}, \tag{1.12}$$

where $h = \frac{n-1}{2}$, n is odd, $T \in \mathcal{BC}(V_n)$ and $p, s \in \rho_{\mathcal{F}}(T)$.

In the introduction of the paper [11] a comparison between all the properties of the resolvent equations of the holomorphic functional calculus, the S-functional calculus and the $\mathcal{F}$-functional calculus, is done.

Moreover, we prove that this equation is fundamental to study the Riesz projectors that are defined by

$$\check{P} = \frac{1}{2\pi \gamma_n} \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T)dp_I p^{n-1} = \frac{1}{2\pi \gamma_n} \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{n-1}ds_I \mathcal{F}_n^R(s, T), \tag{1.13}$$

where γ_n is given in (1.8) and the sets G_1 and G_2 contain part of the $\mathcal{F}$-spectrum.

Outline of the paper: Besides of this introduction the paper consist of 4 sections. In Section 2 we recall basic results of the theories of slice hyperholomorphic and monogenic functions. Moreover, we recall also the notions of S-functional functional calculus and $\mathcal{F}$-functional calculus. In Section 3 we provide an expression of the $\mathcal{F}$-resolvent equation for any n odd. Finally in Section 4 we study the Riesz projectors for the $\mathcal{F}$-functional calculus by using the $\mathcal{F}$-resolvent equation.

2 Preliminary Material

Keeping in mind the notations about the real Clifford algebra $\mathbb{R}_n$, given in the Introduction, we recall the main notions for the slice hyperholomorphic and monogenic functions. These two classes of functions are the ones that appear in the Fueter-Sce-Qian construction and they extend holomorphic functions to quaternionic or Clifford algebra valued-functions.

2.1 Function Theories

We start by recalling the concept of slice hyperholomorphic function. To introduce this notion we need to fix some notations. We denote by $\mathbb{S}$ the sphere of purely imaginary vectors with modulus 1, which is defined by

$$\mathbb{S} = \{\underline{x} = e_1 x_1 + \cdots + e_n x_n \mid x_1^2 + \cdots + x_n^2 = 1\}.$$

We observe that if $I \in \mathbb{S}$, then $I^2 = -1$. This means that I behaves like an imaginary unit, and we denote by

$$\mathbb{C}_I := \{u + Iv \mid u, v \in \mathbb{R}\},$$

an isomorphic copy of the complex numbers. Given a non-real paravector $x = x_0 + \underline{x} = x_0 + J_x|\underline{x}|$, we set $J_x := \underline{x}/|\underline{x}| \in \mathbb{S}$, and we associate to x the sphere defined by

$$[x] = \{x_0 + J|\underline{x}| \mid J \in \mathbb{S}\}.$$

Definition 2.1 Let $U \subseteq \mathbb{R}^{n+1}$. We say that U is axially symmetric if, for every $u + Iv \in U$, all the elements $u + Jv$ for $J \in \mathbb{S}$ are contained in U.

Definition 2.2 Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric open set and let $\mathcal{U} \subseteq \mathbb{R} \times \mathbb{R}$ be such that $x = u + Jv \in U$ for all $(u, v) \in \mathcal{U}$. We say that a function $f : U \to \mathbb{R}_n$ of the form

$$f(x) = f_0(u, v) + J f_1(u, v)$$

is left slice hyperholomorphic if f_0, f_1 are $\mathbb{R}_n$-valued differentiable functions such that

$$f_0(u, v) = f_0(u, -v), \quad f_1(u, v) = -f_1(u, -v) \quad \text{for all } (u, v) \in \mathcal{U}$$

and if f_0 and f_1 satisfy the Cauchy-Riemann system

$$\partial_u f_0 - \partial_v f_1 = 0, \quad \partial_v f_0 + \partial_u f_1 = 0.$$

We recall that right slice hyperholomorphic functions are of the form

$$f(x) = f_0(u, v) + f_1(u, v) J$$

where f_0, f_1 satisfy the above conditions.

Remark 2.3 There are other different notions of slice hyperholomorphic functions. However, the previous definition is the most appropriate for the operator theory see [14, 15].

The set of left (resp. right) slice hyperholomorphic function on U is denoted with the symbol $SH_L(U)$ (resp. $SH_R(U)$). The subset of intrinsic functions consist of those slice hyperholomorphic functions such that f_0, f_1 are real-valued and is denoted by $N(U)$.

Now, we recall the slice hyperholomorphic Cauchy formulas, that are crucial to develop the hyperholomorphic spectral theory on the S-spectrum.

Definition 2.4 Let $x \notin [s]$. We define and

$$S_L^{-1}(s, x) := -(x^2 - 2\text{Re}(s)x + |s|^2)^{-1}(x - \bar{s}) \tag{2.1}$$

$$= (s - \bar{x})(s^2 - 2\text{Re}(x)s + |x|^2)^{-1}, \tag{2.2}$$

$$S_R^{-1}(s, x) := -(x - \bar{s})(x^2 - 2\mathrm{Re}(s)x + |s|^2)^{-1} \tag{2.3}$$

$$= (s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-1}(s - \bar{x}). \tag{2.4}$$

We say that (2.1) is the left Cauchy kernel in form I, while (2.2) is in the form II. Analogously, we say that the right Cauchy kernel in form I, while (2.4) is in the form II.

Lemma 2.5 *Let $s \notin [x]$.*

- *The left slice hyperholomorphic Cauchy kernel $S_L^{-1}(s, x)$ is left slice hyperholomorphic in x and right slice hyperholomorphic in s.*
- *The right slice hyperholomorphic Cauchy kernel $S_R^{-1}(s, x)$ is left slice hyperholomorphic in s and right slice hyperholomorphic in x.*

Theorem 2.6 (The Cauchy formulas for slice monogenic functions) *Let $U \subset \mathbb{R}^{n+1}$ be a bounded slice Cauchy domain, let $J \in \mathbb{S}$ and set $ds_J = ds(-J)$. If f is a (left) slice monogenic function on a set that contains $\overline{U}$ then*

$$f(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} S_L^{-1}(s, x) \, ds_J \, f(s), \quad \text{for any } x \in U. \tag{2.5}$$

If f is a right slice hyperholomorphic function on a set that contains $\overline{U}$, then

$$f(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} f(s) \, ds_J \, S_R^{-1}(s, x), \quad \text{for any } x \in U. \tag{2.6}$$

These integrals depend neither on U nor on the imaginary unit $J \in \mathbb{S}$.

Now, we recall the definition of monogenic functions

Definition 2.7 Let U be an open set in $\mathbb{R}^{n+1}$. A real differentiable function $f : U \to \mathbb{R}_n$ is left monogenic if

$$Df(x) := \frac{\partial}{\partial x_0} f(x) + \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i} f(x) = 0.$$

It is right monogenic if

$$f(x)D := \frac{\partial}{\partial x_0} f(x) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(x)e_i = 0.$$

Also for this class of functions it is possible to have a Cauchy formula, that is the heart of the functional calculus developed by A. McIntosh and collaborators see [36].

A bridge between the theory of slice hyperholomorphic and monogenic functions is the Fueter-Sce theorem. This was proved by R. Fueter in 1934, see [32] for quaternions. More than 20 years later M. Sce, see [39], extended this result to Clifford algebras in

a very general way; see [27] for an English translation of Sce works in hypercomplex analysis.

In the original paper of R. Fueter, [32], holomorphic functions are defined on open sets of the complex upper half plane. However, this condition can be relaxed if one consider functions of the following type

$$g(z) = g_0(u, v) + i g_1(u, v), \quad z = u + iv$$

defined in a set $D \subseteq \mathbb{C}$, symmetric with respect to the real axis such that the functions g_0 and g_1 satisfy the so-called even-odd conditions, namely

$$g_0(u, -v) = g_0(u, v) \quad \text{and} \quad g_1(u, -v) = -g_1(u, v).$$

Furthermore g_0 and g_1 satisfy the Cauchy-Riemann system. The same conditions are required by Sce for higher dimensions.

Theorem 2.8 (Sce [39]) *Consider the Euclidean space $\mathbb{R}^{n+1}$, n odd, whose elements are identified with paravectors $x = x_0 + \underline{x}$. Let $\tilde{f}(z) = f_0(u, v) + i f_1(u, v)$ be a holomorphic function defined in a domain (open and connected) D in the upper-half complex plane and let*

$$\Omega_D = \{x = x_0 + \underline{x} \mid (x_0, |\underline{x}|) \in D\}$$

be the open set induced by D in $\mathbb{R}^{n+1}$. The map

$$f(x) = T_{FS1}(\tilde{f}) := f_0\left(x_0, |\underline{x}|\right) + \frac{\underline{x}}{|\underline{x}|} f_1(x_0, |\underline{x}|)$$

takes the holomorphic functions $\tilde{f}(z)$ and gives the Clifford-valued function $f(x)$. Then the function

$$\check{f}(x) := T_{FS2}\left(f_0(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} f_1(x_0, |\underline{x}|)\right),$$

where $T_{FS2} := \Delta_{n+1}^{\frac{n-1}{2}}$ and Δ_{n+1} is the Laplacian in $n+1$ dimensions, is a monogenic function.

We observe that the operator T_{FS2} is a differential operator when n is odd. On the other side when n is even the operator T_{FS2} is a fractional operator, and it is possible to define T_{FS2} by means of the Fourier multipliers, as Tao Qian did in in [37].

Moreover, we note that a different method to connect slice hyperholomorphic and monogenic functions is the Radon and dual Radon transform, see [19].

In [23] the Fueter-Sce theorem is written in integral form. The main advantage of this approach is that one can obtain a monogenic function by integrating suitable slice hyperholomorphic functions. We recall what happens when we apply the Fueter-Sce map $T_{FS2} = \Delta_{n+1}^{\frac{n-1}{2}}$ to the slice hyperholomorphic Cauchy kernels written in the second form.

Theorem 2.9 *Let n be an odd number and let* $x, s \in \mathbb{R}^{n+1}$. *For* $s \notin [x]$, *we have*

$$\Delta^{\frac{n-1}{2}} S_L^{-1}(s, x) = \gamma_n(s - \bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-\frac{n+1}{2}},$$

and

$$\Delta^{\frac{n-1}{2}} S_R^{-1}(s, x) = \gamma_n(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-\frac{n+1}{2}}(s - \bar{x}),$$

where the constants γ_n *are defined in (1.8).*

Remark 2.10 The previous result has been generalized for all the dimensions in [12].

Proposition 2.11 *Let n be an odd number and let* $x, s \in \mathbb{R}^{n+1}$ *be such that* $x \notin [s]$. *Let* $S_L^{-1}(s, x)$ *and* $S_R^{-1}(s, x)$ *be the slice hyperholomorphic Cauchy kernels in form II. Then:*

- *The function* $\Delta^{\frac{n-1}{2}} S_L^{-1}(s, x)$ *is a left monogenic function in the variable x and right slice hyperholomorphic in s.*
- *The function* $\Delta^{\frac{n-1}{2}} S_R^{-1}(s, x)$ *is a right monogenic function in the variable x and left slice hyperholomorphic in s.*

Definition 2.12 (*The F-kernels*) Let *n* be an odd number and let $x, s \in \mathbb{R}^{n+1}$. We define, for $s \notin [x]$, the $\mathcal{F}_n^L$-kernel as

$$\mathcal{F}_n^L(s, x) := \Delta^{\frac{n-1}{2}} S_L^{-1}(s, x) = \gamma_n(s - \bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-\frac{n+1}{2}},$$

and the $\mathcal{F}_n^R$-kernel as

$$\mathcal{F}_n^R(s, x) := \Delta^{\frac{n-1}{2}} S_R^{-1}(s, x) = \gamma_n(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-\frac{n+1}{2}}(s - \bar{x}),$$

where the constant γ_n are defined in (1.8).

Theorem 2.13 (The Fueter-Sce mapping theorem in integral form) *Let* $U \subset \mathbb{R}^{n+1}$ *be a bounded slice Cauchy domain, let* $J \in \mathbb{S}$ *and set* $ds_J = ds(-J)$.

(a) If f is a (left) slice monogenic function on a set that contains $\overline{U}$, *then the left monogenic function* $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$ *admits the integral representation*

$$\breve{f}(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} \mathcal{F}_n^L(s, x) ds_J f(s). \tag{2.7}$$

(b) If f is a right slice monogenic function on a set that contains $\overline{U}$, *then the right monogenic function* $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$ *admits the integral representation*

$$\breve{f}(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} f(s) ds_J \mathcal{F}_n^R(s, x). \tag{2.8}$$

The integrals depend neither on U and nor on the imaginary unit $J \in \mathbb{S}$.

2.2 Functional Calculi on the *S*-Spectrum

In the sequel, we will consider bounded paravector operators $T = e_1 T_1 + \cdots + e_n T_n$, with commuting components T_ℓ acting on a real vector space V, i.e. $T_\ell \in \mathcal{B}(V)$ for $\ell = 0, 1, \ldots, n$. The set of bounded paravector operators is denoted by $\mathcal{BC}^{0,1}(V_n)$ where $V_n = V \otimes \mathbb{R}_n$. The subset of $\mathcal{B}(V_n)$ given by the operators T with commuting components T_ℓ will be denoted by $\mathcal{BC}(V_n)$.

Let $T \in \mathcal{BC}^{0,1}(V_n)$. We denote by $\mathcal{SM}_L(\sigma_S(T))$, $\mathcal{SM}_R(\sigma_S(T))$ the set of all left (or right) slice hyperholomorphic functions f with $\sigma_S(T) \subset \mathrm{dom}(f)$. We now recall the following definition:

Definition 2.14 (*The S-functional calculus for n-tuples of operators*) Let V_n be a two sided Banach module and $T \in \mathcal{B}^{0,1}(V_n)$. Let $U \subset \mathbb{R}^{n+1}$ be a bounded slice Cauchy domain that contains $\sigma_S(T)$ and set $ds_J = -ds J$. We define

$$f(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} S_L^{-1}(s, T) \, ds_J \, f(s), \ \ for \ \ f \in \mathcal{SM}_L(\sigma_S(T)), \quad (2.9)$$

and

$$f(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} f(s) \, ds_J \, S_R^{-1}(s, T), \ \ for \ \ f \in \mathcal{SM}_R(\sigma_S(T)). \quad (2.10)$$

The definition of the *S*-functional calculus is well posed since the integrals in (2.9) and (2.10) depend neither on U and nor on the imaginary unit $J \in \mathbb{S}$.

Remark 2.15 The *S*-functional calculus has been generalized for fully Clifford operators with non commuting components in [18].

The $\mathcal{F}$-functional calculus is based on the Fueter-Sce theorem in integral form. Since this result is obtained by means of the second form of the slice hyperholomorphic Cauchy kernels the $\mathcal{F}$-functional calculus is limited to paravector operators with commuting components. Moreover, it is based on the commutative version of the *S*-spectrum, the so-called $\mathcal{F}$-spectrum.

Definition 2.16 (*The $\mathcal{F}$-functional calculus for bounded operators*) Let n be an odd number, let $T = e_1 T_1 + \cdots + e_n T_n \in \mathcal{BC}(V_n)$, assume that the operators $T_\ell, \ell = 1, .., n$ have real spectrum and set $ds_J = ds/J$. For any function $f \in \mathcal{SM}_L(\sigma_S(T))$, we define

$$\check{f}(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} \mathcal{F}_n^L(s, T) \, ds_J \, f(s). \quad (2.11)$$

For any $f \in \mathcal{SM}_R(\sigma_S(T))$, we define

$$\check{f}(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} f(s) \, ds_J \, \mathcal{F}_n^R(s, T), \quad (2.12)$$

where $J \in \mathbb{S}$ and U is a slice Cauchy domain U.

The definition of the $\mathcal{F}$-functional calculus is well posed since the integrals in (2.11) and (2.12) depend neither on U and nor on the imaginary unit $J \in \mathbb{S}$.

We can write the left $\mathcal{F}$-resolvent operator in terms of the pseudo S-resolvent operators as follows.

Definition 2.17 ($\mathcal{F}$-*resolvent operators*) Let n be an odd number. Let us consider $T = T_0 + T_1 e_1 + \cdots + T_n e_n \in \mathcal{BC}^{0,1}(V_n)$. For $s \in \rho_{\mathcal{F}}(T)$, we define the left $\mathcal{F}$-resolvent operator as

$$\mathcal{F}_n^L(s, T) = \gamma_n (s\mathcal{I} - \overline{T}) \mathcal{Q}_s(T)^{\frac{n+1}{2}}, \tag{2.13}$$

and the right $\mathcal{F}$-resolvent operator as

$$\mathcal{F}_n^R(s, T) = \gamma_n \mathcal{Q}_s(T)^{\frac{n+1}{2}} (s\mathcal{I} - \overline{T}), \tag{2.14}$$

where γ_n are defined in (1.8).

For these operators hold the following relations, proved in [13, Thm. 5.1]

Theorem 2.18 (The left and right $\mathcal{F}$-resolvent equations) *Let n be an odd number and let $T \in \mathcal{B}^{0,1}(V_n)$. Let $s \in \rho_{\mathcal{F}}(T)$. Then the $\mathcal{F}$-resolvent operators satisfy the equations*

$$\mathcal{F}_n^L(s, T)s - T\mathcal{F}_n^L(s, T) = \gamma_n \mathcal{Q}_s(T)^{\frac{n-1}{2}} \tag{2.15}$$

and

$$s\mathcal{F}_n^R(s, T) - \mathcal{F}_n^R(s, T)T = \gamma_n \mathcal{Q}_s(T)^{\frac{n-1}{2}}, \tag{2.16}$$

where the constants γ_n are given by (1.8).

In [11] a series expansions of the $\mathcal{F}$-resolvent operators is proved in terms of T and $\overline{T}$. To state this result we need to introduce the following notation

$$K_\ell(m, n) := \left[\Gamma\left(\frac{n+1}{2}\right)\right]^2 2^{n-1} (-1)^{\frac{n-1}{2}} \frac{\left(\frac{n+1}{2}\right)_{m+1-n-\ell} \left(\frac{n-1}{2}\right)_\ell}{\ell!(m-n+1-\ell)!} \quad 0 \le \ell \le m-n+1, \; m \ge n-1$$

Theorem 2.19 *Let $s \in \mathbb{R}^{n+1}$. For $\|T\| < |s|$, we have*

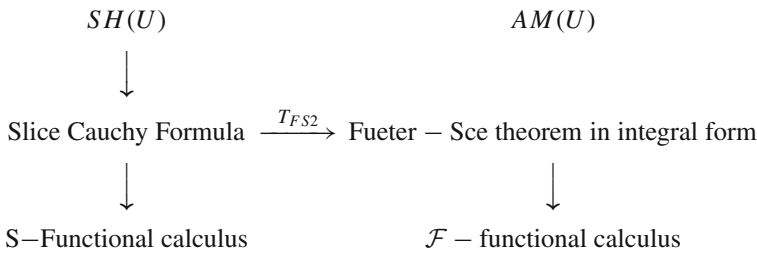$$\sum_{m=n-1}^{+\infty} \sum_{\ell=0}^{m-n+1} K_\ell(m, n) \, T^{m+1-n-\ell} \, \overline{T}^\ell \, s^{-1-m}$$

$$= \gamma_n (s\mathcal{I} - \overline{T})(s^2\mathcal{I} - (T + \overline{T})s + T\overline{T})^{-\frac{n+1}{2}}, \tag{2.17}$$

and

$$\sum_{m=n-1}^{+\infty} \sum_{\ell=0}^{m-n+1} K_\ell(m,n) s^{-1-m} T^{m+1-n-\ell} \bar{T}^\ell$$

$$= \gamma_n (s^2 \mathcal{I} - (T + \overline{T})s + T\overline{T})^{-\frac{n+1}{2}} (s\mathcal{I} - \overline{T}), \qquad (2.18)$$

where γ_n are as in (1.8).

The following diagram sums up all the constructions exposed

$SH(U)$ $\qquad\qquad\qquad\qquad$ $AM(U)$

$\downarrow$

Slice Cauchy Formula $\xrightarrow{T_{FS2}}$ Fueter $-$ Sce theorem in integral form

$\downarrow$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $\downarrow$

S$-$Functional calculus $\qquad\qquad\qquad$ $\mathcal{F}$ $-$ functional calculus

Remark 2.20 Observe that in the above diagram the arrow from the space of axially monogenic function $AM(U)$ is missing because the $\mathcal{F}$-functional calculus is deduced from the slice hyperholomorphic Cauchy formula.

3 The $\mathcal{F}$-Resolvent Equation for *n* Odd

Inspired from the $\mathcal{F}$-resolvent equations in the cases $n = 3,5,7$ studied in [11], we get a general $\mathcal{F}$-resolvent equation for any n odd. From the case $n = 7$, see (1.11), it is clear that writing the $\mathcal{F}$-resolvent equation only in terms of $\mathcal{F}$-resolvent operators is too much complicated. For generic n odd we write the $\mathcal{F}$-resolvent equation in terms of S-resolvent operators and pseudo S-resolvent operators. We start by showing a technical result.

Lemma 3.1 (The general structure of the $\mathcal{F}$-resolvent equation with the pseudo S-resolvent operators) *Let $n > 3$ be an odd number, and let $h = \frac{n-1}{2}$ be the Sce exponent. Let us consider $T \in \mathcal{BC}^{0,1}(V_n)$. Then for $p, s \in \rho_\mathcal{F}(T)$ the following equation holds*

$$\mathcal{F}_n^R(s,T) S_L^{-1}(p,T) + S_R^{-1}(s,T) \mathcal{F}_n^L(p,T)$$

$$+ \gamma_n \left[\sum_{i=0}^{h-2} \mathcal{Q}_s^{h-i-1}(T) S_R^{-1}(s,T) S_L^{-1}(p,T) \mathcal{Q}_p^{i+1}(T) + \sum_{i=0}^{h-1} \mathcal{Q}_s^{h-i}(T) \mathcal{Q}_p^{i+1}(T) \right]$$

$$= \left\{ \left[\mathcal{F}_n^R(s,T) - \mathcal{F}_n^L(p,T) \right] p - \bar{s} \left[\mathcal{F}_n^R(s,T) - \mathcal{F}_n^L(p,T) \right] \right\} (p^2 - 2s_0 p + |s|^2)^{-1}.$$
$$(3.1)$$

Proof We left multiply the S-resolvent equation (1.3) by $\gamma_n \mathcal{Q}_s^h(T)$

$$
\begin{aligned}
\mathcal{F}_n^R(s,T)S_L^{-1}(p,T) &= \Big\{ [\mathcal{F}_n^R(s,T) - \gamma_n \mathcal{Q}_s^h(T)S_L^{-1}(p,T)]p \\
&\quad - \bar{s}[\mathcal{F}_n^R(s,T) - \gamma_n \mathcal{Q}_s^h(T)S_L^{-1}(p,T)] \Big\} \cdot \\
&\quad \cdot (p^2 - 2s_0 p + |s|^2)^{-1}
\end{aligned}
\tag{3.2}
$$

and we right multiply it by $\gamma_n \mathcal{Q}_p^h(T)$, so we get

$$
\begin{aligned}
S_R^{-1}(s,T)\mathcal{F}_n^L(p,T) &= \Big\{ [\gamma_n S_R^{-1}(s,T)\mathcal{Q}_p^h(T) - \mathcal{F}_n^L(p,T)]p - \bar{s}[\gamma_n S_R^{-1}(s,T)\mathcal{Q}_p^h(T) \\
&\quad - \mathcal{F}_n^L(p,T)] \Big\} \cdot \cdot (p^2 - 2s_0 p + |s|^2)^{-1}.
\end{aligned}
\tag{3.3}
$$

We now multiply S-resolvent equation on the left and on the right by $\mathcal{Q}_s^{h-1-i}(T)$ and $\mathcal{Q}_p^{i+1}(T)$, respectively. Then, we sum on the index $0 \le i \le h-2$ and we obtain

$$
\begin{aligned}
\sum_{i=0}^{h-2} \mathcal{Q}_s^{h-1-i}(T)S_R^{-1}(s,T)S_L^{-1}(p,T)\mathcal{Q}_p^{i+1}(T) &= \Big\{ \Big[\sum_{i=0}^{h-2} \mathcal{Q}_s^{h-1-i}(T)S_R^{-1}(s,T)\mathcal{Q}_p^{i+1}(T) \\
&\quad - \mathcal{Q}_s^{h-1-i}(T)S_L^{-1}(p,T)\mathcal{Q}_p^{i+1}(T) \Big]p - \bar{s}\Big[\sum_{i=0}^{h-2} \mathcal{Q}_s^{h-1-i}(T)S_R^{-1}(s,T)\mathcal{Q}_p^{i+1}(T) \\
&\quad - \mathcal{Q}_s^{h-1-i}(T)S_L^{-1}(p,T)\mathcal{Q}_p^{i+1}(T) \Big] \Big\} \cdot (p^2 - 2s_0 p + |s|^2)^{-1}.
\end{aligned}
\tag{3.4}
$$

Now we sum (3.2), (3.3) and (3.4) multiplied by γ_n, and we get

$$
\begin{aligned}
&\mathcal{F}_n^R(s,T)S_L^{-1}(p,T) + S_R^{-1}(s,T)\mathcal{F}_n^L(p,T) \\
&+ \ \gamma_n \sum_{i=0}^{h-2} \mathcal{Q}_s^{h-i-1}(T)S_R^{-1}(s,T)S_L^{-1}(p,T)\mathcal{Q}_p^{i+1}(T) \\
&= \Big\{ \Big[\mathcal{F}_n^R(s,T) - \gamma_n \mathcal{Q}_s^h(T)S_L^{-1}(p,T) \\
&\quad + \gamma_n S_R^{-1}(s,T)\mathcal{Q}_p^h(T) - \mathcal{F}_n^L(p,T) + \gamma_n \sum_{i=0}^{h-2} \mathcal{Q}_s^{h-1-i}(T)S_R^{-1}(s,T)\mathcal{Q}_p^{i+1}(T) \\
&\quad - \mathcal{Q}_s^{h-1-i}(T)S_L^{-1}(p,T)\mathcal{Q}_p^{i+1}(T) \Big]p \\
&\quad - \bar{s}\Big[\mathcal{F}_n^R(s,T) - \gamma_n \mathcal{Q}_s^h(T)S_L^{-1}(p,T) + \gamma_n S_R^{-1}(s,T)\mathcal{Q}_p^h(T) - \mathcal{F}_n^L(p,T) \\
&\quad + \gamma_n \sum_{i=0}^{h-2} \mathcal{Q}_s^{h-1-i}(T)S_R^{-1}(s,T)\mathcal{Q}_p^{i+1}(T)
\end{aligned}
$$

$$- \mathcal{Q}_s^{h-1-i}(T)S_L^{-1}(p,T)\mathcal{Q}_p^{i+1}(T)\Big]\Big\}$$
$$(p^2 - 2s_0 p + |s|^2)^{-1}. \tag{3.5}$$

Putting in order the terms in the right hand side of the previous equation we get

$$\mathcal{F}_n^R(s,T)S_L^{-1}(p,T) + S_R^{-1}(s,T)\mathcal{F}_n^L(p,T)$$
$$+ \gamma_n \sum_{i=0}^{h-2} \mathcal{Q}_s^{h-i-1}(T)S_R^{-1}(s,T)S_L^{-1}(p,T)\mathcal{Q}_p^{i+1}(T)$$
$$= \Big\{[\mathcal{F}_n^R(s,T) - \mathcal{F}_n^L(p,T)]p - \bar{s}[\mathcal{F}_n^R(s,T) - \mathcal{F}_n^L(p,T)]\Big\}(p^2 - 2s_0 p + |s|^2)^{-1}$$
$$+ \gamma_n\Big\{[S_R^{-1}(s,T)\mathcal{Q}_p^h(T) - \mathcal{Q}_s^h(T)S_L^{-1}(p,T) + \sum_{i=0}^{h-2}\mathcal{Q}_s^{h-1-i}(T)S_R^{-1}(s,T)\mathcal{Q}_p^{i+1}(T)$$
$$- \mathcal{Q}_s^{h-1-i}(T)S_L^{-1}(p,T)\mathcal{Q}_p^{i+1}(T)]p - \bar{s}[S_R^{-1}(s,T)\mathcal{Q}_p^h(T) - \mathcal{Q}_s^h(T)S_L^{-1}(p,T)$$
$$+ \sum_{i=0}^{h-2}\mathcal{Q}_s^{h-1-i}(T)S_R^{-1}(s,T)\mathcal{Q}_p^{i+1}(T) - \mathcal{Q}_s^{h-1-i}(T)S_L^{-1}(p,T)\mathcal{Q}_p^{i+1}(T)]\Big\}$$
$$(p^2 - 2s_0 p + |s|^2)^{-1}. \tag{3.6}$$

Now, using the definition of left and right S-resolvent operators we get

$$S_R^{-1}(s,T)\mathcal{Q}_p^h(T) - \mathcal{Q}_s^h(T)S_L^{-1}(p,T) + \sum_{i=0}^{h-2}\mathcal{Q}_s^{h-1-i}(T)S_R^{-1}(s,T)\mathcal{Q}_p^{i+1}(T)$$
$$- \sum_{i=0}^{h-2}\mathcal{Q}_s^{h-1-i}(T)S_L^{-1}(p,T)\mathcal{Q}_p^{i+1}(T)$$
$$= \mathcal{Q}_s(T)(s\mathcal{I} - \overline{T})\mathcal{Q}_p^h(T) - \mathcal{Q}_s^h(T)(p\mathcal{I} - \overline{T})\mathcal{Q}_p(T) + \sum_{i=0}^{h-2}\mathcal{Q}_s^{h-i}(T)(s\mathcal{I} - \overline{T})\mathcal{Q}_p^{i+1}(T)$$
$$- \sum_{i=0}^{h-2}\mathcal{Q}_s^{h-1-i}(T)(p\mathcal{I} - \overline{T})\mathcal{Q}_p^{i+2}(T)$$
$$= \mathcal{Q}_s(T)(s\mathcal{I} - \overline{T})\mathcal{Q}_p^h(T) - \mathcal{Q}_s^h(T)(p\mathcal{I} - \overline{T})\mathcal{Q}_p(T) + \mathcal{Q}_s^h(T)(s\mathcal{I} - \overline{T})\mathcal{Q}_p(T)$$
$$+ \sum_{i=1}^{h-2}\mathcal{Q}_s^{h-i}(T)(s\mathcal{I} - \overline{T})\mathcal{Q}_p^{i+1}(T)$$
$$- \mathcal{Q}_s(T)(p\mathcal{I} - \overline{T})\mathcal{Q}_p^h(T) - \sum_{i=0}^{h-3}\mathcal{Q}_s^{h-1-i}(T)(p\mathcal{I} - \overline{T})\mathcal{Q}_p^{i+2}(T)$$
$$= \mathcal{Q}_s(T)(s - p)\mathcal{Q}_p^h(T) + \mathcal{Q}_s^h(T)(s - p)\mathcal{Q}_p(T) + \sum_{i=1}^{h-2}\mathcal{Q}_s^{h-i}(T)(s\mathcal{I} - \overline{T})\mathcal{Q}_p^{i+1}(T)$$
$$- \sum_{i=0}^{h-3}\mathcal{Q}_s^{h-1-i}(T)(p\mathcal{I} - \overline{T})\mathcal{Q}_p^{i+2}(T)$$
$$= \mathcal{Q}_s(T)(s - p)\mathcal{Q}_p^h(T) + \mathcal{Q}_s^h(T)(s - p)\mathcal{Q}_p(T) + \sum_{i=1}^{h-2}\mathcal{Q}_s^{h-i}(T)(s\mathcal{I} - \overline{T})\mathcal{Q}_p^{i+1}(T)$$

$$- \sum_{i=1}^{h-2} \mathcal{Q}_s^{h-i}(T)(p\mathcal{I} - \overline{T})\mathcal{Q}_p^{i+1}(T)$$

$$= \mathcal{Q}_s(T)(s - p)\mathcal{Q}_p^h(T) + \mathcal{Q}_s^h(T)(s - p)\mathcal{Q}_p(T) + \sum_{i=1}^{h-2} \mathcal{Q}_s^{h-i}(T)(s - p)\mathcal{Q}_p^{i+1}(T)$$

$$= \sum_{i=0}^{h-1} \mathcal{Q}_s^{h-i}(T)(s - p)\mathcal{Q}_p^{i+1}(T). \tag{3.7}$$

Then we compute

$$\gamma_n \Big\{ \Big[S_R^{-1}(s, T)\mathcal{Q}_p^h(T) - \mathcal{Q}_s^h(T)S_L^{-1}(p, T) + \sum_{i=0}^{h-2} \mathcal{Q}_s^{h-1-i}(T)S_R^{-1}(s, T)\mathcal{Q}_p^{i+1}(T)$$

$$- \mathcal{Q}_s^{h-1-i}(T)S_L^{-1}(p, T)\mathcal{Q}_p^{i+1}(T) \Big] p - \bar{s}\Big[S_R^{-1}(s, T)\mathcal{Q}_p^h(T) - \mathcal{Q}_s^h(T)S_L^{-1}(p, T)$$

$$+ \sum_{i=0}^{h-2} \mathcal{Q}_s^{h-1-i}(T)S_R^{-1}(s, T)\mathcal{Q}_p^{i+1}(T) - \mathcal{Q}_s^{h-1-i}(T)S_L^{-1}(p, T)\mathcal{Q}_p^{i+1}(T) \Big] \Big\}$$

$$(p^2 - 2s_0 p + |s|^2)^{-1}$$

$$= \gamma_n \Big\{ \Big[\sum_{i=0}^{h-1} \mathcal{Q}_s^{h-i}(T)(s - p)\mathcal{Q}_p^{i+1}(T) \Big] p - \bar{s}\Big[\sum_{i=0}^{h-1} \mathcal{Q}_s^{h-i}(T)(s - p)\mathcal{Q}_p^{i+1}(T) \Big] \Big\}$$

$$(p^2 - 2s_0 p + |s|^2)^{-1}$$

$$= \gamma_n \Big[\sum_{i=0}^{h-1} \mathcal{Q}_s^{h-i}(T)(sp - p^2)\mathcal{Q}_p^{i+1}(T) - \sum_{i=0}^{h-1} \mathcal{Q}_s^{h-i}(T)(|s|^2 - \bar{s}p)\mathcal{Q}_p^{i+1}(T) \Big]$$

$$(p^2 - 2s_0 p + |s|^2)^{-1}$$

$$= \gamma_n \Big[\sum_{i=0}^{h-1} \mathcal{Q}_s^{h-i}(T)(sp - p^2 - |s|^2 + \bar{s}p)\mathcal{Q}_p^{i+1}(T) \Big] (p^2 - 2s_0 p + |s|^2)^{-1}$$

$$= -\gamma_n \sum_{i=0}^{h-1} \mathcal{Q}_s^{h-i}(T)\mathcal{Q}_p^{i+1}(T)(p^2 - 2s_0 p + |s|^2)(p^2 - 2s_0 p + |s|^2)^{-1}$$

$$= -\gamma_n \sum_{i=0}^{h-1} \mathcal{Q}_s^{h-i}(T)\mathcal{Q}_p^{i+1}(T).$$

Hence

$$\gamma_n \Big\{ \Big[S_R^{-1}(s, T)\mathcal{Q}_p^h(T) - \mathcal{Q}_s^h(T)S_L^{-1}(p, T) + \sum_{i=0}^{h-2} \mathcal{Q}_s^{h-1-i}(T)S_R^{-1}(s, T)\mathcal{Q}_p^{i+1}(T)$$

$$- \mathcal{Q}_s^{h-1-i}(T)S_L^{-1}(p, T)\mathcal{Q}_p^{i+1}(T) \Big] p - \bar{s}\Big[S_R^{-1}(s, T)\mathcal{Q}_p^h(T) - \mathcal{Q}_s^h(T)S_L^{-1}(p, T)$$

$$+ \sum_{i=0}^{h-2} \mathcal{Q}_s^{h-1-i}(T)S_R^{-1}(s, T)\mathcal{Q}_p^{i+1}(T) - \mathcal{Q}_s^{h-1-i}(T)S_L^{-1}(p, T)\mathcal{Q}_p^{i+1}(T) \Big] \Big\}(p^2 - 2s_0 p + |s|^2)^{-1}$$

$$= -\gamma_n \sum_{i=0}^{h-1} \mathcal{Q}_s^{h-i}(T)\mathcal{Q}_p^{i+1}(T). \tag{3.8}$$

Finally, by substituting (3.8) in (3.6) we get (3.1).

Remark 3.2 The proof of the previous lemma shows that the structure of the resolvent equations of the hyperholomorphic functional calculi is crucial. In fact the term

$$\{[\mathcal{F}_n^R(s, T) - \mathcal{F}_n^L(p, T)]p - \bar{s}[\mathcal{F}_n^R(s, T) - \mathcal{F}_n^L(p, T)]\}(p^2 - 2s_0 p + |s|^2)^{-1}$$

involves the difference of the $\mathcal{F}$-resolvent operators entangled with the Cauchy kernel of slice monogenic functions. This term is equal to a function involving the products of the F-resolvent operators and of the S-resolvent operators that appear in the term

$$\mathcal{F}_n^R(s, T)S_L^{-1}(p, T) + S_R^{-1}(s, T)\mathcal{F}_n^L(p, T)$$

and of a more complicated part that involves the S-resolvent operators and the pseudo S-resolvent operators, namely

$$\gamma_n \left[\sum_{i=0}^{h-2} \mathcal{Q}_s^{h-i-1}(T)S_R^{-1}(s, T)S_L^{-1}(p, T)\mathcal{Q}_p^{i+1}(T) + \sum_{i=0}^{h-1} \mathcal{Q}_s^{h-i}(T)\mathcal{Q}_p^{i+1}(T) \right].$$

Remark 3.3 If in equation (3.1) we consider $n = 5$, then $h = 2$, and $n = 7$, then $h = 3$, we get the equation (1.10) and (1.11).

In order to find a pseudo $\mathcal{F}$-resolvent equation we divide into two cases according to the parity of the Sce exponent $h = \dfrac{n-1}{2}$. To state the following result we introduce this notations

$$\begin{aligned}
\mathcal{A}_0(s, p, T) := & -s^h \mathcal{F}_n^R(s, T)T\mathcal{F}_n^L(p, T)p^{h-1} - s^{h-1}\mathcal{F}_n^R(s, T)T\mathcal{F}_n^L(p, T)p^h \\
& + s^{h-1}\mathcal{F}_n^R(s, T)T^2\mathcal{F}_n^L(p, T)p^{h-1} \\
& + s^h \mathcal{F}_n^R(s, T)\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k}(|T|^2 - 2T_0 p)^k \mathcal{F}_n^L(p, T)p^{h-2k} \\
& - s^h \mathcal{F}_n^R(s, T)\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k}(|T|^2 - 2T_0 p)^k T\mathcal{F}_n^L(p, T)p^{h-1-2k} \\
& - s^{h-1}\mathcal{F}_n^R(s, T)\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k}T(|T|^2 - 2T_0 p)^k \mathcal{F}_n^L(p, T)p^{h-2k} \\
& + s^{h-1}\mathcal{F}_n^R(s, T)\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k}T(|T|^2 - 2T_0 p)^k T\mathcal{F}_n^L(p, T)p^{h-1-2k},
\end{aligned} \tag{3.9}$$

$$
\mathcal{B}_0(s, p, T) := \left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} s^{h-2k} \mathcal{F}_n^R(s, T)(|T|^2 - 2T_0 s)^k \right)
$$

$$
\left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} (|T|^2 - 2T_0 p)^k \mathcal{F}_n^L(p, T) p^{h-2k} \right)
$$

$$
- \left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} s^{h-2k} \mathcal{F}_n^R(s, T)(|T|^2 - 2T_0 s)^k \right)
$$

$$
\left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} (|T|^2 - 2T_0 p)^k T \mathcal{F}_n^L(p, T) p^{h-1-2k} \right)
$$

$$
- \left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} s^{h-2k-1} \mathcal{F}_n^R(s, T) T (|T|^2 - 2T_0 s)^k \right)
$$

$$
\left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} (|T|^2 - 2T_0 p)^k \mathcal{F}_n^L(p, T) p^{h-2k} \right)
$$

$$
+ \left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} s^{h-2k-1} \mathcal{F}_n^R(s, T) T (|T|^2 - 2T_0 s)^k \right)
$$

$$
\left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} (|T|^2 - 2T_0 p)^k T \mathcal{F}_n^L(p, T) p^{h-1-2k} \right), \tag{3.10}
$$

and

$$
\mathcal{C}_0(s, p, T) := \left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} s^{h-2k} \mathcal{F}_n^R(s, T)(|T|^2 - 2T_0 s)^k \right) \mathcal{F}_n^L(p, T) p^h
$$

$$
- \left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} s^{h-2k} \mathcal{F}_n^R(s, T)(|T|^2 - 2T_0 s)^k T \right) \mathcal{F}_n^L(p, T) p^{h-1}
$$

$$
- \left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} s^{h-1-2k} \mathcal{F}_n^R(s, T) T (|T|^2 - 2T_0 s)^k \right) \mathcal{F}_n^L(p, T) p^h
$$

$$+ \left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} s^{h-1-2k} \mathcal{F}_n^R(s, T) T (|T|^2 - 2T_0 s)^k T \right) \mathcal{F}_n^L(p, T) p^{h-1}.$$

$$(3.11)$$

3.1 The General Structure of the Pseudo $\mathcal{F}$-Resolvent Equation for h Odd

The main result of this subsection is the following theorem.

Theorem 3.4 (The general structure of the pseudo $\mathcal{F}$-resolvent equation for h odd number) *Let $n > 3$ be an odd number as well as h. Let $T \in \mathcal{BC}^{0,1}(V_n)$. Then for $p, s \in \rho_{\mathcal{F}}(T)$ the following equation holds*

$$\mathcal{F}_n^R(s, T) S_L^{-1}(p, T) + S_R^{-1}(s, T) \mathcal{F}_n^L(p, T)$$

$$+ \gamma_n \Bigg[s \sum_{i=0}^{h-2} \mathcal{Q}_s^{h-i}(T) \mathcal{Q}_p^{i+2}(T) p - s \sum_{i=0}^{h-2} \mathcal{Q}_s^{h-i}(T) \overline{T} \mathcal{Q}_p^{i+2}(T)$$

$$- \sum_{i=0}^{h-2} \mathcal{Q}_s^{h-i}(T) \overline{T} \mathcal{Q}_p^{i+2}(T) p + \sum_{i=0}^{h-2} \mathcal{Q}_s^{h-i}(T) \overline{T}^2 \mathcal{Q}_p^{i+2}(T) + \sum_{i=0, i \neq \frac{h-1}{2}}^{h-1} \mathcal{Q}_s^{h-i}(T) \mathcal{Q}_p^{i+1}(T) \Bigg]$$

$$+ \gamma_n^{-1} \Bigg[s^h \mathcal{F}_n^R(s, T) \mathcal{F}_n^L(p, T) p^h + \mathcal{A}_0(s, p, T) + \mathcal{B}_0(s, p, T) + \mathcal{C}_0(s, p, T) \Bigg]$$

$$= \{ [\mathcal{F}_n^R(s, T) - \mathcal{F}_n^L(p, T)] p - \bar{s} [\mathcal{F}_n^R(s, T) - \mathcal{F}_n^L(p, T)] \} (p^2 - 2s_0 p + |s|^2)^{-1}, \qquad (3.12)$$

where the three terms $\mathcal{A}_0(s, p, T)$, $\mathcal{B}_0(s, p, T)$ and $\mathcal{C}_0(s, p, T)$ are defined above.

Proof We start by rewriting formula (3.1) as

$$\mathcal{F}_n^R(s, T) S_L^{-1}(p, T) + S_R^{-1}(s, T) \mathcal{F}_n^L(p, T)$$

$$+ \gamma_n \Bigg[\sum_{i=0}^{h-2} \mathcal{Q}_s^{h-i-1}(T) S_R^{-1}(s, T) S_L^{-1}(p, T) \mathcal{Q}_p^{i+1}(T) + \mathcal{Q}_s^{\frac{h+1}{2}}(T) \mathcal{Q}_p^{\frac{h+1}{2}}(T)$$

$$+ \sum_{i=0, i \neq \frac{h-1}{2}}^{h-1} \mathcal{Q}_s^{h-i}(T) \mathcal{Q}_p^{i+1}(T) \Bigg]$$

$$= \{ [\mathcal{F}_n^R(s, T) - \mathcal{F}_n^L(p, T)] p - \bar{s} [\mathcal{F}_n^R(s, T) - \mathcal{F}_n^L(p, T)] \} (p^2 - 2s_0 p + |s|^2)^{-1}.$$

$$(3.13)$$

Now, we focus on the term $\mathcal{Q}_s^{\frac{h+1}{2}}(T) \mathcal{Q}_p^{\frac{h+1}{2}}(T)$ and with some manipulations we obtain

$$\mathcal{Q}_s^{\frac{h+1}{2}}(T) \mathcal{Q}_p^{\frac{h+1}{2}}(T) = \mathcal{Q}_s^{\frac{h+1}{2}}(T) \mathcal{Q}_s^{\frac{h-1}{2}}(T) \mathcal{Q}_s^{\frac{1-h}{2}}(T) \mathcal{Q}_p^{\frac{1-h}{2}}(T) \mathcal{Q}_p^{\frac{h-1}{2}}(T) \mathcal{Q}_p^{\frac{h+1}{2}}(T)$$

$$= \mathcal{Q}_s^h(T) \mathcal{Q}_s^{\frac{1-h}{2}}(T) \mathcal{Q}_p^{\frac{1-h}{2}}(T) \mathcal{Q}_p^h(T). \qquad (3.14)$$

By the binomial formula we get

$$
\mathcal{Q}_s^{\frac{h+1}{2}}(T)\mathcal{Q}_p^{\frac{h+1}{2}}(T) = \mathcal{Q}_s^h(T)(s^2\mathcal{I} - 2sT_0 + T\overline{T})^{\frac{h-1}{2}}(p^2\mathcal{I} - 2pT_0 + T\overline{T})^{\frac{h-1}{2}}\mathcal{Q}_p^h(T)
$$

$$
= \mathcal{Q}_s^h(T)\left(\sum_{k=0}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}s^{h-1-2k}(|T|^2 - 2T_0s)^k\right)
$$

$$
\left(\sum_{k=0}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}(|T|^2 - 2T_0p)^k p^{h-1-2k}\right)\mathcal{Q}_p^h(T)
$$

$$
= \mathcal{Q}_s^h(T)\left(s^{h-1} + \sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}s^{h-1-2k}(|T|^2 - 2T_0s)^k\right)
$$

$$
\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}(|T|^2 - 2T_0p)^k p^{h-1-2k} + p^{h-1}\right)\mathcal{Q}_p^h(T)
$$

$$
= s^{h-1}\mathcal{Q}_s^h(T)\mathcal{Q}_p^h(T)p^{h-1} + s^{h-1}\mathcal{Q}_s^h(T)\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}(|T|^2 - 2T_0p)^k p^{h-1-2k}\right)\mathcal{Q}_p^h(T)
$$

$$
+ \mathcal{Q}_s^h(T)\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}s^{h-1-2k}(|T|^2 - 2T_0s)^k\right)
$$

$$
\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}(|T|^2 - 2T_0p)^k p^{h-1-2k}\right)\mathcal{Q}_p^h(T)
$$

$$
+ \mathcal{Q}_s^h(T)\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}s^{h-1-2k}(|T|^2 - 2T_0s)^k\right)\mathcal{Q}_p^h(T)p^{h-1}. \tag{3.15}
$$

Now, we use the left and right $\mathcal{F}$-resolvent equations, (see Theorem 2.18)

$$
\mathcal{F}_n^L(p, T)p - T\mathcal{F}_n^L(p, T) = \gamma_n \mathcal{Q}_p^h(T)
$$

and

$$
s\mathcal{F}_n^R(s, T) - \mathcal{F}_n^R(s, T)T = \gamma_n \mathcal{Q}_s^h(T).
$$

We go through the computations term by term

$$
s^{h-1}\mathcal{Q}_s^h(T)\mathcal{Q}_p^h(T)p^{h-1}
$$

$$
= \gamma_n^{-2}\Big[s^h\mathcal{F}_n^R(s, T)\mathcal{F}_n^L(p, T)p^h - s^h\mathcal{F}_n^R(s, T)T\mathcal{F}_n^L(p, T)p^{h-1}
$$

$$
- s^{h-1}\mathcal{F}_n^R(s, T)T\mathcal{F}_n^L(p, T)p^h + s^{h-1}\mathcal{F}_n^R(s, T)T^2\mathcal{F}_n^L(p, T)p^{h-1}\Big]. \tag{3.16}
$$

Then we consider

$$
s^{h-1} \mathcal{Q}_s^h (T) \left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} (|T|^2 - 2T_0 p)^k \mathcal{Q}_p^h (T) p^{h-1-2k} \right)
$$

$$
= \gamma_n^{-2} \Bigg[s^h \mathcal{F}_n^R (s, T) \sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} (|T|^2 - 2T_0 p)^k \mathcal{F}_n^L (p, T) p^{h-2k}
$$

$$
- s^h \mathcal{F}_n^R (s, T) \sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} (|T|^2 - 2T_0 p)^k T \mathcal{F}_n^L (p, T) p^{h-1-2k}
$$

$$
- s^{h-1} \mathcal{F}_n^R (s, T) \sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} T (|T|^2 - 2T_0 p)^k \mathcal{F}_n^L (p, T) p^{h-2k}
$$

$$
+ s^{h-1} \mathcal{F}_n^R (s, T) \sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} T (|T|^2 - 2T_0 p)^k T \mathcal{F}_n^L (p, T) p^{h-1-2k} \Bigg]. \quad (3.17)
$$

Then we compute the term

$$
\left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} s^{h-1-2k} \mathcal{Q}_s^h (T) (|T|^2 - 2T_0 s)^k \right) \left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} (|T|^2 - 2T_0 p)^k \mathcal{Q}_p^h (T) p^{h-1-2k} \right)
$$

$$
= \gamma_n^{-2} \Bigg[\left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} s^{h-2k} \mathcal{F}_n^R (s, T) (|T|^2 - 2T_0 s)^k \right)
$$

$$
\left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} (|T|^2 - 2T_0 p)^k \mathcal{F}_n^L (p, T) p^{h-2k} \right)
$$

$$
- \left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} s^{h-2k} \mathcal{F}_n^R (s, T) (|T|^2 - 2T_0 s)^k \right)
$$

$$
\left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} (|T|^2 - 2T_0 p)^k T \mathcal{F}_n^L (p, T) p^{h-1-2k} \right)
$$

$$
- \left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} s^{h-2k-1} \mathcal{F}_n^R (s, T) T (|T|^2 - 2T_0 s)^k \right)
$$

$$
\left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} (|T|^2 - 2T_0 p)^k \mathcal{F}_n^L (p, T) p^{h-2k} \right)
$$

$$
+ \left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} s^{h-2k-1} \mathcal{F}_n^R(s,T) T (|T|^2 - 2T_0 s)^k \right)
$$

$$
\left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} (|T|^2 - 2T_0 p)^k T \mathcal{F}_n^L(p,T) p^{h-1-2k} \right) \Bigg].
\tag{3.18}
$$

We have also

$$
\left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} s^{h-1-2k} \mathcal{Q}_s^h(T) (|T|^2 - 2T_0 s)^k \right) \mathcal{Q}_p^h(T) p^{h-1}
$$

$$
= \gamma_n^{-2} \left[\left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} s^{h-2k} \mathcal{F}_n^R(s,T) (|T|^2 - 2T_0 s)^k \right) \mathcal{F}_n^L(p,T) p^h \right.
$$

$$
- \left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} s^{h-2k} \mathcal{F}_n^R(s,T) (|T|^2 - 2T_0 s)^k T \right) \mathcal{F}_n^L(p,T) p^{h-1}
$$

$$
- \left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} s^{h-1-2k} \mathcal{F}_n^R(s,T) T (|T|^2 - 2T_0 s)^k \right) \mathcal{F}_n^L(p,T) p^h
$$

$$
+ \left. \left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} s^{h-1-2k} \mathcal{F}_n^R(s,T) T (|T|^2 - 2T_0 s)^k T \right) \mathcal{F}_n^L(p,T) p^{h-1} \right].
\tag{3.19}
$$

Finally by using the definition of left and right S-resolvent operators we get

$$
\sum_{i=0}^{h-2} \mathcal{Q}_s^{h-i-1}(T) S_R^{-1}(s,T) S_L^{-1}(p,T) \mathcal{Q}_p^{i+1}(T) = \sum_{i=0}^{h-2} \mathcal{Q}_s^{h-i}(T)(s\mathcal{I} - \overline{T})(p\mathcal{I} - \overline{T}) \mathcal{Q}_p^{i+2}(T)
$$

$$
= s \sum_{i=0}^{h-2} \mathcal{Q}_s^{h-i}(T) \mathcal{Q}_p^{i+2}(T) p - s \sum_{i=0}^{h-2} \mathcal{Q}_s^{h-i}(T) \overline{T} \mathcal{Q}_p^{i+2}(T) - \sum_{i=0}^{h-2} \mathcal{Q}_s^{h-i}(T) \overline{T} \mathcal{Q}_p^{i+2}(T) p
$$

$$
+ \sum_{i=0}^{h-2} \mathcal{Q}_s^{h-i}(T) \overline{T}^2 \mathcal{Q}_p^{i+2}(T),
\tag{3.20}
$$

and this concludes the proof.   □

3.2 The General Structure of the Pseudo $\mathcal{F}$-Resolvent Equation for h Even Number

In this last subsection we consider the case in which $h = (n-1)/2$ is an even number. To state the following result we need these notations

$$
\begin{aligned}
\mathcal{A}_1(s, p, T) := & -s^h \mathcal{F}_n^R(s, T) T \mathcal{F}_n^L(p, T) p^{h-1} - s^{h-1} \mathcal{F}_n^R(s, T) T \mathcal{F}_n^L(p, T) p^h \\
& + s^{h-1} \mathcal{F}_n^R(s, T) T^2 \mathcal{F}_n^L(p, T) p^{h-1}
\end{aligned}
\tag{3.21}
$$

and

$$
\begin{aligned}
\mathcal{B}_1(s, p, T) := & \, s^h \mathcal{F}_n^R(s, T) \sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} (|T|^2 - 2T_0 p)^k \mathcal{F}_n^L(p, T) p^{h-2k} \\
& - s^h \mathcal{F}_n^R(s, T) \sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} (|T|^2 - 2T_0 p)^k T \mathcal{F}_n^L(p, T) p^{h-1-2k} + \\
& - s^{h-1} \mathcal{F}_n^R(s, T) \sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} T (|T|^2 - 2T_0 p)^k \mathcal{F}_n^L(p, T) p^{h-2k} \\
& + s^{h-1} \mathcal{F}_n^R(s, T) \sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} T (|T|^2 - 2T_0 p)^k T \mathcal{F}_n^L(p, T) p^{h-1-2k}
\end{aligned}
\tag{3.22}
$$

and

$$
\begin{aligned}
\mathcal{C}_1(s, p, T) := & \left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} s^{h-2k} \mathcal{F}_n^R(s, T)(|T|^2 - 2T_0 s)^k \right) \\
& \left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} (|T|^2 - 2T_0 p)^k \mathcal{F}_n^L(p, T) p^{h-2k} \right) \\
& - \left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} s^{h-2k} \mathcal{F}_n^R(s, T)(|T|^2 - 2T_0 s)^k \right) \\
& \left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} (|T|^2 - 2T_0 p)^k T \mathcal{F}_n^L(p, T) p^{h-1-2k} \right) \\
& - \left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} s^{h-2k-1} \mathcal{F}_n^R(s, T) T (|T|^2 - 2T_0 s)^k \right)
\end{aligned}
$$

$$
\left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-2}{2}}{k} (|T|^2 - 2T_0 p)^k \mathcal{F}_n^L(p, T) p^{h-2k} \right)
$$

$$
+ \left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} s^{h-2k-1} \mathcal{F}_n^R(s, T) T (|T|^2 - 2T_0 s)^k \right)
$$

$$
\left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} (|T|^2 - 2T_0 p)^k T \mathcal{F}_n^L(p, T) p^{h-1-2k} \right)
$$

$$
+ \left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} s^{h-2k} \mathcal{F}_n^R(s, T)(|T|^2 - 2T_0 s)^k \right) \mathcal{F}_n^L(p, T) p^h
$$

$$
- \left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} s^{h-2k} \mathcal{F}_n^R(s, T)(|T|^2 - 2T_0 s)^k T \right) \mathcal{F}_n^L(p, T) p^{h-1}
$$

$$
- \left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} s^{h-1-2k} \mathcal{F}_n^R(s, T) T (|T|^2 - 2T_0 s)^k \right) \mathcal{F}_n^L(p, T) p^h
$$

$$
+ \left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} s^{h-1-2k} \mathcal{F}_n^R(s, T) T (|T|^2 - 2T_0 s)^k T \right) \mathcal{F}_n^L(p, T) p^{h-1}.
$$
(3.23)

Theorem 3.5 (The general structure of the pseudo $\mathcal{F}$-resolvent equation for h even number) *Let $n > 3$ be an odd number and h be even. Let $T \in \mathcal{BC}^{0,1}(V_n)$. Then for $p, s \in \rho_{\mathcal{F}}(T)$ the following equation holds*

$$
\mathcal{F}_n^R(s, T) S_L^{-1}(p, T) + S_R^{-1}(s, T) \mathcal{F}_n^L(p, T)
$$
$$
+ \gamma_n \left[-s \mathcal{Q}_s^{\frac{h+2}{2}}(T) \overline{T} \mathcal{Q}_p^{\frac{h+2}{2}}(T) - \mathcal{Q}_s^{\frac{h+2}{2}}(T) \overline{T} \mathcal{Q}_p^{\frac{h+2}{2}}(T) p + \mathcal{Q}_s^{\frac{h+2}{2}}(T) \overline{T}^2 \mathcal{Q}_p^{\frac{h+2}{2}}(T) \right.
$$
$$
+ \sum_{i=0}^{h-1} \mathcal{Q}_s^{h-i}(T) \mathcal{Q}_p^{i+1}(T) + s \sum_{i=0, i \neq \frac{h-2}{2}}^{h-2} \mathcal{Q}_s^{h-i}(T) \mathcal{Q}_p^{i+2}(T) p - s \sum_{i=0, i \neq \frac{h-2}{2}}^{h-2} \mathcal{Q}_s^{h-i}(T) \overline{T} \mathcal{Q}_p^{i+2}(T)
$$
$$
\left. - \sum_{i=0, i \neq \frac{h-2}{2}}^{h-2} \mathcal{Q}_s^{h-i}(T) \overline{T} \mathcal{Q}_p^{i+2}(T) p + \sum_{i=0, i \neq \frac{h-2}{2}}^{h-2} \mathcal{Q}_s^{h-i}(T) \overline{T}^2 \mathcal{Q}_p^{i+2}(T) \right]
$$
$$
+ \gamma_n^{-1} \left[\mathcal{A}_1(s, p, T) + \mathcal{B}_1(s, p, T) + \mathcal{C}_1(s, p, T) + s^h \mathcal{F}_n^R(s, T) \mathcal{F}_n^L(p, T) p^h \right]
$$
$$
= \{ [\mathcal{F}_n^R(s, T) - \mathcal{F}_n^L(p, T)] p - \bar{s} [\mathcal{F}_n^R(s, T) - \mathcal{F}_n^L(p, T)] \} (p^2 - 2s_0 p + |s|^2)^{-1}, \tag{3.24}
$$

where the three terms $\mathcal{A}_1(s, p, T)$, $\mathcal{B}_1(s, p, T)$ and $\mathcal{C}_1(s, p, T)$ are defined above.

Proof Let us begin by writing formula (3.1) as

$$
\mathcal{F}_n^R(s, T)S_L^{-1}(p, T) + S_R^{-1}(s, T)\mathcal{F}_n^L(p, T) + \gamma_n \left[\mathcal{Q}_s^{\frac{h}{2}}(T)S_R^{-1}(s, T)S_L^{-1}(p, T)\mathcal{Q}_p^{\frac{h}{2}}(T) \right.
$$

$$
+ \sum_{i=0, i \neq \frac{h-2}{2}}^{h-2} \mathcal{Q}_s^{h-i-1}(T)S_R^{-1}(s, T)S_L^{-1}(p, T)\mathcal{Q}_p^{i+1}(T) + \left. \sum_{i=0}^{h-1} \mathcal{Q}_s^{h-i}(T)\mathcal{Q}_p^{i+1}(T) \right]
$$

$$
= \{[\mathcal{F}_n^R(s, T) - \mathcal{F}_n^L(p, T)]p - \bar{s}[\mathcal{F}_n^R(s, T) - \mathcal{F}_n^L(p, T)]\}(p^2 - 2s_0 p + |s|^2)^{-1}. \quad (3.25)
$$

Now, we focus on the $\mathcal{Q}_s^{\frac{h}{2}}(T)S_R^{-1}(s, T)S_L^{-1}(p, T)\mathcal{Q}_p^{\frac{h}{2}}(T)$. By definition of left and right S-resolvent operators we get

$$
\mathcal{Q}_s^{\frac{h}{2}}(T)S_R^{-1}(s, T)S_L^{-1}(p, T)\mathcal{Q}_p^{\frac{h}{2}}(T) = \mathcal{Q}_s^{\frac{h+2}{2}}(T)(s\mathcal{I} - \overline{T})(p\mathcal{I} - \overline{T})\mathcal{Q}_p^{\frac{h+2}{2}}(T)
$$

$$
= s\mathcal{Q}_s^{\frac{h+2}{2}}(T)\mathcal{Q}_p^{\frac{h+2}{2}}(T)p - s\mathcal{Q}_s^{\frac{h+2}{2}}(T)\overline{T}\mathcal{Q}_p^{\frac{h+2}{2}}(T) - \mathcal{Q}_s^{\frac{h+2}{2}}(T)\overline{T}\mathcal{Q}_p^{\frac{h+2}{2}}(T)p
$$

$$
+ \mathcal{Q}_s^{\frac{h+2}{2}}(T)\overline{T}^2\mathcal{Q}_p^{\frac{h+2}{2}}(T). \quad (3.26)
$$

We continue the calculations only on the term $s\mathcal{Q}_s^{\frac{h+2}{2}}(T)\mathcal{Q}_p^{\frac{h+2}{2}}(T)p$. By the binomial formula we get

$$
s\mathcal{Q}_s^{\frac{h+2}{2}}(T)\mathcal{Q}_p^{\frac{h+2}{2}}(T)p = s\mathcal{Q}_s^h(T)(s^2\mathcal{I} - 2sT_0 + T\overline{T})^{\frac{h-2}{2}}(p^2\mathcal{I} - 2pT_0 + T\overline{T})^{\frac{h-2}{2}}\mathcal{Q}_p^h(T)p
$$

$$
= s\mathcal{Q}_s^h(T)\left(\sum_{k=0}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} s^{h-2-2k}(|T|^2 - 2T_0 s)^k \right)
$$

$$
\left(\sum_{k=0}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} (|T|^2 - 2T_0 p)^k p^{h-2-2k} \right)\mathcal{Q}_p^h(T)p
$$

$$
= s\mathcal{Q}_s^h(T)\left(s^{h-2} + \sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} s^{h-2-2k}(|T|^2 - 2T_0 s)^k \right) \times
$$

$$
\times \left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} (|T|^2 - 2T_0 p)^k p^{h-2-2k} + p^{h-2} \right)\mathcal{Q}_p^h(T)p
$$

$$
= s^{h-1}\mathcal{Q}_s^h(T)\mathcal{Q}_p^h(T)p^{h-1} + s^{h-1}\mathcal{Q}_s^h(T)\left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} (|T|^2 - 2T_0 p)^k p^{h-1-2k} \right)\mathcal{Q}_p^h(T)
$$

$$
+ \mathcal{Q}_s^h(T)\left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-1}{2}}{k} s^{h-1-2k}(|T|^2 - 2T_0 s)^k \right)
$$

$$\left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} (|T|^2 - 2T_0 p)^k p^{h-1-2k} \right) \mathcal{Q}_p^h(T)$$

$$+ \mathcal{Q}_s^h(T) \left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} s^{h-1-2k} (|T|^2 - 2T_0 s)^k \right) \mathcal{Q}_p^h(T) p^{h-1}. \tag{3.27}$$

Now, we use the left and right $\mathcal{F}$-resolvent equations in Theorem 2.18 and we go through the computations term by term

$$s^{h-1} \mathcal{Q}_s^h(T) \mathcal{Q}_p^h(T) p^{h-1} = \gamma_n^{-2} \Big[s^h \mathcal{F}_n^R(s, T) \mathcal{F}_n^L(p, T) p^h - s^h \mathcal{F}_n^R(s, T) T \mathcal{F}_n^L(p, T) p^{h-1}$$

$$- s^{h-1} \mathcal{F}_n^R(s, T) T \mathcal{F}_n^L(p, T) p^h + s^{h-1} \mathcal{F}_n^R(s, T) T^2 \mathcal{F}_n^L(p, T) p^{h-1} \Big], \tag{3.28}$$

and

$$s^{h-1} \mathcal{Q}_s^h(T) \left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} (|T|^2 - 2T_0 p)^k \mathcal{Q}_p^h(T) p^{h-1-2k} \right)$$

$$= \gamma_n^{-2} \Big[s^h \mathcal{F}_n^R(s, T) \sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} (|T|^2 - 2T_0 p)^k \mathcal{F}_n^L(p, T) p^{h-2k}$$

$$- s^h \mathcal{F}_n^R(s, T) \sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} (|T|^2 - 2T_0 p)^k T \mathcal{F}_n^L(p, T) p^{h-1-2k}$$

$$- s^{h-1} \mathcal{F}_n^R(s, T) \sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} T (|T|^2 - 2T_0 p)^k \mathcal{F}_n^L(p, T) p^{h-2k}$$

$$+ s^{h-1} \mathcal{F}_n^R(s, T) \sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} T (|T|^2 - 2T_0 p)^k T \mathcal{F}_n^L(p, T) p^{h-1-2k} \Big], \tag{3.29}$$

then we consider the term

$$\left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} s^{h-1-2k} \mathcal{Q}_s^h(T) (|T|^2 - 2T_0 s)^k \right) \left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} (|T|^2 - 2T_0 p)^k \mathcal{Q}_p^h(T) p^{h-1-2k} \right)$$

$$= \gamma_n^{-2} \Bigg[\left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} s^{h-2k} \mathcal{F}_n^R(s, T) (|T|^2 - 2T_0 s)^k \right)$$

$$\left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} (|T|^2 - 2T_0 p)^k \mathcal{F}_n^L(p, T) p^{h-2k} \right)$$

$$- \left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-1}{2}}{k} s^{h-2k} \mathcal{F}_n^R(s, T)(|T|^2 - 2T_0 s)^k \right)$$

$$\left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} (|T|^2 - 2T_0 p)^k T \mathcal{F}_n^L(p, T) p^{h-1-2k} \right)$$

$$- \left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} s^{h-2k-1} \mathcal{F}_n^R(s, T) T (|T|^2 - 2T_0 s)^k \right)$$

$$\left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} (|T|^2 - 2T_0 p)^k \mathcal{F}_n^L(p, T) p^{h-2k} \right)$$

$$+ \left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} s^{h-2k-1} \mathcal{F}_n^R(s, T) T (|T|^2 - 2T_0 s)^k \right)$$

$$\left. \left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} (|T|^2 - 2T_0 p)^k T \mathcal{F}_n^L(p, T) p^{h-1-2k} \right) \right], \tag{3.30}$$

and the other term

$$\left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} s^{h-1-2k} \mathcal{Q}_s^h(T)(|T|^2 - 2T_0 s)^k \right) \mathcal{Q}_p^h(T) p^{h-1}$$

$$= \gamma_n^{-2} \left[\left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} s^{h-2k} \mathcal{F}_n^R(s, T)(|T|^2 - 2T_0 s)^k \right) \mathcal{F}_n^L(p, T) p^h \right.$$

$$- \left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} s^{h-2k} \mathcal{F}_n^R(s, T)(|T|^2 - 2T_0 s)^k T \right) \mathcal{F}_n^L(p, T) p^{h-1}$$

$$- \left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} s^{h-1-2k} \mathcal{F}_n^R(s, T) T (|T|^2 - 2T_0 s)^k \right) \mathcal{F}_n^L(p, T) p^h$$

$$+ \left. \left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} s^{h-1-2k} \mathcal{F}_n^R(s, T) T (|T|^2 - 2T_0 s)^k T \right) \mathcal{F}_n^L(p, T) p^{h-1} \right]. \tag{3.31}$$

Finally by using the definition of left and right S-resolvent operators we obtain

$$\sum_{i=0, i\neq\frac{h-2}{2}}^{h-2} \mathcal{Q}_s^{h-i-1}(T)S_R^{-1}(s,T)S_L^{-1}(p,T)\mathcal{Q}_p^{i+1}(T)$$

$$= \sum_{i=0, i\neq\frac{h-2}{2}}^{h-2} \mathcal{Q}_s^{h-i}(T)(s\mathcal{I}-\overline{T})(p\mathcal{I}-\overline{T})\mathcal{Q}_p^{i+2}(T)$$

$$= s\sum_{i=0, i\neq\frac{h-2}{2}}^{h-2} \mathcal{Q}_s^{h-i}(T)\mathcal{Q}_p^{i+2}(T)p - s\sum_{i=0, i\neq\frac{h-2}{2}}^{h-2} \mathcal{Q}_s^{h-i}(T)\overline{T}\mathcal{Q}_p^{i+2}(T)$$

$$- \sum_{i=0, i\neq\frac{h-2}{2}}^{h-2} \mathcal{Q}_s^{h-i}(T)\overline{T}\mathcal{Q}_p^{i+2}(T)p + \sum_{i=0, i\neq\frac{h-2}{2}}^{h-2} \mathcal{Q}_s^{h-i}(T)\overline{T}^2\mathcal{Q}_p^{i+2}(T) \quad (3.32)$$

and this concludes the proof. □

4 The Riesz Projectors for the $\mathcal{F}$-Functional Calculus: The General Case of n Odd

In the monogenic functional calculus developed by McIntosh and collaborators, [36], the resolvent equation is missing. They are able to study the Riesz projectors by using another functional calculus: the Weyl calculus. For the $\mathcal{F}$-functional calculus, which is a monogenic functional calculus, the interesting symmetries that appear in the equations of Theorem 3.4 and Theorem 3.5 allow to study the Riesz projectors.

Next result follows as in the case $n = 5$, see [11, Lemma 5.4].

Lemma 4.1 *Let $T \in \mathcal{BC}(V_n)$ and $h = \frac{n-1}{2}$. Suppose that G contains just some points of the $\mathcal{F}$-spectrum of T and assume that the closed smooth curve $\partial(G \cap \mathbb{C}_I)$ belongs to the $\mathcal{F}$-resolvent set of T, for every $I \in \mathbb{S}$. Then*

$$\int_{\partial(G\cap\mathbb{C}_I)} s^m ds_I \mathcal{F}_n^R(s,T) = 0,$$

$$\int_{\partial(G\cap\mathbb{C}_I)} \mathcal{F}_n^L(p,T) dp_I p^m = 0,$$

for all $m \leq 2h-1$.

In order to study the Riesz Projectors for the $\mathcal{F}$-functional calculus we now state the following results.

Lemma 4.2 *Let f and g be left slice monogenic and right slice monogenic functions, respectively, defined on an open set U. For any $I \in \mathbb{S}$ and any open bounded set D_I in*

$U \cap \mathbb{C}_I$ whose boundary is a finite union of continuously differentiable Jordan curves, we have

$$\int_{\partial D_I} g(s) ds_I f(s) = 0.$$

Lemma 4.3 *Let $B \in \mathcal{B}(V_n)$. Let G be a bounded slice Cauchy domain and let f be an intrinsic slice monogenic function whose domain contains G. Then for $p \in G$, and for any $I \in \mathbb{S}$ we have*

$$\frac{1}{2\pi} \int_{\partial(G \cap \mathbb{C}_I)} f(s) ds_I (\bar{s} B - Bp)(p^2 - 2s_0 p + |s|^2)^{-1} = Bf(p).$$

Theorem 4.4 *Let $n > 3$ be an odd number and let $T = \sum_{i=1}^{n} e_i T_i \in \mathcal{BC}^{0,1}(V_n)$. Let $\sigma_{\mathcal{F}}(T) = \sigma_{\mathcal{F},1}(T) \cup \sigma_{\mathcal{F},2}(T)$ with*

$$\mathrm{dist}\left(\sigma_{\mathcal{F},1}(T), \sigma_{\mathcal{F},2}(T)\right) > 0,$$

and

$$\sigma(T_\ell) \subset \mathbb{R} \text{ for all } \ell = 1, \ldots, n.$$

Let G_1, G_2 be two admissible sets for T such that $\sigma_{\mathcal{F},1}(T) \subset G_1$ and $\bar{G}_1 \subset G_2$ and such that $\mathrm{dist}\left(G_2, \sigma_{\mathcal{F},2}(T)\right) > 0$. Then the operator

$$\check{P} = \frac{1}{\gamma_n(2\pi)} \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I p^{n-1} = \frac{1}{\gamma_n(2\pi)} \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{n-1} ds_I \mathcal{F}_n^R(s, T). \quad (4.1)$$

is a projector.

Proof We divide the proof in two cases, according to the parity of $h = \frac{n-1}{2}$.

CASE I: The Sce exponent h is odd.

We start by multiplying the equation of Theorem 3.4 by s^h on the left and p^h on the right, and since $T_0 = 0$ we get

$$s^h \mathcal{F}_n^R(s, T) S_L^{-1}(p, T) p^h + s^h S_R^{-1}(s, T) \mathcal{F}_n^L(p, T) p^h$$
$$+ \gamma_n \Bigg[s^{h+1} \sum_{i=0}^{h-2} \mathcal{Q}_s^{h-i}(T) \mathcal{Q}_p^{i+2}(T) p^{h+1} +$$
$$+ s^{h+1} \sum_{i=0}^{h-2} \mathcal{Q}_s^{h-i}(T) T \mathcal{Q}_p^{i+2}(T) p^h + s^h \sum_{i=0}^{h-2} \mathcal{Q}_s^{h-i}(T) T \mathcal{Q}_p^{i+2}(T) p^{h+1}$$
$$+ s^h \sum_{i=0}^{h-2} \mathcal{Q}_s^{h-i}(T) T^2 \mathcal{Q}_p^{i+2}(T) p^h +$$

$$+ s^h \sum_{i=0, i \neq \frac{h-1}{2}}^{h-1} \mathcal{Q}_s^{h-i}(T) \mathcal{Q}_p^{i+1}(T) p^h \Bigg] + \gamma_n^{-1} \Bigg[s^{2h} \mathcal{F}_n^R(s, T) \mathcal{F}_n^L(p, T) p^{2h}$$

$$+ s^h \mathcal{A}_0(s, p, T) p^h + s^h \mathcal{B}_0(s, p, T) p^h$$

$$+ s^h \mathcal{C}_0(s, p, T) p^h \Bigg] = s^h \{[\mathcal{F}_n^R(s, T) - \mathcal{F}_n^L(p, T)] p - \bar{s}[\mathcal{F}_n^R(s, T) - \mathcal{F}_n^L(p, T)]\}$$

$$(p^2 - 2s_0 p + |s|^2)^{-1} p^h. \tag{4.2}$$

Now, we multiply equation (4.2) by ds_I on the left, integrate it over $\partial(G_2 \cap \mathbb{C}_I)$ with respect to ds_I and then we multiply it by dp_I on the right and integrate over $\partial(G_1 \cap \mathbb{C}_I)$ with respect to dp_I. We obtain

$$\int_{\partial(G_2 \cap \mathbb{C}_I)} s^h ds_I \mathcal{F}_n^R(s, T) \int_{\partial(G_1 \cap \mathbb{C}_I)} S_L^{-1}(p, T) dp_I p^h$$

$$+ \int_{\partial(G_2 \cap \mathbb{C}_I)} s^h ds_I S_R^{-1}(s, T) \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I p^h$$

$$+ \gamma_n \Bigg[\int_{\partial(G_2 \cap \mathbb{C}_I)} s^{h+1} \sum_{i=0}^{h-2} \mathcal{Q}_s^{h-i}(T) ds_I \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{Q}_p^{i+2}(T) dp_I p^{h+1}$$

$$+ \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{h+1} \sum_{i=0}^{h-2} \mathcal{Q}_s^{h-i}(T) ds_I T \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{Q}_p^{i+2}(T) dp_I p^h$$

$$+ \int_{\partial(G_2 \cap \mathbb{C}_I)} s^h \sum_{i=0}^{h-2} \mathcal{Q}_s^{h-i}(T) ds_I T \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{Q}_p^{i+2}(T) dp_I p^{h+1}$$

$$+ \int_{\partial(G_2 \cap \mathbb{C}_I)} s^h \sum_{i=0}^{h-2} \mathcal{Q}_s^{h-i}(T) ds_I T^2 \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{Q}_p^{i+2}(T) dp_I p^h$$

$$+ \int_{\partial(G_2 \cap \mathbb{C}_I)} s^h \sum_{i=0, i \neq \frac{h-1}{2}}^{h-1} \mathcal{Q}_s^{h-i}(T) ds_I \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{Q}_p^{i+1}(T) dp_I p^h \Bigg]$$

$$+ \gamma_n^{-1} \Bigg[\int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h} ds_I \mathcal{F}_n^R(s, T) \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I p^{2h} +$$

$$+ \int_{\partial(G_2 \cap \mathbb{C}_I)} \int_{\partial(G_1 \cap \mathbb{C}_I)} s^h ds_I \mathcal{A}_0(s, p, T) dp_I p^h + s^h ds_I \mathcal{B}_0(s, p, T) dp_I p^h$$

$$+ s^h ds_I \mathcal{C}_0(s, p, T) dp_I p^h \Bigg]$$

$$= \int_{\partial(G_2 \cap \mathbb{C}_I)} ds_I \int_{\partial(G_1 \cap \mathbb{C}_I)} s^h \{[\mathcal{F}_n^R(s, T) - \mathcal{F}_n^L(p, T)] p - \bar{s}[\mathcal{F}_n^R(s, T) - \mathcal{F}_n^L(p, T)]\}$$

$$(p^2 - 2s_0 p + |s|^2)^{-1} dp_I p^h. \tag{4.3}$$

Recalling the definition of $\mathcal{A}_0, \mathcal{B}_0, \mathcal{C}_0$ and the fact that $T_0 = 0$ we have

$$\int_{\partial(G_2 \cap \mathbb{C}_I)} \int_{\partial(G_1 \cap \mathbb{C}_I)} s^h ds_I \mathcal{A}_0(s, p, T) dp_I p^h + s^h ds_I \mathcal{B}_0(s, p, T) dp_I p^h$$

$$+ s^h ds_I \mathcal{C}_0(s, p, T) dp_I p^h$$

$$= - \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h} \mathcal{F}_n^R(s, T) ds_I T \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I p^{2h-1}$$

$$- \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h-1} \mathcal{F}_n^R(s, T) ds_I \, T \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I \, p^{2h}$$

$$+ \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h-1} ds_I \mathcal{F}_n^R(s, T) T^2 \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I \, p^{2h-1}$$

$$+ \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h} ds_I \mathcal{F}_n^R(s, T) \sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} |T|^{2k} \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I \, p^{2h-2k}$$

$$- \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h} ds_I \mathcal{F}_n^R(s, T) \sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} |T|^{2k} T \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) \, p^{2h-1-2k}$$

$$- \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h-1} ds_I \mathcal{F}_n^R(s, T) \sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} T |T|^{2k} \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I \, p^{2h-2k}$$

$$+ \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h-1} \mathcal{F}_n^R(s, T) ds_I \sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} T |T|^{2k} T \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I \, p^{2h-1-2k}$$

$$+ \left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h-2k} ds_I \mathcal{F}_n^R(s, T) |T|^{2k} \right)$$

$$\left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} |T|^{2k} \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I \, p^{2h-2k} \right)$$

$$- \left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h-2k} ds_I \mathcal{F}_n^R(s, T) |T|^{2k} \right)$$

$$\left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} |T|^k T \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I \, p^{2h-1-2k} \right)$$

$$- \left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h-2k-1} ds_I \mathcal{F}_n^R(s, T) T |T|^{2k} \right)$$

$$\left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} |T|^{2k} \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I \, p^{2h-2k} \right)$$

$$+ \left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h-2k-1} ds_I \mathcal{F}_n^R(s, T) T |T|^{2k} \right)$$

$$\left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} |T|^{2k} T \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I \, p^{2h-1-2k} \right)$$

$$+ \left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h-2k} ds_I \mathcal{F}_n^R(s, T) |T|^{2k} \right) \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I \, p^{2h}$$

$$
-\left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h-2k} ds_I \mathcal{F}_n^R(s, T) |T|^{2k} T \right) \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I p^{2h-1}
$$

$$
-\left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h-1-2k} ds_I \mathcal{F}_n^R(s, T) T |T|^{2k} \right) \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I p^{2h}
$$

$$
+\left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h-1-2k} ds_I \mathcal{F}_n^R(s, T) T^2 |T|^{2k} \right) \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I p^{2h-1}.
$$

$$
\tag{4.4}
$$

Now, since $h \leq 2h - 1$ by Lemma 4.1 we get

$$
\int_{\partial(G_2 \cap \mathbb{C}_I)} s^h ds_I \mathcal{F}_n^R(s, T) \int_{\partial(G_1 \cap \mathbb{C}_I)} S_L^{-1}(p, T) dp_I p^h
$$

$$
= \int_{\partial(G_2 \cap \mathbb{C}_I)} s^h ds_I S_R^{-1}(s, T) \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I p^h = 0.
$$

Moreover, since $2h - 2k \leq 2h - 1$ and $2h - 1 - 2k \leq 2h - 1$ we obtain

$$
\int_{\partial(G_2 \cap \mathbb{C}_I)} \int_{\partial(G_1 \cap \mathbb{C}_I)} s^h ds_I \mathcal{A}_0(s, p, T) dp_I p^h + s^h ds_I \mathcal{B}_0(s, p, T) dp_I p^h
$$

$$
+ s^h ds_I \mathcal{C}_0(s, p, T) dp_I p^h = 0.
$$

Now, we focus on the term

$$
\int_{\partial(G_2 \cap \mathbb{C}_I)} s^{h+1} \sum_{i=0}^{h-2} \mathcal{Q}_s^{h-i}(T) ds_I \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{Q}_p^{i+2}(T) dp_I p^{h+1}.
$$

First of all we split the sum in two parts and write

$$
\sum_{i=0}^{h-2} \mathcal{Q}_s^{h-i}(T) \mathcal{Q}_p^{i+2}(T) = \sum_{i=0}^{\lfloor \frac{h-2}{2} \rfloor} \mathcal{Q}_s^{h-i}(T) \mathcal{Q}_p^{i+2}(T) + \sum_{i=\lfloor \frac{h-2}{2} \rfloor+1}^{h-2} \mathcal{Q}_s^{h-i}(T) \mathcal{Q}_p^{i+2}(T),
$$

where $\lfloor . \rfloor$ is the floor of a number. In the first sum the powers of $\mathcal{Q}_s(T)$ are more than the powers of $\mathcal{Q}_p(T)$, and conversely in the second sum.

Since $T_0 = 0$, by the binomial formula we get

$$
\sum_{i=0}^{\lfloor \frac{h-2}{2} \rfloor} \mathcal{Q}_s^{h-i}(T) \mathcal{Q}_p^{i+2}(T) = \mathcal{Q}_s^h(T) \sum_{i=0}^{\lfloor \frac{h-2}{2} \rfloor} \sum_{k=0}^{i} \binom{i}{k} s^{2k} |T|^{2(i-k)} \mathcal{Q}_p^{i+2}(T) +
$$

$$
+ \sum_{i=\lfloor \frac{h-2}{2} \rfloor+1}^{h-2} \mathcal{Q}_s^{h-i}(T) \sum_{k=0}^{h-2-i} \binom{h-2-i}{k} p^{2k} |T|^{2(h-2-i-k)} \mathcal{Q}_p^h(T). \tag{4.5}
$$

Consider the first sum. By the $\mathcal{F}$- resolvent equation, see (2.16) we get

$$\sum_{i=0}^{\lfloor \frac{h-2}{2} \rfloor} \sum_{k=0}^{i} \binom{i}{k} s^{2k} \mathcal{Q}_s^h(T)|T|^{2(i-k)} \mathcal{Q}_p^{i+2}(T)$$

$$= \gamma_n^{-1} \sum_{i=0}^{\lfloor \frac{h-2}{2} \rfloor} \sum_{k=0}^{i} \binom{i}{k} s^{2k} \left(s\mathcal{F}_n^R(s,T) - \mathcal{F}_n^R(s,T)T \right)|T|^{2(i-k)} \mathcal{Q}_p^{i+2}(T).$$

Hence we have to compute the following integrals

$$\gamma_n^{-1} \sum_{i=0}^{\lfloor \frac{h-2}{2} \rfloor} \sum_{k=0}^{i} \binom{i}{k} \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{h+2+2k} ds_I \mathcal{F}_n^R(s,T)|T|^{2(i-k)} \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{Q}_p^{i+2}(T) dp_I p^{h+1},$$

$$\gamma_n^{-1} \sum_{i=0}^{\lfloor \frac{h-2}{2} \rfloor} \sum_{k=0}^{i} \binom{i}{k} \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{h+1+2k} ds_I \mathcal{F}_n^R(s,T)T|T|^{2(i-k)} \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{Q}_p^{i+2}(T) dp_I p^{h+1}.$$

Now, since h is odd then we can write $h = 2N + 1$, with $N \in \mathbb{N}$. This implies that

$$h + 2 + 2k \leq 2i + 2 + 2N + 1 \leq 2\left\lfloor \frac{h-2}{2} \right\rfloor + 2 + 2N + 1$$

$$= 2(N-1) + 2 + 2N + 1 = 4N + 1 = 2h - 1.$$

Similarly we get

$$h + 1 + 2k \leq 2h - 1.$$

Therefore by Lemma 4.1 we get

$$\gamma_n^{-1} \sum_{i=0}^{\lfloor \frac{h-2}{2} \rfloor} \sum_{k=0}^{i} \binom{i}{k} \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{h+2+2k} ds_I \mathcal{F}_n^R(s,T) \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{Q}_p^{i+2}(T) dp_I p^{h+1} = 0$$

$$\gamma_n^{-1} \sum_{i=0}^{\lfloor \frac{h-2}{2} \rfloor} \sum_{k=0}^{i} \binom{i}{k} \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{h+1+2k} ds_I \mathcal{F}_n^R(s,T) \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{Q}_p^{i+2}(T) dp_I p^{h+1} = 0.$$

$$(4.6)$$

Now, we focus on the second sum. By the $\mathcal{F}$- resolvent equation, see (2.15), we get

$$\sum_{i=\lfloor \frac{h-2}{2} \rfloor+1}^{h-2} \mathcal{Q}_s^{h-i}(T) \sum_{k=0}^{h-2-i} \binom{h-2-i}{k} |T|^{2(h-2-i-k)} \mathcal{Q}_p^h(T) p^{2k}$$

$$= \gamma_n^{-1} \sum_{i=\lfloor \frac{h-2}{2} \rfloor+1}^{h-2} \mathcal{Q}_s^{h-i}(T) \sum_{k=0}^{h-2-i} \binom{h-2-i}{k} |T|^{2(h-2-i-k)} \left(\mathcal{F}_n^R(p,T)p - T\mathcal{F}_n^R(p,T) \right) p^{2k}.$$

$$(4.7)$$

Hence we have to compute the following integrals

$$\gamma_n^{-1} \sum_{i=\lfloor \frac{h-2}{2} \rfloor+1}^{h-2} \sum_{k=0}^{2+i-h} \binom{h-2-i}{k} \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{h+1} ds_I \mathcal{Q}_s^{h-i}(T)|T|^{2(h-2-i-k)}$$

$$\int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^R(p, T) dp_I \, p^{h+2k+2},$$

$$\gamma_n^{-1} \sum_{i=\lfloor \frac{h-2}{2} \rfloor + 1}^{h-2} \sum_{k=0}^{h-2-i} \binom{h-2-i}{k} \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{h+1} ds_I \, \mathcal{Q}_s^{h-i}(T) |T|^{2(h-2-i-k)} T$$

$$\int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^R(p, T) dp_I \, p^{h+2k+1}.$$

Since $h = 2N + 1$, with $N \in \mathbb{N}$ we get

$$2k + 2 + h \le 2(h - 2 - i) + 2 + h = 2h - 4 - 2i + 2 + h$$
$$\le 4N + 2 - 4 - 2 \left(\lfloor \frac{h-2}{2} \rfloor + 1 \right)$$
$$+ 2 + 2N + 1 = 4N + 1 = 2h - 1,$$

and similarly

$$2k + 1 + h \le 2h - 1,$$

together with Lemma 4.1 we get

$$\gamma_n^{-1} \sum_{i=\lfloor \frac{h-2}{2} \rfloor + 1}^{h-2} \sum_{k=0}^{h-2-i} \binom{h-2-i}{k} \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{h+1} ds_I \, \mathcal{Q}_s^{h-i}(T) |T|^{2(h-2-i-k)}$$

$$\int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^R(p, T) dp_I \, p^{h+2k+2} = 0$$

$$\gamma_n^{-1} \sum_{i=\lfloor \frac{h-2}{2} \rfloor + 1}^{h-2} \sum_{k=0}^{h-2-i} \binom{h-2-i}{k} \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{h+1} ds_I \, \mathcal{Q}_s^{h-i}(T) |T|^{2(h-2-i-k)} T$$

$$\int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^R(p, T) dp_I \, p^{h+2k+1} = 0. \tag{4.8}$$

Similar arguments applied to the other members of (4.3) lead to

$$\gamma_n^{-1} \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h} ds_I \mathcal{F}_n^R(s, T) \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I \, p^{2h}$$
$$= \int_{\partial(G_2 \cap \mathbb{C}_I)} ds_I \int_{\partial(G_1 \cap \mathbb{C}_I)} s^h \{ [\mathcal{F}_n^R(s, T) - \mathcal{F}_n^L(p, T)] p - \bar{s} [\mathcal{F}_n^R(s, T) - \mathcal{F}_n^L(p, T)] \}$$
$$(p^2 - 2s_0 p + |s|^2)^{-1} dp_I \, p^h. \tag{4.9}$$

Since $h = \frac{n-1}{2}$, by formula (4.1) we get

$$\frac{(2\pi)^2}{\gamma_n^{-1}} \check{P}^2 = \int_{\partial(G_2 \cap \mathbb{C}_I)} ds_I \int_{\partial(G_1 \cap \mathbb{C}_I)} s^h \{[\mathcal{F}_n^R(s, T) - \mathcal{F}_n^L(p, T)] p$$
$$- \bar{s}[\mathcal{F}_n^R(s, T) - \mathcal{F}_n^L(p, T)]\}(p^2 - 2s_0 p + |s|^2)^{-1} p^h dp_I.$$

Now, we work on the integral on the right hand side. As $\bar{G}_1 \subset G_2$, for any $s \in \partial(G_2 \cap \mathbb{C}_I)$ the functions

$$p \mapsto p(p^2 - 2s_0 p + |s|^2)^{-1} p^h,$$
$$p \mapsto (p^2 - 2s_0 p + |s|^2)^{-1} p^h$$

are slice monogenic on $\bar{G}_1$. By Lemma 4.2 we have

$$\int_{\partial(G_1 \cap \mathbb{C}_I)} p(p^2 - 2s_0 p + |s|^2)^{-1} dp_I p^h = 0,$$
$$\int_{\partial(G_1 \cap \mathbb{C}_I)} (p^2 - 2s_0 p + |s|^2)^{-1} p^h dp_I = 0.$$

This implies that

$$\int_{\partial(G_2 \cap \mathbb{C}_I)} ds_I \int_{\partial(G_1 \cap \mathbb{C}_I)} s^h \mathcal{F}_n^R(s, T) p(p^2 - 2s_0 p + |s|^2)^{-1} dp_I p^h = 0$$

and

$$\int_{\partial(G_2 \cap \mathbb{C}_I)} ds_I \int_{\partial(G_1 \cap \mathbb{C}_I)} s^h \bar{s} \mathcal{F}_n^R(s, T)(p^2 - 2s_0 p + |s|^2)^{-1} dp_I p^h = 0.$$

Then we have

$$\frac{(2\pi)^2}{\gamma_n^{-1}} \check{P}^2 = \int_{\partial(G_2 \cap \mathbb{C}_I)} s^h ds_I \int_{\partial(G_1 \cap \mathbb{C}_I)} [\bar{s} \mathcal{F}_n^L(p, T) - \mathcal{F}_7^L(p, T) p](p^2 - 2s_0 p + |s|^2)^{-1} dp_I p^h.$$

From Lemma 4.3 with $B =: \mathcal{F}_n^L(p, T)$ and $f(s) := s^h$ we get

$$\check{P}^2 = \frac{1}{(2\pi)\gamma_n} \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I p^{2h} = \check{P}.$$

CASE II: The Sce exponent h is even.

We multiply the equation of Theorem 3.5 by s^h left and p^h on the right, and since $T_0 = 0$ we get

$$
s^h \mathcal{F}_n^R(s, T) S_L^{-1}(p, T) p^h + s^h S_R^{-1}(s, T) \mathcal{F}_n^L(p, T) p^h
$$
$$
+ \gamma_n \left[s^{h+1} \mathcal{Q}_s^{\frac{h+2}{2}}(T) T \mathcal{Q}_p^{\frac{h+2}{2}}(T) p^h + s^h \mathcal{Q}_s^{\frac{h+2}{2}}(T) T \mathcal{Q}_p^{\frac{h+2}{2}}(T) p^{h+1} \right.
$$
$$
+ s^h \mathcal{Q}_s^{\frac{h+2}{2}}(T) T^2 \mathcal{Q}_p^{\frac{h+2}{2}}(T) p^h + s^h \sum_{i=0}^{h-1} \mathcal{Q}_s^{h-i}(T) \mathcal{Q}_p^{i+1}(T) p^h
$$
$$
+ s^{h+1} \sum_{i=0, i \neq \frac{h-2}{2}}^{h-2} \mathcal{Q}_s^{h-i}(T) \mathcal{Q}_p^{i+2}(T) p^{h+1}
$$
$$
+ s^{h+1} \sum_{i=0, i \neq \frac{h-2}{2}}^{h-2} \mathcal{Q}_s^{h-i}(T) T \mathcal{Q}_p^{i+2}(T) p^h + s^h \sum_{i=0, i \neq \frac{h-2}{2}}^{h-2} \mathcal{Q}_s^{h-i}(T) \mathcal{Q}_p^{i+2}(T) p^{h+1}
$$
$$
\left. + s^h \sum_{i=0, i \neq \frac{h-2}{2}}^{h-2} \mathcal{Q}_s^{h-i}(T) T^2 \mathcal{Q}_p^{i+2}(T) p^h \right]
$$
$$
+ \gamma_n^{-1} \left[s^h \mathcal{A}_1(s, p, T) p^h + s^h \mathcal{B}_1(s, p, T) p^h + s^h \mathcal{C}_1(s, p, T) p^h \right.
$$
$$
\left. + s^{2h} \mathcal{F}_n^R(s, T) \mathcal{F}_n^L(p, T) p^{2h} \right]
$$
$$
= s^h \{ [\mathcal{F}_n^R(s, T) - \mathcal{F}_n^L(p, T)] p - \bar{s} [\mathcal{F}_n^R(s, T) - \mathcal{F}_n^L(p, T)] (p^2 - 2s_0 p + |s|^2)^{-1} p^h. \quad (4.10)
$$

Now, we multiply by ds_I on the left, integrate it over $\partial(G_2 \cap \mathbb{C}_I)$ with respect to ds_I and then we multiply it by dp_I on the right and integrate over $\partial(G_1 \cap \mathbb{C}_I)$ with respect to dp_I, and we obtain

$$
\int_{\partial(G_2 \cap \mathbb{C}_I)} s^h \mathcal{F}_n^R(s, T) ds_I \int_{\partial(G_1 \cap \mathbb{C}_I)} S_L^{-1}(p, T) dp_I p^h
$$
$$
+ \int_{\partial(G_2 \cap \mathbb{C}_I)} s^h ds_I S_R^{-1}(s, T) \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) p^h +
$$
$$
+ \gamma_n \left[\int_{\partial(G_2 \cap \mathbb{C}_I)} s^{h+1} ds_I \mathcal{Q}_s^{\frac{h+2}{2}}(T) T \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{Q}_p^{\frac{h+2}{2}}(T) dp_I p^h \right.
$$
$$
+ \int_{\partial(G_2 \cap \mathbb{C}_I)} s^h ds_I \mathcal{Q}_s^{\frac{h+2}{2}}(T) T \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{Q}_p^{\frac{h+2}{2}}(T) dp_I p^{h+1} +
$$
$$
+ \int_{\partial(G_2 \cap \mathbb{C}_I)} s^h ds_I \mathcal{Q}_s^{\frac{h+2}{2}}(T) T^2 \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{Q}_p^{\frac{h+2}{2}}(T) dp_I p^h
$$
$$
+ \int_{\partial(G_2 \cap \mathbb{C}_I)} s^h ds_I \sum_{i=0}^{h-1} \mathcal{Q}_s^{h-i}(T) \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{Q}_p^{i+1}(T) dp_I p^h +
$$
$$
+ \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{h+1} ds_I \sum_{i=0, i \neq \frac{h-2}{2}}^{h-2} \mathcal{Q}_s^{h-i}(T) \int_{\partial(G_2 \cap \mathbb{C}_I)} \mathcal{Q}_p^{i+2}(T) dp_I p^{h+1}
$$

$$+ \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{h+1} \sum_{i=0, i \neq \frac{h-2}{2}}^{h-2} \mathcal{Q}_s^{h-i}(T) T \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{Q}_p^{i+2}(T) dp_I \, p^h$$

$$+ \int_{\partial(G_2 \cap \mathbb{C}_I)} s^h ds_I \sum_{i=0, i \neq \frac{h-2}{2}}^{h-2} \mathcal{Q}_s^{h-i}(T) \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{Q}_p^{i+2}(T) dp_I \, p^{h+1}$$

$$+ \int_{\partial(G_2 \cap \mathbb{C}_I)} s^h ds_I \sum_{i=0, i \neq \frac{h-2}{2}}^{h-2} \mathcal{Q}_s^{h-i}(T) T^2 \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{Q}_p^{i+2}(T) dp_I \, p^h \Bigg] +$$

$$+ \gamma_n^{-1} \Bigg[\int_{\partial(G_2 \cap \mathbb{C}_I)} \int_{\partial(G_1 \cap \mathbb{C}_I)} s^h ds_I \mathcal{A}_1(s, p, T) dp_I \, p^h$$

$$+ s^h ds_I \mathcal{B}_1(s, p, T) dp_I \, p^h + s^h ds_I \mathcal{C}_1(s, p, T) dp_I \, p^h$$

$$+ \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h} ds_I \mathcal{F}_n^R(s, T) \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I \, p^{2h} \Bigg]$$

$$= \int_{\partial(G_2 \cap \mathbb{C}_I)} ds_I \int_{\partial(G_1 \cap \mathbb{C}_I)} s^h \Big\{ \big[\mathcal{F}_n^R(s, T) - \mathcal{F}_n^L(p, T) \big] p$$

$$- \bar{s} \big[\mathcal{F}_n^R(s, T) - \mathcal{F}_n^L(p, T) \big] (p^2 - 2s_0 p + |s|^2)^{-1} dp_I \, p^h. \tag{4.11}$$

From the definition of $\mathcal{A}_1, \mathcal{B}_1, \mathcal{C}_1$ and recalling that $T_0 = 0$ we have

$$\int_{\partial(G_2 \cap \mathbb{C}_I)} \int_{\partial(G_1 \cap \mathbb{C}_I)} s^h ds_I \mathcal{A}_1(s, p, T) dp_I \, p^h + s^h ds_I \mathcal{B}_1(s, p, T) dp_I \, p^h$$

$$+ s^h ds_I \mathcal{C}_1(s, p, T) dp_I \, p^h$$

$$= - \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h} \mathcal{F}_n^R(s, T) ds_I T \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I \, p^{2h-1} +$$

$$- \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h-1} \mathcal{F}_n^R(s, T) ds_I T \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I \, p^{2h}$$

$$+ \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h-1} ds_I \mathcal{F}_n^R(s, T) T^2 \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I \, p^{2h-1} +$$

$$+ \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h} ds_I \mathcal{F}_n^R(s, T) \sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} |T|^{2k} \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I \, p^{2h-2k}$$

$$- \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h} ds_I \mathcal{F}_n^R(s, T) \sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} T |T|^{2k} \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) p^{2h-1-2k}$$

$$- \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h-1} ds_I \mathcal{F}_n^R(s, T) \sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} T |T|^{2k} \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I \, p^{2h-2k}$$

$$+ \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h-1} \mathcal{F}_n^R(s, T) ds_I \sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} T^2 |T|^{2k} \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I \, p^{2h-1-2k} +$$

$$+ \left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h-2k} ds_I \mathcal{F}_n^R(s, T) |T|^{2k} \right)$$

$$\left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} |T|^{2k} \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I p^{2h-2k} \right)$$

$$- \left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h-2k} ds_I \mathcal{F}_n^R(s, T) |T|^{2k} \right)$$

$$\left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} |T|^k T \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I p^{2h-1-2k} \right)$$

$$- \left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h-2k-1} ds_I \mathcal{F}_n^R(s, T) T |T|^{2k} \right)$$

$$\left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} |T|^{2k} \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I p^{2h-2k} \right)$$

$$+ \left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h-2k-1} ds_I \mathcal{F}_n^R(s, T) T |T|^{2k} \right)$$

$$\left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} |T|^{2k} T \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I p^{2h-1-2k} \right)$$

$$+ \left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h-2k} ds_I \mathcal{F}_n^R(s, T) |T|^{2k} \right) \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I p^{2h}$$

$$- \left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h-2k} ds_I \mathcal{F}_n^R(s, T) |T|^{2k} T \right) \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I p^{2h-1}$$

$$- \left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h-1-2k} ds_I \mathcal{F}_n^R(s, T) T |T|^{2k} \right) \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I p^{2h}$$

$$+ \left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{2h-1-2k} ds_I \mathcal{F}_n^R(s, T) T^2 |T|^{2k} \right) \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I p^{2h-1}.$$

$$(4.12)$$

Now we observe that since $h \leq 2h - 1$ by Lemma 4.1 we have

$$\int_{\partial(G_2 \cap \mathbb{C}_I)} s^h ds_I \mathcal{F}_n^R(s, T) \int_{\partial(G_1 \cap \mathbb{C}_I)} S_L^{-1}(p, T) dp_I p^h$$

$$= \int_{\partial(G_2 \cap \mathbb{C}_I)} s^h ds_I S_R^{-1}(s, T) \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I p^h = 0.$$

Moreover, since $2h - 2k \leq 2h - 1$ and $2h - 1 - 2k \leq 2h - 1$ we get

$$\int_{\partial(G_2 \cap \mathbb{C}_I)} \int_{\partial(G_1 \cap \mathbb{C}_I)} s^h ds_I \mathcal{A}_1(s, p, T) dp_I p^h$$
$$+ s^h ds_I \mathcal{B}_1(s, p, T) dp_I p^h + s^h ds_I \mathcal{C}_1(s, p, T) dp_I p^h = 0.$$

Now, we focus on computing the integral

$$\int_{\partial(G_2 \cap \mathbb{C}_I)} s^{h+1} ds_I \mathcal{Q}_s^{\frac{h+2}{2}}(T) T \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{Q}_p^{\frac{h+2}{2}}(T) dp_I p^h.$$

By the binomial formula and recalling that $T_0 = 0$ we get

$$\mathcal{Q}_p^{\frac{h+2}{2}}(T) = \mathcal{Q}_p^{\frac{2-h}{2}}(T) \mathcal{Q}_p^{\frac{h-2}{2}}(T) \mathcal{Q}_p^{\frac{h+2}{2}}(T) = (p^2 + |T|^2)^{\frac{h-2}{2}} \mathcal{Q}_p^h(T)$$
$$= \sum_{k=0}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} |T|^{2\left(\frac{h-2}{2}-k\right)} \mathcal{Q}_p^h(T) p^{2k}.$$

By the $\mathcal{F}$-resolvent, see (2.15), we deduce that

$$\int_{\partial(G_2 \cap \mathbb{C}_I)} s^{h+1} ds_I \mathcal{Q}_s^{\frac{h+2}{2}}(T) T \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{Q}_p^{\frac{h+2}{2}}(T) dp_I p^h$$

$$= \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{h+1} ds_I \mathcal{Q}_s^{\frac{h+2}{2}}(T) T \sum_{k=0}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} |T|^{2\left(\frac{h-2}{2}-k\right)} \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{Q}_p^h(T) dp_I p^{2k+h}$$

$$= \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{h+1} ds_I \mathcal{Q}_s^{\frac{h+2}{2}}(T) T \sum_{k=0}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} |T|^{2\left(\frac{h-2}{2}-k\right)} \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I p^{2k+h+1}$$

$$- \int_{\partial(G_2 \cap \mathbb{C}_I)} s^{h+1} ds_I \mathcal{Q}_s^{\frac{h+2}{2}}(T) T^2 \sum_{k=0}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} |T|^{2\left(\frac{h-2}{2}-k\right)} \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{F}_n^L(p, T) dp_I p^{2k+h}.$$
$$(4.13)$$

We observe that

$$h + 1 + 2k \leq h + 1 + h - 2 = 2h - 1,$$

similarly we have $h + 2k \leq 2h - 1$, so formula (4.13) together with Lemma 4.1 imply that

$$\int_{\partial(G_2 \cap \mathbb{C}_I)} s^{h+1} ds_I \mathcal{Q}_s^{\frac{h+2}{2}}(T) T \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{Q}_p^{\frac{h+2}{2}}(T) dp_I p^h = 0.$$

Using similar arguments, we obtain

$$\int_{\partial(G_2 \cap \mathbb{C}_I)} s^h ds_I \mathcal{Q}_s^{\frac{h+2}{2}}(T) T \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{Q}_p^{\frac{h+2}{2}}(T) dp_I p^{h+1} = 0$$

$$\int_{\partial(G_2 \cap \mathbb{C}_I)} s^h ds_I \, \mathcal{Q}_s^{\frac{h+2}{2}}(T) T^2 \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{Q}_p^{\frac{h+2}{2}}(T) dp_I \, p^h = 0. \qquad (4.14)$$

By similar computations made when h is odd we get

$$\int_{\partial(G_2 \cap \mathbb{C}_I)} s^h ds_I \sum_{i=0}^{h-1} \mathcal{Q}_s^{h-i}(T) \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{Q}_p^{i+1}(T) dp_I \, p^h = 0,$$

$$\int_{\partial(G_2 \cap \mathbb{C}_I)} s^{h+1} ds_I \sum_{i=0, i \neq \frac{h-2}{2}}^{h-2} \mathcal{Q}_s^{h-i}(T) \int_{\partial(G_2 \cap \mathbb{C}_I)} \mathcal{Q}_p^{i+2}(T) dp_I \, p^{h+1} = 0,$$

$$\int_{\partial(G_2 \cap \mathbb{C}_I)} s^{h+1} \sum_{i=0, i \neq \frac{h-2}{2}}^{h-2} \mathcal{Q}_s^{h-i}(T) T \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{Q}_p^{i+2}(T) dp_I \, p^h = 0,$$

$$\int_{\partial(G_2 \cap \mathbb{C}_I)} s^h ds_I \sum_{i=0, i \neq \frac{h-2}{2}}^{h-2} \mathcal{Q}_s^{h-i}(T) \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{Q}_p^{i+2}(T) dp_I \, p^{h+1} = 0,$$

$$\int_{\partial(G_2 \cap \mathbb{C}_I)} s^h ds_I \sum_{i=0, i \neq \frac{h-2}{2}}^{h-2} \mathcal{Q}_s^{h-i}(T) T^2 \int_{\partial(G_1 \cap \mathbb{C}_I)} \mathcal{Q}_p^{i+2}(T) dp_I \, p^h = 0.$$

By formula (4.1) we get

$$\frac{(2\pi)^2}{\gamma_n^{-1}} \check{P}^2 = \int_{\partial(G_2 \cap \mathbb{C}_I)} ds_I \int_{\partial(G_1 \cap \mathbb{C}_I)} s^h \{ [\mathcal{F}_n^R(s, T) - \mathcal{F}_n^L(p, T)]$$
$$p - \bar{s}[\mathcal{F}_n^R(s, T) - \mathcal{F}_n^L(p, T)] \}(p^2 - 2s_0 p + |s|^2)^{-1} p^h dp_I.$$

Finally, by following exactly the same steps done when h is odd we get

$$\check{P}^2 = \check{P}.$$

$\square$

Data Availability There are no data involved in the research in this manuscript.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Alpay, D., Colombo, F., Gantner, J., Sabadini, I.: A new resolvent equation for the S-functional calculus. J. Geom. Anal. **25**(3), 1939–1968 (2015)
2. Alpay, D., Colombo, F., Kimsey, D.P.: The spectral theorem for quaternionic unbounded normal operators based on the S-spectrum. J. Math. Phys. **57**(2), 023503, 27 (2016)
3. Alpay, D., Colombo, F., Kimsey, D.P., Sabadini, I.: The spectral theorem for unitary operators based on the S-spectrum. Milan J. Math. **84**(1), 41–61 (2016)
4. Alpay, D., Colombo, F., Sabadini, I.: Slice Hyperholomorphic Schur Analysis, Operator Theory: Advances and Applications, 256. Birkhäuser/Springer, Cham, . xii+362 pp (2016)
5. Alpay, D., Colombo, F., Sabadini, I.: Quaternionic de Branges spaces and characteristic operator function, Springer Briefs in Mathematics, Springer, Cham (2020/21)
6. Birkhoff, G., Von Neumann, J.: The logic of quantum mechanics. Ann. Math. **37**, 823–843 (1936)
7. Brackx, F., Delanghe, R., Sommen, F.: Clifford analysis, Research Notes in Mathematics, 76. Pitman (Advanced Publishing Program), Boston, MA, x+308 pp (1982)
8. Cerejeiras, P., Colombo, F., Kähler, U., Sabadini, I.: Perturbation of normal quaternionic operators. Trans. Am. Math. Soc. **372**(5), 3257–3281 (2019)
9. Colombo, F., Deniz-Gonzales, D., Pinton, S.: Fractional powers of vector operators with first order boundary conditions. J. Geom. Phys., **151**, 103618, 18 pp (2020)
10. Colombo, F., Deniz-Gonzales, D., Pinton, S.: Non commutative fractional Fourier law in bounded and unbounded domains. Complex Anal. Oper. Theory (2021)
11. Colombo, F., De Martino, A., Sabadini, I.: Towards a general $\mathcal{F}$-resolvent equation and Riesz projectors. J. Math. Anal. Appl. **517**(2), 126652 (2023)
12. Colombo, F., De Martino, A., Qian, T., Sabadini, I.: The Poisson kernel and the Fourier transform of the slice monogenic Cauchy kernels. J. Math. Anal. Appl. **512**(1), 126115 (2022)
13. Colombo, F., Gantner, J.: Formulations of the $\mathcal{F}$- functional calculus and some consequences. Proc. R. Soc. Edinb. **146 A**, 509–545 (2016)
14. Colombo, F., Gantner, J.: Quaternionic Closed Operators, Fractional Powers and Fractional Diffusion Processes, Operator Theory: Advances and Applications, 274. Birkhäuser/Springer, Cham, viii+322 pp (2019)
15. Colombo, F., Gantner, J., Kimsey, D.P.: Spectral Theory on the S-Spectrum for Quaternionic Operators, Operator Theory: Advances and Applications, 270. Birkhäuser/Springer, Cham, ix+356 pp (2018)
16. Colombo, F., Gantner, J., Kimsey, D.P., Sabadini, I.: Universality property of the S-functional calculus, noncommuting matrix variables and Clifford operators. Preprint (2020)
17. Colombo, F., Kimsey, D.P.: The spectral theorem for normal operators on a Clifford module. Preprint (2020)
18. Colombo, F., Kimsey, D.P., Pinton, S., Sabadini, I.: Slice monogenic functions of a Clifford variable. Proc. Am. Math. Soc. Ser. B **8**, 281–296 (2021)
19. Colombo, F., Lavicka, R., Sabadini, I., Soucek, V.: The Radon transform between monogenic and generalized slice monogenic functions. Math. Ann. **363**(3–4), 733–752 (2015)
20. Colombo, F., Peloso, M., Pinton, S.: The structure of the fractional powers of the noncommutative Fourier law. Math. Methods Appl. Sci. **42**, 6259–6276 (2019)
21. Colombo, F., Sabadini, I.: The F-functional calculus for unbounded operators. J. Geom. Phys. **86**, 392–407 (2014)
22. Colombo, F., Sabadini, I.: The F-spectrum and the SC-functional calculus. Proc. R. Soc. Edinb. Sect. A **142**(3), 479–500 (2012)

23. Colombo, F., Sabadini, I., Sommen, F.: The Fueter mapping theorem in integral form and the F-functional calculus. Math. Methods Appl. Sci. **33**, 2050–2066 (2010)
24. Colombo, F., Sabadini, I., Sommen, F., Struppa, D.C.: Analysis of Dirac systems and computational algebra, Progress in Mathematical Physics, 39. Birkhäuser Boston, Inc., Boston, MA, xiv+332 pp (2004)
25. Colombo, F., Sabadini, I., Struppa, D.C.: Entire slice regular functions, Springer Briefs in Mathematics. Springer, Cham, v+118 pp (2016)
26. Colombo, F., Sabadini, I., Struppa, D.C.: Noncommutative Functional Calculus. Theory and Applications of Slice Hyperholomorphic Functions, Progress in Mathematics, 289. Birkhäuser/Springer Basel AG, Basel (2011)
27. Colombo, F., Sabadini, I., Struppa, D.C.: Michele Sce's works in hypercomplex analysis, p. 122. Birkhäuser/Springer, Cham, A translation with commentaries (2020)
28. Delanghe, R., Sommen, F., Soucek, V.: Clifford algebra and spinor-valued functions. A function theory for the Dirac operator, Related REDUCE software by F. Brackx and D. Constales. With 1 IBM-PC floppy disk (3.5 inch). Mathematics and its Applications, 53. Kluwer Academic Publishers Group, Dordrecht (1992)
29. De Martino, A.: On the Clifford short-time Fourier transform and its properties. Appl. Math. Comput. **418**, 126812 (2022)
30. De Martino, A., Diki, K.: On the quaternionic short-time Fourier and Segal–Bargmann transforms. Mediterr. J. Math. **18**(3) (2021)
31. De Martino, A., Diki, K.: On the polyanalytic short-time Fourier transform in the quaternionic setting. Commun. Pure Appl. Anal **21**(11), 3629–3665 (2022)
32. Fueter, R.: Die Funktionentheorie der Differentialgleichungen $\Delta u = 0$ und $\Delta \Delta u = 0$ mit vier reellen Variablen. Comment. Math. Helv., **7**, 307–330 (1934-35)
33. Gal, S., Sabadini, I.: Quaternionic approximation. With application to slice regular functions, Frontiers in Mathematics. Birkhäuser/Springer, Cham, x+221 pp (2019)
34. Gentili, G., Stoppato, C., Struppa, D.C.: Regular functions of a quaternionic variable, Springer Monographs in Mathematics. Springer, Heidelberg, x+185 pp (2013)
35. Gürlebeck, K., Habetha, K., Sprößig, W.: Application of holomorphic functions in two and higher dimensions, Birkhääuser/Springer, [Cham], xv+390 pp (2016)
36. Jefferies, B.: Spectral properties of noncommuting operators. Lecture Notes in Mathematics, vol. 1843. Springer-Verlag, Berlin (2004)
37. Qian, T.: Generalization of Fueter's result to R^{n+1}. Rend. Mat. Acc. Lincei **9**, 111–117 (1997)
38. Qian, T., Li, P.: Singular integrals and Fourier theory on Lipschitz boundaries, Science Press Beijing, Beijing; Springer, Singapore, xv+306 pp (2019)
39. Sce, M.: Osservazioni sulle serie di potenze nei moduli quadratici. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **23**, 220–225 (1957)