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A-Isometries and Hilbert-*A*-Modules Over Product Domains

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Abstract

For a compact set $K \subset \mathbb{C}^n$, let $A \subset C(K)$ be a function algebra containing the polynomials $\mathbb{C}[z_1, \cdots, z_n]$. Assuming that a certain regularity condition holds for A, we prove a commutant-lifting theorem for A-isometries that contains the known results for isometric subnormal tuples in its different variants as special cases, e.g., Mlak (Studia Math. 43(3): 219–233, 1972) and Athavale (J. Oper. Theory 23(2): 339–350, 1990; Rocky Mt. J. Math. 48(1): 2018; Complex Anal. Oper. Theory 2(3): 417–428, 2008; New York J. Math. 25: 934–948, 2019). In the context of Hilbert-A-modules, our result implies the existence of an extension map $\varepsilon : \mathrm{Hom}_A(\mathcal{S}_1, \mathcal{S}_2) \to \mathrm{Hom}_{C(\partial_A)}(\mathcal{H}_1, \mathcal{H}_2)$ for hypo-Shilov-modules $\mathcal{S}_i \subset \mathcal{H}_i$ $(i = 1, 2)$. By standard arguments, we obtain an identification $\mathrm{Hom}_A(\mathcal{S}_1, \mathcal{S}_2) \cong \mathrm{Hom}_A(\mathcal{H}_1 \ominus \mathcal{S}_1, \mathcal{H}_2 \ominus \mathcal{S}_2)$ where $\mathcal{H}_i$ is the minimal $C(\partial_A)$-extension of $\mathcal{S}_i$ $(i = 1, 2)$, provided that $\mathcal{H}_1$ is projective and $\mathcal{S}_2$ is pure. Using embedding techniques, we show that these results apply in particular to the domain algebra $A = A(D) = C(\overline{D}) \cap \mathcal{O}(D)$ over a product domain $D = D_1 \times \cdots \times D_k \subset \mathbb{C}^n$ where each factor D_i is either a smoothly bounded, strictly pseudoconvex domain or a bounded symmetric and circled domain in some $\mathbb{C}^{d_i}$ $(1 \leq i \leq k)$. This extends known results from the ball and polydisc-case, Guo (Studia Math. 135(1): 1–12, 1999) and Chen and Guo (J. Oper. Theory 43: 69–81, 2000).

Keywords Commutant lifting for subnormal isometric tuples · Analytic Hilbert modules over product domains

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Dedicated to the memory of Jörg Eschmeier, in deep gratitude for his guidance and inspiration.

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1 *A*-Isometries

Let $\mathcal{H}$ be a separable complex Hilbert space. A spherical isometry is a commuting tuple $T = (T_1, \cdots, T_n) \in B(\mathcal{H})^n$ of bounded linear operators on $\mathcal{H}$ that satisfies the condition $T_1^* T_1 + \cdots + T_n^* T_n = 1_{\mathcal{H}}$. Athavale proved in [3] that spherical isometries are subnormal with normal spectrum $\sigma_n(T)$ conained in $\partial\mathbb{B}_n$, a fact which provides them with a natural $A(\mathbb{B}_n)$-functional calculus, where $A(\mathbb{B}_n) = \{f \in C(\overline{\mathbb{B}}_n) : f|\mathbb{B}_n \text{ is holomorphic}\}$ denotes the ball algebra. Replacing $A(\mathbb{B}_n)$ with a general function algebra $A \subset C(K)$ on a compact set $K \subset \mathbb{C}^n$ and $\partial\mathbb{B}_n$ with its Shilov-boundary ∂_A, one arrives at the notion of A-isometry as introduced by Eschmeier in [16] (see below for a precise definition). It is this general setting in which we will formulate most of our results. We use the same notations as in [14] and [17]. A reader familiar with one of these works may readily proceed with Sect. 2.

Subnormal tuples

A tuple $T \in B(\mathcal{H})^n$ is called *subnormal*, if there is a commuting tuple of normal operators $U \in B(\widehat{\mathcal{H}})^n$ on a Hilbert space $\widehat{\mathcal{H}} \supset \mathcal{H}$ such that $\mathcal{H}$ is invariant for the components of U and $T = U|\mathcal{H}$. We suppose in the following that U is minimal in the sense that the only reducing subspace for U containing $\mathcal{H}$ is $\widehat{\mathcal{H}}$. The *normal spectrum* of T is defined as $\sigma_n(T) = \sigma(U)$, where $\sigma(U)$ denotes the Taylor spectrum of U. This is independent of the special choice of U. By a result of Putinar, the spectral inclusion $\sigma_n(T) \subset \sigma(T)$ holds. Spectral theory for normal tuples asserts the existence of a so-called *scalar spectral measure*, i. e., a finite positive Borel measure μ on $\sigma(U)$ with the property that there exists an isomorphism of von Neumann algebras

$$\Psi_U : L^\infty(\mu) \to W^*(U) \subset B(\widehat{\mathcal{H}}) \quad \text{mapping} \quad z_i \mapsto U_i \quad (i = 1, \cdots, n),$$

called the L^∞-*functional calculus* of U. (Note that such a measure μ is unique up to mutual absolute continuity.) One then defines the *restriction algebra*

$$\mathcal{R}_T = \{f \in L^\infty(\mu) : \Psi_U(f)\mathcal{H} \subset \mathcal{H}\},$$

which is a w^*-closed subalgebra of $L^\infty(\mu)$ containing the polynomials $\mathbb{C}[z]$ in n complex vairables $z = (z_1, \cdots, z_n)$. By [10, Proposition 1.1], the induced w^*-continuous algebra homomorphism

$$\gamma_T : \mathcal{R}_T \to B(\mathcal{H}), \quad f \mapsto \Psi_U(f)|\mathcal{H}$$

is isometric again. It satisfies the following uniqueness property:

$$\gamma_T(f|\sigma_n(T)) = f(T) \qquad (\text{whenever } f \in \mathcal{O}(W),\ W \subset \mathbb{C}^n \text{ open and } W \supset \sigma(T)), \tag{1.1}$$

where the symbol $f(T)$ on the right-hand side is meant in the sense of Taylor's holomorphic functional calculus (see, e.g., [18], Chapter 2). To see this, recall that the L^∞-calculus Ψ_U satisfies a spectral mapping theorem of the form $\sigma(\Psi_U(f_1), \cdots, \Psi_U(f_k)) = F(\sigma(U))$ for every k-tuple $F = (f_1, \cdots, f_k) \in C(K)^k$ with $k \in \mathbb{N}$ (see, e. g., [29], Section 0.3 and the references therein). A well-known uniqueness result for Taylor's functional calculus (cp. Theorem 5.2.4 in [18]) then guarantees that $\Psi_U(f) = f(U)$ for every $f \in \mathcal{O}(W)$ if $W \subset \mathbb{C}^n$ is any open set containing $\sigma(U)$. If we even assume that $W \supset \sigma(T)$, we can apply Lemma 2.5.8 in [18] to the inclusion map $i : \mathcal{H} \to \widehat{\mathcal{H}}$ as intertwiner of T and U to deduce that $f(U)|\mathcal{H} = f(T)$ for $f \in \mathcal{O}(W)$. Hence the asserted uniqueness (1.1) follows.

A-subnormal tuples, *A*-isometries

Fix a compact set $K \subset \mathbb{C}^n$ and let $C(K)$ denote the algebra of all continuous complex-valued functions equipped with the supremum norm. For a closed subalgebra $A \subset C(K)$, we write ∂_A for the Shilov boundary of A, i.e. the smallest closed subset of K such that $\|f\|_{\infty,K} = \|f\|_{\infty,\partial_A}$. Following Eschmeier, [16], we define:

Definition 1.1 Suppose that $A \subset C(K)$ is a closed subalgebra containing $\mathbb{C}[z]$. A subnormal tuple $T \in B(H)^n$ is said to be A-*subnormal*, if $\sigma_n(T) \subset K$ and $\mathcal{R}_T \supset A$. If, in addition, $\sigma_n(T) \subset \partial_A$, then T is said to be an A-*isometry*. An A-isometry consisting of normal operators is called A-*unitary*.

Natural choices for A to consider are algebras of disc-algebra type, that is,

$$A(D) = \{f \in C(\overline{D}) : f|D \text{ is holomorphic}\} \subset C(\overline{D}),$$

where $D \subset \mathbb{C}^n$ is a suitably chosen bounded open set. As pointed out above, setting $A = A(\mathbb{B}_n)$ yields exactly the class of spherical isometries, whereas $A(\mathbb{D}^n)$-isometries ($\mathbb{D} \subset \mathbb{C}$ stands for the open unit disc) are precisely commuting tuples whose components are isometric operators on $\mathcal{H}$.

Let now $T \in B(\mathcal{H})^n$ be an A-subnormal tuple for $A \subset C(K)$ as in the definition, and fix a scalar spectral measure μ of its minimal normal extension. Via trivial extension, we may regard μ as an element of $M^+(K)$, the set of all finite regular Borel measures on K. Since $\mathcal{R}_T$ is w^*-closed and assumed to contain A, the restriction of γ_T to the algebra

$$H^\infty_A(\mu) = \overline{A}^{w^*} \subset L^\infty(\mu)$$

is well defined. By the properties of γ_T stated above, it therefore induces an isometric and w^*-continuous isomorphism onto its image $\mathcal{T}_a(T) = \gamma_T(H^\infty_A(\mu))$ (the set of all so-called *analytic T-Toeplitz operators*). To keep the notation simple, we denote the induced map again by γ_T, and call it the *canonical H^∞-functional calculus* of T:

$$\gamma_T : H^\infty_A(\mu) \to \mathcal{T}_a(T) \subset B(H), \qquad f \mapsto \Psi_U(f)|\mathcal{H}.$$

A-unitary dilations

Let again $T \in B(\mathcal{H})^n$ be A-subnormal with minimal normal extension $U \in B(\widehat{\mathcal{H}})^n$. As a $*$-homomorphism, the map $\Psi_U|C(K) : C(K) \to B(\widehat{\mathcal{H}})$ is completely contractive, and hence so is $\gamma_T|A : A \to B(\mathcal{H})$, if A is regarded as an operator subspace of $C(K)$. Since both embeddings $A \subset C(K)$ and $A \subset C(\partial_A)$ induce the same operator space structure on A (see [27, Theorem 3.9]), a theorem of Arveson (see, e.g., [27, Corollary 7.7]) asserts that there exists an A-unitary tuple $V \in B(\mathcal{K})^n$ on a Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that

$$\gamma_T(f) = P_H \Psi_V(f)|\mathcal{H} \qquad (f \in A).$$

We call such a tuple an *A-unitary dilation* of T. When restricted to the smallest closed subspace that contains $\mathcal{H}$ and reduces V, it is said to be a *minimal A-unitary dilation*.

2 Regularity

If $T \in B(\mathcal{H})^n$ is an A-subnormal tuple, then one can use its canonical functional calculus γ_T to carry over approximation properties of the underlying function algebra A to the algebra $\mathcal{T}_a(T)$ of analytic Toeplitz operators. As it turns out, of particular interest are approximation properties of A that are related to the so-called abstract inner function problem. In its classical form, the inner function problem asks if there are non-constant bounded analytic functions $\theta : \mathbb{B}_n \to \mathbb{C}$ on the unit ball $\mathbb{B}_n$ such that their radial limit $\tilde{\theta} : \partial\mathbb{B}_n \to \mathbb{C}$ satisfies $|\tilde{\theta}| = 1$ σ-a.e. (so-called inner functions). Here, σ denotes the normalized surface measure on $\partial\mathbb{B}_n$. In his celebrated work [1], Aleksandrov succeded to solve the inner function problem for what he called "regular triples" (A, K, μ) (cf. Corollary 29 therein and Proposition 2.6 below for a suitable formulation in our context). The triple $(A(\mathbb{B}_n), \overline{\mathbb{B}}_n, \sigma)$ models the classical context. The precise definition of regularity reads as follows:

Definition 2.1 Let $K \subset \mathbb{C}^n$ compact and $A \subset C(K)$ a closed subalgebra.

(a) For $\mu \in M^+(K)$, the triple (A, K, μ) is called *regular* (in the sense of Aleksandrov [1]), if for every $\varphi \in C(K)$ with $\varphi > 0$, there is a sequence (f_k) in A satisfying $|f_k| < \varphi$ (on K) for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} |f_k| = \varphi$ (μ-a.e. on K).
(b) We call the algebra $A \subset C(K)$ itself *regular*, if (A, K, μ) is regular for every measure $\mu \in M^+(K)$ with $\mathrm{supp}(\mu) \subset \partial_A$.

Note that the condition on the support of μ in part (b) of the preceding definition results from the fact that the stated inclusion necessarily holds for regular triples. A concrete source of regular triples is the following embedding criterion due to Aleksandrov:

Theorem 2.2 (Aleksandrov [2, Theorem 3]) *Let $A \subset C(K)$ be a function algebra on a compact set K. Suppose that, for some $m \in \mathbb{N}$, there exists an injective map $F \in A^m$ such that $F(\partial_A) \subset \partial\mathbb{B}_m$. Then, $A \subset C(K)$ is regular.*

Using this criterion, one can show that the algebra $A(D)$ is regular for various kinds of open sets $D \subset \mathbb{C}^n$, among them bounded symmetric and circled domains, and relatively compact, strictly pseudoconvex open subsets of $\mathbb{C}^n$, or even of a Stein submanifold of $\mathbb{C}^n$ (cp. [1, 2] and [11, Corollary 2.1.3 (e)]). By constructing a suitable embedding, we show the following result for product domains:

Theorem 2.3 *Let $k \in \mathbb{N}$ and $D_i \subset \mathbb{C}^{d_i}$ be open sets with $d_i \in \mathbb{N}$ for $1 \leq i \leq k$, each of which is either a strictly pseudoconvex domain with C^2 boundary or a bounded symmetric and circled domain. Then the algebra $A(D_1 \times \cdots \times D_k) \subset C(\overline{D}_1 \times \cdots \times \overline{D}_k)$ is regular.*

We first recall some basic facts about bounded symmetric domains: By definition, a bounded domain $D \subset \mathbb{C}^d$ is symmetric if, for each $z \in D$, there exists a biholomorphic map $s_z : D \to D$ possessing z as isolated fixed point such that $s_z \circ s_z = \mathrm{id}_D$. We assume that D is circled at the origin, i.e., $0 \in D$ and $zD \subset D$ for $z \in \mathbb{C}$ with $|z| = 1$. Every set of this type is convex [24, Corollary 4.6] and the Shilov boundary of $A(D)$ consists precisely of those points in $\overline{D}$ with maximal Euclidean distance from the origin in $\mathbb{C}^d$ [24, Theorem 6.5]. Since every finite product of bounded symmetric and circled domains is again of this type, we only have to consider one such factor in Theorem 2.3. To keep the notations in the proof simple, we restrict ourselves to the case

$$D = D_1 \times D_2 \times D_3,$$

where D_1 and D_2 are strictly pseudoconvex sets and D_3 is the circled and bounded symmetric factor. This is no restriction, since the result is well-known for $k = 1$ (see the references given above), whereas the cases $k = 2$ and $k > 3$ can be handled by straight-forward modifications of our construction described below. In the following, we write $S_D = \partial_{A(D)}$.

To start with, let us fix real numbers $r_1, r_2, r_3 > 0$ such that

$$\overline{D}_i \subset r_i \cdot \mathbb{B}_{d_i} \quad (1 \leq i \leq 2) \quad \text{and} \quad D_3 \subset r_3 \cdot \mathbb{B}_{d_3}, \quad S_{D_3} = \overline{D}_3 \cap (r_3 \cdot \partial \mathbb{B}_{d_3}).$$

We first calculate the Shilov boundary of D. From the existence of peaking functions for every boundary point of a strictly pseudoconvex domain (see Range [28, Corollary VI.1.14]), it follows that $S_{D_i} = \partial D_i$ for the strictly pseudoconvex factors $i = 1, 2$.

Lemma 2.4 $S_D = \partial D_1 \times \partial D_2 \times S_{D_3}$.

Proof Fix an arbitrary $z = (z_1, z_2, z_3) \in D_1 \times D_2 \times D_3$. Since $f(\cdot, z_2, z_3) \in A(D_1)$, there is a $w_1 \in \partial D_1$ such that $|f(z_1, z_2, z_3)| \leq |f(w_1, z_2, z_3)|$. Repeating the argument twice, we obtain $w_2 \in \partial D_2$ and $w_3 \in S_{D_3}$ such that $|f(z_1, z_2, z_3)| \leq |f(w_1, w_2, w_3)|$. This shows that $\partial D_1 \times \partial D_2 \times S_{D_3}$ is a boundary for $A(D)$. Now fix $w = (w_1, w_2, w_3)$ in the latter product set. To finish the proof, it suffices to observe that $h(z_1, z_2, z_3) = h_1(z_1) \cdot h_2(z_2) \cdot h_3(z_3)$ is a peaking function for w, if $h_i \in A(D_i)$ is a peaking functions for w_i $(i = 1, 2, 3)$. For the strictly pseudoconvex factors, such functions exist by the reference given above. For D_3, we may take the restriction of a peaking function for w_3 in $A(r_3 \cdot \mathbb{B}_{d_3})$. □

The key ingredient in the proof of Theorem 2.3, is an embedding theorem of Løw [25, Theorem 3]. Reduced to what we need, this theorem says the following: Given a strictly pseudoconvex domain $D \subset \mathbb{C}^n$ with C^2-boundary and a strictly positive continuous function $\phi : \partial D \to (0, \infty)$, there exists a mapping $g \in A(D)^l$ for some natural number l such that $|g(z)| = \phi(z)$ for every $z \in \partial D$.

Proof of Theorem 2.3 Since, by assumption, $\overline{D}_i \subset r_i \cdot \mathbb{B}_{d_i}$ for $i = 1, 2$, the functions $\phi_i(z) = (r_i^2 - |z|^2)^{\frac{1}{2}}$ $(z \in \partial D_i,\ i = 1, 2)$ are strictly positive if $|\cdot|$ denotes the Euclidean norm on $\mathbb{C}^{d_i}$. By the cited theorem of Løw, we find $g_i \in A(D_i)^{l_i}$ $(i = 1, 2)$ with $|g_i(z)| = \phi_i(z)$ $(z \in \partial D_i)$. Then we set $r = \sqrt{r_1^2 + r_2^2 + r_3^2}$ and define

$$F(z) = (1/r) \cdot \big(z, g_1(z_1), g_2(z_2)\big) \in \mathbb{C}^m \qquad (z = (z_1, z_2, z_3) \in \overline{D}_1 \times \overline{D}_2 \times \overline{D}_3 = \overline{D}),$$

to obtain an injective mapping $F \in A(D)^m$ with $m = d_1 + d_2 + d_3 + l_1 + l_2$ which, for $z = (z_1, z_2, z_3) \in \partial D_1 \times \partial D_2 \times r_3 \cdot \partial\mathbb{B}_{d_3}$, satisfies

$$|F(z)|^2 = (1/r^2) \cdot \Big(|z_1|^2 + |z_2|^2 + r_3^2 + (r_1^2 - |z_1|^2) + (r_2^2 - |z_2|^2)\Big) = 1.$$

Together with Lemma 2.4 and the fact that $S_{D_3} \subset r_3 \cdot \partial\mathbb{B}_{d_3}$, this yields the inclusion $F(S_D) \subset \partial\mathbb{B}_m$. The desired regularity now follows as an application of Aleksandrov's embedding criterion stated as Theorem 2.2 above. □

Remark 2.5 Let $F \in A(D)^m$, and let $|\cdot|$ denote the Euclidean norm and $\langle\cdot,\cdot\rangle$ the Euclidean scalar-product on $\mathbb{C}^m$. By considering scalar-valued functions of the form $\langle F(\cdot), F(w)\rangle$ with suitably chosen $w \in \overline{D}$, one can show that $|F|$ takes its maximum value on the Shilov boundary S_D, and that F must be constant if $|F|$ takes its maximum at some point inside D. In view if this, we can – for further reference – state some additional properties of the map F constructed in the proof of Theorem 2.3, namely $F(\overline{D}) \subset \overline{\mathbb{B}}_m$ and $F(D) \subset \mathbb{B}_m$. Moreover, we remark that the map $F : \overline{D} \to F(\overline{D})$ is a homeomorphism, as it is a continuous bijection from a compact space onto a Hausdorff space.

As for the scope of Theorem 2.3, we should remark that every bounded open set $D \subset \mathbb{C}$ with C^2 boundary is strictly pseudoconvex, cp. the remark at the end of Section 1.5 in [22]. So, in particular $A(D) \subset C(\overline{D})$, is regular for polydomains $D = D_1 \times \cdots \times D_k$ with C^2-bounded domains $D_i \subset \mathbb{C}$ $(i = 1, \cdots, k)$.

Now, let (A, K, μ) be a regular triple. In analogy with the classical case we define a *μ-inner function* to be an element of the set

$$I_\mu = \left\{\theta \in H_A^\infty(\mu) : |\theta| = 1 \quad (\mu\text{-a.e. on } \partial_A)\right\}.$$

Our study of A-isometries in the following sections relies on a density result for such abstract inner functions taken from [14]. It is a slight variation of Aleksandrov's corresponding result [1, Corollary 29]. In its original form, it says that, given the regularity of (A, K, μ), the w^*-closure of the set I_μ in $L^\infty(\mu)$ contains the set

$\{[f]_\mu : f \in A,\ |f| \leq 1$ everywhere on $K\}$ supposed that μ is a *continuous* measure. Continuity means in this context that every one-point set has μ-measure zero. The key point here is that the following result holds *without any continuity assumption* on μ.

Proposition 2.6 (see [14, Proposition 2.4 and Corollary 2.5]) *If* (A, K, μ) *is a regular triple and* $A \supset \mathbb{C}[z]$. *Then we have*

$$H^\infty_A(\mu) = \overline{LH}^{w^*}(I_\mu) \quad \text{and} \quad L^\infty(\mu) = \overline{LH}^{w^*}(\{\overline{\theta}\eta : \theta, \eta \in I_\mu\}),$$

where $\overline{LH}^{w^*}$ *denotes the* w^**-closed linear hull.*

Note that, in the context of the preceding proposition, the first of these density relations implies the second one: The w^*-closed linear hull on the right is easily seen to be a w^*-closed subalgebra of $L^\infty(\mu)$ that is closed under complex conjugation. Since it contains the restrictions of all polynomials (by the first density relation), it actually must be all of $L^\infty(\mu)$.

It has turned out (cp. [14, 17, 19]) that this kind of density is exactly what is needed for operator-theoretic applications. So we define:

Definition 2.7 Let $K \subset \mathbb{C}^n$ compact, $A \subset C(K)$ a closed subalgebra.

(a) Given a measure $\mu \in M^+(K)$, the triple (A, K, μ) will be called w^**-regular*, if $H^\infty_A(\mu) = \overline{LH}^{w^*}(I_\mu)$, and $L^\infty(\mu) = \overline{LH}^{w^*}(\{\overline{\theta}\eta : \theta, \eta \in I_\mu\})$ hold.
(b) We simply call the algebra A itself w^**-regular*, if the triple (A, K, μ) is w^*-regular for every measure $\mu \in M^+(K)$ with $\mathrm{supp}(\mu) \subset \partial_A$.
(c) If $A \supset \mathbb{C}[z]$, then an A-isometry is called w^**-regular*, if the triple (A, K, μ) is w^*-regular, where $\mu \in M^+(\partial_A)$ denotes the scalar-valued spectral measure of a minimal normal extension of T.

Theorem 2.3 provides natural examples of w^*-regular function algebras. Further examples are the so called *unit-modulus algebras* introduced by Guo and Chen in [9, Definition 2.4].

3 Commutant Lifting for *A*-Isometries

The aim of this section is to prove the following commutant-lifting theorem that applies by the preceding section in particular to all algebras of the form $A(D_1 \times \cdots \times D_n)$ where each of the factors D_i is either a bounded strictly pseudoconvex domain with C^2-boundary, or a bounded sysmmetric and circled domain.

Theorem 3.1 *Let* $K \subset \mathbb{C}^n$ *be a compact set and* $A \subset C(K)$ *a closed,* w^**-regular subalgebra that contains* $\mathbb{C}[z]$. *Suppose that* $T_1 \in B(\mathcal{H}_1)^n$ *is an* A*-subnormal tuple with minimal* A*-unitary dilation* $U_1 \in B(\widehat{\mathcal{H}}_1)^n$ *and that* $T_2 \in B(\mathcal{H}_2)^n$ *is an* A*-isometry with minimal normal extension* $U_2 \in B(\widehat{\mathcal{H}}_2)^n$. *Then every operator* $X \in B(\mathcal{H}_1, \mathcal{H}_2)$ *satisfying*

$$X\gamma_{T_1}(f) = \gamma_{T_2}(f)X \qquad (f \in A)$$

possesses a unique extension to an operator $\widehat{X} \in B(\widehat{\mathcal{H}}_1, \widehat{\mathcal{H}}_2)$ *with the property that*

$$\widehat{X}\Psi_{U_1}(f) = \Psi_{U_2}(f)\widehat{X} \qquad (f \in C(\partial_A)).$$

This extension satisfies $\|\widehat{X}\| = \|X\|$. *If* X *has dense range, then so has* $\widehat{X}$. *If* U_1 *is even an* A*-unitary extension of* T_1 *and, in addition,* X *is isometric, then so is* $\widehat{X}$.

The above theorem contains several known intertwining results for subnormal isometric tuples as special cases (note that all but the last one require T_1 to be an A-isometry):

- A result of Guo (see [21, Lemma 3.6]) from the context of normal Hilbert modules. Actually our proof is inspired by Guo's idea. But his hypotheses are rather special: Roughly speaking, both tuples are assumed to be spherical isometries, and the spectral measures of their minimal normal extensions are assumed to be absolutely continuous with respect to the surface measure of the sphere.
- An intertwining result for spherical isometries by Athavale (Proposition 8 in [3]). More generally, the analogous result for so called ∂D-isometries introduced by Athavale in [4], where $D \subset \mathbb{C}^n$ is a relatively compact strictly pseudoconvex domain: Our Theorem 3.1 implies Athavale's Theorem 3.2.
  We carry out the details for the convenience of the reader: First of all, note that $A(D)$ is w^*-regular according to the remark preceding Theorem 2.3. Now, if we choose a domain $\Omega \supset \overline{D}$ in such a way that $\mathcal{O}(\Omega)|\overline{D}$ is dense in $A(D)$ (called *HKL-superdomain* in [4], see Remark 2.1 therein; for the existence, cp. [11, Corollary 2.1.3 (b)] and the references therein), then every ∂D-isometry with $\sigma(T) \subset \Omega$ is actually an $A(D)$-isometry: To see this, note that $\mathcal{R}_T$ contains $\mathcal{O}(\Omega)$, since γ_T extends the $\mathcal{O}(\Omega)$-functional calculus for T as observed in (1.1), Sect. 1, and hence $\mathcal{R}_T \supset A(D)$ for density reasons. In Athavales result, two ∂D-isometries $T_k = (T_{k,1}, \cdots, T_{k,n}) \in B(\mathcal{H}_k)^n$ $(k = 1, 2)$ are assumed to satisfy the inclusion $\sigma(T_1) \cup \sigma(T_2) \subset \Omega$ and an intertwining relation of the form

  $$XT_{1,i} = T_{2,i}X \qquad (i = 1, \cdots, n).$$

  To apply our theorem, it remains to check that $X\gamma_{T_1}(f) = \gamma_{T_2}(f)X$ for every $f \in A(D)$ hold in this case. But the assumed intertwining relation for the components immediately implies that $Xf(T_1) = f(T_2)X$ for $f \in \mathcal{O}(\Omega)$, where $f(T_k)$ is built using the holomorphic functional calculus. Again, the uniqueness of the canonical H^∞-calculi of T_1 and T_2 then implies that $X\gamma_{T_1}(f) = \gamma_{T_2}(f)X$ for all $f \in \mathcal{O}(\Omega)$. So by the density of $\mathcal{O}(\Omega)|\overline{D}$ in $A(D)$, the hypothesis of Theorem 3.1 is indeed satisfied.
- An intertwining result for so-called S_Ω-isometries [6, Theorem 2.1]. We sketch the main arguments: For a relatively compact, convex set $\Omega \subset \mathbb{C}^n$ with $0 \in \Omega$, it follows from the Oka-Weil theorem and radial approximation that $\mathbb{C}[z]$ is dense in $A(\Omega)$. So in this case, the condition $\mathcal{R}_T \supset A(\Omega)$ is always fulfilled. This shows in particular that S_Ω-isometries (where Ω is a Cartan domain of type I, II, III, or IV) as defined by Athavale in [6] are A-isometries with $A = A(\Omega)$, which is known to be regular for bounded symmetric and circled domains. Again, for uniqueness

reasons of the involved functional calculi, a componentwise intertwining relation implies the one assumed in our theorem. For another class of convex domains, where the domain algebra is regular, cp. [5] (see Proposition 2.5 therein for the regularity, and Proposition 4.6 for the corresponding commutant-lifting result). The sets considered there are special complex ellipsoids.

- Réolon's analogous result for K-isometries (cp. Proposition 3.5.18 in [29]) which are by definition A-isometries with $A = \overline{\mathbb{C}[z]} \subset C(K)$, where $K \subset \mathbb{C}^n$ compact and A is assumed to be regular in our sense. This class contains in particular finite commuting tuples of isometries, which were first treated by Mlak [26, Proposition 5.2].

We should further remark that all but the first cited result rely on a lemma of Mlak on spectral dilations [26, Lemma 4.1]. In contrast, we give an alternative and elementary proof by writing down an explicit formula for $\widehat{X}$, based on an idea of Guo [21, Lemma 3.6].

For the proof of Theorem 3.1, we need an operator-theoretic version of the measure-theoretic density assumption from Definition 2.7. To formulate it appropriately, let us fix the following notation: Given two measures $\mu, \nu \in M^+(\partial_A)$ with $\mu \ll \nu$, let

$$r_\mu^\nu : L^\infty(\nu) \to L^\infty(\mu)$$

denote the canonical map defined by $[f]_\nu \mapsto [f]_\mu$ for every bounded measurable function $f : \partial_A \to \mathbb{C}$. Clearly, every r_μ^ν is a w^*-continuous and contractive $*$-homomorphism.

Given an A-unitary tuple $U \in B(\widehat{\mathcal{H}})^n$ with scalar-valued spectral measure $\mu \ll \nu$, we denote by Ψ_U^ν the composition

$$\Psi_U^\nu = \Psi_U \circ r_\mu^\nu : L^\infty(\nu) \to B(\widehat{\mathcal{H}}).$$

As a consequence of Proposition 2.6, we have:

Lemma 3.2 *Let (A, K, ν) be a w^*-regular triple and $U \in B(\widehat{\mathcal{H}})^n$ an A-unitary tuple with scalar-valued spectral measure $\mu \in M^+(\partial_A)$ satisfying $\mu \ll \nu$. Then, for every subspace $\mathcal{H} \subset \widehat{\mathcal{H}}$, the set*

$$\widehat{\mathcal{H}}_0 = \overline{LH}\{\Psi_U^\nu(\overline{\theta}\eta)h : \theta, \eta \in I_\nu,\ h \in \mathcal{H}\} \subset \widehat{\mathcal{H}}$$

is the smallest closed subspace of $\widehat{\mathcal{H}}$ that contains $\mathcal{H}$ and reduces U. (Here, $\overline{LH}$ stands for the norm-closed linear hull in $\widehat{\mathcal{H}}$.)

Proof We first show that $\widehat{\mathcal{H}}_0$ reduces U. Towards this, let $\theta_0, \eta_0 \in I_\nu$. Obviously, $\Psi_U^\nu(\overline{\theta}_0\eta_0)$ maps the set

$$\{\Psi_U^\nu(\overline{\theta}\eta)h : \theta, \eta \in I_\nu,\ h \in \mathcal{H}\}$$

into itself. By linearity and continuity, we may pass to $\overline{LH}$, so

$$\Psi_U^\nu(\overline{\theta}\eta)\widehat{\mathcal{H}}_0 \subset \widehat{\mathcal{H}}_0 \qquad (\text{whenever } \theta, \eta \in I_\nu).$$

Since invariant subspaces are preserved under w^*-limits, the w^*-regularity assumption (cf. Definition 2.7) allows us to conclude that $\Psi_U^\nu(f)\widehat{\mathcal{H}}_0 \subset \widehat{\mathcal{H}}_0$ for every $f \in L^\infty(\nu)$. By taking f to be z_i and $\overline{z}_i$ ($i = 1, \cdots, n$), we see that $\widehat{\mathcal{H}}_0$ is actually reducing for U. Standard arguments based on the theorems of Stone-Weierstraß and Lusin yield the stated minimality. □

Besides the above density result, an essential ingredient of the proof is the simple observation that, for every $\theta \in I_\nu$, the operator $\Psi_U^\nu(\theta)$ is isometric, since it evidently satisfies $\Psi_U^\nu(\theta)^*\Psi_U^\nu(\theta) = \Psi_U^\nu(\overline{\theta}\theta) = 1_{\mathcal{H}}$. (Actually, it is even unitary as $\overline{\theta}$ is the inverse of θ in $L^\infty(\nu)$, but isometry will suffice for our purposes.)

Proof of Theorem 3.1 For $k = 1, 2$, let $\mu_k \in M^+(K)$ denote the trivial extensions of scalar spectral measures of U_k to elements in $M^+(K)$. The proof will actually work if we replace the hypothesis on A to be a w^*-regular function algebra with the following weaker but more technical one: There exists a measure $\nu \in M^+(\partial_A)$ such that (A, K, ν) is w^*-regular and $\mu_1, \mu_2 \ll \nu$. Let us fix such a measure ν for the rest of the proof. (Under the original hypothesis of the theorem, $\nu = \mu_1 + \mu_2$ evidently has the desired properties.) Throughout the proof, we use the abbreviations $\Psi_{U_k}^\nu = \Psi_{U_k} \circ r_{\mu_k}^\nu : L^\infty(\nu) \to B(\widehat{\mathcal{H}}_k)$ and $\gamma_{T_k}^\nu = \gamma_{T_k} \circ r_{\mu_k}^\nu : H_A^\infty(\nu) \to B(\mathcal{H})$ for $k = 1, 2$.

We first define $\widehat{X}$ on a dense subset of $\widehat{\mathcal{H}}_1$ (cf. Lemma 3.2) in the obvious way, namely

$$\widehat{X}\left(\sum_{i\in F} \Psi_{U_1}^\nu(\overline{\theta}_i\eta_i)h_i\right) = \sum_{i\in F} \Psi_{U_2}^\nu(\overline{\theta}_i\eta_i)Xh_i, \tag{3.1}$$

where $F \subset \mathbb{N}$ denotes a finite set and $\theta_i, \eta_i \in I_\nu, h_i \in \mathcal{H}$ are arbitrary elements for $i \in F$. To see that this is well defined, we estimate the norm of the right-hand side. If we define $\pi = \prod_{i\in F} \theta_i$, then $\pi \in I_\nu$, and so the operator $\Psi_{U_2}^\nu(\pi)$ is isometric. This yields

$$\left\|\sum_{i\in F} \Psi_{U_2}^\nu(\overline{\theta}_i\eta_i)Xh_i\right\| = \left\|\sum_{i\in F} \Psi_{U_2}^\nu(\pi\overline{\theta}_i\eta_i)Xh_i\right\| \tag{3.2}$$

From the very definition of π, the functions $\pi\overline{\theta}_i$ belong to I_ν for every choice of $i \in F$. Since, in addition, X maps into $\mathcal{H}_2$, and U_2 is an A-unitary extension of T_2, we may replace $\Psi_{U_2}^\nu$ with $\gamma_{T_2}^\nu$. Then we use the intertwining relation $\gamma_{T_2}(f)X = X\gamma_{T_1}(f) = XP_{\mathcal{H}}\Psi_{U_1}(f)|\mathcal{H}$ ($f \in A$) from the hypothesis. We restate this as

$$\langle\gamma_{T_2}(f)Xh, k\rangle = \langle\Psi_{U_1}(f)h, X^*k\rangle \qquad (f \in A, \quad h, k \in \mathcal{H}).$$

to see that it extends by w^*-WOT-continuity to

$$\gamma_{T_2}^\nu(f)X = XP_{\mathcal{H}}\Psi_{U_1}^\nu(f)|\mathcal{H} \qquad (f \in H_A^\infty(\nu)).$$

These considerations allow us to continue expression (3.2) as follows

$$\begin{aligned} \cdots &= \left\| \sum_{i \in F} \gamma^{\nu}_{T_2}(\pi \overline{\theta}_i \eta_i) X h_i \right\| \leq \|X\| \cdot \left\| \sum_{i \in F} \Psi^{\nu}_{U_1}(\pi \overline{\theta}_i \eta_i) h_i \right\| \\ &= \|X\| \cdot \left\| \sum_{i \in F} \Psi^{\nu}_{U_1}(\overline{\theta}_i \eta_i) h_i \right\|, \end{aligned} \tag{3.3}$$

where we again used the isometry of $\Psi^{\nu}_{U_1}(\pi)$ in the last step. First of all, this shows that $\widehat{X}$ is indeed well defined (and then, by construction, linear) on the space $\mathcal{F}$ of all finite linear combinations of vectors of the form $\Psi^{\nu}_{U_1}(\overline{\theta}_i \eta_i) h_i$ with $\theta_i, \eta_i \in I_{\nu}, h_i \in \mathcal{H}_1$. Moreover, the estimate derived in (3.2)–(3.3) briefly says that

$$\|\widehat{X} x\| \leq \|X\| \cdot \|x\| \qquad (x \in \mathcal{F}). \tag{3.4}$$

In particular, $\widehat{X}$ maps Cauchy-sequences in $\mathcal{F} \subset \widehat{\mathcal{H}}_1$ to Cauchy-sequences in $\widehat{\mathcal{H}}_2$ and thus can be continuously extended to a linear map on the closure $\overline{\mathcal{F}}$, which coincides with $\widehat{\mathcal{H}}_1$ by Lemma 3.2. This map has obviuously norm less than $\|X\|$, and a look at the assumed density (Definition 2.7 (b)) and the defining formula (3.1) for $\widehat{X}$ shows that it actually has the desired intertwining property.

In view of Lemma 3.2 and (3.1), $\widehat{X}$ has dense range if so has X. Moreover, if U_1 is even an A-unitary extension of T_1, then we may drop the operator $P_{\mathcal{H}}$ in the considerations preceding estimate (3.3). Thus, if X is isometric, then equality holds in (3.3) and (3.4) with $\|X\| = 1$.

To finish the proof, note that an operator $\widehat{X}$ satisfying the intertwining relation from the statement of the theorem necessarily fulfills $\widehat{X} \Psi^{\nu}_{U_1}(f) = \Psi^{\nu}_{U_1}(f) \widehat{X}$, because $C(\partial_A)$ is w^*-dense in $L^{\infty}(\nu)$. Consequently, it satisfies (3.1), and therefore coincides with the extension constructed above. □

When specialized to one A-isometry, Theorem 3.1 reads as follows:

Corollary 3.3 *Let $T \in B(\mathcal{H})^n$ be a w^*-regular A-isometry with minimal normal extension $U \in B(\widehat{\mathcal{H}})^n$. Then, every element $X \in \mathcal{T}_a(T)'$ possesses a unique extension to an element $\widehat{X} \in (U)'$.*

Combining Theorem 3.1 and Lemma 1 from [3]), we can draw another standard conclusion in this context (cp. Athavale [3, Proposition 9]):

Corollary 3.4 *If two w^*-regular A-isometries are quasi-similar, then their minimal unitary extensions are unitarily equivalent.*

4 Lifting of Module Homomorphisms

Let $A \subset C(K)$ be function algebra, i. e., a closed subalgebra of $C(K)$ that separates the points of K. Recall that a *Hilbert-A-module* is a Hilbert space $\mathcal{H}$ together with a continuous bilinear map

$$A \times \mathcal{H} \to \mathcal{H}, \quad (f, h) \mapsto f \cdot h$$

turning $\mathcal{H}$ into an A-module in the algebraical sense. Given $f \in A$, we write M_f (or $M_f^{\mathcal{H}}$, if we want to emphasize the underlying space) for the continuos linear multiplication operator $\mathcal{H} \to \mathcal{H},\ h \mapsto f \cdot h$ with $f \in A$. The continuous linear map $\Phi_{\mathcal{H}} : A \to B(\mathcal{H}),\ f \mapsto M_f$ will be referred to as the underlying representation of $\mathcal{H}$. The Hilbert module $\mathcal{H}$ is called *contractive*, if so is $\Phi_{\mathcal{H}}$.

Definition 4.1 Let $A \subset C(\partial_A)$ be a function algebra. A Hilbert-A-module $\mathcal{S}$ is called a *hypo-Shilov-module*, if there exists a Hilbert-$C(\partial_A)$-module $\mathcal{H}$ such that $\mathcal{S} \subset \mathcal{H}$ is an A-submodule. In this case, $\mathcal{H}$ is called a $C(\partial_A)$-extension of $\mathcal{H}$. If $\mathcal{H}$ is contractive, $\mathcal{S}$ is called a *Shilov module* over A.

Given this definition, the following further notations are natural: The extension $\mathcal{H}$ of a (hypo-)Shilov-module is called *minimal*, if $C(\partial_A) \cdot \mathcal{S}$ is dense in $\mathcal{H}$. A hypo-Shilov-module is called *reductive*, if it is a $C(\partial_A)$-submodule of $\mathcal{H}$, and *pure*, if no non-zero subspace of it is reductive.

The key observation for applying Theorem 3.1 in the context of Hilbert modules is the following simple lemma which says that A-isometries and Shilov-A-modules are essentially the same.

Lemma 4.2 *Let $A \subset C(K)$ be a closed subalgebra containing $\mathbb{C}[z]$.*

(a) *If $\mathcal{S} \subset \mathcal{H}$ is a Shilov-A-module with minimal contractive $C(\partial_A)$-extension $\mathcal{H}$, then $M_z \in B(\mathcal{S})^n$ is an A-isometry with minimal normal extension $M_z \in B(\mathcal{H})^n$.*
(b) *Conversely, if $T \in B(\mathcal{S})^n$ is an A-isometry with minimal normal extension $U \in B(\mathcal{H})^n$, then, introducing the multiplications $A \times \mathcal{S} \to \mathcal{S},\ (f, h) \mapsto f \cdot h = \gamma_T(f)h$ on $\mathcal{H}$ and $C(\partial_A) \times \mathcal{H} \to \mathcal{H},\ (f, h) \mapsto f \cdot h = \Psi_U(f)h$ on $\mathcal{H}$ turns $\mathcal{S} \subset \mathcal{H}$ into a Shilov module over A with minimal $C(\partial_A)$-extension $\mathcal{H}$.*

Proof Since, in part (a), the representation $\Phi_{\mathcal{H}} : C(\partial_A) \to B(\mathcal{H}),\ f \mapsto M_f$ is assumed to be contractive, it is a $*$-homomorphism (see [15, Theorem 1.12]) and thus gives rise to a normal tuple $M_z = (M_{z_1}, \cdots, M_{z_n}) \in B(\mathcal{H})^n$ with Taylor spectrum $\sigma(M_z) \subset \partial_A$. Since $\mathcal{S}$ is an A-submodule of $\mathcal{H}$, we have the inclusion

$$M_f \mathcal{S} \subset \mathcal{S} \qquad \text{(for every } f \in A). \tag{4.1}$$

This implies at first that $T = M_z|\mathcal{S}$ is subnormal with normal extension $U = M_z \in B(\mathcal{H})^n$. Since, by assumption, $C(\partial_A) \cdot \mathcal{S}$ is dense in $\mathcal{H}$, it follows that U is a minimal normal extension of T. By a Stone-Weiertraß argument, the identity $\Phi_{\mathcal{H}}(f) = \Psi_U(f)$, where Ψ_U is the canonical L^∞-calculus of U (see Sect. 1) holds for all $f \in C(\partial_A)$. Hence the inclusion (4.1) from above actually says that $\mathcal{R}_T \supset A$ as required in the definition of an A-isometry. This observation completes the proof of part (a). Part (b) is obvious. □

By an *A-module homomorphism* $X : \mathcal{H} \to \mathcal{K}$ between two Hilbert-A-modules we mean a bounded linear map that respects the module multiplication in the sense that $X(f \cdot h) = f \cdot Xh$ whenever $f \in A$ and $h \in \mathcal{H}$. We write $\mathrm{Hom}_A(\mathcal{H}, \mathcal{K})$ to denote the set of all A-module homomorphisms from $\mathcal{H}$ to $\mathcal{K}$.

Let now two Shilov modules $\mathcal{S}_1, \mathcal{S}_2$ with minimal $C(\partial_A)$-extensions $\mathcal{H}_1, \mathcal{H}_2$ be given. Using the identifications from the preceding proof, namely $\gamma_{T_k}(f) = M_f^{\mathcal{H}_k}|\mathcal{S}_k$ $(f \in A)$ and $\Psi_{U_k}(f) = M_f^{\mathcal{H}_k}$ $(f \in C(\partial_A))$, the following identities are evident:

$$\mathrm{Hom}_A(\mathcal{S}_1, \mathcal{S}_2) = \{X \in B(\mathcal{S}_1, \mathcal{S}_2) : \gamma_{T_1}(f)X = X\gamma_{T_2}(f) : f \in A\}$$
$$\mathrm{Hom}_{C(\partial_A)}(\mathcal{H}_1, \mathcal{H}_2) = \{X \in B(\mathcal{H}_1, \mathcal{H}_2) : \Psi_{U_1}(f)X = X\Psi_{U_2}(f) : f \in C(\partial_A)\}.$$

As a last ingredient for the extension theorem we aim for (Theorem 4.4), we need the following well-known fact:

Theorem 4.3 (cp. [15, Theorem 1.9]) *Let $\mathcal{H}$ be a Hilbert-$C(K)$-module. Then $\mathcal{H}$ is similar to a contractive $C(K)$-module $\mathcal{K}$, i. e., there exists an invertible $C(K)$-module map (a similarity map) $X : \mathcal{H} \to \mathcal{K}$.*

Now we are ready to prove the main result of this section. In the case $A = A(\mathbb{B}_n)$, this appears as Lemma 3.6 in [21] (where it is stated for Shilov-modules only, and under an additional continuity assumption). For a unit modulus algebra A, the corresponding result appears as Proposition 2.5 in [9]. Our theorem contains both as special cases.

Theorem 4.4 *Let $\mathcal{S}_k$ be hypo-Shilov modules over A with $C(\partial_A)$-extensions $\mathcal{H}_k$ ($k =$ 1, 2). Suppose that $A \supset \mathbb{C}[z]$ is w^*-regular. Then there exists a map*

$$\varepsilon : \mathrm{Hom}_A(\mathcal{S}_1, \mathcal{S}_2) \to \mathrm{Hom}_{C(\partial_A)}(\mathcal{H}_1, \mathcal{H}_2) \quad \textit{with } \varepsilon(X)|\mathcal{S}_1 = X.$$

If $\mathcal{H}_1$ is minimal, then ε is unique.

Proof The proof is divided into three steps.
Step I: Reduction to the minimal case. To prove the existence of ε, we may assume that $\mathcal{H}_k$ are minimal $C(\partial_A)$-extensions of $\mathcal{S}_k$ $(k = 1, 2)$. To see this, let $\mathcal{K}_k$ are arbitrary $C(\partial_A)$-extensions of $\mathcal{S}_k$, and set $\mathcal{H}_k = \overline{C(\partial_A) \cdot_{\mathcal{K}_k} \mathcal{S}_k}$, which is minimal. Then $\mathcal{H}_1$ is a reducing submodule of $\mathcal{K}_1$, whence the projection $P_{\mathcal{H}_1} \in B(\mathcal{K}_1)$ is a $C(\partial_A)$-module homomorphism, as is the inclusion map $i_{\mathcal{K}_2} : \mathcal{H}_2 \to \mathcal{K}_2$ (for trivial reasons). Assuming the existence-assertion of the theorem to hold for the minimal extensions $\mathcal{H}_k$, we may set

$$\varepsilon'(X) = i_{\mathcal{K}_2} \circ \varepsilon(X) \circ P_{\mathcal{K}_1} \in \mathrm{Hom}_{C(\partial_A)}(\mathcal{K}_1, \mathcal{K}_2) \qquad (X \in \mathrm{Hom}_A(\mathcal{S}_1, \mathcal{S}_2))$$

to obtain an extension mapping $\varepsilon' : \mathrm{Hom}_A(\mathcal{S}_1, \mathcal{S}_2) \to \mathrm{Hom}_{C(\partial_A)}(\mathcal{K}_1, \mathcal{K}_2)$ in the general case. To finish the first step, we have to justify that replacing $\mathcal{K}_2$ with $\mathcal{H}_2$ does not impose any restriction on the uniqueness assertion either: But if $\mathcal{H}_1$ is assumed to be minimal (as it is in the uniqueness part), then every $\widehat{X} \in \mathrm{Hom}_{C(\partial_A)}(\mathcal{H}_1, \mathcal{K}_2)$ satisfies $\widehat{X}(C(\partial_A) \cdot \mathcal{S}_1) \subset C(\partial_A) \cdot \mathcal{S}_2$ and thus has range in $\mathcal{H}_2 = \overline{C(\partial_A) \cdot \mathcal{S}_2}$, and may

actually considered an element of $\mathrm{Hom}_{C(\partial_A)}(\mathcal{H}_1, \mathcal{H}_2)$. So in what follows, we may and shall assume $\mathcal{H}_k$ to be a minimal extension of $\mathcal{S}_k$ $(k = 1, 2)$.
Step II: Reduction to the Shilov-case. According to Theorem 4.3, we find invertible Hilbert-$C(\partial_A)$-module maps

$$S_k : \mathcal{H}_k \to \mathcal{K}_k \qquad (k = 1, 2)$$

onto contractive Hilbert-$C(\partial_A)$-modules. Since restrictions $S_k|\mathcal{S}_k : \mathcal{S}_k \to S_k\mathcal{S}_k$ are also invertible A-module maps, the vertical maps in the following diagram

$$\begin{array}{ccc} \mathrm{Hom}_A(\mathcal{S}_1, \mathcal{S}_2) & \xrightarrow{\varepsilon} & \mathrm{Hom}_{C(\partial_A)}(\mathcal{H}_1, \mathcal{H}_2) \\ \downarrow{\scriptstyle C} & & \downarrow{\scriptstyle \widehat{C}} \\ \mathrm{Hom}_A(S_1\mathcal{S}_1, S_2\mathcal{S}_2) & \xrightarrow{\varepsilon_S} & \mathrm{Hom}_{C(\partial_A)}(\mathcal{K}_1, \mathcal{K}_2), \end{array}$$

which are defined as "conjugations"

$$\begin{array}{ll} C(X) = (S_2|\mathcal{S}_2) \circ X \circ (S_1|\mathcal{S}_1)^{-1} & (X \in \mathrm{Hom}_A(\mathcal{S}_1, \mathcal{S}_2)) \\ \widehat{C}(\widehat{X}) = S_2 \circ \widehat{X} \circ S_1^{-1} & (\widehat{X} \in \mathrm{Hom}_{C(\partial_A)}(\mathcal{H}_1, \mathcal{H}_2)) \end{array}$$

are bijections. Assuming the theorem to hold for Shilov-modules and their minimal extensions, there is an extension map ε_S from $\mathrm{Hom}_A(S_1\mathcal{S}_1, S_2\mathcal{S}_2)$ to $\mathrm{Hom}_{C(\partial_A)}(\mathcal{K}_1, \mathcal{K}_2)$ as stated in the theorem, which we take as the lower horizontal map of the diagram. We define the upper horizontal map so to make the diagram commutative. To state it explicitly,

$$\varepsilon(X) = S_2^{-1} \circ \varepsilon_S\big((S_2|\mathcal{S}_2) \circ X \circ (S_1|\mathcal{S}_1)^{-1}\big) \circ S_1 \quad (X \in \mathrm{Hom}_A(\mathcal{S}_1, \mathcal{S}_2)).$$

The stated extension property $\varepsilon(X)|\mathcal{S}_1 = X$ for $X \in \mathrm{Hom}_A(\mathcal{S}_1, \mathcal{S}_2)$ is obvious. Moreover, using the diagram, it is easy to see that ε is unique with the stated extension property if and only if so is ε_S with the corresponding property.
Step III: The Shilov-case with minimal extensions. So we may finally assume that $\mathcal{S}_k$ are Shilov-modules with minimal $C(\partial_A)$-extensions $\mathcal{H}_k$ for $k = 1, 2$. By Lemma 4.2 (a) and the subsequent remarks on A-module homomorphisms, the statement of the theorem is nothing but a reformulation of Theorem 3.1. Thus, the proof is complete. □

The existence of a lifting map of the above type implies by standard arguments the existence of an "Hom-isomorphism theorem" for hypo-Shilov-modules and their complements (cp. [21] and [9]). To give an appropriate formulation (see Theorem 6.1 below), we need the concept of projectivity, which is introduced and studied for $A(D)$-modules over product domains in the next section.

5 Projectivity and Injectivity Over Product Domains

Let $\mathcal{H}_A$ denote the category of Hilbert-A-modules. An object $\mathcal{H} \in \mathcal{H}_A$ is called *projective*, if for every pair of objects $\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{H}_A$ together with A-module maps $X_2 : \mathcal{H} \to \mathcal{H}_2$ and $X_2^1 : \mathcal{H}_1 \to \mathcal{H}_2$ with X_2^1 onto, there exists an A-module map $X_1 : \mathcal{H} \to \mathcal{H}_1$ such that the following diagram is commutative

$$\begin{tikzcd} & \mathcal{H}_1 \arrow[d, "X_2^1"] \\ \mathcal{H} \arrow[ur, dashed, "\exists X_1"] \arrow[r, "X_2"] & \mathcal{H}_2. \end{tikzcd}$$

A module $\mathcal{H} \in \mathcal{H}_A$ is called *injective*, if for every pair $\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{H}_A$ with A-module maps $Y_1 : \mathcal{H}_1 \to \mathcal{H}$ and $Y_1^2 : \mathcal{H}_1 \to \mathcal{H}_2$ such that Y_1^2 is injective with closed range, there exists an A-module map $Y_2 : \mathcal{H}_2 \to \mathcal{H}$ such that $Y_1 = Y_2 \circ Y_1^2$.

We refer to [7] for the homological background. Proposition 2.1.5 therein relates the characterization of injective and projective objects to the vanishing of cohomology groups $\mathrm{Ext}^1_{\mathcal{H}_A}(\mathcal{K}, \mathcal{H})$ defined as follows: Two short exact sequences

$$E : 0 \longrightarrow \mathcal{H} \xrightarrow{A} \mathcal{J} \xrightarrow{B} \mathcal{K} \longrightarrow 0 \quad \text{and} \quad E' : 0 \longrightarrow \mathcal{H} \xrightarrow{A'} \mathcal{J}' \xrightarrow{B'} \mathcal{K} \longrightarrow 0$$

in the category $\mathcal{H}_A$ are called *equivalent* if there exists a morphism $X \in \mathrm{Hom}_A(\mathcal{J}, \mathcal{J}')$ making the diagram

$$\begin{tikzcd} E : 0 \arrow[r] & \mathcal{H} \arrow[r, "A"] \arrow[d, equal] & \mathcal{J} \arrow[r, "B"] \arrow[d, "X"] & \mathcal{K} \arrow[r] \arrow[d, equal] & 0 \\ E' : 0 \arrow[r] & \mathcal{H} \arrow[r, "A'"] & \mathcal{J}' \arrow[r, "B'"] & \mathcal{K} \arrow[r] & 0 \end{tikzcd}$$

commutative. The first cohomology group is then defined by

$$\mathrm{Ext}^1_{\mathcal{H}_A}(\mathcal{K}, \mathcal{H}) = \left\{[E];\; E : 0 \to \mathcal{H} \xrightarrow{A} \mathcal{J} \xrightarrow{B} \mathcal{K} \to 0 \text{ is an exact sequence in } \mathcal{H}_A\right\}.$$

The zero element of $\mathrm{Ext}^1_{\mathcal{H}_A}(\mathcal{K}, \mathcal{H})$ is the split extension

$$0 \longrightarrow \mathcal{H} \xrightarrow{i_{\mathcal{H}}} \mathcal{H} \oplus \mathcal{K} \xrightarrow{P_{\mathcal{K}}} \mathcal{K} \longrightarrow 0,$$

where $i_{\mathcal{H}}$ and $P_{\mathcal{K}}$ denote the canonical inclusion and projection, respectively. In terms of Ext-groups, an element $\mathcal{K} \in \mathcal{H}_A$ is projective, if and only if $\mathrm{Ext}^1_{\mathcal{H}_A}(\mathcal{K}, \mathcal{H}) = 0$ for every $\mathcal{H} \in \mathcal{H}_A$, and injective, if and only if $\mathrm{Ext}^1_{\mathcal{H}_A}(\mathcal{H}, \mathcal{K}) = 0$ for every $\mathcal{H} \in \mathcal{H}_A$.

It was shown by Carlson and Clark [8] for $D = \mathbb{D}^n$, by Guo [21] for $D = \mathbb{B}_n$ (under an additional continuity assumption on the underlying modules related to the surface measure on $\partial\mathbb{B}_n$) and by Eschmeier and the author [13] in full generality for every strictly pseudoconvex bounded open set and every bounded symmetric domain

$D \subset \mathbb{C}^n$ that $\mathrm{Ext}^1_{\mathcal{H}_A}(\mathcal{K}, \mathcal{H}) = 0$ and $\mathrm{Ext}^1_{\mathcal{H}_A}(\mathcal{H}, \mathcal{K}) = 0$, whenever $\mathcal{H}$ is a Hilbert-$C(S_D)$-module. The main result of this section says that this remains true in the product setting we studied in Sect. 2:

Theorem 5.1 *Let $k \in \mathbb{N}$ and $D_i \subset \mathbb{C}^{d_i}$ be bounded open sets with $d_i \in \mathbb{N}$ for $1 \le i \le k$, each of which is either a strictly pseudoconvex domain with C^5 boundary or a bounded symmetric and circled domain, and let $D = D_1 \times \cdots \times D_k$ be the product domain. If $\mathcal{H}$ is a Hilbert-$C(S_D)$-module, then we have*

$$\mathrm{Ext}^1_{\mathcal{H}_{A(D)}}(\mathcal{K}, \mathcal{H}) = 0 \quad \textit{and} \quad \mathrm{Ext}^1_{\mathcal{H}_{A(D)}}(\mathcal{H}, \mathcal{K}) = 0$$

for every Hilbert-$A(D)$-module $\mathcal{K}$.

To prove this, we basically use an emedding argument similar to the one used to deduce the bounded symmetric case from the strictly pseudoconvex one in [13] (see the remarks preceding Corollary 3.2 therein). While the embedding used in the latter case was the trivial one (namely, the inclusion of a bounded symmetric and circled domain into its envelopping ball), the construction of a suitable embedding is the non-trivial part in the product setting under consideration.

In preparation of the main construction we show the following auxiliary lemma:

Lemma 5.2 *Let $D_1 \subset \mathbb{C}^m$ and $D_2 \subset \mathbb{C}^k$ be bounded open sets and assume that D_2 is convex. Then, the mapping*

$$S : A(D_1 \times D_2) \longrightarrow A(D_2, A(D_1)), \qquad (Sf)(z_2) = f(\cdot, z_2) \quad \textit{for } z_2 \in D_2$$

is an isometric isomorphism. Here, $A(D_2, A(D_1)) \subset C(\overline{D}_2, A(D_1))$ is the closed subspace consisting of those functions that are analytic on D_2, viewed as functions with values in the Banach space $A(D_1)$.

Proof It is elementary to check that S, viewed as a map $A(D_1 \times D_2) \to C(\overline{D}_2, A(D_1))$ is a well-defined isometry. To see that it actually maps into $A(D_2, A(D_1))$, we have to make sure that $Sf|D_2$ is analytic as a function with values in $A(D_1)$. Since $A(D_1)$ isometrically embeds as a subspace into $H^\infty(D_1)$ (by restriction), it suffices to check the holomorphy of $Sf|D_2$, viewed as a map $D_2 \to H^\infty(D_1)$. But this follows from [12, Lemma 5.4], which settles the corresponding identification $H^\infty(D_1 \times D_2) \cong H^\infty(D_2, H^\infty(D_1))$. So the map S from the statement of the theorem is actually a well-defined isometry. To conclude the proof, it suffices to check that S is surjective. Towards this, let $f \in A(D_2, A(D_1))$ be given. Applying a suitable translation, we may assume that D_2 contains the origin. Using the uniform continuity of f, one shows that, for $r \uparrow 1$ $(0 < r < 1)$, the functions f_r defined by $f_r(z) = f(rz)$ which are holomorphic on the open set $D_r = (1/r) \cdot D_2 \supset \overline{D}_2$ converge to f uniformly on $\overline{D}_2$. Thus we can choose a sequence (r_n) in such a way that the functions

$$f_n = f_{r_n} \in \mathcal{O}(D_{r_n}, A(D_1)) \quad \text{satisfy } \|f - f_n\|_{\infty, \overline{D}_2} < \frac{1}{n} \text{ for } n \ge 1.$$

In view of the well known identification $\mathcal{O}(D_{r_n})\widehat{\otimes}A(D_1) \cong \mathcal{O}(D_{r_n}, A(D_1))$ (see, e. g., [30]), where the latter space carries the topology of uniform convergence on compact subsets, we can choose, for each $n \geq 1$, a natural number k_n and functions $\varphi_i^n \in \mathcal{O}(D_{r_n})$ and $h_i^n \in A(D_1)$ $(1 \leq i \leq k_n)$ such that

$$\left\| f_n - \sum_{i=1}^{k_n} \varphi_i^n \otimes h_i^n \right\|_{\infty,\overline{D}_2} < \frac{1}{n},$$

where $(\varphi_i^n \otimes h_i^n) \in \mathcal{O}(D_{r_n}, A(D_1))$ denotes the function acting as $(\varphi_i^n \otimes h_i^n)(z) = \varphi_i^n(z) \cdot h_i^n$ for $z \in D_{r_n}$). In a final approximation step, we make use of the (polynomial) convexity of $\overline{D}_2$ and use the Oka-Weil theorem to find polynomials $(p_i^n)_{1 \leq i \leq k_n}$ in $\mathbb{C}[z]$ such that

$$\|p_i^n - \varphi_i^n\|_{\infty,\overline{D}_2} < \left(n \cdot k_n \cdot \max_{i=1,\cdots,k_n} \|h_i^n\|_{\infty,\overline{D}_2}\right)^{-1} \qquad (1 \leq i \leq k_n)$$

for each fixed $n \geq 1$. By construction, we have

$$\left\| f - \sum_{i=1}^{k_n} p_i^n \otimes h_i^n \right\|_{\infty,\overline{D}_2} < \frac{2}{n} + \sum_{i=1}^{k_n} \|p_i^n - \varphi_i^n\|_{\infty,\overline{D}_2} \cdot \|h_i^n\|_{\infty,\overline{D}_1} < \frac{3}{n} \xrightarrow{n} 0.$$

To finish the proof, it suffices to observe that the functions g_i^n defined by $g_i^n(z_1, z_2) = p_i^n(z_1) \cdot h_i^n(z_2)$ $(z_1 \in \overline{D}_1, z_2 \in \overline{D}_2)$ obviously belong to $A(D_1 \times D_2)$ and are mapped to $p_i^n \otimes h_i^n$ via S. This shows that S has dense range and thus is surjective, as desired. □

The previous lemma helps us to handle the symmetric factor in the following embedding theorem. The strictly pseudoconvex parts will be handled by combining the embedding theorems of Fornaess [20, Theorem 9] and Løw [25] with a result of Jacóbczak [23, Theorem 1] on the existence of extension operators.

Proposition 5.3 *Let $D = D_1 \times \cdots \times D_k \subset \mathbb{C}^n$ be a product domain as in the statement of Theorem 5.1. Then there exist constants $N \in \mathbb{N}$, $r > 0$ and a mapping $F \in A(D)^N$ such that, with $B = r \cdot \mathbb{B}_N$, the following assertions hold:*

(a) *$F : \overline{D} \to F(\overline{D}) \subset \mathbb{C}^N$ is a homeomorphism.*
(b) *$F(D) \subset B$ and $F(S_D) \subset \partial B$.*
(c) *The composition operator $C_F : A(B) \to A(D)$, $f \mapsto f \circ F$, has dense range.*

Proof As in the proof of Theorem 2.3, we may restrict ourselves to the case

$$D = D_1 \times D_2 \times D_3 \subset \mathbb{C}^{d_1} \times \mathbb{C}^{d_2} \times \mathbb{C}^{d_3},$$

where D_1 and D_2 are strictly pseudoconvex factors and D_3 is a bounded symmetric and circled open set. The proof is divided into three steps:
Step I: Reduction to the case of convex domains. All of the following assertions hold for $i = 1, 2$: Since $\overline{D}_i$ is a Stein compactum and hence holomorphically convex, we may apply Fornaess' embedding theorem [20, Theorem 9] (in the special case $X = \mathbb{C}^{d_i}$),

to obtain holomorphic maps $\psi_i : \mathbb{C}^{d_i} \to \mathbb{C}^{m_i}$ and strictly convex bounded domains $C_i \subset \mathbb{C}^{m_i}$ with C^5-boundary such that

- ψ_i is biholomorphic onto a closed subvariety $M_i' = \psi(\mathbb{C}^{d_i}) \subset \mathbb{C}^{m_i}$,
- $\psi_i(D_i) \subset C_i$ and $\psi_i(\mathbb{C}^{d_i} \setminus \overline{D}_i) \subset \mathbb{C}^{m_i} \setminus \overline{C_i}$ (and consequently[1] $\psi_i(\partial D_i) \subset \partial C_i$), and
- M_i' intersects ∂C transversally.

If we set $M_i = \psi_i(D_i)$ and note that $M_i = M_i' \cap C_i$ (with transversal intersection at ∂C_i), then these sets fulfill exactly the hypotheses of [23, Theorem 1] (cp. also the Remark preceding Section 4 therein). We write $\psi = (\psi_1, \psi_2) : \overline{D}_1 \times \overline{D}_2 \to \overline{M}_1 \times \overline{M}_2$ and consider the following commuting diagram

$$\begin{array}{ccccc} A(C_1 \times C_2 \times D_3) & & \xrightarrow{R=C_{(\psi,id)}} & & A(D_1 \times D_2 \times D_3) \\ \downarrow\cong & & & & \downarrow\cong \\ A(D_3, A(C_1 \times C_2)) & \xrightarrow{1\otimes R_M} & A(D_3, A(M_1 \times M_2)) & \xrightarrow{1\otimes C_\psi} & A(D_3, A(D_1 \times D_2)), \end{array} \tag{5.1}$$

where

- the vertical maps are given by the identification established in the previous lemma,
- the map $1 \otimes R_M$ acts by pointwise restriction to $M = \overline{M}_1 \times \overline{M}_2$, i.e., $(1 \otimes R_M f)(z) = f(z)|M$ for $z \in \overline{D}_3$ and $f \in A(D_2, A(C_1 \times C_2))$,
- the mapping $1 \otimes C_\psi$ acts by pointwise application of the composition operator $C_\psi : A(M_1 \times M_2) \to A(D_1 \times D_2),\ f \mapsto f \circ \psi$,
- the upper horizontal map R (which we think of as a restriction map) is by definition the composition operator with the map $(\psi, id) : \overline{D}_1 \times \overline{D}_2 \times \overline{D}_3 \to \overline{C}_1 \times \overline{C}_2 \times \overline{D}_3$.

It is elementary to check that the diagram commutes. We claim that R is surjective, which lifts our problem to a product of convex sets (in the upper left corner). Since $1 \otimes C_\psi$ is obviously a topological isomorphism (with inverse $1 \otimes C_{\psi^{-1}}$), the difficulty lies in checking that $1 \otimes R_M$ is surjective. By a result of Jacóbczak [23, Theorem 1] (and the remark at the end of Section 3 therein), there is a continuous linear extension operator

$$E : A(M_1 \times M_2) \to A(C_1 \times C_2) \quad \text{such that} \quad (Ef)|M = f \quad (f \in A(M_1 \times M_2)).$$

Defining $1 \otimes E : A(D_3, A(M_1 \times M_2)) \to A(D_3, A(C_1 \times C_2))$ by the pointwise action of E on $A(D_3, A(M_1 \times M_2))$-functions, we obtain a bounded linear right inverse for $1 \otimes R_M$, and the claimed surjectivity follows.

Step II: Embedding a product of convex domains into a ball. Towards an application of Løw's embedding theorem to the sets C_i as in the proof of Theorem 2.3 on p. 4, Let us choose $r_i > 0$ such that $\overline{C}_i \subset r_i \cdot \mathbb{B}_{m_i}$ $(i = 1, 2)$ and $D_3 \subset r_3 \cdot \mathbb{B}_{d_3}$ with

[1] For a strictly pseudoconvex bounded open set $D \subset \mathbb{C}^n$, one can show that $\partial D = \partial\overline{D}$: While "$\supset$" trivially holds for all open sets, the reverse inclusion follows as an application of the maximum principle for a strictly plurisubharmonic defining function. From the two inclusions stated in the original version of Fornaess' theorem we can therefore deduce that $\psi_i(\partial D_i) \subset \psi_i(\overline{D}_i) \subset \overline{C}_i$ and $\psi_i(\partial D_i) = \psi_i(\partial\overline{D}_i) \subset \psi_i\left(\overline{\mathbb{C}^{d_i} \setminus \overline{D}_i}\right) \subset \overline{\mathbb{C}^{m_i} \setminus \overline{C}_i}$. Together, these two inclusions yield $\psi_i(\partial D_i) \subset \partial\overline{C}_i = \partial C_i$, as asserted above.

$S_{D_3} = \overline{D}_3 \cap r_3 \cdot \partial\mathbb{B}_{d_3}$. Then, exactly as in the cited proof, the application of Løw's theorem yields a map

$$
\begin{aligned}
&G : \overline{C}_1 \times \overline{C}_2 \times \overline{D}_3 \to \mathbb{C}^N, \\
&\quad z \mapsto (z, g_1(z_1), g_2(z_2)) \quad (z = (z_1, z_2, z_3) \in \overline{C}_1 \times \overline{C}_2 \times \overline{D}_3)
\end{aligned}
\tag{5.2}
$$

with $g_i \in A(C_i)^{l^i}$ for suitable $l_i \in \mathbb{N}$, such that with $r = \sqrt{r_1^2 + r_2^2 + r_3^2}$ and $B = r \cdot \mathbb{B}_N$ with $N = m_1 + m_2 + d_3 + l_1 + l_2$ the following assertions hold (consider Remark 2.5):

- $G : \overline{C}_1 \times \overline{C}_2 \times \overline{D}_3 \to G(\overline{C}_1 \times \overline{C}_2 \times \overline{D}_3)$ is a homeomorphism,
- $G(S_{C_1 \times C_2 \times D_3}) \subset \partial B$,
- $G(C_1 \times C_2 \times D_3) \subset B$.

To argue as in the referenced proof, note that the sets C_i are strictly pseudoconvex (as they are strictly convex, cp. p. 529 of [20]), so $S_{C_1 \times C_2 \times D_3} = \partial C_1 \times \partial C_2 \times (\overline{D}_3 \cap r_3 \cdot \partial\mathbb{B}_{d_3})$. To finally obtain a map $F \in A(D)^N$ as in the statement of the theorem, we set

$$
F = G \circ (\psi, id) : \overline{D}_1 \times \overline{D}_2 \times \overline{D}_3 \xrightarrow{(\psi_1, \psi_2, id)} \overline{C}_1 \times \overline{C}_2 \times \overline{D}_3 \xrightarrow{\ G\ } \overline{B} \subset \mathbb{C}^N.
$$

Properties (a) and (b) from the assertion of the theorem are valid by construction, so it remains to check that (c) holds.

Step III: The range of $C_F : A(B) \to A(D)$, $f \mapsto f \circ F$ is dense. By construction, C_F is the composition

$$
C_F : A(B) \xrightarrow{C_G} A(C_1 \times C_2 \times D_3) \xrightarrow{R} A(D),
$$

where $C_G(f) = f \circ G$ and R is the upper horizontal map from (5.1). To complete the proof, it suffices to check that $C_G : A(B) \to A(C_1 \times C_2 \times D_3)$ has dense range. Assuming that, for a moment, it follows immediately that $\overline{C_F(A(B))} \supset R(\overline{C_G(A(B))}) = R(A(C_1 \times C_2 \times D_3)) = A(D)$, since R is surjective as we have seen in Step I.

So let us finally aim for the density of C_G. After a suitable translation, we may of course assume that the set $C_1 \times C_2 \times D_3 \subset \mathbb{C}^m$ contains the origin. Using radial approximation and the fact that the set $\overline{C}_1 \times \overline{C}_2 \times \overline{D}_3$ is polynomially convex, it suffices to check that the range of C_G contains the polynomials in $z = (z_1, \cdots, z_m)$. Since C_G is a unital algebra homomorphism, it suffices to check that the range of C_G contains the (restrictions) of the coordinate projections $\pi_1, \cdots, \pi_m : \mathbb{C}^m \to \mathbb{C}$, i. e., $\pi_i(z) = z_i$. But, by construction (see (5.2) above), G acts as the identity on the first m components,

$$
G(z_1, \cdots, z_m) = (z_1, \cdots, z_m, \cdots) \in \mathbb{C}^N \qquad (z \in \overline{C}_1 \times \overline{C}_2 \times \overline{D}_3 \subset \mathbb{C}^m),
$$

so if we denote the coordinate functions on $\mathbb{C}^N$ with $\pi_j^N : \mathbb{C}^N \to \mathbb{C}$, then we clearly have $\pi_j^N | \overline{B} \in A(B)$ and

$$(\pi_j^N|\overline{B}) \circ G(z) = z_j = \pi_j(z)$$
$$\text{for} \quad j = 1, \cdots, m \ \text{ and } \ z = (z_1, \cdots, z_m) \in \overline{C}_1 \times \overline{C}_2 \times \overline{D}_3.$$

This observation finishes the proof. □

Proof of Theorem 5.1 Let a short exact sequence $E : 0 \longrightarrow \mathcal{H} \xrightarrow{A} \mathcal{J} \xrightarrow{B} \mathcal{K} \longrightarrow 0$ in the category $\mathcal{H}_{A(D)}$ be given, where $\mathcal{H}$ extends to a $C(S_D)$-module. For the first statement, we have to show that there is an $A(D)$-module map $X : \mathcal{J} \to \mathcal{H} \oplus \mathcal{K}$ making the diagram

$$\begin{array}{ccccccccc}
E : 0 & \longrightarrow & \mathcal{H} & \xrightarrow{\quad A \quad} & \mathcal{J} & \xrightarrow{\quad B \quad} & \mathcal{K} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle X} & & \| & & \\
E' : 0 & \longrightarrow & \mathcal{H} & \xrightarrow{i_{\mathcal{H}}} & \mathcal{H} \oplus \mathcal{K} & \xrightarrow{p_{\mathcal{K}}} & \mathcal{K} & \longrightarrow & 0
\end{array}$$

commutative in $\mathcal{H}_{A(D)}$. Towards this, let $F : \overline{D} \to \overline{B}$ be the embedding constructed in Proposition 5.3. Note that $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{J}$ can be viewed as $A(B)$-modules when we introduce the module multipliation

$$A(B) \times \mathcal{H} \to \mathcal{H}, \quad (f, h) \mapsto f \bullet h = (f \circ F) \cdot h,$$

analogously for $\mathcal{K}$, $\mathcal{J}$. Since $F(S_D) \subset \partial B$, this module structure extends to a $C(S_D)$-module structure for $\mathcal{H}$. As the Ext^1-group is known to vanish in the ball case by [13, Theorem 3.1], there is an $A(B)$-module homomorphism X making the above diagram commuting in the category of $A(B)$-modules. But since the image of the composition operator $C_F : A(B) \to A(D)$, $f \mapsto f \circ F$, has dense range, the map $X \in \mathrm{Hom}_{A(B)}(\mathcal{J}, \mathcal{H} \oplus \mathcal{K})$ actually turns out to be an $A(D)$-module homomorphism by the continuity of the (original) module multiplication. This proves that $[E] = 0$ and thus $\mathrm{Ext}^1_{\mathcal{H}_{A(D)}}(\mathcal{K}, \mathcal{H}) = 0$, as we claimed. The second half of the assertion follows analogously. □

As a direct consequence, by [7, Proposition 2.1.5], we have:

Corollary 5.4 *Let $k \in \mathbb{N}$ and $D_i \subset \mathbb{C}^{d_i}$ be bounded open sets with $d_i \in \mathbb{N}$ for $1 \leq i \leq k$, each of which is either a strictly pseudoconvex domain with C^5 boundary or a bounded symmetric and circled domain, and let $D = D_1 \times \cdots \times D_k$. Then every Hilbert-$C(S_D)$-module $\mathcal{H}$ is projective and injective in the category $\mathcal{H}_{A(D)}$.*

6 A Hom-Isomorphism Theorem

Let $\mathcal{H}$ be a Hilbert-A-module and $\mathcal{S} \subset \mathcal{H}$ a closed A-submodule, that is, a closed subspace satisfying $M_f^{\mathcal{H}}\mathcal{S} \subset \mathcal{S}$ for $f \in A$. If we write $P_{\mathcal{S}}, P_{\mathcal{S}^\perp} \in B(\mathcal{H})$ for the orthogonal projections with ranges $\mathcal{S}$ and $\mathcal{S}^\perp$, respectively, then the identity

$$P_{\mathcal{S}^\perp} M_f^{\mathcal{H}} P_{\mathcal{S}^\perp} M_g^{\mathcal{H}} - P_{\mathcal{S}^\perp} M_{fg}^{\mathcal{H}} = P_{\mathcal{S}^\perp} M_f^{\mathcal{H}} (P_{\mathcal{S}^\perp} - 1) M_g^{\mathcal{H}} = -P_{\mathcal{S}^\perp} M_f^{\mathcal{H}} P_{\mathcal{S}} M_g^{\mathcal{H}} = 0,$$

valid for arbitrary $f, g \in A$ shows that $\mathcal{S}^\perp$ carries a natural A-module structure, where the module multiplication is given by

$$f \cdot_{\mathcal{S}^\perp} h = P_{\mathcal{S}^\perp}(f \cdot h) \qquad (f \in A,\ h \in \mathcal{S}^\perp).$$

Furthermore, the space $\mathrm{Hom}_A(\mathcal{H}_1, \mathcal{H}_2)$ can be equipped with an A-module structure by setting $f \cdot X = X \circ M_f^{\mathcal{H}_1}$ $(f \in A,\ X \in \mathrm{Hom}_A(\mathcal{H}_1, \mathcal{H}_2))$.

Theorem 6.1 *Let $A \subset C(\partial_A)$ be a w^*-regular function algebra containing the polynomials $\mathbb{C}[z]$, and let $\mathcal{S}_1, \mathcal{S}_2$ be hypo-Shilov modules over A with minimal $C(\partial_A)$-extensions $\mathcal{H}_1, \mathcal{H}_2$. If $\mathcal{H}_1$ is projective and $\mathcal{S}_2$ is pure, then the map*

$$\beta : \mathrm{Hom}_A(\mathcal{S}_1, \mathcal{S}_2) \to \mathrm{Hom}_A(\mathcal{S}_1^\perp, \mathcal{S}_2^\perp), \quad X \mapsto P_{\mathcal{S}_2^\perp} \varepsilon(X)|\mathcal{S}_1^\perp$$

is an isomorphism of A-modules, where $\varepsilon : \mathrm{Hom}_{C(\partial_A)}(\mathcal{H}_1, \mathcal{H}_2) \to \mathrm{Hom}_A(\mathcal{H}_1, \mathcal{H}_2)$ denotes the extension map from Theorem 4.4.

Proof For the convenience of the reader, we repeat the known argument from the proof of [21, Theorem 3.7]. The map β from the statement of the theorem is defined in such a way that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{S}_1 \arrow[r, "i_1"] \arrow[d, "X"] & \mathcal{H}_1 \arrow[r, "P_{\mathcal{S}_1^\perp}"] \arrow[d, "\varepsilon(X)"] & \mathcal{S}_1^\perp \arrow[r] \arrow[d, "\beta(X)"] & 0 \\
0 \arrow[r] & \mathcal{S}_2 \arrow[r, "i_2"] & \mathcal{H}_2 \arrow[r, "P_{\mathcal{S}_2^\perp}"] & \mathcal{S}_2^\perp \arrow[r] & 0
\end{tikzcd}$$

Only the bijectivity of β requires an argument. To see that β is injective, suppose that $\beta(X) = 0$ for some $X \in \mathrm{Hom}_A(\mathcal{S}_1, \mathcal{S}_2)$. It follows immediately that $\overline{\varepsilon(X)\mathcal{H}_1} \subset \mathcal{S}_2$ is a reducing submodule. Since $\mathcal{S}_2$ is assumed to be pure, we conclude that $\varepsilon(X) = 0$ and thus $X = 0$, as desired.

For the surjectivity, let an arbitrary $Y \in \mathrm{Hom}_A(\mathcal{S}_1^\perp, \mathcal{S}_2^\perp)$ be given. By the assumed projectvity of $\mathcal{H}_1$, we may choose an A-module homomorphism $X' : \mathcal{H}_1 \to \mathcal{H}_2$ such that $P_{\mathcal{S}_2^\perp} X' = Y P_{\mathcal{S}_1^\perp}$. Setting $X = X'|\mathcal{S}_1$, it follows that $\varepsilon(X) = X'$ from uniqueness, and so $\beta(X) = P_{\mathcal{S}_2^\perp} X'|\mathcal{S}_1^\perp = Y$. $\square$

The limiting factor for an application of the preceding theorem is the projectivity of $\mathcal{H}_1$. In view of the fact that every $C(S_D)$-extension of a Hilbert-$A(D)$-module is projective for $D \subset \mathbb{C}^n$ if D is either (i) a bounded strictly pseudoconvex set or (ii) a circled bounded symmetric domain (see [13, Theorem 3.1]) or (iii) a product of finitely many such domains (Theorem 5.1), the above result allows us to extend Guo [21, Theorem 3.7] and Chen and Guo [9, Theorem 3.5] to these kinds of sets. In particular, the restriction to Shilov-modules and the additional continuity assumption made in [21] to handle the ball case are not necessary.

Corollary 6.2 *Let $k \in \mathbb{N}$ and $D_i \subset \mathbb{C}^{d_i}$ be bounded open sets with $d_i \in \mathbb{N}$ for $1 \le i \le k$, each of which is either a strictly pseudoconvex domain with C^5 boundary or* *a bounded symmetric and circled domain, and let* $A = A(D)$ *with* $D = D_1 \times \cdots \times D_k$. *Assume that* $\mathcal{S}_i$ *are hypo-Shilov modules with minimal* $C(S_D)$*-extensions* $\mathcal{H}_i$ $(i = 1, 2)$ *and that* $\mathcal{H}_2$ *is pure. Then we have an isomorphism* $\beta : \mathrm{Hom}_{A(D)}(\mathcal{S}_1, \mathcal{S}_2) \to \mathrm{Hom}_{A(D)}(\mathcal{H}_1, \mathcal{H}_2)$ *as stated in the above theorem.*

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