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Norm Estimates for Selfadjoint Toeplitz Operators on the Fock Space

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Abstract

An estimate for the norm of selfadjoint Toeplitz operators with a radial, bounded and integrable symbol is obtained. This emphasizes the fact that the norm of such operator is strictly less than the supremum norm of the symbol. Consequences for time-frequency localization operators are also given.

Keywords Fock space · Toeplitz operator · Localization operator · Time-frequency

Mathematics Subject Classification 47B35 · 47G30

1 Introduction

The Bargmann-Fock space $\mathcal{F}^2(\mathbb{C})$ is the Hilbert space consisting of those analytic functions $f \in H(\mathbb{C})$ such that

$$\|f\|_{\mathcal{F}}^2 = \int_{\mathbb{C}} |f(z)|^2 e^{-\pi|z|^2}\, dA(z) < +\infty,$$

where $dA(z)$ denotes the Lebesgue measure. $\mathcal{F}^2(\mathbb{C})$ admits a reproducing kernel $K_w(z) = e^{\pi\overline{w}z}$, which means that

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$$f(w) = \langle f, K_w \rangle, \quad f \in \mathcal{F}^2(\mathbb{C}).$$

The normalized monomials

$$e_n(z) = \left(\frac{\pi^n}{n!}\right)^{\frac{1}{2}} z^n, \quad n \geq 0,$$

form an orthonormal basis. For a fixed $a \in \mathbb{C}$ the translation operator

$$W_a : \mathcal{F}^2(\mathbb{C}) \to \mathcal{F}^2(\mathbb{C}), \ (W_a f)(z) = f(z-a) e^{-\frac{\pi}{2}|a|^2 + \pi z \overline{a}},$$

is an isometry (see [14, Proposition 2.38]). We denote $d\lambda(z) = e^{-\pi|z|^2} dA(z)$, so $\mathcal{F}^2(\mathbb{C})$ is a closed subspace of $L^2(\mathbb{C}, d\lambda)$. The orthogonal projection

$$P : L^2(\mathbb{C}, d\lambda) \to \mathcal{F}^2(\mathbb{C})$$

is the integral operator

$$(Pf)(z) = \int_{\mathbb{C}} f(w) K_w(z) \, d\lambda(w).$$

For a measurable and bounded function F on $\mathbb{C}$ the Toeplitz operator with symbol F is defined as

$$T_F(f)(z) = P(Ff)(z) = \int_{\mathbb{C}} F(w) f(w) K_w(z) \, d\lambda(w).$$

The systematic study of Toeplitz operators on the Fock space started in [3,4]. Since then it has been a very active research area. We refer to [14, Chapter 6], where boundedness and membership in the Schatten classes is discussed.

It is obvious that

$$T_F : \mathcal{F}^2(\mathbb{C}) \to \mathcal{F}^2(\mathbb{C})$$

is a bounded operator and

$$\|T_F(f)\| \leq \|Ff\|_{L^2(\mathbb{C}, d\lambda)} \leq \|F\|_\infty \cdot \|f\|.$$

In particular, $\|T_F\| \leq 1$ whenever $\|F\|_\infty \leq 1$. If moreover T_F is compact, which happens for instance when $F \in L^1(\mathbb{C})$, then $\|T_F\|$ is strictly less than 1 but, as far as we know, no precise estimate for the norm is known. The main result of the paper gives a bound for $\|T_F\|$ in the case that the symbol F is radial, real-valued, and satisfies some integrability condition. For Toeplitz operators with radial symbols we refer to [11]. Besides Toeplitz operators on the Fock space we consider time-frequency localization operators with Gaussian window, also known as anti-Wick operators. They were introduced by Daubechies [7] as filters in signal analysis and can be obtained from Toeplitz operators on the Fock space after applying Bargmann transform.

2 Toeplitz Operators on the Fock Space

The Toeplitz operator defined by a real valued symbol F is self-adjoint. This is immediate from the identity

$$\langle T_F(f), g\rangle = \int_{\mathbb{C}} F(z) f(z)\overline{g(z)} d\lambda(z)$$

for all $f, g \in \mathcal{F}^2(\mathbb{C})$. In this case we have

$$\|T_F\| = \sup_{\|f\|=1} |\langle T_F(f), f\rangle| \le \sup_{\|f\|=1} \int_{\mathbb{C}} |F(z)| \cdot |f(z)|^2 d\lambda(z).$$

A symbol F is said to be radial with respect to $a \in \mathbb{C}$ if $F(z) = g(|z-a|)$ for some bounded and measurable function g on $[0, +\infty)$. The main result of the paper is as follows.

Theorem 1 *Let $F \in L^1(\mathbb{C}) \cap L^\infty(\mathbb{C})$ be a real-valued and radial symbol with respect to $a \in \mathbb{C}$. Then*

$$\|T_F\| \le \|F\|_\infty \left(1 - \exp\left(-\frac{\|F\|_1}{\|F\|_\infty}\right)\right).$$

An expression for the norm of Toeplitz operators with radial symbols can be found in [11] but it is unclear how the estimate provided by Theorem 1 can be obtained from it.

For the proof we will need some auxiliary results. First we observe that for $|F(z)| = g(|z|)$ and $f = \sum_{n=0}^{\infty} b_n e_n$ we have, after changing to polar coordinates,

$$\begin{aligned}\int_{\mathbb{C}} |F(z)| \cdot |f(z)|^2 d\lambda(z) &= \sum_{n=0}^{\infty} |b_n|^2 \int_{\mathbb{C}} g(|z|)|e_n(z)|^2 \, d\lambda(z) \\ &= \sum_{n=0}^{\infty} |b_n|^2 2\pi \int_0^\infty g(r)\pi^n \frac{r^{2n+1}}{n!} e^{-\pi r^2} \, dr \\ &= \sum_{n=0}^{\infty} |b_n|^2 \int_0^\infty g\left(\sqrt{\frac{t}{\pi}}\right)\frac{t^n}{n!} e^{-t} \, dt.\end{aligned}$$

The d-dimensional Lebesgue measure of a set $\Omega \subset \mathbb{R}^d$ is denoted $|\Omega|$ both for $d = 1$ and $d = 2$.

Lemma 1 *Let $I \subset [0, +\infty)$ be a measurable set with finite Lebesgue measure. Then*

$$\frac{1}{n!}\int_I s^n e^{-s} \, ds \le 1 - e^{-|I|}.$$

Proof (a) We first assume that I is a finite union of bounded intervals. The function $h(s) = \frac{s^n}{n!}e^{-s}$ attains its absolute maximum at $s = n$. Then h increases on $[0, n]$ and decreases on $[n, +\infty)$. We consider $a \le n \le b$ such that

$$n - a = |I \cap [0, n]| \, , \; b - n = |I \cap [n, +\infty)|.$$

Then

$$\begin{aligned} \frac{1}{n!}\int_I s^n e^{-s}\, ds &\le \int_a^b h(s)\, ds = \frac{e^{-a}}{n!}\int_0^{b-a} (t+a)^n e^{-t}\, dt \\ &= \sum_{k=0}^{n} \binom{n}{k} \frac{a^{n-k}}{n!} e^{-a} \int_0^{|I|} t^k e^{-t}\, dt \\ &= \sum_{k=0}^{n} \frac{a^{n-k}}{(n-k)!} e^{-a} \frac{1}{k!} \int_0^{|I|} t^k e^{-t}\, dt \\ &\le \sup_{0\le k\le n} \frac{1}{k!} \int_0^{|I|} t^k e^{-t}\, dt = \int_0^{|I|} e^{-t}\, dt. \end{aligned}$$

For the last identity observe that

$$\frac{1}{k!}\int_0^s t^k e^{-t}\, dt = 1 - e^{-s}\sum_{j=0}^{k} \frac{s^j}{j!}.$$

(b) For a general measurable set I with finite measure the conclusion follows from part (a) and the fact that for every $\varepsilon > 0$ there is a set J, finite union of bounded intervals, with the property that

$$|J \setminus I| + |I \setminus J| \le \varepsilon.$$

□

Lemma 2 *Let $(I_k)_{k=1}^N$ be disjoint sets with finite measure and $0 \le \varepsilon_k \le 1$ for every $1 \le k \le N$. Then, for every $p \in \mathbb{N}_0$ we have*

$$\sum_{k=1}^{N} \varepsilon_k \int_{I_k} \frac{t^p}{p!} e^{-t}\, dt \le 1 - \exp\left(-\sum_{k=1}^{N} \varepsilon_k |I_k| \right).$$

Proof We denote by n the number of indexes k such that $0 < \varepsilon_k < 1$ and we proceed by induction on n. For $n = 0$ this is the content of Lemma 1. Let us now assume $n = 1$. Let $1 \le j \le N$ be the coordinate with the property that $0 < \varepsilon_j <$ and check that

$$\psi(\varepsilon) := \sum_{k\ne j} \int_{I_k} \frac{t^p}{p!} e^{-t}\, dt + \varepsilon \int_{I_j} \frac{t^p}{p!} e^{-t}\, dt + \exp\left(-\sum_{k\ne j} |I_k| - \varepsilon |I_j| \right) \le 1$$

for every $0 \le \varepsilon \le 1$. In fact, $\psi(0) \le 1$ and $\psi(1) \le 1$ follow from Lemma 1. Moreover, the critical point ε_0 of ψ satisfies

$$\int_{I_j} \frac{t^p}{p!} e^{-t}\, dt = |I_j| \exp\left(-\sum_{k \neq j} |I_k| - \varepsilon_0 |I_j|\right).$$

Hence

$$\begin{aligned}\psi(\varepsilon_0) &= \sum_{k \neq j} \int_{I_k} \frac{t^p}{p!} e^{-t}\, dt + \varepsilon_0 |I_j| \exp\left(-\sum_{k \neq j} |I_k| - \varepsilon_0 |I_j|\right) \\ &\quad + \exp\left(-\sum_{k \neq j} |I_k| - \varepsilon |I_j|\right) \\ &= \sum_{k \neq j} \int_{I_k} \frac{t^p}{p!} e^{-t}\, dt + \left(1 + \varepsilon_0 |I_j|\right) \exp\left(-\sum_{k \neq j} |I_k| - \varepsilon |I_j|\right).\end{aligned}$$

Since

$$1 + \varepsilon_0 |I_j| \le \exp\left(\varepsilon_0 |I_j|\right)$$

we conclude

$$\psi(\varepsilon_0) \le \sum_{k \neq j} \int_{I_k} \frac{t^p}{p!} e^{-t}\, dt + \exp\left(-\sum_{k \neq j} |I_k|\right) \le 1.$$

Let us assume that the Lemma holds for $n = \ell$ $(0 \le \ell < N)$ and let $n = \ell + 1$. We consider the function $\psi : [0, 1]^{\ell+1} \to \mathbb{R}$ defined by

$$\psi(\boldsymbol{\varepsilon}) := \sum_{k=1}^{\ell+1} \varepsilon_k \int_{I_k} \frac{t^p}{p!} e^{-t}\, dt + \sum_j \int_{J_j} \frac{t^p}{p!} e^{-t}\, dt + \exp\left(-\sum_k \varepsilon_k |I_k| - \sum_j |J_j|\right)$$

for $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_{\ell+1})$. The induction hypothesis means that $\psi(\boldsymbol{\varepsilon}) \le 1$ whenever $\boldsymbol{\varepsilon}$ is in the boundary of $[0, 1]^{\ell+1}$. The lemma is proved after checking that $\psi(\boldsymbol{\varepsilon}_0) \le 1$, where $\boldsymbol{\varepsilon}_0$ is a critical point of ψ. Proceeding as before,

$$\begin{aligned}\psi(\boldsymbol{\varepsilon}_0) &= \left(\sum_{k=1}^{\ell+1} \varepsilon_k |I_k| + 1\right) e^{-\sum_k \varepsilon_k |I_k|} e^{-\sum_j |J_j|} + \sum_j \int_{J_j} \frac{t^p}{p!} e^{-t}\, dt \\ &\le \exp\left(-\sum_j |J_j|\right) + \sum_j \int_{J_j} \frac{t^p}{p!} e^{-t}\, dt \le 1.\end{aligned}$$

□

Proof of Theorem 1 We first assume $a = 0$, that is, F is radial. After replacing F by $G = \frac{F}{\|F\|_\infty}$ if necessary we can assume that $\|F\|_\infty = 1$. Since F is radial we have $F(z) = g\big(|z|\big)$. We aim to prove that

$$\int_{\mathbb{C}} \big|g\big(|z|\big)\big| \cdot |f(z)|^2 \, e^{-\pi|z|^2} \, dA(z) \leq 1 - \exp\left(-2\pi \int_0^\infty r\,|g(r)| \, dr\right)$$

for every entire function $f(z) = \sum_{p=0}^\infty b_p e_p$ such that $\sum_{p=0}^\infty |b_p|^2 = 1$. We have

$$\int_{\mathbb{C}} \big|g\big(|z|\big)\big| \cdot |F(z)|^2 \, e^{-\pi|z|^2} \, dA(z) = \sum_{p=0}^\infty |b_p|^2 \int_0^\infty \left|g\left(\sqrt{\frac{t}{\pi}}\right)\right| \cdot \frac{t^p}{p!} e^{-t} \, dt.$$

Let us first assume

$$g = \sum_{k=1}^N \varepsilon_k \chi_{I_k}, \ \ |\varepsilon_k| \leq 1, \tag{1}$$

where $(I_k)_{k=1}^N$ are disjoint intervals. Then, Lemma 1 gives

$$\begin{aligned}\sum_{p=0}^\infty |b_p|^2 \int_0^\infty \left|g\left(\sqrt{\frac{t}{\pi}}\right)\right| \cdot \frac{t^p}{p!} e^{-t} \, dt &\leq 1 - \exp\left(-\sum_{k=1}^N |\varepsilon_k||J_k|\right) \\ &= 1 - \exp\left(-2\pi \int_0^\infty r|g(r)| \, dr\right) \\ &= 1 - \exp\left(-\|F\|_1\right).\end{aligned}$$

We used $J_k = \left\{t : \sqrt{\frac{t}{\pi}} \in I_k\right\}$ and $|J_k| = 2\pi \int_{I_k} r dr$. Theorem 1 is proved for g as in (1). Let us now assume that $\|g\|_\infty \leq 1$ and $g \in L^1(\mathbb{R}^+, rdr) \cap L^\infty(\mathbb{R}^+)$. Then there is a sequence $(g_n)_n$ of step functions as in (1) such that

$$\lim_{n\to\infty} \int_0^\infty |g_n(r) - g(r)| \, rdr = 0.$$

We put $F_n(z) := g_n(|z|)$. According to [12, Theorem 3.5] there is a constant $K > 0$ such that

$$\|T_G\| \leq K \sup_{z\in\mathbb{C}} \int_{D(z,1)} |G| \, dA \tag{2}$$

for every bounded symbol G, which implies

$$\lim_{n\to\infty} \|T_F - T_{F_n}\| \leq K \lim_{n\to\infty} \|F_n - F\|_1 = 0.$$

We finally conclude

$$\|T_F\| \leq 1 - \exp\big(-\|F\|_1\big).$$

In the case $a \neq 0$, the identity

$$\int_{\mathbb{C}} g\big(|z-a|\big)|f(z)|^2 d\lambda(z) = \int_{\mathbb{C}} g\big(|u|\big)\, |(W_{-a}f)(u)|^2\, d\lambda(u)$$

and the fact that W_{-a} is an isometry gives the conclusion. We can also argue from the fact that $W_{-a} \circ T_F = T_G \circ W_{-a}$, where $G(z) = g\big(|z|\big)$. □

In particular, if $\Omega \subset \mathbb{C}$ presents radial symmetry with respect to some point then

$$\int_{\Omega} |f(z)|^2\, d\lambda(z) \leq \left(1 - e^{-|\Omega|}\right) \cdot \int_{\mathbb{C}} |f(z)|^2\, d\lambda(z) \tag{3}$$

for every $f \in \mathcal{F}^2(\mathbb{C})$.

The question arises whether inequality (3) holds for every subset Ω. This is related to a conjecture by Abreu and Speckbacher in [1] (see the next section). We do not have an answer to this question except for monomials or its translates.

Example 1 Let $k_w = e^{-\frac{\pi}{2}|w|^2} K_w$ be the normalized reproducing kernel of $\mathcal{F}^2(\mathbb{C})$. Then, for every set $\Omega \subset \mathbb{C}$ with finite measure we have

$$\int_{\Omega} |k_w(z)|^2\, d\lambda(z) \leq 1 - e^{-|\Omega|}.$$

Proof In fact, $k_w = W_w\,(e_0)$. Hence

$$\int_{\Omega} |k_w(z)|^2 d\lambda(z) = \int_{\Omega - w} d\lambda(z)$$

and the conclusion follows from the fact that the last integral attains its maximum when Ω is a disc centered at w (see the comment after [1, Conjecture 1]). □

It is easy to check that when Ω is a disc centered at point ω the inequality in Example 1 is an identity.

Proposition 1 *Let $\Omega \subset \mathbb{R}^2$ be a set with finite measure. Then, for every $n \in \mathbb{N}$ and $a \in \mathbb{C}$,*

$$\int_{\Omega} |W_a(e_n)(z)|^2 \, d\lambda(z) \leq 1 - e^{-|\Omega|}.$$

Proof Since

$$\int_{\Omega} |W_a(e_n)(z)|^2 \, d\lambda(z) = \int_{\Omega - a} |e_n(z)|^2 \, d\lambda(z)$$

we can assume that $a = 0$. For every $\theta \in [0, 2\pi]$ we denote

$$\Omega_\theta = \left\{ r \geq 0 : \; re^{i\theta} \in \Omega \right\}.$$

Then

$$\begin{aligned} \int_{\Omega} |e_n(z)|^2 \, d\lambda(z) &= \frac{\pi^n}{n!} \int_{\Omega} \left| z^n \right|^2 e^{-\pi|z|^2} \, dA(z) \\ &= \frac{\pi^n}{n!} \int_0^{2\pi} \left(\int_{\Omega_\theta} r^{2n} e^{-\pi r^2} 2\pi r \, dr \right) \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} \left(\int_{I_\theta} \frac{t^n}{n!} e^{-t} \, dt \right) \frac{d\theta}{2\pi}, \end{aligned}$$

where

$$I_\theta = \left\{ t = \pi r^2 : \; r \in \Omega_\theta \right\}.$$

Since $|\Omega| < \infty$ then a.e. $\theta \in [0, 2\pi]$ we have

$$|I_\theta| = 2\pi \int_{\Omega_\theta} r dr < +\infty.$$

Moreover, by Lemma 1,

$$\int_0^{2\pi} \left(\int_{I_\theta} \frac{t^n}{n!} e^{-t} \, dt \right) \frac{d\theta}{2\pi} \leq \int_0^{2\pi} \left(1 - e^{-|I_\theta|} \right) \frac{d\theta}{2\pi}.$$

Finally we consider the convex function $f(t) = e^{-t} - 1$ and the probability measure $\frac{d\theta}{2\pi}$ and put $h(\theta) = |I_\theta|$. Jensen's inequality gives

$$f \left(\int_0^{2\pi} h(\theta) \frac{d\theta}{2\pi} \right) \leq \int_0^{2\pi} f\left(h(\theta)\right) \frac{d\theta}{2\pi},$$

which means

$$\int_0^{2\pi} \left(1 - e^{-|I_\theta|}\right) \frac{d\theta}{2\pi} \le 1 - \exp\left(-\int_0^{2\pi} |I_\theta| \frac{d\theta}{2\pi}\right)$$
$$= 1 - \exp\left(-\int_0^{2\pi} \left(\int_{\Omega_\theta} r\,dr\right) d\theta\right)$$
$$= 1 - e^{-|\Omega|}.$$

□

We finish the section with some examples of sets Ω with infinite Lebesgue measure for which the Toeplitz operator with symbol $F = \chi_\Omega$ has norm as small as we want.

Proposition 2 *For every $\varepsilon > 0$ there exists Ω with infinite Lebesgue measure such that*

$$\int_\Omega |f(z)|^2 d\lambda(z) \le \varepsilon \int_{\mathbb{C}} |f(z)|^2 d\lambda(z)$$

for every $f \in \mathcal{F}^2$.

Proof Let $K > 0$ as in (2) and let $(\Omega_n)_n$ be a sequence of bounded sets with Lebesgue measure $|\Omega_n| = \frac{\varepsilon}{K}$ and such that $\mathrm{dist}(\Omega_n, \Omega_m) > 2$ whenever $n \neq m$, and take $\Omega = \cup_{n\in\mathbb{N}} \Omega_n$. Since each disc $D(z, 1)$ meets at most one set Ω_n we have $|\Omega \cap D(z, 1)| \le \frac{\varepsilon}{K}$. The estimate (2) turns out $\|T_{\chi_\Omega}\| \le \varepsilon$, which gives the conclusion. □

3 Time-Frequency Localization Operators

For $F \in L^1(\mathbb{C})$ we denote by $H_F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ the localization operator

$$H_F f = \int_{\mathbb{C}} F(z)\, \langle f, \pi(z)h_0\rangle\, \pi(z)h_0\, dA(z).$$

Here $h_0(t) = 2^{1/4} e^{-\pi t^2}$ is the Gaussian and $\pi(z)$ is the time-frequency shift, defined for $z = x + i\omega$ as

$$\big(\pi(z)f\big)(t) = e^{2\pi i\omega t} f(t - x),\ \ f \in L^2(\mathbb{R}).$$

In case F is the characteristic function of a set Ω we write H_Ω instead of H_{χ_Ω}. We refer to [5] or [6, Chapter 4] for general facts concerning localization operators.

For $f, g \in L^2(\mathbb{R})$, the expression

$$\big(V_g f\big)(z) := \langle f, \pi(z)g\rangle$$

is the short time Fourier transform of f with window g, known as Gabor transform in the case where the window $g = h_0$ is the Gaussian.

If F is real-valued then H_F is a selfadjoint operator on $L^2(\mathbb{R})$, hence

$$\|H_F\| = \sup_{\|f\|_2=1} |\langle H_F f,\ f\rangle| \le \sup_{\|f\|_2=1} \int_{\mathbb{C}} |F(z)| \cdot \left|\left(V_{h_0} f\right)(z)\right|^2 dA(z).$$

There is a connection between localization operators and Toeplitz operators on the Fock space via the Bargmann transform.

The Bargmann transform is the surjective and unitary operator

$$\mathcal{B} : L^2(\mathbb{R}) \to \mathcal{F}^2(\mathbb{C})$$

defined as

$$\big(\mathcal{B}f\big)(z) = 2^{1/4} \int_{\mathbb{R}} f(t) e^{2\pi t z - \pi t^2 - \frac{\pi}{2} z^2}\, dt.$$

It was introduced in [2] and has the important property that the Hermite functions are mapped into normalized analytic monomials. More precisely, $\mathcal{B}(h_n) = e_n$, where h_n is defined via the so called Rodrigues formula as

$$h_n(t) = \frac{2^{1/4}}{\sqrt{n!}} \left(\frac{-1}{2\sqrt{\pi}}\right)^n e^{\pi t^2} \frac{d^n}{dt^n}\left(e^{-2\pi t^2}\right), \quad n \geq 0.$$

Then $(h_n)_{n\geq 0}$ forms an orthonormal basis for $L^2(\mathbb{R})$. The Gabor transform of Hermite functions is well-known (see for instance [9, Chapter 1.9]). In fact, for $z = x + i\xi$,

$$\langle h_n, \pi(z) h_0 \rangle = e^{-i\pi x\xi - \frac{\pi}{2}|z|^2} \sqrt{\frac{\pi^n}{n!}}\, \overline{z}^n. \tag{4}$$

Since for $z = x + i\xi$ we have ([10, 3.4.1])

$$\big(V_{h_0} f\big)(x, -\xi) = e^{i\pi x\xi} \cdot \big(\mathcal{B}f\big)(z) \cdot e^{-\frac{\pi|z|^2}{2}}$$

then, for every $f \in L^2(\mathbb{R})$ and $F \in L^1(\mathbb{C}) \cap L^\infty(\mathbb{C})$ we obtain

$$\int_{\mathbb{C}} |F(z)| \cdot \left|\big(V_{h_0} f\big)(z)\right|^2 dA(z) = \int_{\mathbb{C}} |F(z)| \cdot \left|\big(Bf\big)(z)\right|^2 \, d\lambda(z).$$

Consequently, all the estimates in the previous section can be translated into estimates concerning localization operators.

Abreu, Speckbacher conjecture in [1] that, among all the sets with a given measure, $\|H_\Omega\|$ attains its maximum when Ω is a disc, up to perturbations of Lebesgue measure zero. This turns out to be equivalent to the validity of inequality (3) for every function in the Fock space or, equivalently, to the fact that

$$\|f\|_2^2 \leq e^{|\Omega|} \int_{\mathbb{C}\setminus\Omega} \left|\big(V_{h_0} f\big)(z)\right|^2 dA(z) \;\; \forall f \in L^2(\mathbb{R}).$$

In this regard it is worth noting that Nazarov [13] proved the existence of two absolute constants A, B such that

$$\|f\|_2^2 \leq Ae^{B\cdot|S|\cdot|\Sigma|}\left(\int_{\mathbb{R}\setminus S}|f|^2+\int_{\mathbb{R}\setminus\Sigma}|\widehat{f}|^2\right)$$

for every $f \in L^2(\mathbb{R})$ and for any pair (S, Σ) of sets with finite measure. Also, it follows from [8, Theorem 4.1] that for every set $\Omega \subset \mathbb{R}^2$ thin at infinity and for every $g \in L^2(\mathbb{R})$ there exist a constant $C > 0$ such that

$$\|f\|_2^2 \leq C\int_{\mathbb{C}\setminus\Omega}\left|\left(V_g f\right)(z)\right|^2 dA(z) \;\; \forall f \in L^2(\mathbb{R}).$$

From Theorem 1 and Proposition 1 we get the following.

Corollary 1 *Let $F \in L^1(\mathbb{C}) \cap L^\infty(\mathbb{C})$ be a real-valued and radial symbol with respect to $a \in \mathbb{C}$. Then*

$$\|H_F\| \leq \|F\|_\infty\left(1-\exp\left(-\frac{\|F\|_1}{\|F\|_\infty}\right)\right).$$

Corollary 2 *Let $\Omega \subset \mathbb{R}^2$ be a set with finite measure. Then, for every $n \in \mathbb{N}$,*

$$|\langle H_\Omega h_n,\ h_n\rangle| \leq 1-e^{-|\Omega|}.$$

We fix a non-zero window $g \in L^2(\mathbb{R})$. The modulation space $M^1(\mathbb{R})$, also known as Feichtinger algebra, is the set of tempered distributions $f \in \mathcal{S}'(\mathbb{R})$ such that

$$\|f\|_{M^1} := \int_{\mathbb{C}}|\langle f, \pi(z)g\rangle|\, dA(z) < +\infty.$$

The use of different windows g in the definition of $M^1(\mathbb{R})$ yields the same spaces with equivalent norms. It is well known that $M^1(\mathbb{R})$ is continuously included in $L^2(\mathbb{R})$ and

$$\|f\|_2 = \|V_g f\|_2 \leq \|V_g f\|_1$$

whenever $f \in M^1(\mathbb{R})$ and $\|g\|_2 = 1$. See for instance [10, 3.2.1] for the first identity.

Proposition 3 *Let $\Omega \subset \mathbb{R}^2$ be a set with finite measure. Then, for every $f \in M^1(\mathbb{R})$ and $n \in \mathbb{N}_0$ we have*

$$\int_\Omega\left|\left(V_{h_0}f\right)(z)\right|^2 dA(z) \leq \|V_{h_n}f\|_1^2\cdot\left(1-e^{-|\Omega|}\right).$$

Proof It suffices to prove the proposition under the additional assumption that $\|V_{h_n}f\|_1 = 1$. Fixed $n \in \mathbb{N}_0$ we consider the set

$$B := \{\pi(z)h_n :\ z \in \mathbb{C}\} \subset L^2(\mathbb{R}).$$

Then

$$B^\circ := \left\{ g \in L^2(\mathbb{R}) : \ |\langle g, \pi(z) h_n\rangle| \le 1 \right\} = \left\{ g \in L^2(\mathbb{R}) : \ \|V_{h_n} g\|_\infty \le 1 \right\}.$$

We have

$$|\langle f, g\rangle| = \left|\langle V_{h_n} f, V_{h_n} g\rangle\right| \le \|V_{h_n} f\|_1 \cdot \|V_{h_n} g\|_\infty \le 1$$

for every $g \in B^\circ$, which means that $f \in B^{\circ\circ}$. According to the bipolar theorem,

$$f = L^2 - \lim_{k\to\infty} f_k$$

where each f_k is in the absolutely convex hull of B. For each $k \in \mathbb{N}$ we can find scalars $(\alpha_j)_{j=1}^N$ and points $(z_j)_{j=1}^N$ such that $f_k = \sum_{j=1}^N \alpha_j \pi(z_j) h_n$ and $\sum_{j=1}^N |\alpha_j| \le 1$. Then

$$\begin{aligned}
\left(\int_\Omega \left| \left(V_{h_0} f_k\right)(z)\right|^2 dA(z) \right)^{\frac{1}{2}} &= \left(\int_\Omega |\langle f_k, \pi(z)\varphi\rangle|^2 \, dA(z) \right)^{\frac{1}{2}} \\
&\le \sum_{j=1}^N |\alpha_j| \left(\int_\Omega \left|\langle \pi(z_j) h_n, \pi(z)\varphi\rangle\right|^2 dA(z) \right)^{\frac{1}{2}} \\
&= \sum_{j=1}^N |\alpha_j| \left(\int_\Omega \left|\langle h_n, \pi(z - z_j)\varphi\rangle\right|^2 dA(z) \right)^{\frac{1}{2}} \\
&= \sum_{j=1}^N |\alpha_j| \left(\int_{\Omega - z_j} |\langle h_n, \pi(z)\varphi\rangle|^2 \, dA(z) \right)^{\frac{1}{2}} \\
&= \sum_{j=1}^N |\alpha_j| \left|\langle H_{\Omega - z_j} h_n, h_n\rangle\right|^{\frac{1}{2}} \le \left(1 - e^{-|\Omega|}\right)^{\frac{1}{2}}.
\end{aligned}$$

Finally,

$$\int_\Omega \left|\left(V_{h_0} f\right)(z)\right|^2 dA(z) = \lim_{k\to\infty} \int_\Omega \left|\left(V_{h_0} f_k\right)(z)\right|^2 dA(z) \le 1 - e^{-|\Omega|}.$$

□

The next result is a direct consequence of Proposition 2.

Corollary 3 *For every $\varepsilon > 0$ there exists Ω with infinite Lebesgue measure such that*

$$\|H_\Omega\| \le \varepsilon.$$

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Declarations

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