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Global Properties of Eigenvalues of Parametric Rank One Perturbations for Unstructured and Structured Matrices II

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Abstract

We present in this note a correction to Theorem 17 in Ran and Wojtylak (Compl. Anal. Oper. Theory 15:44, 2021) and sharpen the estimates for eigenvalues of parametric rank one perturbations given in that theorem.

Keywords Eigenvalues of matrices · Perturbation theory · Parametric dependence of eigenvalues · Puiseux series

Mathematics Subject Classification Primary 15A18 · 47A55

1 Introduction and Preliminaries

This note concerns an erratum and addendum to [4], in particular to Theorem 17. One of the main points of the theorem is to show the asymptotic behavior of the eigenvalues of $B(\tau) = A + \tau uv^*$ when $|\tau| \to \infty$. The statement is that these eigenvalues trace out a set of curves which asymptotically approximates a set of non-intersecting circles. The statement of the theorem needs a small correction. The proof contains some independent miscalculations, which fortunately, had no effect on the validity of the

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corresponding statements. Therefore, we find it necessary to present a fully corrected version with a new proof. The detailed list of corrections can be found in Sect. 3. The results are illustrated by an example.

First we repeat the setting of [4], to make this erratum and addendum independently readable. Let A be an $n \times n$ complex matrix, and let u, v be two vectors in $\mathbb{C}^n$. We consider the eigenvalues of the parametric rank one perturbation $B(\tau) = A + \tau uv^*$ of A. Denote by $m_A(\lambda)$ the minimal polynomial of A, and define

$$p_{uv}(\lambda) = v^* m_A(\lambda)(\lambda I_n - A)^{-1}u.$$

Observe that this is a polynomial (see [3]), and

$$\begin{aligned}\det(\lambda I_n - B(\tau)) &= \det(\lambda I_n - A) \cdot (1 - \tau v^*(\lambda I_n - A)^{-1}u) \\ &= \frac{\det(\lambda I_n - A)}{m_A(\lambda)}(m_A(\lambda) - \tau p_{uv}(\lambda)). \end{aligned} \tag{1}$$

Also introduce $Q(\lambda) = v^*(\lambda I_n - A)^{-1}u$. By Proposition 2 in [4], if λ_0 is not an eigenvalue of A, then it is an eigenvalue of $B(\tau_0)$ of multiplicity $\kappa \geq 1$, if and only if $\tau_0 Q(\lambda_0) = 1$, $Q'(\lambda_0) = 0, \ldots, Q^{(\kappa-1)}(\lambda_0) = 0$, $Q^{(\kappa)}(\lambda_0) \neq 0$. In this case λ_0 has geometric multiplicity one.

2 Main Results

We are interested in the behavior of the eigenvalues of $B(\tau)$, where $\tau = te^{\mathrm{i}\theta}$ as functions of θ for fixed t, and then in particular in what happens as $t \to \infty$.

For that reason, introduce for $t > 0$ the set

$$\sigma(A, u, v; t) = \bigcup_{0 \leq \theta \leq 2\pi} \sigma(A + te^{\mathrm{i}\theta}uv^*) \setminus \sigma(A).$$

Theorem 17 in [4] describes the asymptotic behavior of (parts of) these sets as $t \to \infty$. We correct, complete and extend the result.

Theorem 1 *Let $A \in \mathbb{C}^{n\times n}$, $u, v \in \mathbb{C}^n$ and let $l \in \mathbb{N}$ denote the degree of the minimal polynomial $m_A(\lambda)$. If*

$$v^*u = \cdots = v^*A^{l-1}u = 0 \tag{2}$$

then $p_{uv}(\lambda) \equiv 0$ and $\sigma(A + \tau uv^*) = \sigma(A)$ for any $\tau \in \mathbb{C}$. If

$$v^*u = \cdots = v^*A^{\kappa-1}u = 0, \quad v^*A^{\kappa}u \neq 0, \tag{3}$$

for some $\kappa \in \{0, \ldots, l-1\}$ then the following statements hold.

(i) *$p_{uv}(\lambda)$ is of degree $l - \kappa - 1$;*
(ii) *$l - \kappa - 1$ eigenvalues of $B(\tau)$ converge to the roots of $p_{uv}(\lambda)$ as $\tau \to \infty$;*

(iii) *there are* $\kappa+1$ *eigenvalues* $\lambda_1(\tau),\dots,\lambda_{\kappa+1}(\tau)$ *of* $A+\tau uv^*$ *having the following Puiseux expansion at* $\tau=\infty$

$$\lambda_j(\tau) = c_{-1}\tau^{\frac{1}{\kappa+1}} + c_0 + c_1\tau^{-\frac{1}{\kappa+1}} + \cdots, \quad j=1,\dots\kappa+1, \tag{4}$$

where

$$\begin{aligned} c_{-1} &= (v^*A^\kappa u)^{\frac{1}{\kappa+1}},\\ c_0 &= \frac{1}{\kappa+1}\cdot\frac{v^*A^{\kappa+1}u}{v^*A^\kappa u},\\ c_1 &= \frac{1}{\kappa+1}\cdot\frac{1}{(v^*A^\kappa u)^{1+\frac{1}{\kappa+1}}}\cdot\left(v^*A^{\kappa+2}u - \frac{\kappa+2}{2(\kappa+1)}\cdot\frac{(v^*A^{\kappa+1}u)^2}{v^*A^\kappa u}\right); \end{aligned}$$

(iv) *if* ζ *is a root of the polynomial* $p_{uv}(\lambda)$ *of multiplicity* k *and is not a root of* $m_A(\lambda)$, *then there are* k *eigenvalues of* $A+\tau uv^*$ *converging to* ζ *with* $\tau\to\infty$ *having the following Puiseux expansion at* $\tau=\infty$

$$\lambda_j(\tau) = \zeta - b_1\tau^{-\frac{1}{k}} - b_2\tau^{-\frac{2}{k}} - b_3\tau^{-\frac{3}{k}} - \cdots, \quad j=1,\dots,k, \tag{5}$$

where, using $a_m = a_m(\zeta) = v^*(\zeta I_n - A)^{-m}u$ *for* $m\geq 0$,

$$\begin{aligned} b_1 &= b_1(\zeta) = a_{k+1}^{-\frac{1}{k}}, \qquad a_{k+1}\neq 0,\\ b_2 &= b_2(\zeta) = -\frac{1}{k}\cdot\frac{b_1^2\,a_{k+2}}{a_{k+1}}. \end{aligned}$$

The complex roots in Eqs. (4) and (5) should be understood as in the theory of Puiseux series: each particular root determines uniquely the eigenvalue λ_j, see also Remark 4 below.

Remark 2 Let us comment on some genericity issues appearing in Theorem 1. It was shown in [3] that for generic u, v (i.e., u, v with arbitrary complex entries except an algebraic subset of $\mathbb{C}^{2n}$, depending possibly on A) we have that $v^*u\neq 0$ and the roots of $p_{uv}(\lambda)$ are all simple and disjoint with the roots of $m_A(\lambda)$. Hence, in Theorem 1 we have generically $\kappa=1$ and in part (iv) for each root ζ of $p_{uv}(\lambda)$ we have $k=1$ and $m_A(\zeta)\neq 0$. However, in general, many different situations might occur. For example, a root of $p_{uv}(\lambda)$ might be a root of $m_A(\lambda)$, as can be seen in Example 5 below.

Further, note that the multiplicities of the roots of $p_{uv}(\lambda)$ and the number κ are in some relation, e.g., due to Theorem 1(i). In particular, if $\kappa=l-1$ then $p_{uv}(\lambda)$ is a constant, nonzero polynomial and the only limit point of eigenvalues is infinity. There are, however, some other hidden constraints relating κ and the multiplicities of $p_{uv}(\lambda)$ and their nature needs to be studied more intensively in future work.

All these comments explain the role of the assumptions in Corollary 3 below.

Corollary 3 *With the notation of Theorem 1, if (3) holds and none of the roots $\zeta_1, \ldots, \zeta_\nu$ of $p_{uv}(\lambda)$ is a root of $m_A(\lambda)$, then for sufficiently large $t = |\tau|$ the set $\sigma(A, u, v; t)$ can be parametrized by disjoint curves $\Gamma_1(\theta), \ldots, \Gamma_{\nu+1}(\theta)$, where the $\kappa + 1$ eigenvalues which go to infinity together trace out a curve*

$$\Gamma_{\nu+1}(\theta) = c_{-1} t^{\frac{1}{\kappa+1}} e^{\mathrm{i}\theta} + c_0 + c_1 t^{-\frac{1}{\kappa+1}} e^{-\mathrm{i}\theta} + O(t^{-\frac{2}{\kappa+1}}), \quad 0 \leq \theta \leq 2\pi,$$

while the k_j eigenvalues near ζ_j together trace out a curve $\Gamma_j(\theta)$ which is of the form

$$\Gamma_j(\theta) = \zeta_j - b_1(\zeta_j) t^{-\frac{1}{k_j}} e^{\mathrm{i}\theta} - b_2(\zeta_j) t^{-\frac{2}{k_j}} e^{2\mathrm{i}\theta} + O(t^{-\frac{3}{k_j}}), \;\; j \in \{1, \ldots, \nu\}.$$

In both cases above the O is with respect to $t = |\tau| \to \infty$ and is uniform in $\theta \in [0, 2\pi]$.

Proof of Theorem 1 Assume first (2) holds. Expanding the resolvent at infinity we get

$$\begin{aligned} p_{uv}(\lambda) &= m_A(\lambda) v^* (\lambda I_n - A)^{-1} u \\ &= m_A(\lambda) v^* \left(\sum_{j=1}^{\infty} \lambda^{-j-1} A^j \right) u \\ &= \sum_{j=l}^{\infty} \lambda^{-j-1} m_A(\lambda) v^* A^j u. \end{aligned}$$

However, recall that $p_{uv}(\lambda)$ is a polynomial, hence $v^* A^j u = 0$ for all $j = 0, 1, 2, \ldots$ and $p_{uv}(\lambda) \equiv 0$. In consequence, $\det(\lambda I_n - B(\tau)) = \det(\lambda I_n - A)$ for all $\tau \in \mathbb{C}$ by (1).

Assume now that (3) holds. Statements (i) and (ii) were proved in [4], the proof did not contain errors. Let us now show (iii). For large values of $|\tau|$ consider the eigenvalues of $A + \tau u v^*$ which are not eigenvalues of A. These are among the roots of $m_A(\lambda) - \tau p_{uv}(\lambda)$. Dividing by τ, and viewing $s = 1/\tau$ as a variable, they are also roots of $s m_A(\lambda) - p_{uv}(\lambda)$, and hence we have by general theory concerning the behavior of roots of a polynomial under a perturbation such as this that the roots are given by Puiseux series, see e.g., [1], Part II, Chapter V, and [2],Theorem 9.1.1. For the large eigenvalues of $B(\tau)$ we can make this more precise as follows. Recall that the eigenvalues of $B(\tau)$ which are not eigenvalues of A satisfy $\tau v^* (\lambda I - A)^{-1} u = 1$. For $|\lambda| > \|A\|$ this may be rewritten as

$$1 = \tau v^* \sum_{j=0}^{\infty} \frac{A^j}{\lambda^{j+1}} u = \tau \sum_{j=\kappa}^{\infty} \frac{v^* A^j u}{\lambda^{j+1}},$$

where in the last step we used the definition of κ. Hence

$$\frac{\lambda^{\kappa+1}}{\tau} = v^* A^\kappa u + \frac{1}{\lambda} v^* A^{\kappa+1} u + \frac{1}{\lambda^2} v^* A^{\kappa+2} u + \cdots . \tag{6}$$

We can write λ as a Puiseux series as

$$\lambda = c_{-1}\tau^{\frac{1}{\kappa+1}} + c_0 + c_1\tau^{-\frac{1}{\kappa+1}} + \cdots , \tag{7}$$

or equivalently,

$$\frac{\lambda}{\tau^{\frac{1}{\kappa+1}}} = c_{-1} + c_0\tau^{-\frac{1}{\kappa+1}} + c_1\tau^{-\frac{2}{\kappa+1}} + \cdots . \tag{8}$$

Taking the $(\kappa+1)$-th power of this we arrive at

$$\begin{aligned}\frac{\lambda^{\kappa+1}}{\tau} &= c_{-1}^{\kappa+1} + (\kappa+1)c_{-1}^{\kappa}c_0\tau^{-\frac{1}{\kappa+1}} \\ &\quad + \left((\kappa+1)c_{-1}^{\kappa}c_1 + \binom{\kappa+1}{2}c_{-1}^{\kappa-1}c_0^2\right)\tau^{-\frac{2}{\kappa+1}} + O(\tau^{-\frac{3}{\kappa+1}}).\end{aligned} \tag{9}$$

From the leading terms in (6) and (9) we see that

$$c_{-1}^{\kappa+1} = v^*A^{\kappa}u. \tag{10}$$

Now consider $\left(\frac{\lambda^{\kappa+1}}{\tau} - v^*A^{\kappa}u\right)\lambda$. By (6) this is equal to

$$\left(\frac{\lambda^{\kappa+1}}{\tau} - v^*A^{\kappa}u\right)\lambda = v^*A^{\kappa+1}u + \frac{1}{\lambda}v^*A^{\kappa+2}u + O(\lambda^{-2}). \tag{11}$$

On the other hand, inserting (10) into (9), and then inserting (4) we obtain

$$\begin{aligned}&\left(\frac{\lambda^{\kappa+1}}{\tau} - v^*A^{\kappa}u\right)\lambda \\ &= \lambda\left((\kappa+1)c_{-1}^{\kappa}c_0\tau^{-\frac{1}{\kappa+1}} + \left((\kappa+1)c_{-1}^{\kappa}c_1 + \binom{\kappa+1}{2}c_{-1}^{\kappa-1}c_0^2\right)\tau^{-\frac{2}{\kappa+1}} + O(\tau^{-\frac{3}{\kappa+1}})\right) \\ &= \left(c_{-1}\tau^{\frac{1}{\kappa+1}} + c_0 + c_1\tau^{-\frac{1}{\kappa+1}} + O(\tau^{-\frac{2}{\kappa+1}})\right) \\ &\quad\cdot\left((\kappa+1)c_{-1}^{\kappa}c_0\tau^{-\frac{1}{\kappa+1}} + \left((\kappa+1)c_{-1}^{\kappa}c_1 + \binom{\kappa+1}{2}c_{-1}^{\kappa-1}c_0^2\right)\tau^{-\frac{2}{\kappa+1}} + O(\tau^{-\frac{3}{\kappa+1}})\right) \\ &= (\kappa+1)c_{-1}^{\kappa+1}c_0 + (\kappa+1)\left(c_{-1}^{\kappa}c_0^2 + c_{-1}^{\kappa+1}c_1 + \frac{\kappa}{2}c_{-1}^{\kappa}c_0^2\right)\tau^{-\frac{1}{\kappa+1}} + O(\tau^{-\frac{2}{\kappa+1}}) \\ &= (\kappa+1)c_{-1}^{\kappa+1}c_0 + (\kappa+1)\left(\frac{\kappa+2}{2}c_{-1}^{\kappa}c_0^2 + c_{-1}^{\kappa+1}c_1\right)\tau^{-\frac{1}{\kappa+1}} + O(\tau^{-\frac{2}{\kappa+1}})\end{aligned} \tag{12}$$

Comparing formulas (11) and (12) we see

$$(\kappa+1)c_{-1}^{\kappa+1}c_0 = v^*A^{\kappa+1}u. \tag{13}$$

Using (10) we obtain

$$c_0 = \frac{1}{\kappa+1}\cdot\frac{v^*A^{\kappa+1}u}{v^*A^{\kappa}u}. \tag{14}$$

In addition, subtracting the constant term in (11) and then multiplying by λ we obtain

$$\left(\left(\frac{\lambda^{\kappa+1}}{\tau} - v^*A^\kappa u\right)\lambda - v^*A^{\kappa+1}u\right)\lambda = v^*A^{\kappa+2}u + O(\lambda^{-1}). \tag{15}$$

On the other hand, subtracting the constant term in (12) and then multiplying by λ we obtain, also using (13),

$$\begin{aligned}&\left(\left(\frac{\lambda^{\kappa+1}}{\tau} - v^*A^\kappa u\right)\lambda - v^*A^{\kappa+1}u\right)\lambda\\ &\quad= \lambda\cdot(\kappa+1)\left(\frac{\kappa+2}{2}c_{-1}^{\kappa}c_0^2 + c_{-1}^{\kappa+1}c_1\right)\tau^{-\frac{1}{\kappa+1}} + O(\tau^{-\frac{2}{\kappa+1}}).\end{aligned}$$

Now use again (4) to see that

$$\begin{aligned}&\left(\left(\frac{\lambda^{\kappa+1}}{\tau} - v^*A^\kappa u\right)\lambda - v^*A^{\kappa+1}u\right)\lambda\\ &\quad= \left(c_{-1}\tau^{\frac{1}{\kappa+1}} + c_0 + O(\tau^{-\frac{1}{\kappa+1}})\right)\\ &\qquad\cdot\left((\kappa+1)\left(\frac{\kappa+2}{2}c_{-1}^{\kappa}c_0^2 + c_{-1}^{\kappa+1}c_1\right)\tau^{-\frac{1}{\kappa+1}} + O(\tau^{-\frac{2}{\kappa+1}})\right)\\ &\quad= (\kappa+1)\left(\frac{\kappa+2}{2}c_{-1}^{\kappa+1}c_0^2 + c_{-1}^{\kappa+2}c_1\right) + O(\tau^{-\frac{1}{\kappa+1}}).\end{aligned} \tag{16}$$

Comparing the constant terms in (15) and (16) we see that

$$(\kappa+1)\left(\frac{\kappa+2}{2}c_{-1}^{\kappa+1}c_0^2 + c_{-1}^{\kappa+2}c_1\right) = v^*A^{\kappa+2}u.$$

Solving this equation for c_1 using the formulas (10) and (14), one finds after some computation

$$c_1 = \frac{1}{\kappa+1}\cdot\frac{1}{(v^*A^\kappa u)^{1+\frac{1}{\kappa+1}}}\cdot\left(v^*A^{\kappa+2}u - \frac{\kappa+2}{2(\kappa+1)}\cdot\frac{(v^*A^{\kappa+1}u)^2}{v^*A^\kappa u}\right),$$

as stated in the theorem.

(iv) Since ζ is a root of $p_{uv}(\lambda)$ and by assumption is not a root of $m_A(\lambda)$ we have $v^*(\zeta I_n - A)^{-1}u = 0$. Hence, for λ near ζ we expand

$$\begin{aligned}v^*(\lambda I_n - A)^{-1}u &= v^*((\lambda-\zeta)I_n + (\zeta I_n - A))^{-1}u\\ &= v^*((\lambda-\zeta)(\zeta I_n - A)^{-1} + I_n)^{-1}(\zeta I_n - A)^{-1}u\end{aligned}$$

$$= v^* \sum_{j=0}^{\infty} (-1)^j (\lambda - \zeta)^j (\zeta I_n - A)^{-(j+1)} u$$
$$= v^* \sum_{j=1}^{\infty} (\zeta - \lambda)^j (\zeta I_n - A)^{-(j+1)} u.$$

Recall that any eigenvalue of $A + \tau u v^*$ which is not an eigenvalue of A satisfies

$$\frac{1}{\tau} = v^* (\lambda I_n - A)^{-1} u.$$

As ζ is a root of $p_{uv}(\lambda)$ with multiplicity k, we have

$$\frac{1}{\tau} = (\zeta - \lambda)^k a_{k+1} + (\zeta - \lambda)^{k+1} a_{k+2} + \cdots, \qquad a_{k+1} \neq 0. \tag{17}$$

We express now λ in a a Puiseux series in τ^{-1}, this is possible because λ is a root of $m_A(\lambda) - \tau p_{uv}(\lambda) = 0$. Since ζ has multiplicity k we have that

$$\lambda = \zeta - b_1 \tau^{-\frac{1}{k}} - b_2 \tau^{-\frac{2}{k}} - b_3 \tau^{-\frac{3}{k}} - \cdots$$

for some $b_1, b_2, \ldots$. Then $\zeta - \lambda = b_1 \tau^{-\frac{1}{k}} + b_2 \tau^{-\frac{2}{k}} + \cdots$, and inserting that in the equation (17) we obtain

$$\begin{aligned} \frac{1}{\tau} &= \frac{1}{\tau} b_1^k a_{k+1} + k b_1^{k-1} \tau^{-\frac{k-1}{k}} \cdot b_2 \tau^{-\frac{2}{k}} a_{k+1} \\ &\quad + b_1^{k+1} \tau^{-\frac{k+1}{k}} a_{k+2} + \text{smaller order terms} \\ &= \frac{1}{\tau} b_1^k a_{k+1} + \tau^{-\frac{k+1}{k}} \left(k b_1^{k-1} b_2 a_{k+1} + b_1^{k+1} a_{k+2} \right) + \cdots. \end{aligned} \tag{18}$$

Equating terms of equal powers in τ, for the terms $\frac{1}{\tau}$ on the left and right hand sides gives

$$b_1 = a_{k+1}^{-\frac{1}{k}} = \left(\frac{1}{v^* (\zeta I_n - A)^{-(k+1)} u} \right)^{\frac{1}{k}}.$$

The term on the right hand side with power $\tau^{-\frac{k+1}{k}}$ gives

$$k b_2 a_{k+1} + b_1^2 a_{k+2} = 0,$$

i.e.,

$$b_2 = -\frac{1}{k} \cdot \frac{b_1^2 a_{k+2}}{a_{k+1}}.$$

This completes the proof. ◻

Using the formula for b_1 we can derive an alternative formula for b_2 completely in terms of a_{k+1} and a_{k+2}, which after some computation, and with proper care for the kth roots, becomes

$$b_2 = -\frac{1}{k} \cdot \frac{a_{k+2}}{a_{k+1}^{1+\frac{2}{k}}} = -\frac{1}{k} \cdot \frac{v^*(\zeta I_n - A)^{-(k+2)}u}{\left(v^*(\zeta I_n - A)^{-(k+1)}u\right)^{1+\frac{2}{k}}}.$$

Remark 4 As stated in the Corollary 3, the $\kappa + 1$ eigenvalues going to infinity together trace out the curve $\Gamma_{\nu+1}(\theta)$. Let us number the eigenvalues so that these are $\lambda_1(\tau), \ldots, \lambda_{\kappa+1}(\tau)$. After possibly renumbering these eigenvalues, one derives from the theory of Puiseux series, see e.g. [1],

$$\lambda_j(te^{\mathrm{i}\theta}) = \sqrt[\kappa+1]{t\,r}\, e^{\mathrm{i}(\frac{1}{\kappa+1}(\theta+\theta_0)+\frac{2j}{\kappa+1}\pi)} + O(1), \qquad j = 1, 2, \ldots, \kappa+1,$$

where $v^* A^\kappa u = re^{\mathrm{i}\theta_0}$. As $\theta \to 2\pi$ one has that

$$\lambda_j(te^{\mathrm{i}\theta}) \to \lambda_{j+1}(t), \quad j = 1, \ldots, \kappa, \quad \lambda_{\kappa+1}(te^{\mathrm{i}\theta}) \to \lambda_1(t).$$

A similar statement holds for the k_j eigenvalues near ζ_j tracing out the curve $\Gamma_j(\theta)$.

3 List of Corrections

- The analysis of the case indicated in formula (2) above was missing in [4]. For completeness, we have included it in the current version.
- The eigenvalues ζ_j in point (v) of Theorem 17 of [4] were not assumed to be disjoint with the roots of $m_A(\lambda)$. If some ζ_j is a root of $m_A(\lambda)$ several things might occur, which need an independent work. A reformulation of the Theorem, including that assumption, was necessary.
- A more detailed Puiseux expansion for the eigenvalues for $\tau \to \infty$ was given, both in the case of eigenvalues converging to infinity and to a root of $p_{uv}(\lambda)$. The version in [4] contained only the first term.
- All results on the set $\sigma(A, u, v, t)$ were moved to a separate Corollary. This is partially due to the two previous items, and partially due to presentation issues.
- On page 17, line 13 in [4] the formula given there for c_1 is wrong.
- On page 17, last three lines, and page 18, the first line in [4] the display formula contains a mistake which has an effect on the remainder of the proof. There is a factor $(-1)^k$ missing in the summation.
- The formula at the bottom of page 18 in [4] contains an error.

4 Examples

Let us begin with the promised example when $m_A(\lambda)$ and $p_{uv}(\lambda)$ have a common root.

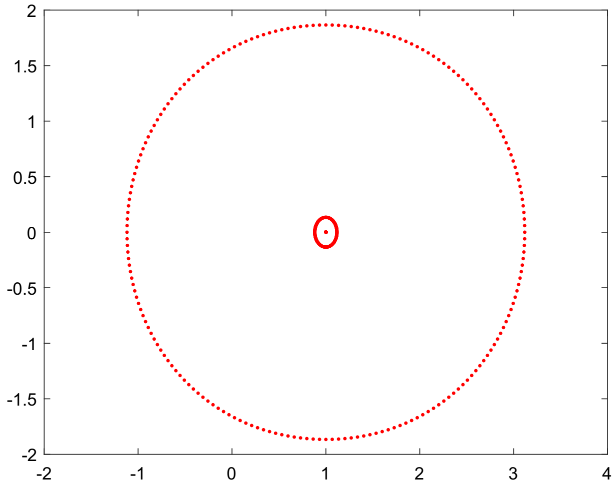

Fig. 1 Eigenvalues for $t = 1$ in Example 5

Example 5 Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}, \quad u = v = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Then $\kappa = 1$ and $p_{uv}(\lambda) = 2(\lambda - 1)^2$. Hence, $A + \tau uv^*$ has one eigenvalue converging to infinity and two eigenvalues converging to 1 with $\tau \to \infty$. However, unlike in Theorem 1(iv), in the plot of $\sigma(A, u, v, t)$ we do not have a circle around 1 formed by two eigenvalues. In the current situation one eigenvalue remains at 1 for all $\tau \in \mathbb{C}$, while the other one forms the full circle, see Fig. 1. The plot in Fig. 1 shows these circles for $t = 1$, together with the eigenvalues of $B(te^{\mathrm{i}\theta})$ for $t = 1$ and $\theta = \frac{2\pi j}{200}$ for $j = 1, 2, \ldots, 200$.

Remark 6 The formulas in Theorem 1(iii) take an especially nice form in the generic case, when $\kappa = 0$. In that case we have

$$c_{-1} = v^*u, \quad c_0 = \frac{v^*Au}{v^*u}, \quad c_1 = \frac{1}{(v^*u)^2}\left(v^*A^2u - \frac{(v^*Au)^2}{v^*u}\right).$$

As a first approximation we obtain the circle

$$\Gamma(\theta) \approx v^*u \cdot te^{\mathrm{i}\theta} + \frac{v^*Au}{v^*u},$$

while a further refinement is the curve

$$\Gamma(\theta) \approx v^*u \cdot te^{\mathrm{i}\theta} + \frac{v^*Au}{v^*u} + \frac{1}{(v^*u)^2}\left(v^*A^2u - \frac{(v^*Au)^2}{v^*u}\right) \cdot \frac{1}{t}e^{-\mathrm{i}\theta}.$$

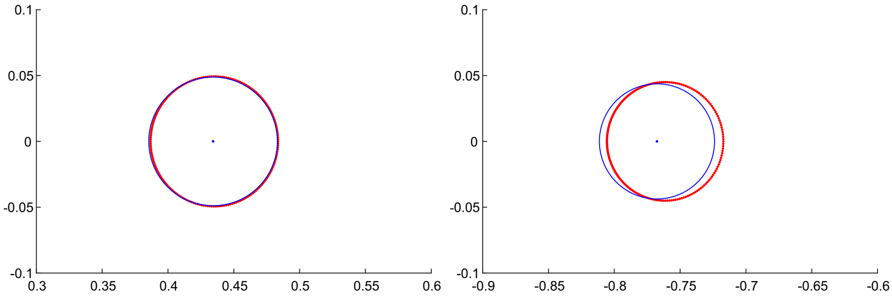

Fig. 2 Eigenvalues for $t = 1$ in red, the circular approximation in blue. On the left around ζ_1, on the right around ζ_2

Let us also specialize the formulas in Theorem 1(iv) for the generic case $k = 1$. In this case we have

$$b_1 = \frac{1}{v^*(\zeta I_n - A)^{-2}u}, \quad b_2 = -\frac{v^*(\zeta I_n - A)^{-3}u}{(v^*(\zeta I_n - A)^{-2}u)^3}.$$

As a first approximation we obtain the circle

$$\Gamma(\theta) \approx \zeta - b_1 e^{\mathrm{i}\theta}\tfrac{1}{t}$$

as a second approximation we obtain the curve

$$\Gamma(\theta) \approx \zeta - b_1 e^{\mathrm{i}\theta}\tfrac{1}{t} - b_2 e^{2\mathrm{i}\theta}\tfrac{1}{t^2}.$$

Example 7 Consider $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Then $p_{uv}(\lambda) = 6\lambda^2 + 2\lambda - 2$, with zeroes $\zeta_1 = -\frac{1}{6} + \frac{\sqrt{13}}{6} \approx 0.4343$ and $\zeta_2 = -\frac{1}{6} - \frac{\sqrt{13}}{6} \approx -0.7676$. One computes that $b_1(\zeta_1) \approx 0.0489$ and $b_1(\zeta_2) \approx 0.0437$. So the first approximation of the curves $\Gamma_j(\theta)$ are given by $\Gamma_j(\theta) \approx \zeta_j - b_1(\zeta_j)e^{\mathrm{i}\theta}\frac{1}{t}$, which are circles with centers at ζ_j and radii $b_1(\zeta_j)$. The plots in Fig. 2 show these circles for $t = 1$, together with the eigenvalues of $B(te^{\mathrm{i}\theta})$ for $t = 1$ and $\theta = \frac{2\pi j}{200}$ for $j = 1, 2, \ldots, 200$.

It is obvious from the graphs that the circle around ζ_1 is already a fairly good approximation for the eigenvalues of $B(\tau)$. However, the circle around ζ_2 is definitely not a very nice approximation. We further compute that $b_2(\zeta_1) \approx -9.5594 \times 10^{-4}$ and $b_2(\zeta_2) \approx -0.062$. Focusing on the next approximation of $\Gamma_2(\theta)$ we get as an approximation $\Gamma_2(\theta) \approx -0.7676 - 0.0437e^{\mathrm{i}\theta} + 0.062e^{2\mathrm{i}\theta}$. Incorporating this extra term in the approximation leads to a much better approximation as is shown in Fig. 3.

For the large eigenvalues of $B(\tau)$, already for $\tau = 1$ the circular approximation is fairly good, the extra term in the approximation only makes it even better. In this case we have $c_{-1} = 6$, $c_0 = \frac{1}{3}$ and $c_1 = \frac{23}{216}$. See Fig. 4, where for 200 equally spaces values of θ the eigenvalues of $B(e^{\mathrm{i}\theta})$ are plotted, together with the circular approximation $\frac{1}{3} + 6e^{\mathrm{i}\theta}$ and the second approximation $\frac{1}{3} + 6e^{\mathrm{i}\theta} + \frac{23}{216}e^{-\mathrm{i}\theta}$.

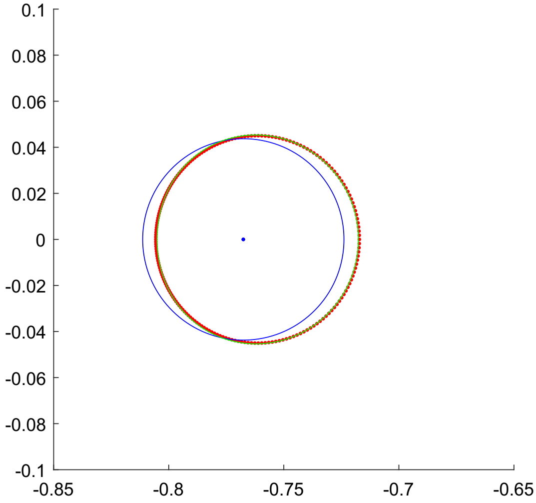

Fig. 3 Eigenvalues for $t = 1$ close to ζ_2 in red, the circular approximation in blue, in green the more accurate approximation with an additional term

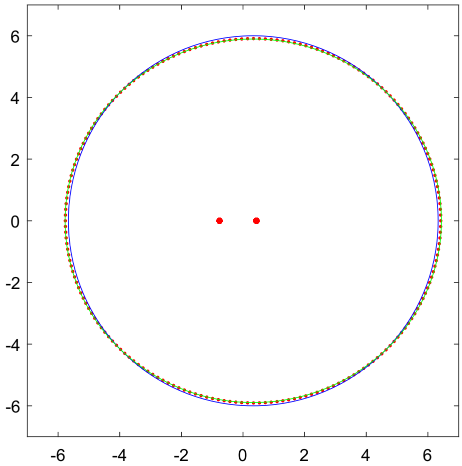

Fig. 4 The eigenvalues of $B(e^{\mathrm{i}\theta})$ are plotted in red, the circular approximation is plotted in blue and the second approximation is plotted in green

For $t = \frac{1}{2}$ the picture is clearer. The approximating circle in this case is $\frac{1}{3} + 3e^{\mathrm{i}\theta}$, the second approximation is $\frac{1}{3} + 3e^{\mathrm{i}\theta} + \frac{46}{216}e^{-\mathrm{i}\theta}$, as is shown in Fig. 5

Example 8 Take A and u as in the previous example, but let $v = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$. Then $v^*u = 0$ and $v^*Au = -\frac{3}{2}$, so $\kappa = 1$. Furthermore, $v^*A^2u = \frac{1}{2}$ and $v^*A^3u = -\frac{3}{2}$. In that case, $c_{-1}^2 = -\frac{3}{2}$, so $c_{-1} = \sqrt{\frac{3}{2}}\,\mathrm{i}$, $c_0 = -\frac{1}{6}$ and $c_1 = \frac{-5\sqrt{2}}{24\sqrt{3}}\,\mathrm{i}$. With $\tau = te^{\mathrm{i}\theta}$ we obtain for the circular approximation of the two eigenvalues going to infinity

$$\pm\sqrt{\frac{3}{2}}\sqrt{t}\left(-\sin\left(\frac{1}{2}\theta\right) + \mathrm{i}\cos\left(\frac{1}{2}\theta\right)\right) - \frac{1}{6}.$$

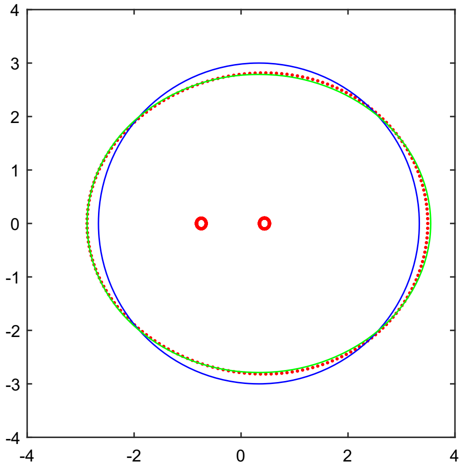

Fig. 5 The eigenvalues of $B(\frac{1}{2}e^{\mathrm{i}\theta})$ are plotted in red, the circular approximation is plotted in blue and the second approximation is plotted in green

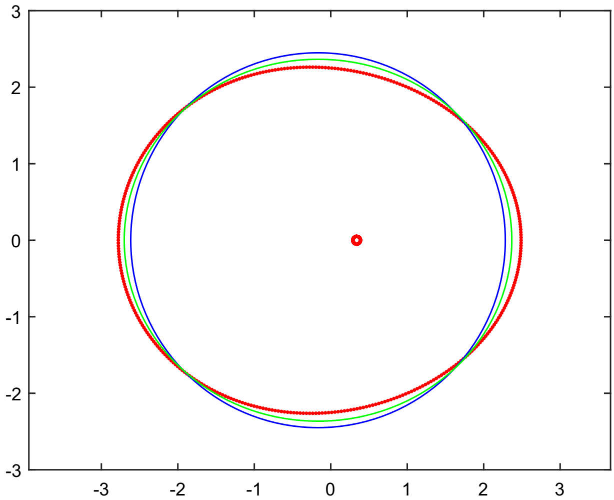

Fig. 6 The eigenvalues of $B(4e^{\mathrm{i}\theta})$ plotted in red, the circular approximation in blue and the second approximation in green

Adding the extra terms with c_1 is a bit more involved; it gives

$$-\frac{1}{6} \pm \left(\sqrt{\frac{3}{2}}\sqrt{t} \left(-\sin\left(\frac{1}{2}\theta\right) + \mathrm{i}\cos\left(\frac{1}{2}\theta\right) \right) - \frac{1}{\sqrt{t}} \frac{5\sqrt{2}}{24\sqrt{3}} \left(\sin\left(-\frac{1}{2}\theta\right) - \mathrm{i}\cos\left(-\frac{1}{2}\theta\right) \right) \right).$$

For $t = 4$ Fig. 6 shows the situation, and also illustrates that the fit for the circle is not satisfactory, while the fit with the next term is essentially better. Obviously, for larger t this will improve even further.

The results of Theorem 1 and Corollary 3 show how the curves Γ_j can be described not only qualitatively, but as the examples show, also quantitatively the results are fairly sharp, certainly if we take into account the correction term to the circular approximation.

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Code availability Not applicable.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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