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Radon Inversion Problem for Holomorphic Functions on Circular, Strictly Convex Domains

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Abstract
In this paper we study the so-called Radon inversion problem in bounded, circular, strictly convex domains with $\mathcal{C}^2$ boundary. We show that given $p > 0$ and a strictly positive, continuous function Φ on $\partial\Omega$, by use of homogeneous polynomials it is possible to construct a holomorphic function $f \in \mathcal{O}(\Omega)$ such that $\int_0^1 |f(zt)|^p dt = \Phi(z)$ for all $z \in \partial\Omega$. In our approach we make use of so-called lacunary K-summing polynomials (see definition below) that allow us to construct solutions with in some sense extremal properties.

Keywords Radon inversion problem · Divergent Taylor series · Boundary behaviour of holomorphic functions of several complex variables · Inner functions

Mathematics Subject Classification Primary 32A40 · Secondary 32A0

1 Introduction

In general, Radon inversion problem is to reconstruct a function on the basis of the values of its integrals over some subset of submanifolds of its domain. Here we consider bounded, circular, strictly convex domains and we are interested in finding a holomorphic function such that its radial integrals are equal to the values of some given strictly positive, continuous function on the boundary. More precisely, let $\Omega \subset \mathbb{C}^\nu$ be a bounded, circular, strictly convex domain with $\mathcal{C}^2$ boundary. Fix $p > 0$. For a

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holomorphic function $f \in \mathcal{O}(\Omega)$ we may consider the integral operator $\mathcal{R}^p$ defined as follows

$$\mathcal{R}^p(f)(z) := \int_0^1 |f(zt)|^p dt, \quad z \in \overline{\Omega}$$

and called *Radon operator*. Then for a given strictly positive, continuous function $\Phi\colon \partial\Omega \longmapsto \mathbb{R}_+$ we look for a function $f \in \mathcal{O}(\Omega)$ such that

$$\mathcal{R}^p(f) = \Phi \quad \text{on} \quad \partial\Omega. \tag{1}$$

Slightly different formulation of the Radon inversion problem in terms of considered integrals for the operator $\mathcal{R}^p$ was described in [7] and solved at almost all boundary points with respect to a given probability measure on the boundary of the domain. Moreover, in the same paper it was observed that Radon inversion problem for holomorphic functions is similar to construction of inner functions in several variables. The radial limits that exist almost everywhere on the boundary are replaced by the Radon operator. However, the Radon operator $\mathcal{R}^p$ is well-defined at all boundary points, so the aim of the present paper is to solve (1) for all $z \in \partial\Omega$. For the first two independent constructions of a non-constant inner function in several variables see [1] and [9].

To achieve our aim we make use of homogeneous polynomials constructed in [6]. Historically they are a generalization of Ryll-Wojtaszczyk polynomials (see [11]). For our purpose we add some properties to the polynomials from [6] to obtain, as we call them, lacunary K-summing polynomials (see definition below). They will form generators that enable us to construct by induction a solution to (1).

Then we intend to study properties of the obtained solution. In our approach, the solution is clearly unbounded on the boundary of the domain. Nevertheless, more interesting properties might be seen by considering a linear operator $\mathcal{S}^p$ for holomorphic functions, $p > 0$. If $\{u_k\}_{k\in\mathbb{N}}$ is a sequence of homogeneous polynomials of degree $n_k \in \mathbb{N}$ respectively, then the operator $\mathcal{S}^p$ is defined as follows

$$\mathcal{S}^p\colon\ \mathcal{O}(\Omega) \ni \sum_{k=1}^{\infty} u_k \longmapsto \sum_{k=1}^{\infty} \frac{u_k}{\sqrt[p]{pn_k+1}} \in \mathcal{O}(\Omega).$$

First of all, we construct a solution f to (1) such that $\mathcal{S}^p(f)$ is continuous up to the boundary for any $p > 0$. Then we show that one may find another solution g such that all slice functions of $\mathcal{S}^p(g)$ have divergent series of Taylor coefficients with every exponent $s < \min\{1, p\}$. Moreover, if $p \leq 1$, then $\mathcal{S}^p(g) \in \mathcal{C}(\overline{\Omega})$ and for $p \in (1, 2]$, function $\mathcal{S}^p(g)$ is square-integrable on all circles $z\partial\mathbb{D}$, $z \in \partial\Omega$. This is a similar result to the ones obtained for functions in the ball algebra in [12] and for inner functions in [2] and [3]. In some sense these two approaches give us flexibility and diversity in solving Radon inversion problem and might be useful in view of constructing holomorphic functions with prescribed boundary behaviour.

2 Notation and Preliminaries

In this paper we set the following notation. Let σ, μ and λ be normalized Lebesgue measures on $\partial\Omega$, Ω and $\partial\mathbb{D}$, respectively, i.e. $\sigma(\partial\Omega) = 1$, $\mu(\Omega) = 1$ and $\lambda(\partial\mathbb{D}) = 1$. Let $\mathcal{C}_M(\overline{\Omega})$ stand for the space of continuous functions in $\overline{\Omega}$ and bounded by M. Let us denote L_1 norm on circles $z\partial\mathbb{D}$, $z \in \partial\Omega$, by $||f||_{1,z\partial\mathbb{D}} := \int\limits_{\partial\mathbb{D}} |f(z\omega)|d\lambda(\omega)$ and supremum norm $||f||_T := \sup\limits_{z\in T}|f(z)|$. For $z \in \partial\Omega$ denote by $f_z\colon \mathbb{D} \ni \omega \longmapsto f(\omega z)$ the slice function for f. Recall that a polynomial P is said to be of order k if $\dfrac{\partial^{|\alpha|}}{\partial z^\alpha}P(0) = 0$ for any multi-index α such that $|\alpha| < k$ and we write $\operatorname{ord}(P) = k$. Finally, for $p > 0$ and $q = \max\{p, 1\}$ we define the space

$$\mathcal{HR}^p(\Omega) := \left\{ f \in \mathcal{O}(\Omega)\colon \sup_{z\in\partial\Omega} \mathcal{R}^p(f)(z) < \infty \right\}$$

equipped with the metric $d(f, g) := \sup\limits_{z\in\partial\Omega} \Big(\mathcal{R}^p(f-g)(z)\Big)^{\frac{1}{q}}$, where $f, g \in \mathcal{HR}^p(\Omega)$. This space is complete as we shall show it in the following proposition.

Proposition 2.1 *The space $\big(\mathcal{HR}^p(\Omega), d\big)$ is complete for any $p > 0$. Moreover, if $f_n \xrightarrow[n\to\infty]{d} f$ in $\mathcal{HR}^p(\Omega)$, then $\mathcal{R}^p(f_n) \rightrightarrows \mathcal{R}^p(f)$ on $\partial\Omega$ as $n \to \infty$.*

Proof Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{HR}^p(\Omega)$. Set $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that

$$\Big(\mathcal{R}^p(f_m - f_n)(z)\Big)^{\frac{1}{q}} < \varepsilon \quad \forall_{m,n\geq N} \;\; \forall_{z\in\partial\Omega}$$

which implies that

$$\int_0^1 |(f_m - f_n)(zt)|^p dt < \varepsilon^q \quad \forall_{m,n\geq N} \;\; \forall_{z\in\partial\Omega}. \tag{2}$$

Now we may integrate (2) over $\partial\Omega$ to obtain that

$$\int_{\partial\Omega}\int_0^1 |(f_m - f_n)(zt)|^p dt d\sigma(z) < \varepsilon^q \quad \forall_{m,n\geq N}.$$

Since $f_m - f_n$ is a holomorphic function, on the basis of [10] (Prop. 1.5.4.), function $|f_m - f_n|^p$ is subharmonic. Let K be a compact subset of Ω, $w \in K$ and $r = \frac{1}{2}\operatorname{dist}(K, \partial\Omega)$, so $\overline{\mathbb{B}(w, r)} \subset \Omega$. The sub-mean value theorem applied to the function

$|f_m - f_n|^p$ and $\mathbb{B}(w, r)$ gives the following estimate

$$|(f_m - f_n)(w)|^p \leq \frac{1}{\mu(\overline{\mathbb{B}(w,r)})} \int_{\mathbb{B}(w,r)} |(f_m - f_n)(\zeta)|^p d\mu(\zeta)$$
$$\leq \frac{1}{r^{2\nu}} \int_{\partial\Omega} \int_0^1 |(f_m - f_n)(zt)|^p dt d\sigma(z) < \frac{\varepsilon^q}{r^{2\nu}}$$

for any $m, n \geq N$. In particular,

$$\sup_{w \in K} |(f_m - f_n)(w)|^p < \frac{\varepsilon^q}{r^{2\nu}} \quad \forall_{m,n \geq N}.$$

This implies that $\{f_n\}_{n\in\mathbb{N}}$ converges locally uniformly on compact subsets of Ω. Let f be the limit function for the sequence $\{f_n\}_{n\in\mathbb{N}}$. Since f_n are holomorphic, so is f.

Now we shall show that $f_n \xrightarrow{d} f$ and $f \in \mathcal{HR}^p(\Omega)$. It follows from (2) that

$$\int_0^\delta |(f_m - f_n)(zt)|^p dt < \varepsilon^q \quad \forall_{\delta\in(0,1)} \ \forall_{m,n\geq N} \ \forall_{z\in\partial\Omega}. \tag{3}$$

Since $f_m \longrightarrow f$ locally uniformly we may pass with m to the limit in (3) to obtain that

$$\int_0^\delta |(f - f_n)(zt)|^p dt \leq \varepsilon^q \quad \forall_{\delta\in(0,1)} \ \forall_{n\geq N} \ \forall_{z\in\partial\Omega}.$$

Above inequality holds for any $\delta \in (0, 1)$, so we get that

$$\int_0^1 |(f - f_n)(zt)|^p dt \leq \varepsilon^q \quad \forall_{n\geq N} \ \forall_{z\in\partial\Omega}.$$

Moreover, by triangle inequality,

$$\sup_{z\in\partial\Omega} \left(\mathcal{R}^p(f)(z)\right)^{\frac{1}{q}} \leq \sup_{z\in\partial\Omega} \left(\mathcal{R}^p(f - f_n)(z)\right)^{\frac{1}{q}} + \sup_{z\in\partial\Omega} \left(\mathcal{R}^p(f_n)(z)\right)^{\frac{1}{q}} < \infty \quad \forall_{n\geq N}.$$

Therefore $f_n \xrightarrow[n\to\infty]{d} f$ and $f \in \mathcal{HR}^p(\Omega)$.

If $f_n \xrightarrow[n\to\infty]{d} f$, then $d(f, f_n) \xrightarrow[n\to\infty]{} 0$. Triangle inequality implies that

$$\sup_{z\in\partial\Omega} \left(\left(\mathcal{R}^p(f)(z)\right)^{\frac{1}{q}} - \left(\mathcal{R}^p(f_n)(z)\right)^{\frac{1}{q}} \right) \leq \sup_{z\in\partial\Omega} \left(\mathcal{R}^p(f - f_n)(z)\right)^{\frac{1}{q}} \xrightarrow[n\to\infty]{} 0.$$

Hence $\left(\mathcal{R}^p(f_n)\right)^{\frac{1}{q}} \rightrightarrows \left(\mathcal{R}^p(f)\right)^{\frac{1}{q}}$ on $\partial\Omega$ as $n \to \infty$ and consequently that $\mathcal{R}^p(f_n) \rightrightarrows \mathcal{R}^p(f)$ on $\partial\Omega$ as $n \to \infty$. □

In the construction of a solution to the Radon inversion problem we will use the following theorem from [4]. It provides a collection of holomorphic and continuous up to the boundary functions that in some sense together approximate a given strictly positive, continuous function on $\partial\Omega$. It is important to emphasise that the number of elements in this set of functions does not depend on the function we want to approximate.

Theorem 2.2 ([4], Theorem 3.2) *There exists a natural number $N = N(\partial\Omega)$ such that if $\varepsilon \in (0, 1)$, T is a compact subset of Ω, h is a continuous strictly positive function on $\partial\Omega$, then there exist holomorphic functions $f_1, \dots, f_N \in \mathcal{O}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ such that*

1. $||f_m||_T \leq \varepsilon, \quad m = 1, \dots, N$
2. $\dfrac{h}{2} < \max\limits_{1\leq m\leq N} |f_m| < h$ *on* $\partial\Omega$.

As it is observed in [8], functions in Theorem 2.2 may be replaced with polynomials.

3 Lacunary *K*-Summing Polynomials

As we said before, the main tool in our approach to solve the Radon inversion problem is lacunary K-summing polynomials. Hereafter, we give a precise definition and then we study useful properties of lacunary K-summing polynomials, in particular how they behave composed with operators $\mathcal{R}^p$ and $\mathcal{S}^p$ and what relation there is between these operators.

Definition 3.1 For a given $K \in \mathbb{N}$, a polynomial Q is called a K-summing polynomial, if it has a homogeneous expansion $\sum_{j=1}^{K} u_j$, where $\deg(u_j) = n_j \in \mathbb{N}, \quad j = 1, 2, \dots, K$, and satisfies the following conditions

i) $\max\limits_{1\leq j\leq K} |u_j(z)| \leq 1 \quad \text{for } z \in \partial\Omega$

ii) $\dfrac{1}{2}\deg(Q) \leq \operatorname{ord}(Q) = n_1 < n_2 < \cdots < n_K = \deg(Q)$.

Moreover, Q is said to be a lacunary K-summing polynomial, if it additionally possesses properties $iii)$ and $iv)$

iii) $\max\limits_{1\leq j\leq K} |u_j(z)| \geq \dfrac{1}{2} \quad \text{for } z \in \partial\Omega$

iv) $\sqrt[K]{2} < \dfrac{n_{j+1}}{n_j} < 2, \quad j = 1, 2, \dots, K-1.$

To prove the existence of lacunary K-summing polynomials we need the following theorem from [6].

Theorem 3.2 ([6], Theorem 2.5) *There exists $K = K(\partial\Omega) \in \mathbb{N}$ such that there exists $N_0 \in \mathbb{N}$ such that for all integers $N \geq N_0$ and $n_1, \dots, n_K \in \mathbb{N}$ with $N \leq n_1 \leq \cdots \leq n_K \leq 2N$ there exist homogeneous polynomials $u_1, \dots, u_K$ of degrees $n_1, \dots, n_K$, respectively, such that $\dfrac{1}{2} < \max\limits_{1\leq j\leq K} |u_j(z)| < 1$ for all $z \in \partial\Omega$.*

Corollary 3.3 *There exists $K = K(\partial\Omega) \in \mathbb{N}$ for which there exists $N_K \in \mathbb{N}$ such that for any $N \geq N_K$ there exists a lacunary K-summing polynomial of degree N.*

Proof Take $K = K(\partial\Omega) \in \mathbb{N}$ and $N_0 \in \mathbb{N}$ from Theorem 3.2. Choose such a big $N_K \geq 2N_0$ that for $N \geq N_K$ there exists a K-tuple of integers $\frac{1}{2}N < n_1 < n_2 < \cdots < n_K = N$ that satisfy $\sqrt[K]{2} < \frac{n_{j+1}}{n_j} < 2$ for $j = 1, 2, \ldots, K-1$. For such N and $n_1, n_2, \ldots, n_K$ by Theorem 3.2, there exist homogeneous polynomials $u_1, u_2, \ldots, u_K$ of degrees $n_1, n_2, \ldots, n_K$, respectively, such that $\frac{1}{2} < \max\limits_{1 \leq j \leq K} |u_j| < 1$ on $\partial\Omega$. Then observe that $\sum_{j=1}^{K} u_j$ is a lacunary K-summing polynomial of degree N. □

Lemma 3.4 *Let $p > 0,\ \varepsilon > 0,\ M > 0$ and $K \in \mathbb{N}$. There exists $N_0 \in \mathbb{N}$ such that for any K-summing polynomial Q of degree $N \geq N_0$ and $f, g \in \mathcal{C}_M(\overline{\Omega})$ the following inequalities*

$$\mathcal{R}^p\big(fQ\big) + \mathcal{R}^p(g) - \varepsilon \ < \ \mathcal{R}^p\big(fQ + g\big) \ < \ \mathcal{R}^p\big(fQ\big) + \mathcal{R}^p(g) + \varepsilon$$

hold in $\overline{\Omega}$.

Proof Let $q = \max\{p, 1\}$ and $k \in \mathbb{N}$ be such that $k \geq p$. We may choose $N_0 \in \mathbb{N}$ such that for any $N \geq N_0$ we have $\dfrac{M^p K^q}{\frac{1}{2}pN + 1} < \dfrac{\varepsilon}{8}$ and $\sum_{m=1}^{k-1} \binom{k}{m}^{\frac{p}{k}} M^p K^m \dfrac{k}{\frac{1}{2}Np + k} < \dfrac{\varepsilon}{2}$. We will show that such N_0 fulfills the requirements to be chosen for Lemma 3.4.

Since Q is a K-summing polynomial of degree $N \geq N_0$, let $Q = \sum_{j=1}^{K} u_j$ be its homogeneous expansion of degrees $n_1, \ldots, n_K$, respectively, such that $\frac{1}{2}N \leq n_1 < \cdots < n_K = N$. For simplicity denote by $F := fQ$. Let $\delta \in (0, 1)$ be such that $\int_\delta^1 |g(zt)|^p dt \leq \dfrac{\varepsilon}{8}$. Then observe that for $z \in \overline{\Omega}$ we have

$$\begin{aligned}
\left(\int_0^\delta |F(zt)|^p dt\right)^{\frac{1}{q}} &= \left(\int_0^\delta \left|f(zt)\sum_{j=1}^{K} t^{n_j} u_j(z)\right|^p dt\right)^{\frac{1}{q}} \\
&\leq M^{\frac{p}{q}} \sum_{j=1}^{K} \left(|u_j(z)|^p \int_0^\delta t^{pn_j} dt\right)^{\frac{1}{q}} \\
&\leq M^{\frac{p}{q}} \sum_{j=1}^{K} \left(\frac{\delta^{pn_j+1}}{pn_j + 1}\right)^{\frac{1}{q}} \leq \left(M^p K^q \frac{\delta^{\frac{1}{2}pN+1}}{\frac{1}{2}pN + 1}\right)^{\frac{1}{q}} < \left(\frac{\varepsilon}{8}\right)^{\frac{1}{q}}.
\end{aligned}$$

Hence $\int_0^\delta |F(zt)|^p dt < \frac{\varepsilon}{8}$. Since $\frac{p}{k} \leq 1$, we are able to make use of triangle inequality to obtain the following estimates

$$\begin{aligned}
\mathcal{R}^{\frac{p}{k}}\big(F^k + g^k\big)(z) &= \int_0^1 \big|\big(F^k + g^k\big)(zt)\big|^{\frac{p}{k}} dt \leq \int_0^1 \big|F(zt)\big|^p dt + \int_0^1 \big|g(zt)\big|^p dt \\
&= \mathcal{R}^p(F)(z) + \mathcal{R}^p(g)(z)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{R}^{\frac{p}{k}}(F^k + g^k)(z) &= \int_0^1 |(F^k + g^k)(zt)|^{\frac{p}{k}} dt = \int_0^\delta |(F^k + g^k)(zt)|^{\frac{p}{k}} dt \\
&\quad + \int_\delta^1 |(F^k + g^k)(zt)|^{\frac{p}{k}} dt \\
&\geq \int_\delta^1 |F(zt)|^p dt - \int_0^\delta |F(zt)|^p dt \\
&\quad + \int_0^\delta |g(zt)|^p dt - \int_\delta^1 |g(zt)|^p dt \\
&= \mathcal{R}^p(F)(z) + \mathcal{R}^p(g)(z) - 2\int_0^\delta |F(zt)|^p dt - 2\int_\delta^1 |g(zt)|^p dt \\
&> \mathcal{R}^p(F)(z) + \mathcal{R}^p(g)(z) - \frac{\varepsilon}{2}.
\end{aligned}$$

If $0 < p \leq 1$, then $k = 1$ does the job. For $p > 1$ we need a little bit more effort to make. For shortness, denote by

$$\mathcal{I}_k^p(F, g)(z) := \int_0^1 \Big|\sum_{m=1}^{k-1} \binom{k}{m} \big(F(zt)\big)^m \big(g(zt)\big)^{k-m}\Big|^{\frac{p}{k}} dt.$$

Then observe that for $z \in \overline{\Omega}$

$$\begin{aligned}
\mathcal{I}_k^p(F, g)(z) &\leq \sum_{m=1}^{k-1} \binom{k}{m}^{\frac{p}{k}} \int_0^1 \Big|f(zt) \sum_{j=1}^K u_j(zt)\Big|^{\frac{mp}{k}} |g(zt)|^{\frac{(k-m)p}{k}} dt \\
&\leq \sum_{m=1}^{k-1} \binom{k}{m}^{\frac{p}{k}} M^{\frac{mp}{k}} M^{\frac{p(k-m)}{k}} \Big(\sum_{j=1}^K |u_j(z)|^{\frac{p}{k}} \int_0^1 t^{\frac{pn_j}{k}} dt\Big)^m \\
&\leq \sum_{m=1}^{k-1} \binom{k}{m}^{\frac{p}{k}} M^p \Big(\sum_{j=1}^K \frac{k}{pn_j + k}\Big)^m \leq \sum_{m=1}^{k-1} \binom{k}{m}^{\frac{p}{k}} M^p \Big(K \frac{k}{\frac{1}{2}pN + k}\Big)^m \\
&\leq \sum_{m=1}^{k-1} \binom{k}{m}^{\frac{p}{k}} M^p K^m \frac{k}{\frac{1}{2}Np + k} < \frac{\varepsilon}{2}.
\end{aligned}$$

This implies that for $z \in \overline{\Omega}$

$$\begin{aligned}\mathcal{R}^p(F+g)(z) &= \int_0^1 \left(|(F+g)(zt)|^k\right)^{\frac{p}{k}} dt = \int_0^1 \Big| \sum_{m=0}^{k} \binom{k}{m} \big(F(zt)\big)^m \big(g(zt)\big)^{k-m} \Big|^{\frac{p}{k}} dt \\ &\le \int_0^1 |F(zt)|^p dt + \int_0^1 |g(zt)|^p dt + \mathcal{I}_k^p(F,g)(z) \\ &< \mathcal{R}^p(F)(z) + \mathcal{R}^p(g)(z) + \frac{\varepsilon}{2}\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}^p(F+g)(z) = \int_0^1 |(F+g)(zt)|^p dt = \ & \int_0^1 \Big| \sum_{m=0}^{k} \binom{k}{m} \big(F(zt)\big)^m \big(g(zt)\big)^{k-m} \Big|^{\frac{p}{k}} dt \\ [5mm] \ge \int_0^1 & |(F^k+g^k)(zt)|^{\frac{p}{k}} dt - \mathcal{I}_k^p(F,g)(z) \\ > \int_0^1 & |F(zt)|^p dt + \int_0^1 |g(zt)|^p dt - \varepsilon \\ = \mathcal{R}^p & (F)(z) + \mathcal{R}^p(g)(z) - \varepsilon.\end{aligned}$$

Above estimates complete the proof. □

Lemma 3.5 *Let $p > 0$, $\varepsilon > 0$, $M > 0$ and $K \in \mathbb{N}$. There exists $N_0 \in \mathbb{N}$ such that for any K-summing polynomial Q of degree $N \ge N_0$ and $f \in \mathcal{C}_M(\overline{\Omega})$ the following inequalities*

$$|f|^p \mathcal{R}^p(Q) - \varepsilon \ \le\ \mathcal{R}^p\big(fQ\big) \ \le\ |f|^p \mathcal{R}^p(Q) + \varepsilon$$

are satisfied in $\overline{\Omega}$.

Proof There exists $N_0 \in \mathbb{N}$ such that for any $N \ge N_0$ the following inequalities hold true $\dfrac{M^p K^{p+1}}{\frac{1}{2}Np+1} \le \dfrac{\varepsilon}{2}$ and $\frac{1}{2}Np + 1 > K^{p+1}$. We will show that such N_0 fulfills the requirements to be chosen for Lemma 3.5.

Let $\sum_{j=1}^{K} u_j$ be the homogeneous expansion of the polynomial Q, where $u_1, \ldots, u_K$ are homogeneous polynomials of degrees $n_1, \ldots, n_K$, respectively, such that $\frac{1}{2}N \le n_1 < \cdots < n_K = N$. Since f is continuous in $\overline{\Omega}$, there exists $\delta \in (0,1)$ such that for $t \in [\delta, 1]$, if $z \in \overline{\Omega}$, then

$$|f(z)|^p - \frac{\varepsilon}{2} \ \le\ |f(zt)|^p \ \le\ |f(z)|^p + \frac{\varepsilon}{2}. \tag{4}$$

Let $q = \max\{p, 1\}$. Then observe that by triangle inequality,

$$\begin{aligned}\left(\int_\delta^1 |Q(zt)|^p dt\right)^{\frac{1}{q}} &= \left(\int_\delta^1 \Big|\sum_{j=1}^K u_j(zt)\Big|^p dt\right)^{\frac{1}{q}} \\ &\le \sum_{j=1}^K \left(\int_\delta^1 t^{pn_j} dt\right)^{\frac{1}{q}} \le K\left(\frac{1}{\frac{1}{2}Np+1}\right)^{\frac{1}{q}} < 1 \end{aligned} \tag{5}$$

and

$$\begin{aligned}\left(|f(z)|^p \int_0^\delta |Q(zt)|^p dt\right)^{\frac{1}{q}} &\le M^{\frac{p}{q}} \sum_{j=1}^K \left(\int_0^\delta |u_j(z)|^p t^{pn_j} dt\right)^{\frac{1}{q}} \\ &\le M^{\frac{p}{q}} \sum_{j=1}^K \left(\frac{\delta^{pn_j+1}}{pn_j+1}\right)^{\frac{1}{q}} \\ &\le \left(M^p K^q \frac{\delta^{pN+1}}{\frac{1}{2}pN+1}\right)^{\frac{1}{q}} \le \left(\frac{\varepsilon}{2}\right)^{\frac{1}{q}}. \end{aligned} \tag{6}$$

Above inequalities yield the conclusion

$$\begin{aligned}\mathcal{R}^p(fQ)(z) &= \int_0^1 \big|f(zt)Q(zt)\big|^p dt \ge \int_\delta^1 \big|f(zt)Q(zt)\big|^p dt \\ &\overset{(4)}{\ge} \left(|f(z)|^p - \frac{\varepsilon}{2}\right)\int_\delta^1 \big|Q(zt)\big|^p dt \\ &\overset{(5)}{\ge} |f(z)|^p \int_0^1 \big|Q(zt)\big|^p dt - |f(z)|^p \int_0^\delta \big|Q(zt)\big|^p dt - \frac{\varepsilon}{2} \\ &\overset{(6)}{\ge} |f(z)|^p \mathcal{R}^p(Q)(z) - \varepsilon \end{aligned}$$

and

$$\begin{aligned}\mathcal{R}^p(fQ)(z) &= \int_0^1 \big|f(zt)Q(zt)\big|^p dt = \int_\delta^1 \big|f(zt)Q(zt)\big|^p dt + \int_0^\delta \big|f(zt)Q(zt)\big|^p dt \\ &\overset{(4),(6)}{\le} \left(|f(z)|^p + \frac{\varepsilon}{2}\right)\int_\delta^1 \big|Q(zt)\big|^p dt \\ &\quad + \frac{\varepsilon}{2} \overset{(5)}{\le} |f(z)|^p \int_0^1 \big|Q(zt)\big|^p dt + \varepsilon \\ &= |f(z)|^p \mathcal{R}^p(Q)(z) + \varepsilon \end{aligned}$$

for $z \in \overline{\Omega}$. This finishes the proof. □

Lemma 3.6 *Let $p > 0$, $m \in \mathbb{N}$, $\eta \in (1,2)$, $\delta \in (0,1)$. There exist $c_m \in (0,\delta)$ and $l_m \in \mathbb{N}$ such that for any polynomial of the form $P_m(t) = \sum_{j=1}^m a_j t^{n_j}$, where $\delta \leq \max\limits_{1\leq j\leq m} |a_j| \leq 1$, $n_1 \geq l_m$ and $\eta < \frac{n_{j+1}}{n_j} < 2$ for $j = 1, \ldots, m-1$, the following inequality holds true*

$$\int_0^1 pn_m |P_m(t)|^p dt \geq c_m.$$

Proof Let $q = \max\{p, 1\}$. We proceed inductively. Case $m = 1$ is trivial. Consider the polynomial $P_1(t) = a_1 t^{n_1}$, where $|a_1| \geq \delta$. There exists $l_1 \in \mathbb{N}$ such that $pn_1 \geq 1$ for any $n_1 \geq l_1$. Hence

$$\int_0^1 pn_1 |P_1(t)|^p dt = \tfrac{pn_1}{pn_1+1} |a_1|^p \geq \tfrac{1}{2}\delta^p =: c_1.$$

For the case $m = 2$ firstly we choose a suitable natural number l_2. To do this let us observe that there exists $\alpha > 0$ such that $e^{-\frac{p\alpha(\eta-1)}{q}} < \frac{\delta}{10}$. Select $l_2 \in \mathbb{N}$ such that $\frac{pn_1}{pn_1+1} \geq \frac{9}{10}$ and $\left(1 - \frac{\alpha}{n_1}\right)^{\frac{pn_1+1}{q}} \geq \frac{5}{9} e^{-\frac{p\alpha}{q}}$ for any $n_1 > l_2$. Then we may consider the polynomial $P_2(t) = a_1 t^{n_1} + a_2 t^{n_2}$, where $\eta < \frac{n_2}{n_1} < 2$, $n_1 > l_2$ and $\delta \leq \max\left\{|a_1|, |a_2|\right\} \leq 1$. Assume first that $|a_1| < \left(\frac{2}{5}\delta\right)^{\frac{q}{p}}$ and so $|a_2| \geq \delta$. We may estimate

$$\begin{aligned}\left(\int_0^1 pn_2 |P_2(t)|^p dt\right)^{\frac{1}{q}} \geq{}& \left(\frac{pn_2}{pn_2+1}\right)^{\frac{1}{q}} |a_2|^{\frac{p}{q}} \\ &- \left(\frac{pn_2}{pn_1+1}\right)^{\frac{1}{q}} |a_1|^{\frac{p}{q}} \quad \geq \frac{9}{10}\delta - 2\left(\frac{pn_1}{pn_1+1}\right)^{\frac{1}{q}} |a_1|^{\frac{p}{q}} \\ &\geq \frac{9}{10}\delta - \frac{4}{5}\delta = \frac{\delta}{10}.\end{aligned}$$

If $|a_1| \geq \left(\frac{2}{5}\delta\right)^{\frac{q}{p}}$, then we obtain what follows

$$\begin{aligned}\left(\int_0^1 pn_2 |P_2(t)|^p dt\right)^{\frac{1}{q}} &\geq \left(\int_0^{1-\frac{\alpha}{n_1}} pn_2 |P_2(t)|^p dt\right)^{\frac{1}{q}} \\ &\geq \left(\int_0^{1-\frac{\alpha}{n_1}} pn_2 |a_1|^p t^{pn_1} dt\right)^{\frac{1}{q}} \\ &\quad - \left(\int_0^{1-\frac{\alpha}{n_1}} pn_2 |a_2|^p t^{pn_2} dt\right)^{\frac{1}{q}} \\ &\geq \left(\frac{pn_2}{pn_1+1}\right)^{\frac{1}{q}} \left(1 - \frac{\alpha}{n_1}\right)^{\frac{pn_1+1}{q}} |a_1|^{\frac{p}{q}}\end{aligned}$$

$$
\begin{aligned}
&- \left(\frac{pn_2}{pn_2+1}\right)^{\frac{1}{q}} \left(1-\frac{\alpha}{n_1}\right)^{\frac{pn_2+1}{q}} |a_2|^{\frac{p}{q}} \\
&\geq \eta^{\frac{1}{q}} \frac{pn_1}{pn_1+1} \left(1-\frac{\alpha}{n_1}\right)^{\frac{pn_1+1}{q}} |a_1|^{\frac{p}{q}} - \left(1-\frac{\alpha}{n_1}\right)^{\frac{pn_2+1}{q}} \\
&\geq \frac{\delta}{5} e^{-\frac{p\alpha}{q}} - e^{-\frac{p\eta\alpha}{q}} = e^{-\frac{p\alpha}{q}} \left(\frac{\delta}{5} - e^{-\frac{p\alpha(\eta-1)}{q}}\right) \geq \frac{\delta}{10} e^{-\frac{p\alpha}{q}}.
\end{aligned}
$$

Now it is enough to define $c_2 := \frac{\delta}{10} e^{-\frac{p\alpha}{q}}$.

Proceeding inductively assume that the statement of the lemma holds true for $m-1$ with constants $c_{m-1} \in (0, \delta)$ and $l_{m-1} \in \mathbb{N}$. There exists $\beta > 0$ such that $e^{-\frac{p\beta(\eta-1)}{q}} < \frac{9}{10} \frac{c_{m-1}}{(m-1)2^{2m-1}}$. Then select an integer $l_m \geq l_{m-1}$ such that $\frac{pn_1}{pn_1+1}\left(1-\frac{\beta}{n_1}\right)^{\frac{pn_1+1}{q}} \geq \frac{9}{10} e^{-\frac{p\beta}{q}}$ for any $n_1 > l_m$. Consider the polynomial $P_m(t) = \sum_{j=1}^m a_j t^{n_j}$, where $\delta \leq \max\limits_{1 \leq j \leq m} |a_j| \leq 1$ and $\eta < \frac{n_{j+1}}{n_j} < 2$ for $j = 1, \ldots, m-1$ with $n_1 > l_m$. Since the statement of the lemma holds true for $m-1$ with the constant c_{m-1}, this means in particular that if $\max\limits_{2 \leq j \leq m} |a_j| \geq \delta$, then

$$
\int_0^1 pn_m \left| \sum_{j=2}^m a_j t^{n_j} \right|^p dt \geq c_{m-1}.
$$

If $|a_1| < \left(\frac{c_{m-1}}{2^m}\right)^{\frac{q}{p}} < \delta$, then $\max\limits_{2 \leq j \leq m} |a_j| \geq \delta$ and from the induction hypothesis we may estimate

$$
\begin{aligned}
\left(\int_0^1 pn_m |P_m(t)|^p dt\right)^{\frac{1}{q}} &= \left(\int_0^1 pn_m \left|\sum_{j=1}^m a_j t^{n_j}\right|^p dt\right)^{\frac{1}{q}} \\
&\geq \left(\int_0^1 pn_m \left|\sum_{j=2}^m a_j t^{n_j}\right|^p dt\right)^{\frac{1}{q}} \\
&\quad - \left(\int_0^1 pn_m |a_1|^p t^{pn_1} dt\right)^{\frac{1}{q}} \\
&\geq c_{m-1}^{\frac{1}{q}} - \left(\frac{pn_m}{pn_1+1}\right)^{\frac{1}{q}} |a_1|^{\frac{p}{q}} \\
&\geq c_{m-1} - 2^{m-1} \left(\frac{pn_1}{pn_1+1}\right)^{\frac{1}{q}} |a_1|^{\frac{p}{q}} \\
&\geq \frac{1}{2} c_{m-1}
\end{aligned}
$$

If $|a_1| \geq \left(\frac{c_{m-1}}{2^m}\right)^{\frac{q}{p}}$, then we obtain

$$\begin{aligned}
\left(\int_0^1 pn_m|P_m(t)|^p dt\right)^{\frac{1}{q}} &\geq \left(\int_0^{1-\frac{\beta}{n_1}} pn_m\Big|\sum_{j=1}^m a_j t^{n_j}\Big|^p dt\right)^{\frac{1}{q}} \\
&\geq \left(\int_0^{1-\frac{\beta}{n_1}} pn_m|a_1|^p t^{pn_1} dt\right)^{\frac{1}{q}} - \left(\sum_{j=2}^m \int_0^{1-\frac{\beta}{n_1}} pn_m|a_j|^p t^{pn_j} dt\right)^{\frac{1}{q}} \\
&\geq \left(\int_0^{1-\frac{\beta}{n_1}} pn_m|a_1|^p t^{pn_1} dt\right)^{\frac{1}{q}} - \sum_{j=2}^m \left(\int_0^{1-\frac{\beta}{n_1}} pn_m|a_j|^p t^{pn_j} dt\right)^{\frac{1}{q}} \\
&\geq \left(\frac{pn_m}{pn_1+1}\right)^{\frac{1}{q}} \left(1-\frac{\beta}{n_1}\right)^{\frac{pn_1+1}{q}} |a_1|^{\frac{p}{q}} - \sum_{j=2}^m \left(\frac{pn_m}{pn_j+1}\right)^{\frac{1}{q}} \left(1-\frac{\beta}{n_1}\right)^{\frac{pn_j+1}{q}} |a_j|^{\frac{p}{q}} \\
&\geq \left(\frac{pn_1}{pn_1+1}\right)^{\frac{1}{q}} \left(1-\frac{\beta}{n_1}\right)^{\frac{pn_1+1}{q}} |a_1|^{\frac{p}{q}} - \sum_{j=2}^m \left(\frac{2^{m-j}pn_j}{pn_j+1}\right)^{\frac{1}{q}} \left(1-\frac{\beta}{n_1}\right)^{\frac{pn_j+1}{q}} \\
&\geq \frac{pn_1}{pn_1+1}\left(1-\frac{\beta}{n_1}\right)^{\frac{pn_1+1}{q}} |a_1|^{\frac{p}{q}} - \sum_{j=2}^m 2^{m-j} e^{-\frac{\beta\eta^{j-1}}{q}} \\
&\geq \frac{9}{10} e^{-\frac{p\beta}{q}} \frac{c_{m-1}}{2^m} - (m-1)2^{m-2} e^{-\frac{p\beta\eta}{q}} \\
&\geq e^{-\frac{p\beta}{q}} \left(\frac{9}{10}\frac{c_{m-1}}{2^m} - (m-1)2^{m-2} e^{-\frac{p\beta(\eta-1)}{q}}\right) \geq 0.9 e^{-\frac{p\beta}{q}} \frac{c_{m-1}}{2^{m+1}}.
\end{aligned}$$

Now from the above inequalities, we may define suitably the constant c_m and this completes the proof. □

Corollary 3.7 *Let $p > 0$, $K \in \mathbb{N}$. There exist constants $c_K \in (0, \frac{1}{2})$, $C_K > 1$ and $l_K \in \mathbb{N}$ such that for any lacunary K-summing polynomial Q of degree $N \geq l_K$ the following inequalities hold*

$$c_K \leq \int_0^1 pN\big|Q(zt)\big|^p dt \leq C_K, \quad z \in \partial\Omega.$$

Proof Let $C_K := 2K^{\max\{p,1\}}$ and $c_K \in (0, \frac{1}{2})$, $l_K \in \mathbb{N}$ be constants from Lemma 3.6 with $m := K$, $\eta := \sqrt[K]{2}$ and $\delta := \frac{1}{2}$. Let $\sum_{j=1}^K u_k$ be the homogeneous expansion of Q, where each u_j is a homogeneous polynomial of degree $n_j \in \mathbb{N}$. Observe that since Q is a lacunary K-summing polynomial of degree $N \geq l_K$, for $z \in \partial\Omega$ polynomial Q satisfies Lemma 3.6 with $a_j := u_j(z)$, i.e.

$$\int_0^1 pN\big|Q(zt)\big|^p dt = \int_0^1 pN\Big|\sum_{j=1}^K u_j(z)t^{n_j}\Big|^p dt \geq c_K.$$

To prove the second inequality let $q = \max\{p, 1\}$. Then we may estimate

$$
\begin{aligned}
\Big(\int_0^1 pN\big|Q(zt)\big|^p dt\Big)^{\frac{1}{q}} &= \Big(\int_0^1 pN\Big|\sum_{j=1}^{K} u_j(z)t^{n_j}\Big|^p dt\Big)^{\frac{1}{q}} \\
&\le \sum_{j=1}^{K}\Big(\int_0^1 pN|u_j(z)|^p t^{pn_j} dt\Big)^{\frac{1}{q}} \\
&\le \sum_{j=1}^{K}\big(\tfrac{pN}{pn_j+1}\big)^{\frac{1}{q}} \le K\big(\tfrac{pN}{\frac{1}{2}pN+1}\big)^{\frac{1}{q}} \le K\sqrt[q]{2}.
\end{aligned}
$$

Hence $\int_0^1 pN\big|Q(zt)\big|^p dt \le C_K$. □

Lemma 3.8 *Let $p > 0,\ \varepsilon > 0$ and $m \in \mathbb{N}$. There exists $N \in \mathbb{N}$ such that for $n \in \mathbb{N}$ if $n \ge N$, then*

$$
(2pn)^{\frac{1}{p}} \frac{(pn + pm + 1)^{\frac{1}{p}} - (pn + 1)^{\frac{1}{p}}}{(pn + pm + 1)^{\frac{1}{p}}(pn + 1)^{\frac{1}{p}}} < \varepsilon.
$$

Proof Select $k \in \mathbb{N}$ such that $k \ge \frac{1}{p}$, so $\frac{1}{pk} \le 1$. Then by triangle inequality we may estimate

$$
\begin{aligned}
\Big((pm + pn + 1)^{\frac{1}{pk}}\Big)^k &\le \Big((pm)^{\frac{1}{pk}} + (pn + 1)^{\frac{1}{pk}}\Big)^k = \sum_{j=0}^{k}\binom{k}{j}(pm)^{\frac{j}{pk}}(pn + 1)^{\frac{k-j}{pk}} \\
&= (pn + 1)^{\frac{1}{p}} + \sum_{j=1}^{k}\binom{k}{j}(pm)^{\frac{j}{pk}}(pn + 1)^{\frac{k-j}{pk}}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
(2pn)^{\frac{1}{p}} \frac{(pn + pm + 1)^{\frac{1}{p}} - (pn + 1)^{\frac{1}{p}}}{(pn + pm + 1)^{\frac{1}{p}}(pn + 1)^{\frac{1}{p}}} &\le (2pn)^{\frac{1}{p}} \frac{\sum_{j=1}^{k}\binom{k}{j}(pm)^{\frac{j}{pk}}(pn + 1)^{\frac{k-j}{pk}}}{(pn + pm + 1)^{\frac{1}{p}}(pn + 1)^{\frac{1}{p}}} \\
&= \frac{(2pn)^{\frac{1}{p}}}{(pn + pm + 1)^{\frac{1}{p}}} \sum\nolimits_{j=1}^{k}\binom{k}{j}(pm)^{\frac{j}{pk}}(pn + 1)^{\frac{-j}{pk}} \\
&\xrightarrow[n\to\infty]{} 0.
\end{aligned}
$$

Hence there exists $N \in \mathbb{N}$ such that for any $n \geq N$ we have

$$(2pn)^{\frac{1}{p}} \frac{(pn+pm+1)^{\frac{1}{p}} - (pn+1)^{\frac{1}{p}}}{(pn+pm+1)^{\frac{1}{p}}(pn+1)^{\frac{1}{p}}} < \varepsilon.$$

□

Lemma 3.9 *Let $p > 0$, $K \in \mathbb{N}$ and $\varepsilon > 0$. For every polynomial f there exists $N_1 \in \mathbb{N}$ such that if Q is a K-summing polynomial of degree $N \geq N_1$, then*

$$\left|\mathcal{S}^p\left(\sqrt[p]{pN} f Q\right) - f\mathcal{S}^p\left(\sqrt[p]{pN} Q\right)\right| < \varepsilon \ \text{ in } \overline{\Omega}.$$

Proof Let $\sum_{i=1}^{L} v_i$ be the homogeneous expansion of f with $\deg(v_i) = m_i \in \mathbb{N}$, $i = 1, 2, \ldots, L$. By Cauchy estimates in the expansion of f, there exists a constant C such that $||v_i||_\Omega < C||f||_\Omega$, $i = 1, 2, \ldots, L$. Lemma 3.8 implies that there exists $N_1 \in \mathbb{N}$ such that if $N \geq N_1$, then for each m_i and all $n \in \left[\frac{1}{2}N, N\right]$ we have

$$\begin{aligned}
&(pN)^{\frac{1}{p}} \frac{(pn+pm_i+1)^{\frac{1}{p}} - (pn+1)^{\frac{1}{p}}}{(pn+pm_i+1)^{\frac{1}{p}}(pn+1)^{\frac{1}{p}}} \\
&\quad \leq (2pn)^{\frac{1}{p}} \frac{(pn+pm_i+1)^{\frac{1}{p}} - (pn+1)^{\frac{1}{p}}}{(pn+pm_i+1)^{\frac{1}{p}}(pn+1)^{\frac{1}{p}}} < \frac{\varepsilon}{CKL||f||_\Omega}.
\end{aligned} \tag{7}$$

Let $\sum_{j=1}^{K} u_j$ be the homogeneous expansion of Q with $\deg(u_j) = n_j \in \mathbb{N}$, $j = 1, 2, \ldots, K$. If Q is a K-summing polynomial of degree $N \geq N_1$, then $n_j \in \left[\frac{1}{2}N, N\right]$, $j = 1, 2, \ldots, K$. Hence by (7), in $\overline{\Omega}$ we obtain what follows

$$\begin{aligned}
\left|\mathcal{S}^p\left(\sqrt[p]{pN} f Q\right) - f\mathcal{S}^p\left(\sqrt[p]{pN} Q\right)\right| &= \left|\mathcal{S}^p\left(\sqrt[p]{pN} \sum_{i=1}^{L}\sum_{j=1}^{K} v_i u_j\right)\right. \\
&\quad \left. - \sum_{i=1}^{L} v_i \mathcal{S}^p\left(\sqrt[p]{pN} \sum_{j=1}^{K} u_j\right)\right| \\
&= \left|\sqrt[p]{pN} \sum_{i=1}^{L}\sum_{j=1}^{K} v_i u_j \left(\frac{1}{\sqrt[p]{p(m_i+n_j)+1}} - \frac{1}{\sqrt[p]{pn_j+1}}\right)\right|
\end{aligned}$$

$$\leq \sum_{i=1}^{L}\sum_{j=1}^{K}|v_i||u_j|\left|\frac{\sqrt[p]{pN}\left(\sqrt[p]{pn_j+1}-\sqrt[p]{p(m_i+n_j)+1}\right)}{\sqrt[p]{p(m_i+n_j)+1}\,\sqrt[p]{pn_j+1}}\right|$$
$$< \sum_{i=1}^{L}\sum_{j=1}^{K}|v_i||u_j|\frac{\varepsilon}{CKL||f||_{\Omega}} < \varepsilon.$$

This completes the proof. □

Remark 3.10 Let $p > 0$ and $K \in \mathbb{N}$. If Q is a K-summing polynomial, then for $z \in \partial\Omega$

$$\left|\mathcal{S}^p\left(\sqrt[p]{p\deg(Q)}Q\right)(z)\right| < K\sqrt[p]{2}.$$

Proof Let $\sum_{j=1}^{K} u_j$ be the homogeneous expansion of Q with $\deg(u_j) = n_j \in \mathbb{N}$, $j = 1, 2, \dots, K$. If Q is a K-summing polynomial, then $\frac{1}{2}\deg(Q) \leq n_j \leq \deg(Q)$, $j = 1, 2, \dots, K$. By properties of K-summing polynomials, we obtain the required estimate on $\partial\Omega$

$$\left|\mathcal{S}^p\left(\sqrt[p]{p\deg(Q)}Q\right)\right| = \left|\mathcal{S}^p\left(\sqrt[p]{p\deg(Q)}\sum_{j=1}^{K}u_j\right)\right| = \left|\sum_{j=1}^{K}\sqrt[p]{\frac{p\deg(Q)}{pn_j+1}}u_j\right|$$
$$\leq \sqrt[p]{\frac{p\deg(Q)}{\frac{1}{2}p\deg(Q)+1}}\sum_{j=1}^{K}|u_j| < K\sqrt[p]{2}.$$

□

4 Radon Inversion Problem

As we saw in the previous section, lacunary K-summing polynomials possess some interesting properties. Herein, we shall show how we can use them to solve by induction the Radon inversion problem with given a strictly positive, continuous function Φ. First of all, in Theorem 4.1 we construct generators which at each induction step allow us to come closer to Φ by a constant fraction of what is still missing. Moreover, the operator $\mathcal{R}^p$ is "almost additive" in the space of generators with respect to a given $\varepsilon > 0$. From this, one concludes that the sum of constructed generators converges in $\mathcal{HR}^p(\Omega)$ and solves the Radon inversion problem with Φ as we will see it in Theorem 4.2.

Theorem 4.1 *Let $p > 0$. There exist $\theta \in (0, 1)$ and $\kappa > 0$ such that if Ψ is a strictly positive continuous function on $\partial\Omega$ and ε, $N_0 \in \mathbb{N}$, $M_0 > 0$, then there exists a polynomial F of order greater than N_0 with the following properties*

$(c1)$ $\theta\Psi < \mathcal{R}^p(F) < \Psi$ *on* $\partial\Omega$
$(c2)$ $\mathcal{R}^p(\phi) + \mathcal{R}^p(F) - \varepsilon \leq \mathcal{R}^p(\phi + F) \leq \mathcal{R}^p(\phi) + \mathcal{R}^p(F) + \varepsilon$ *on* $\partial\Omega$ *for any* $\phi \in \mathcal{C}_{M_0}(\overline{\Omega})$

$(c3)$ $\left|\mathcal{S}^p(F)\right| < \kappa(\Psi)^{\frac{1}{p}}$ *on* $\partial\Omega$.

Proof Take $N = N(\partial\Omega)$, $K = K(\partial\Omega)$ and $N_K \in \mathbb{N}$ from Theorem 2.2 and Corollary 3.3, respectively. For such K we may choose $c_K \in \left(0, \frac{1}{2}\right)$, $C_K > 1$ and $l_K \in \mathbb{N}$ from Corollary 3.7. We shall show that $\theta := \frac{c_K}{8NC_K}$ and $\kappa := \frac{N(K+1)}{\sqrt[p]{C_K}}$ satisfy the statement of Theorem 4.1.

First of all, by Theorem 2.2, there exist polynomials $f_1, \ldots, f_N$ such that

$$\frac{\Psi}{2N} < \max_{1\leq m\leq N} |f_m|^p < \frac{\Psi}{N} \quad \text{on } \partial\Omega. \tag{8}$$

Let $f_0 \equiv 0$, $\widehat{\varepsilon} := \min\left\{\varepsilon, \theta \inf_{z\in\partial\Omega} \Psi(z)\right\}$ and $\alpha := \frac{1}{\sqrt[p]{C_K}} \inf_{z\in\partial\Omega} \left(\Psi(z)\right)^{\frac{1}{p}}$. Then by Lemma 3.5 and Lemma 3.9, there exists an integer $\overline{N} \geq \max\left\{2N_0, l_K, N_K\right\}$ such that for each $m = 0, 1, \ldots, N$ any K-summing polynomial Q of degree $n \geq \overline{N}$ satisfies the following inequalities for $z \in \partial\Omega$

$$\left|\mathcal{S}^p\left(\sqrt[p]{pN} f_m Q\right)(z) - f_m \mathcal{S}^p\left(\sqrt[p]{pN} Q\right)(z)\right| < \frac{\alpha}{N}, \tag{9}$$

$$|f_m(z)|^p \mathcal{R}^p(Q)(z) - \frac{\widehat{\varepsilon}}{N} \leq \mathcal{R}^p\left(f_m Q\right)(z) \leq |f_m(z)|^p \mathcal{R}^p(Q)(z) + \frac{\widehat{\varepsilon}}{N}, \tag{10}$$

$$c_K \leq \int_0^1 p \deg(Q)\left|Q(zt)\right|^p dt \leq C_K. \tag{11}$$

Inductively we shall choose integers $n_0, n_1, \ldots, n_N$ and construct lacunary K-summing polynomials $Q_0, Q_1, \ldots, Q_N$ of orders at least $n_0, n_1, \ldots, n_N$, respectively, such that for $m = 0, 1, \ldots, N$ and for any $\phi \in \mathcal{C}_{M_0}(\overline{\Omega})$ we have on $\partial\Omega$

$$\begin{aligned} &\mathcal{R}^p(\phi) + \mathcal{R}^p\left(\sum_{j=0}^m f_j Q_j\right) - \frac{m\widehat{\varepsilon}}{N} \\ &\quad \leq \mathcal{R}^p\left(\phi + \sum_{j=0}^m f_j Q_j\right) \leq \mathcal{R}^p(\phi) + \mathcal{R}^p\left(\sum_{j=0}^m f_j Q_j\right) + \frac{m\widehat{\varepsilon}}{N}. \end{aligned} \tag{12}$$

Let $n_0 := 0$ and $Q_0 \equiv 1$ in $\overline{\Omega}$. For $m = 1$, we begin by selecting an integer $n_1 \geq \overline{N}$ from Lemma 3.4 applied with $\varepsilon := \frac{\widehat{\varepsilon}}{N}$, $M := M_0$, $f := f_1$. Then by Corollary 3.3, there exists a lacunary K-summing polynomial Q_1 of order greater than n_1 such that

$$\mathcal{R}^p(\phi) + \mathcal{R}^p\left(f_1 Q_1\right) - \frac{\widehat{\varepsilon}}{N} \leq \mathcal{R}^p\left(\phi + f_1 Q_1\right) \leq \mathcal{R}^p(\phi) + \mathcal{R}^p\left(f_1 Q_1\right) + \frac{\widehat{\varepsilon}}{N} \quad \text{on } \partial\Omega.$$

Proceeding inductively, assume that we have already chosen $n_0, n_1, \ldots, n_m$ and constructed lacunary K-summing polynomials $Q_0, Q_1, \ldots, Q_m$ of orders greater than $n_1, n_2, \ldots, n_m$, respectively, with desired properties, $0 < m < N$. Then again apply

Lemma 3.4 with

$$\varepsilon := \frac{\widehat{\varepsilon}}{2N}, \quad M := \sup_{z\in\partial\Omega}\Big(|\phi(z)| + \Big|\sum_{j=0}^{m} f_j(z)Q_j(z)\Big|\Big), \quad f := f_{m+1}$$

to obtain an integer $n_{m+1} \geq \overline{N}$. Then Corollary 3.3 produces a lacunary K-summing polynomial Q_{m+1} of order greater than n_{m+1} such that on $\partial\Omega$ the following inequalities are satisfied

$$\begin{aligned}
\mathcal{R}^p\Big(\phi + \sum_{j=0}^{m+1} f_jQ_j\Big) &\geq \mathcal{R}^p\Big(\phi + \sum_{j=0}^{m} f_jQ_j\Big) + \mathcal{R}^p\big(f_{m+1}Q_{m+1}\big) - \frac{\widehat{\varepsilon}}{2N}\\
&\geq \mathcal{R}^p(\phi) + \mathcal{R}^p\Big(\sum_{j=0}^{m} f_jQ_j\Big) + \mathcal{R}^p\big(f_{m+1}Q_{m+1}\big) - \frac{m\widehat{\varepsilon}}{N} - \frac{\widehat{\varepsilon}}{2N}\\
&\geq \mathcal{R}^p(\phi) + \mathcal{R}^p\Big(\sum_{j=0}^{m+1} f_jQ_j\Big) - \frac{(m+1)\widehat{\varepsilon}}{N},\\
\mathcal{R}^p\Big(\phi + \sum_{j=0}^{m+1} f_jQ_j\Big) &\leq \mathcal{R}^p\Big(\phi + \sum_{j=0}^{m} f_jQ_j\Big) + \mathcal{R}^p\big(f_{m+1}Q_{m+1}\big) - \frac{\widehat{\varepsilon}}{2N}\\
&\leq \mathcal{R}^p(\phi) + \mathcal{R}^p\Big(\sum_{j=0}^{m} f_jQ_j\Big) + \mathcal{R}^p\big(f_{m+1}Q_{m+1}\big) + \frac{m\widehat{\varepsilon}}{N} + \frac{\widehat{\varepsilon}}{2N}\\
&\leq \mathcal{R}^p(\phi) + \mathcal{R}^p\Big(\sum_{j=0}^{m+1} f_jQ_j\Big) + \frac{(m+1)\widehat{\varepsilon}}{N}.
\end{aligned}$$

Define the polynomial $F(z) = \sum_{m=1}^{N} \sqrt[p]{\frac{p\deg(Q_m)}{KC_K}}\, f_m(z)Q_m(z)$. Observe that by (12), the polynomial F satisfies condition $(c2)$. For each $z \in \partial\Omega$ one may choose an index $m_z \in \{1,\ldots,N\}$ for which $|f_{m_z}(z)|^p = \max\limits_{1\leq m\leq N} |f_m(z)|^p > \frac{\Psi(z)}{2N}$. Then for $z \in \partial\Omega$ we obtain

$$\begin{aligned}
\mathcal{R}^p(F)(z) &= \int_0^1 \Big|\sum_{m=1}^{N} \sqrt[p]{\frac{p\deg(Q_m)}{2C_K}}\, f_m(zt)Q_m(zt)\Big|^p dt\\
&\overset{(12)}{\geq} \sum_{m=1}^{N} |f_m(z)|^p \frac{p\deg(Q_m)}{2C_K}\int_0^1 \big|Q_m(zt)\big|^p dt - \frac{c_K}{8NC_K}\inf_{z\in\partial\Omega}\Psi(z)\\
&\geq |f_{m_z}(z)|^p \frac{p\deg(Q_{m_z})}{2C_K}\int_0^1 \big|Q_{m_z}(zt)\big|^p dt - \frac{c_K}{8NC_K}\inf_{z\in\partial\Omega}\Psi(z)
\end{aligned}$$

$$\begin{aligned}
&\overset{(11)}{\geq} \frac{c_K}{2C_K}|f_{m_z}(z)|^p - \frac{c_K}{8NC_K}\inf_{z\in\partial\Omega}\Psi(z) \overset{(8)}{>} \frac{c_K}{4NC_K}\Psi(z) - \frac{c_K}{8NC_K}\Psi(z)\\
&= \frac{c_K}{8NC_K}\Psi(z) = \theta\Psi(z).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\mathcal{R}^p(F)(z) = &\int_0^1 \Big|\sum_{m=1}^N \sqrt[p]{\frac{p\deg(Q_m)}{2C_K}} f_m(zt)Q_m(zt)\Big|^p dt\\
&\overset{(12)}{\leq} \sum_{m=1}^N |f_m(z)|^p \frac{p\deg(Q_m)}{2C_K}\int_0^1 \big|Q_m(zt)\big|^p dt + \frac{c_K}{8NC_K}\inf_{z\in\partial\Omega}\Psi(z)\\
&\overset{(8)}{<} \frac{\Psi(z)}{2C_K}C_K + \frac{1}{2}\Psi(z) = \Psi(z).
\end{aligned}$$

Since any Q_m satisfies (9), for $z \in \partial\Omega$ we have the following estimate

$$\begin{aligned}
\big|\mathcal{S}^p(F)(z)\big| &= \Big|\mathcal{S}^p\Big(\sum_{m=1}^N \sqrt[p]{\frac{p\deg(Q_m)}{2C_K}} f_m Q_m\Big)(z)\Big|\\
&\leq \frac{1}{\sqrt[p]{2C_K}}\sum_{m=1}^N \Big|\mathcal{S}^p\Big(\sqrt[p]{p\deg(Q_m)} f_m Q_m\Big)(z)\Big|\\
&\overset{(9)}{<} \frac{1}{\sqrt[p]{2C_K}}\sum_{m=1}^N |f_m(z)|\Big|\mathcal{S}^p\Big(\sqrt[p]{p\deg(Q_m)} Q_m\Big)(z)\Big| + \frac{N}{\sqrt[p]{C_K}}\inf_{z\in\partial\Omega}\big(\Psi(z)\big)^{\frac{1}{p}}\\
&\leq \frac{NK\sqrt[p]{2}}{\sqrt[p]{2C_K}}|f_{m_z}(z)| + \frac{N}{\sqrt[p]{C_K}}\big(\Psi(z)\big)^{\frac{1}{p}} < \kappa\big(\Psi(z)\big)^{\frac{1}{p}}.
\end{aligned}$$

This completes the proof. □

Once we proved Theorem 4.1, we are ready to give the first construction of a solution G to the Radon inversion problem such that $\mathcal{S}^p(G)$ is continuous up to the boundary of the domain.

Theorem 4.2 *Let $p > 0$ and Φ be a strictly positive, continuous function on $\partial\Omega$. There exists a function $G \in \mathcal{O}(\Omega)$ such that*

$$\mathcal{R}^p(G)(z) = \Phi(z) \ \textit{for} \ z \in \partial\Omega.$$

Moreover, $\mathcal{S}^p(G) \in \mathcal{C}(\overline{\Omega})$.

Proof Take $\theta \in (0, 1)$ and $\kappa > 0$ from Theorem 4.1. We will show that it is possible to construct inductively a sequence of polynomials $\{F_j\}_{j=0}^{\infty}$ such that

$$\left(1-\frac{\theta}{2}\right)\left(\Phi(z)-\mathcal{R}^p\left(\sum_{i=0}^{j-1}F_i\right)(z)\right) > \Phi(z)-\mathcal{R}^p\left(\sum_{i=0}^{j}F_i\right)(z) > 0 \quad (13)$$

and

$$\left|\mathcal{S}^p(F_j)(z)\right| < \kappa\left(1-\frac{\theta}{2}\right)^{\frac{j}{p}}||\Phi||_{\partial\Omega}^{\frac{1}{p}}, \quad z \in \partial\Omega, \quad j = 1, 2, \ldots \quad (14)$$

Set $F_0 \equiv 0$ and observe that (14) holds. For $j = 1$ we construct a polynomial F_1 by applying Theorem 4.1 with $\Psi := \Phi$. Notice that F_1 satisfies (13) and (14). Then assume that we have constructed polynomials $F_0, F_1, \ldots, F_j$ for $j \geq 1$. Again from Theorem 4.1 applied to

$$\Psi := \frac{3}{4}\left(\Phi-\mathcal{R}^p\left(\sum_{i=0}^{j}F_i\right)\right), \quad \varepsilon := \frac{\theta}{4}\inf_{z\in\partial\Omega}\left(\Phi(z)-\mathcal{R}^p\left(\sum_{i=0}^{j}F_i\right)(z)\right),$$

$$M_0 := \sup_{z\in\partial\Omega}\left|\sum_{i=0}^{j}F_i(z)\right|,$$

there exists a polynomial F_{j+1} such that on $\partial\Omega$

1. $\frac{3}{4}\theta\left(\Phi-\mathcal{R}^p\left(\sum_{i=0}^{j}F_i\right)\right) < \mathcal{R}^p(F_{j+1}) < \frac{3}{4}\left(\Phi-\mathcal{R}^p\left(\sum_{i=0}^{j}F_i\right)\right)$
2. $\mathcal{R}^p\left(\sum_{i=0}^{j}F_i\right)+\mathcal{R}^p(F_{j+1})-\varepsilon < \mathcal{R}^p\left(\sum_{i=0}^{j+1}F_i\right) < \mathcal{R}^p\left(\sum_{i=0}^{j}F_i\right)+\mathcal{R}^p(F_{j+1})+\varepsilon$
3. $\left|\mathcal{S}^p(F_{j+1})\right| < \kappa\left(\frac{3}{4}\left(\Phi-\mathcal{R}^p\left(\sum_{i=0}^{j}F_i\right)\right)\right)^{\frac{1}{p}}$.

Hence on $\partial\Omega$ we have the following estimates

$$\begin{aligned}\mathcal{R}^p\left(\sum_{i=0}^{j+1}F_i\right) &> \mathcal{R}^p\left(\sum_{i=0}^{j}F_i\right)+\mathcal{R}^p(F_{j+1})-\varepsilon \\ &> \mathcal{R}^p\left(\sum_{i=0}^{j}F_i\right)+\frac{3}{4}\theta\left(\Phi-\mathcal{R}^p\left(\sum_{i=0}^{j}F_i\right)\right)-\varepsilon\end{aligned}$$

$$
\begin{aligned}
&> \frac{3}{4}\theta\Phi + (1-\frac{3}{4}\theta)\mathcal{R}^p\Big(\sum_{i=0}^{j} F_i\Big) - \frac{\theta}{4}\inf_{z\in\partial\Omega}\Big(\Phi(z) - \mathcal{R}^p\Big(\sum_{i=0}^{j} F_i\Big)(z)\Big)\\
&> \frac{\theta}{2}\Phi + \Big(1-\frac{\theta}{2}\Big)\mathcal{R}^p\Big(\sum_{i=0}^{j} F_i\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{R}^p\Big(\sum_{i=0}^{j+1} F_i\Big) &< \mathcal{R}^p\Big(\sum_{i=0}^{j} F_i\Big) + \mathcal{R}^p(F_{j+1}) + \varepsilon\\
&< \mathcal{R}^p\Big(\sum_{i=0}^{j} F_i\Big) + \frac{3}{4}\Big(\Phi - \mathcal{R}^p\Big(\sum_{i=0}^{j} F_i\Big)\Big)\\
&\quad + \frac{\theta}{4}\inf_{z\in\partial\Omega}\Big(\Phi(z) - \mathcal{R}^p\Big(\sum_{i=0}^{j} F_i\Big)(z)\Big)\\
&\le \mathcal{R}^p\Big(\sum_{i=0}^{j} F_i\Big) + \frac{3}{4}\Big(\Phi - \mathcal{R}^p\Big(\sum_{i=0}^{j} F_i\Big)\Big) + \frac{1}{4}\Big(\Phi - \mathcal{R}^p\Big(\sum_{i=0}^{j} F_i\Big)\Big)\\
&\le \Phi.
\end{aligned}
$$

Therefore $\Big(1-\frac{\theta}{2}\Big)\Big(\Phi - \mathcal{R}^p\Big(\sum_{i=0}^{j} F_i\Big)\Big) > \Phi - \mathcal{R}^p\Big(\sum_{i=0}^{j+1} F_i\Big) > 0$, which implies that

$$
\Big(1-\frac{\theta}{2}\Big)^{j+1}\Phi > \Phi - \mathcal{R}^p\Big(\sum_{i=0}^{j+1} F_i\Big) > 0 \text{ on } \partial\Omega.
$$

From this it follows that

$$
\lim_{n\to\infty}\mathcal{R}^p\Big(\sum\nolimits_{j=0}^{n} F_j\Big)(z) = \Phi(z), \quad z\in\partial\Omega.
$$

In addition, $\big|\mathcal{S}^p(F_{j+1})\big| < \kappa\big(\Phi - \mathcal{R}^p\big(\sum_{i=0}^{j} F_i\big)\big)^{\frac{1}{p}} < \kappa\big(1-\frac{\theta}{2}\big)^{\frac{j+1}{p}}||\Phi||_{\partial\Omega}^{\frac{1}{p}}$ on $\partial\Omega$.

Let $q = \max\{p, 1\}$ and observe that if $n \geq m$, then

$$\begin{aligned}
&\sup_{z\in\partial\Omega} \left(\mathcal{R}^p\Big(\sum_{j=0}^{m} F_j - \sum_{j=0}^{n} F_j\Big)(z)\right)^{\frac{1}{q}} \\
&= \sup_{z\in\partial\Omega} \left(\mathcal{R}^p\Big(\sum_{j=n+1}^{m} F_j\Big)(z)\right)^{\frac{1}{q}} \leq \sup_{z\in\partial\Omega} \left(\sum_{j=n+1}^{m} \mathcal{R}^p(F_j)(z)\right)^{\frac{1}{q}} \\
&\leq \sup_{z\in\partial\Omega} \left(\sum_{j=n+1}^{m} \left(\Phi(z) - \mathcal{R}^p\Big(\sum_{i=0}^{j-1} F_i\Big)(z)\right)^{\frac{1}{q}}\right) \\
&< \sum_{j=n+1}^{m} \left(1 - \frac{\theta}{2}\right)^{\frac{j}{q}} ||\Phi||_{\partial\Omega}^{\frac{1}{q}} \underset{n,m\to\infty}{\longrightarrow} 0,
\end{aligned}$$

so $\Big\{\sum_{j=0}^{n} F_j\Big\}$ is a Cauchy sequence in $\mathcal{HR}^p(\Omega)$ and by Proposition 2.1, it is convergent in $\mathcal{HR}^p(\Omega)$. Hence we may define the function $G := \sum_{j=0}^{\infty} F_j$ and again by Proposition 2.1, G is holomorphic in Ω. Moreover, $\mathcal{R}^p\big(\sum_{j=0}^{n} F_j\big) \rightrightarrows \mathcal{R}^p(G)$ as $n \to \infty$ on $\partial\Omega$, so $\mathcal{R}^p(G) = \lim_{n\to\infty} \mathcal{R}^p\big(\sum_{j=0}^{n} F_j\big) = \Phi$.

Since for $z \in \overline{\Omega}$ we have the following estimate

$$\sum_{j=0}^{\infty} \left|\mathcal{S}^p(F_j)(z)\right| \leq \sum_{j=0}^{\infty} \sup_{z\in\partial\Omega} \left|\mathcal{S}^p(F_j)(z)\right| < \sum_{j=0}^{\infty} \kappa\left(1 - \frac{\theta}{2}\right)^{\frac{j}{p}} ||\Phi||_{\partial\Omega}^{\frac{1}{p}} < \infty,$$

then $\mathcal{S}^p(G)$ is a continuous function in $\overline{\Omega}$. This completes the proof. □

5 Divergent Taylor Series

In what follows we will show that it is possible to construct a solution f to the Radon inversion problem such that the function $\mathcal{S}^p(f)$ in contrast to the solution obtained in Theorem 4.2 has in some sense extremal properties, i.e. every slice function of $\mathcal{S}^p(f)$ has a divergent series of Taylor coefficients with every exponent $s < \min\{1, p\}$. Moreover, for $p \leq 1$ function $\mathcal{S}^p(f)$ is continuous up to the boundary and for $p \in (1, 2]$ is square-integrable on all unit circles $z\partial\mathbb{D}$, $z \in \partial\Omega$. To prove this we need the following technical lemma.

Lemma 5.1 *There exist $\theta \in (0, 1)$ and $C > 0$ such that if $h \in \mathcal{C}(\partial\Omega)$ is a strictly positive function on $\partial\Omega$, g is a bounded, continuous function in $\overline{\Omega}$, $M > 0$, $a, \gamma, \tau \in (0, 1)$, then there exist $N \in \mathbb{N}$ and a sequence of orthogonal polynomials $\{P_k\}_{k=0}^{\infty}$ such that*

$(w1)$ $\operatorname{ord}(P_0) \geq M$

(w2) $\mathcal{R}^p(P_k) < ah$ *on* $\partial\Omega$

(w3) $\mathrm{ord}(P_k) > \deg(P_{k-1})$

(w4) $\mathcal{R}^p\big(\sum_{j=0}^{k} P_j\big) < \mathcal{R}^p\big(\sum_{j=0}^{k-1} P_j\big) + a\big(h - \mathcal{R}^p\big(\sum_{j=0}^{k-1} P_j\big)\big) < h$ *on* $\partial\Omega$

(w5) $\mathcal{R}^p\big(\sum_{j=0}^{k} P_j\big) > \mathcal{R}^p\big(\sum_{j=0}^{k-1} P_j\big) + \frac{1}{2}\theta a\big(h - \mathcal{R}^p\big(\sum_{j=0}^{k-1} P_j\big)\big)$ *on* $\partial\Omega$

(w6) $\Big|\mathcal{R}^p\big(\sum_{j=0}^{k} P_j\big) - \sum_{j=0}^{k} \mathcal{R}^p(P_j)\Big| < \tau$ *on* $\partial\Omega$

(w7) $\Big|\mathcal{R}^p\big(g + \sum_{j=0}^{k} P_j\big) - \sum_{j=0}^{k} \mathcal{R}^p(P_j) - \mathcal{R}^p(g)(z)\Big| < \tau$ *on* $\partial\Omega$

(w8) $\big|\mathcal{S}^p(P_k)\big| < C\big(\mathcal{R}^p(P_k)\big)^{\frac{1}{p}}$ *on* $\partial\Omega$

(w9) $\mathcal{R}^p\big(\sum_{j=0}^{N} P_j\big) > \gamma h$ *on* $\partial\Omega$

(w10) *if* $p \in (0, 1]$, *then* $\Big|\mathcal{S}^p\big(\sum_{j=0}^{N} P_j\big)\Big| < \frac{2C}{\theta} h^{\frac{1}{p}} a^{\frac{1}{p}-1}$ *on* $\partial\Omega$.

Proof Take constants $\theta \in (0, 1)$ and $\kappa > 0$ from Theorem 4.1. Let $C := \frac{\kappa}{\sqrt[p]{\theta}}$. We proceed inductively. For $k = 0$ apply Theorem 4.1 with $\Psi := \frac{1}{2}ah$, $M_0 := ||g||_\Omega$, $N_0 := M$, $\varepsilon := \tau$ to obtain a polynomial P_0 such that $\mathrm{ord}(P_0) \geq M$ and on $\partial\Omega$ the following statements hold

a) $\frac{1}{2}\theta ah < \mathcal{R}^p(P_0) < \frac{1}{2}ah$

b) $\mathcal{R}^p(P_0) + \mathcal{R}^p(g) - \tau < \mathcal{R}^p\big(g + P_0\big) < \mathcal{R}^p(P_0) + \mathcal{R}^p(g) - \tau$

c) $\big|\mathcal{S}^p(P_0)\big| < \kappa\big(\frac{1}{2}ah\big)^{\frac{1}{p}} < C\big(\mathcal{R}^p(P_0)\big)^{\frac{1}{p}}$.

Hence P_0 satisfies conditions $(w1) - (w2)$ and $(w6) - (w8)$. Assuming that $b)$ holds, there exists $\tau_0 \in (0, 1)$ such that $\big|\mathcal{R}^p\big(g + P_0\big) - \mathcal{R}^p(P_0) - \mathcal{R}^p(g)\big| < \tau_0 < \tau$ on $\partial\Omega$. For $k = 1$ we apply Theorem 4.1 to

$$\Psi := \frac{3}{4}a\big(h - \mathcal{R}^p(P_0)\big), \quad M_0 := ||P_0||_\Omega + ||g||_\Omega, \quad N_0 := \deg(P_0) + 1,$$
$$\varepsilon := \min\left\{\frac{\theta}{4} \inf_{z\in\partial\Omega} a\Big(h(z) - \mathcal{R}^p(P_0)(z)\Big), \tau - \tau_0\right\}.$$

This produces a polynomial P_1 such that $\mathrm{ord}(P_1) > \deg(P_0)$ and on $\partial\Omega$ the following conditions are satisfied

- $\frac{3}{4}\theta a\Big(h - \mathcal{R}^p(P_0)\Big) < \mathcal{R}^p(P_1) < \frac{3}{4}a\Big(h - \mathcal{R}^p(P_0)\Big) < ah$
- $\mathcal{R}^p(P_0) + \mathcal{R}^p(P_1) - \varepsilon < \mathcal{R}^p\big(P_0 + P_1\big) < \mathcal{R}^p(P_0) + \mathcal{R}^p(P_1) + \varepsilon$
- $\mathcal{R}^p(g + P_0) + \mathcal{R}^p(P_1) - \varepsilon < \mathcal{R}^p\big(g + P_0 + P_1\big) < \mathcal{R}^p(g + P_0) + \mathcal{R}^p(P_1) + \varepsilon$

- $\left|\mathcal{S}^p(P_1)\right| < \kappa\Big(\frac{3}{4}a\big(h-\mathcal{R}^p(P_0)\big)\Big)^{\frac{1}{p}} < C\big(\mathcal{R}^p(P_1)\big)^{\frac{1}{p}}.$

Then simple calculations give the following estimates on $\partial\Omega$

$$\mathcal{R}^p\big(P_0+P_1\big) < \mathcal{R}^p(P_0)+\mathcal{R}^p(P_1)+\varepsilon < \mathcal{R}^p(P_0)+a\Big(h-\mathcal{R}^p(P_0)\Big)$$

and

$$\mathcal{R}^p\big(P_0+P_1\big) > \mathcal{R}^p(P_0)+\mathcal{R}^p(P_1)-\varepsilon > \mathcal{R}^p(P_0)+\frac{1}{2}\theta a\Big(h-\mathcal{R}^p(P_0)\Big).$$

Also since $\varepsilon < \tau$, it is clear that $\mathcal{R}^p(P_0)+\mathcal{R}^p(P_1)-\tau < \mathcal{R}^p\big(P_0+P_1\big) < \mathcal{R}^p(P_0)+\mathcal{R}^p(P_1)+\tau$ on $\partial\Omega$. Moreover,

$$\begin{aligned}\mathcal{R}^p\big(P_0+P_1+g\big) &< \mathcal{R}^p\big(P_0+g\big)+\mathcal{R}^p(P_1)+\varepsilon\\ &< \mathcal{R}^p(g)+\mathcal{R}^p(P_0)+\mathcal{R}^p(P_1)+\tau_0+\varepsilon\\ &= \mathcal{R}^p(g)+\mathcal{R}^p(P_0)+\mathcal{R}^p(P_1)+\tau\end{aligned}$$

and similarly $\mathcal{R}^p\big(P_0+P_1+g\big) > \mathcal{R}^p(g)+\mathcal{R}^p(P_0)+\mathcal{R}^p(P_1)-\tau$ on $\partial\Omega$. It follows from the above that P_1 is orthogonal to P_0 and has properties $(w1)-(w8)$.

We then proceed inductively as follows: Suppose that orthogonal polynomials $P_0, P_1, \ldots, P_k$ have been found such that conditions $(w1)-(w8)$ hold. Observe that by statements $(w6)$ and $(w7)$ there exist $\tau_1, \tau_2 \in (0,\tau)$ such that

- $\left|\mathcal{R}^p\Big(\sum_{j=0}^{k}P_j\Big)-\sum_{j=0}^{k}\mathcal{R}^p(P_j)\right| < \tau_1 < \tau$ on $\partial\Omega$
- $\left|\mathcal{R}^p\Big(g+\sum_{j=0}^{k}P_j\Big)-\sum_{j=0}^{k}\mathcal{R}^p(P_j)-\mathcal{R}^p(g)\right| < \tau_2 < \tau$ on $\partial\Omega$.

Theorem 4.1 applied to

$$\Psi := \frac{3}{4}a\Big(h-\mathcal{R}^p\Big(\sum_{j=0}^{k}P_j\Big)\Big),\quad M_0 := \Big|\Big|\sum_{j=0}^{k}P_j\Big|\Big|_{\Omega}+||g||_{\Omega},\quad N_0 := \deg(P_k)+1,$$

$$\varepsilon := \min\left\{\frac{\theta}{4}\inf_{z\in\partial\Omega} a\Big(h(z)-\mathcal{R}^p\Big(\sum_{j=0}^{k}P_j\Big)(z)\Big), \tau-\tau_1, \tau-\tau_2\right\}$$

gives a polynomial P_{k+1} such that $\operatorname{ord}(P_{k+1}) > \deg(P_k)$ and on $\partial\Omega$ the following inequalities hold

- $\frac{3}{4}\theta a\big(h-\mathcal{R}^p\big(\sum_{j=0}^{k}P_j\big)\big) < \mathcal{R}^p(P_{k+1}) < \frac{3}{4}a\big(h-\mathcal{R}^p\big(\sum_{j=0}^{k}P_j\big)\big) < ah$
- $\left|\mathcal{R}^p\big(\sum_{j=0}^{k+1}P_j\big)-\mathcal{R}^p\big(\sum_{j=0}^{k+1}P_j\big)-\mathcal{R}^p(P_{k+1})\right| < \varepsilon$

- $\left|\mathcal{R}^p\big(g+\sum_{j=0}^{k+1} P_j\big)-\mathcal{R}^p\big(g+\sum_{j=0}^{k+1} P_j\big)-\mathcal{R}^p(P_{k+1})\right| \ < \ \varepsilon$
- $\left|\mathcal{S}^p(P_{k+1})\right| \ < \ \kappa\big(\frac{3}{4}a\big(h-\mathcal{R}^p\big(\sum_{j=0}^{k} P_j\big)\big)\big)^{\frac{1}{p}} \ < \ C\big(\mathcal{R}^p(P_{k+1})\big)^{\frac{1}{p}}.$

Above inequalities imply that on $\partial\Omega$

$$\begin{aligned}\mathcal{R}^p\big(\sum_{j=0}^{k+1} P_j\big) &< \ \mathcal{R}^p\big(\sum_{j=0}^{k} P_j\big)+\mathcal{R}^p(P_{k+1})\\ &+\varepsilon < \ \mathcal{R}^p\big(\sum_{j=0}^{k} P_j\big)+a\big(h-\mathcal{R}^p\big(\sum_{j=0}^{k} P_j\big)\big)\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}^p\big(\sum_{j=0}^{k+1} P_j\big) &> \ \mathcal{R}^p\big(\sum_{j=0}^{k} P_j\big)+\mathcal{R}^p(P_{k+1})-\varepsilon\\ [5mm] &> \ \mathcal{R}^p\big(\sum_{j=0}^{k} P_j\big)+\frac{3}{4}\theta a\big(h-\mathcal{R}^p\big(\sum_{j=0}^{k} P_j\big)\big)\\ &-\frac{\theta}{4}a\big(h-\mathcal{R}^p\big(\sum_{j=0}^{k} P_j\big)\big)\\ [5mm] &= \ \mathcal{R}^p\big(\sum_{j=0}^{k} P_j\big)+\frac{1}{2}\theta a\big(h-\mathcal{R}^p\big(\sum_{j=0}^{k} P_j\big)\big).\end{aligned}$$

By induction,

$$\begin{aligned}\mathcal{R}^p\big(\sum_{j=0}^{k+1} P_j\big) &< \mathcal{R}^p\big(\sum_{j=0}^{k} P_j\big)+\mathcal{R}^p(P_{k+1})\\ &+\varepsilon < \sum_{j=0}^{k+1}\mathcal{R}^p(P_j)+\tau_1+\varepsilon=\sum_{j=0}^{k+1}\mathcal{R}^p(P_j)+\tau \ \text{ on } \partial\Omega\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}^p\big(\sum_{j=0}^{k+1} P_j+g\big) &< \ \mathcal{R}^p\big(\sum_{j=0}^{k} P_j+g\big)+\mathcal{R}^p(P_{k+1})\\ &+\varepsilon < \ \mathcal{R}^p(g)+\sum_{j=0}^{k+1}\mathcal{R}^p(P_j)+\tau_2+\varepsilon\\ &< \ \mathcal{R}^p(g)+\mathcal{R}^p\big(\sum_{j=0}^{k+1} P_j\big)+\tau \ \text{ on } \partial\Omega.\end{aligned}$$

Similarly we get $\mathcal{R}^p\Big(\sum_{j=0}^{k+1} P_j\Big) \ > \ \sum_{j=0}^{k+1}\mathcal{R}^p(P_j)-\tau$ and $\mathcal{R}^p\Big(\sum_{j=0}^{k+1} P_j+g\Big) > \ \mathcal{R}^p(g)+\mathcal{R}^p\Big(\sum_{j=0}^{k+1} P_j\Big)-\tau$ on $\partial\Omega$. All the above implies that the polynomial P_{k+1} is orthogonal to $P_0, P_1, \ldots, P_k$ and it satisfies conditions $(w1)-(w8)$.

Combining properties $(w4)$ and $(w5)$ with Dini's theorem one may conclude that $\mathcal{R}^p\Big(\sum_{j=0}^{n} P_j\Big)$ converges uniformly to h. Hence there exists $N \in \mathbb{N}$ such that

$\mathcal{R}^p\Big(\sum_{j=0}^N P_j\Big) > \gamma h$ on $\partial\Omega$. Moreover, statement $(w5)$ implies that

$$h - \mathcal{R}^p\Big(\sum_{j=0}^{k} P_j\Big) \;<\; \Big(1-\frac{1}{2}\theta a\Big)\Big(h - \mathcal{R}^p\Big(\sum_{j=0}^{k-1} P_j\Big)\Big) \;<\; \Big(1-\frac{1}{2}\theta a\Big)^k h \;\text{ on } \partial\Omega.$$

Therefore, if $p \in (0, 1]$, we have the following estimate on $\partial\Omega$

$$\begin{aligned}
\Big|\mathcal{S}^p\Big(\sum_{j=0}^N P_j\Big)\Big| &\leq \sum_{j=0}^N \big|\mathcal{S}^p(P_j)\big| \overset{(w9)}{<} C\sum_{j=0}^N \big(\mathcal{R}^p(P_j)\big)^{\frac{1}{p}}\\
&< C\sum_{j=0}^N \Big(\frac{3}{4}a\Big(h - \mathcal{R}^p\Big(\sum_{j=0}^k P_j\Big)\Big)\Big)^{\frac{1}{p}}\\
&< C\sum_{j=0}^N \Big(\frac{3}{4}a\Big(1-\frac{1}{2}\theta a\Big)^k h\Big)^{\frac{1}{p}}\\
&< C\Big(\frac{3}{4}ah\Big)^{\frac{1}{p}} \frac{1}{1-\big(1-\frac{1}{2}\theta a\big)^{\frac{1}{p}}} < C\Big(\frac{3}{4}ah\Big)^{\frac{1}{p}}\frac{2}{\theta a}\\
&< \frac{2C}{\theta} h^{\frac{1}{p}} a^{\frac{1}{p}-1}
\end{aligned}$$

and this finishes the proof. □

Theorem 5.2 *Let Φ be a strictly positive continuous function on $\partial\Omega$. There exists a holomorphic function $f = \sum_{n=0}^{\infty} p_n$, where p_n are homogeneous polynomials, such that*

1. $\mathcal{R}^p(f) = \Phi$ *on* $\partial\Omega$
2. *every slice function of $\mathcal{S}^p(f)$ has a divergent series of Taylor coefficients with every exponent $s < \min\{1, p\}$, i.e.* $\sum_{n=0}^{\infty}\Big(\frac{|p_n(z)|}{\sqrt[p]{pn+1}}\Big)^s = \infty,\;\; s < \min\{1,p\},\; z\in\partial\Omega$
3. *if $p \leq 1$, then* $\mathcal{S}^p(f) \in \mathcal{C}(\overline{\Omega})$
4. *if $p \in (1, 2]$, then* $\mathcal{S}^p(f) \in L^2(z\partial\mathbb{D}),\; z \in \partial\Omega.$

Proof Without loss of generality assume that $\sup\limits_{z\in\partial\Omega} |\Phi(z)| < 1$. Take constants $\theta \in (0, 1)$ and $C > 0$ from Lemma 5.1. Inductively we will construct a sequence of orthogonal polynomials $\big\{Q_k\big\}_{k=0}^{\infty}$ with the following properties

$(p1)$ $Q_k = \sum_{j=0}^{N_k} P_{k,j}$, where $P_{k,0}, P_{k,1}, \ldots, P_{k,N_k}$ are orthogonal polynomials

$(p2)$ $\mathcal{R}^p\Big(\sum_{m=0}^k Q_m\Big) < \Phi$ on $\partial\Omega$

$(p3)$ $\mathcal{R}^p\Big(\sum_{m=0}^k Q_m\Big) > \frac{1}{4}\Big(\Phi - \mathcal{R}^p\Big(\sum_{m=0}^{k-1} Q_m\Big)\Big) + \mathcal{R}^p\Big(\sum_{m=0}^{k-1} Q_m\Big)$ on $\partial\Omega$

$(p4)$ $\mathcal{R}^p(P_{k,j}) < \Big(\Phi - \mathcal{R}^p\Big(\sum_{m=0}^{k-1} Q_m\Big)\Big)^k$ on $\partial\Omega$, $\;j = 0, 1, \ldots, N_k$

$(p5)$ $\left|\mathcal{S}^p(P_{k,j})\right| < C\big(\mathcal{R}^p(P_{k,j})\big)^{\frac{1}{p}}$ on $\partial\Omega$, $j = 0, 1, \ldots, N_k$

$(p6)$ $\mathcal{R}^p(Q_k) - \left|\left|\Phi - \mathcal{R}^p\Big(\sum_{m=0}^{k-1} Q_m\Big)\right|\right|_{\partial\Omega}^k < \sum_{j=0}^{N_k} \mathcal{R}^p(P_{k,j}) < \mathcal{R}^p(Q_k) + \left|\left|\Phi - \mathcal{R}^p\Big(\sum_{m=0}^{k-1} Q_m\Big)\right|\right|_{\partial\Omega}^k$ on $\partial\Omega$

$(p7)$ $\left|\left|\mathcal{R}^p(Q_k)\right|\right|_{1,z\partial\mathbb{D}} > \frac{1}{2}\left|\left|\Phi - \mathcal{R}^p\Big(\sum_{m=0}^{k-1} Q_m\Big)\right|\right|_{1,z\partial\mathbb{D}}$

$(p8)$ if $p \leq 1$, then $\left|\mathcal{S}^p(Q_k)\right| < \frac{2C}{\theta}\left(\Phi - \mathcal{R}^p\Big(\sum_{m=0}^{k-1} Q_m\Big)\right)^{\frac{k}{p}-k+1}$ on $\partial\Omega$.

We begin by selecting $Q_0 \equiv 0$, $N_0 := 0$, $P_{0,0} \equiv 0$. For $k = 1$ we apply Lemma 5.1 with

$$a := \inf_{z\in\partial\Omega} \Phi(z), \quad h := \Phi, \quad \tau := \inf_{z\in\partial\Omega} \Phi(z), \quad \gamma := \frac{1}{2}.$$

to obtain $N_1 \in \mathbb{N}$ and orthogonal polynomials $P_{1,0}, P_{1,1}, \ldots, P_{1,N_1}$ such that the polynomial $Q_1 := \sum_{j=0}^{N_1} P_{1,j}$ has the following properties

- $\frac{1}{2}\Phi < \mathcal{R}^p(Q_1) < \Phi$ on $\partial\Omega$
- $\mathcal{R}^p(P_{1,j}) < \Phi$ on $\partial\Omega$, $j = 0, 1, \ldots, N_1$
- $\mathcal{R}^p(Q_1) - ||\Phi||_{\partial\Omega} < \sum\limits_{j=0}^{N_1} \mathcal{R}^p(P_{1,j}) < \mathcal{R}^p(Q_1) + ||\Phi||_{\partial\Omega}$, on $\partial\Omega$
- $\left|\mathcal{S}^p(P_{1,j})\right| < C\big(\mathcal{R}^p(P_{1,j})\big)^{\frac{1}{p}}$ on $\partial\Omega$, $j = 0, 1, \ldots, N_1$
- if $p \leq 1$, then $\left|\mathcal{S}^p(Q_1)\right| < \frac{2C}{\theta}(\Phi)^{\frac{2}{p}-1} \leq \frac{2C}{\theta}\Phi^{\frac{1}{p}}$ on $\partial\Omega$
- $\left|\left|\mathcal{R}^p(Q_1)\right|\right|_{1,z\partial\mathbb{D}} > \frac{1}{2}||\Phi||_{1,z\partial\mathbb{D}}$.

Observe that Q_1 satisfies conditions $(p1) - (p8)$. Then assume that the orthogonal polynomials $Q_0, Q_1, \ldots, Q_k$ with desired properties $(p1) - (p8)$ have been found. Lemma 5.1 applied to

$$M := \deg(Q_k) + 1, \quad a := \inf_{z\in\partial\Omega}\left(\Phi(z) - \mathcal{R}^p\Big(\sum_{m=0}^{k} Q_m\Big)(z)\right)^k,$$
$$h := \frac{3}{4}\left(\Phi - \mathcal{R}^p\Big(\sum_{m=0}^{k} Q_m\Big)\right),$$
$$\gamma := \frac{2}{3}, \quad \tau := \frac{1}{8}\inf_{z\in\partial\Omega}\left(\Phi(z) - \mathcal{R}^p\Big(\sum_{m=0}^{k} Q_m\Big)(z)\right)^{k+1}$$

produces $N_{k+1} \in \mathbb{N}$ and orthogonal polynomials $P_{k+1,0}, P_{k+1,1}, \ldots, P_{k+1,N_{k+1}}$ that are also orthogonal to the polynomials $Q_0, Q_1, \ldots, Q_k$. If we define $Q_{k+1} := \sum\limits_{j=0}^{N_{k+1}} P_{k+1,j}$, then on $\partial\Omega$ it satisfies the following conditions

a) $\operatorname{ord}(Q_{k+1}) \geq \deg(Q_k) + 1$

b) $\mathcal{R}^p(P_{k+1,j}) < \left(\Phi - \mathcal{R}^p\left(\sum_{m=0}^{k} Q_m\right)\right)^{k+1}, \quad j = 0, \ldots, N_{k+1}$

c) $\left|\mathcal{S}^p(P_{k+1,j})\right| < C\left(\mathcal{R}^p(P_{k+1,j})\right)^{\frac{1}{p}}, \quad j = 0, \ldots, N_{k+1}$

d) $\mathcal{R}^p(Q_{k+1}) - \left\|\Phi - \mathcal{R}^p\left(\sum_{m=0}^{k} Q_m\right)\right\|_{\partial\Omega}^{k+1} < \sum_{j=0}^{N_{k+1}} \mathcal{R}^p(P_{k+1,j}) < \mathcal{R}^p(Q_{k+1}) + \left\|\Phi - \mathcal{R}^p\left(\sum_{m=0}^{k} Q_m\right)\right\|_{\partial\Omega}^{k+1}$

e) $\mathcal{R}^p\left(Q_{k+1}\right) < h = \frac{3}{4}\left(\Phi - \mathcal{R}^p\left(\sum_{m=0}^{k} Q_m\right)\right)$

f) $\mathcal{R}^p\left(Q_{k+1}\right) > \frac{1}{2}\left(\Phi - \mathcal{R}^p\left(\sum_{m=0}^{k} Q_m\right)\right)$

g) if $p \leq 1$, then $\left|\mathcal{S}^p(Q_{k+1})\right| < \frac{2C}{\theta}\left(\Phi - \mathcal{R}^p\left(\sum_{m=0}^{k} Q_m\right)\right)^{\frac{k+1}{p} - k}$

h)
$$\begin{aligned} \mathcal{R}^p\left(\sum_{m=0}^{k+1} Q_m\right) &< \mathcal{R}^p\left(\sum_{m=0}^{k} Q_m\right) + \mathcal{R}^p(Q_{k+1}) + 2\tau \\ &< \mathcal{R}^p\left(\sum_{m=0}^{k} Q_m\right) + \Phi - \mathcal{R}^p\left(\sum_{m=0}^{k} Q_m\right) = \Phi \end{aligned}$$

k)
$$\begin{aligned} \mathcal{R}^p\left(\sum_{m=0}^{k+1} Q_m\right) &> \mathcal{R}^p\left(\sum_{m=0}^{k} Q_m\right) + \mathcal{R}^p(Q_{k+1}) - 2\tau \\ &\geq \mathcal{R}^p\left(\sum_{m=0}^{k} Q_m\right) + \frac{1}{4}\left(\Phi - \mathcal{R}^p\left(\sum_{m=0}^{k} Q_m\right)\right). \end{aligned}$$

Hence Q_{k+1} has properties $(p1) - -(p8)$.

Next observe that by $(p2)$ and $(p3)$, $\lim_{n\to\infty} \mathcal{R}^p\left(\sum_{k=0}^{n} Q_k\right) = \Phi$ on $\partial\Omega$. In addition, property $(p3)$ on $\partial\Omega$ implies the following estimate

$$\begin{aligned} \Phi - \mathcal{R}^p\left(\sum_{m=0}^{k} Q_m\right) &< \Phi - \frac{1}{4}\left(\Phi - \mathcal{R}^p\left(\sum_{m=0}^{k-1} Q_m\right)\right) - \mathcal{R}^p\left(\sum_{m=0}^{k-1} Q_m\right) \\ &= \frac{3}{4}\left(\Phi - \mathcal{R}^p\left(\sum_{m=0}^{k-1} Q_m\right)\right) < \left(\frac{3}{4}\right)^k \Phi. \end{aligned}$$

Therefore it is clear that

$$\mathcal{R}^p(Q_k) < \left(\frac{3}{4}\right)^k ||\Phi||_{\partial\Omega} \text{ on } \partial\Omega. \tag{15}$$

Let $q = \max\{1, p\}$. Observe that if $n \geq k$, then

$$
\begin{aligned}
\sup_{z\in\partial\Omega}\left(\mathcal{R}^p\Big(\sum\nolimits_{m=0}^{n} Q_m - \sum_{m=0}^{k} Q_m\Big)(z)\right)^{\frac{1}{q}} &= \sup_{z\in\partial\Omega}\left(\mathcal{R}^p\Big(\sum\nolimits_{m=k+1}^{n} Q_m\Big)(z)\right)^{\frac{1}{q}} \\
&\leq \sup_{z\in\partial\Omega}\left(\sum\nolimits_{m=k+1}^{n}\big(\mathcal{R}^p(Q_m)(z)\big)^{\frac{1}{q}}\right) \\
&\overset{(15)}{<} \sum\nolimits_{m=k+1}^{n}\Big(\frac{3}{4}\Big)^{\frac{m}{q}}||\Phi||_{\partial\Omega}^{\frac{1}{q}} \underset{n,k\to\infty}{\longrightarrow} 0,
\end{aligned}
$$

so $\left\{\sum_{m=0}^{n} Q_m\right\}_{n=0}^{\infty}$ is a Cauchy sequence in $\mathcal{HR}^p(\Omega)$ which, by Proposition 2.1, is convergent in $\mathcal{HR}^p(\Omega)$. Hence we may define the function $f := \sum_{k=0}^{\infty} Q_k = \sum_{k=0}^{\infty}\sum_{j=0}^{N_k} P_{k,j}$. Proposition 2.1 implies that f is holomorphic in Ω. Moreover, $\mathcal{R}^p\Big(\sum_{k=0}^{n} Q_k\Big) \rightrightarrows \mathcal{R}^p(f)$ on $\partial\Omega$ as $n\to\infty$, so

$$
\mathcal{R}^p(f)(z) = \lim_{n\to\infty}\mathcal{R}^p\Big(\sum\nolimits_{k=0}^{n} Q_k\Big)(z) = \Phi(z), \quad z\in\partial\Omega.
$$

It follows from the property $(p4)$ that for $z \in \partial\Omega$

$$
\Big|\Big|\Phi - \mathcal{R}^p\Big(\sum\nolimits_{m=0}^{k-1} Q_m\Big)\Big|\Big|_{1,z\partial\mathbb{D}}^{-k}\big|\big|\mathcal{R}^p(P_{k,j})\big|\big|_{1,z\partial\mathbb{D}} < 1, \quad j = 0,\dots,N_k.
$$

Thus for $z \in \partial\Omega$ and $s < \min\{1, p\}$ we get that

$$
\begin{aligned}
\big|\big|\mathcal{R}^p(P_{k,j})\big|\big|_{1,z\partial\mathbb{D}}^{s} &> \big|\big|\mathcal{R}^p(P_{k,j})\big|\big|_{1,z\partial\mathbb{D}}^{s}\left(\Big|\Big|\Phi - \mathcal{R}^p\Big(\sum\nolimits_{m=0}^{k-1} Q_m\Big)\Big|\Big|_{1,z\partial\mathbb{D}}^{-k}\big|\big|\mathcal{R}^p(P_{k,j})\big|\big|_{1,z\partial\mathbb{D}}\right)^{p-s} \\
[5mm] &= \Big|\Big|\Phi - \mathcal{R}^p\Big(\sum\nolimits_{m=0}^{k-1} Q_m\Big)\Big|\Big|_{1,z\partial\mathbb{D}}^{-k(p-s)}\big|\big|\mathcal{R}^p(P_{k,j})\big|\big|_{1,z\partial\mathbb{D}}^{p}, \quad j = 0,\dots,N_k.
\end{aligned}
\tag{16}
$$

Let $P_{k,j}(z) = \sum_{n\in I_{k,j}} p_{k,j,n}(z)$ be the homogeneous expansion of the polynomial $P_{k,j}$, where $I_{k,j}$ is the set of monomials' degrees of $P_{k,j}$. By construction, $I_{m,j}\cap I_{k,i} =$

$\emptyset$ for $(m, j) \neq (k, i)$. Moreover,

$$\begin{aligned}
\left|\left|\mathcal{R}^p(P_{k,j})\right|\right|_{1,z\partial\mathbb{D}}^{\frac{1}{q}} &= \left(\int_{\partial\mathbb{D}}\int_0^1 \Big|\sum_{n\in I_{k,j}} p_{k,j,n}(z\omega t)\Big|^p dt d\lambda(\omega)\right)^{\frac{1}{q}} \\
&\leq \sum_{n\in I_{k,j}} \left(\int_{\partial\mathbb{D}}\int_0^1 |p_{k,j,n}(z)|^p t^{pn} dt d\lambda(\omega)\right)^{\frac{1}{q}} \\
&= \sum_{n\in I_{k,j}} \frac{|p_{k,j,n}(z)|^{\frac{p}{q}}}{\sqrt[q]{pn+1}}. \qquad (17)
\end{aligned}$$

Observe that if $s < \min\{1, p\}$, then $\frac{sq}{p} < 1$. Hence for $z \in \partial\Omega$ and $s < \min\{1, p\}$ we have the following estimate

$$\begin{aligned}
&\sum_{k,j,n}\left(\frac{|p_{k,j,n}(z)|}{\sqrt[p]{pn+1}}\right)^s = \sum_{k,j,n}\left(\frac{|p_{k,j,n}(z)|^{\frac{p}{q}}}{\sqrt[q]{pn+1}}\right)^{\frac{sq}{p}} \\
&\geq \sum_{k,j}\left(\sum_{n\in I_{k,j}} \frac{|p_{k,j,n}(z)|^{\frac{p}{q}}}{\sqrt[q]{pn+1}}\right)^{\frac{sq}{p}} \geq \sum_{k,j}\left|\left|\mathcal{R}^p(P_{k,j})\right|\right|_{1,z\partial\mathbb{D}}^{\frac{s}{p}} \\
&\overset{(16)}{>} \sum_{k=1}^{\infty}\left|\left|\Phi - \mathcal{R}^p\Big(\sum_{m=0}^{k-1} Q_m\Big)\right|\right|_{1,z\partial\mathbb{D}}^{-\frac{k(p-s)}{p}} \sum_{j=0}^{N_k}\left|\left|\mathcal{R}^p(P_{k,j})\right|\right|_{1,z\partial\mathbb{D}} \\
&= \sum_{k=1}^{\infty}\left|\left|\Phi - \mathcal{R}^p\Big(\sum_{m=0}^{k-1} Q_m\Big)\right|\right|_{1,z\partial\mathbb{D}}^{-\frac{k(p-s)}{p}} \sum_{j=0}^{N_k}\int_{\partial\mathbb{D}} \mathcal{R}^p(P_{k,j})(\omega z) d\lambda(\omega) \\
&\overset{(p6)}{>} \sum_{k=1}^{\infty}\left|\left|\Phi - \mathcal{R}^p\Big(\sum_{m=0}^{k-1} Q_m\Big)\right|\right|_{1,z\partial\mathbb{D}}^{-\frac{k(p-s)}{p}} \left(\int_{\partial\mathbb{D}} \mathcal{R}^p(Q_k)(\omega z) d\lambda(\omega)\right. \\
&\quad \left. - \left|\left|\Phi - \mathcal{R}^p\Big(\sum_{m=0}^{k-1} Q_m\Big)\right|\right|_{\partial\Omega}^{k}\right) \\
&\geq \sum_{k=1}^{\infty}\left|\left|\Phi - \mathcal{R}^p\Big(\sum_{m=0}^{k-1} Q_m\Big)\right|\right|_{1,z\partial\mathbb{D}}^{-\frac{k(p-s)}{p}} \left|\left|\mathcal{R}^p(Q_k)\right|\right|_{1,z\partial\mathbb{D}} - \sum_{k=1}^{\infty}\left(\frac{3}{4}\right)^{\frac{(k-1)ks}{p}} \\
&\overset{(p7)}{>} \sum_{k=1}^{\infty}\frac{1}{2}\left|\left|\Phi - \mathcal{R}^p\Big(\sum_{m=0}^{k-1} Q_m\Big)\right|\right|_{1,z\partial\mathbb{D}}^{-\frac{k(p-s)}{p}+1} - \sum_{k=1}^{\infty}\left(\frac{3}{4}\right)^{\frac{(k-1)ks}{p}} \\
&\geq \sum_{k>\frac{p}{p-s}} \frac{1}{2} - \sum_{k=1}^{\infty}\left(\frac{3}{4}\right)^{\frac{(k-1)ks}{p}} = \infty.
\end{aligned}$$

Since the last series diverges, so does the series $\sum_{k,j,n}\left(\frac{|p_{k,j,n}(z)|}{\sqrt[p]{pn+1}}\right)^s$.

If $p \leq 1$, then property $(p8)$ gives the estimate in $\overline{\Omega}$

$$\sum_{k=1}^{\infty} \left|\mathcal{S}^p(Q_k)\right| \leq \frac{2C}{\theta} \sum_{k=1}^{\infty} \left\|\Phi - \mathcal{R}^p\Big(\sum_{m=0}^{k-1} Q_m\Big)\right\|_{\partial\Omega}^{\frac{k}{p}-k+1} \leq \frac{2C}{\theta} \sum_{k=1}^{\infty} \Big(\frac{3}{4}\Big)^{k-1} ||\Phi||_{\partial\Omega} < \infty$$

which implies that $\mathcal{S}^p(f) \in \mathcal{C}(\overline{\Omega})$.

Since polynomials $P_{k,0}, P_{k,1}, \ldots, P_{k,N_k}$ are orthogonal, so are $\mathcal{S}^p(P_{k,0})$, $\mathcal{S}^p(P_{k,1}), \ldots, \mathcal{S}^p(P_{k,N_k})$. Then for $p \in (1, 2]$ consider the integral

$$\begin{aligned}
\int_{\partial\mathbb{D}} \left|\mathcal{S}^p(f)(\omega z)\right|^2 d\lambda(\omega) &= \int_{\partial\mathbb{D}} \Big|\sum_{k=0}^{\infty}\sum_{j=0}^{N_k} \mathcal{S}^p(P_{k,j})(\omega z)\Big|^2 d\lambda(\omega) \\
&= \sum_{k=0}^{\infty}\sum_{j=0}^{N_k} \int_{\partial\mathbb{D}} \left|\mathcal{S}^p(P_{k,j})(\omega z)\right|^2 d\lambda(\omega) \\
&\overset{(p5)}{\leq} C^2 \sum_{k=0}^{\infty}\sum_{j=0}^{N_k} \int_{\partial\mathbb{D}} \big(\mathcal{R}^p(P_{k,j})(\omega z)\big)^{\frac{2}{p}} d\lambda(\omega) \\
&\leq C^2 \sum_{k=0}^{\infty}\sum_{j=0}^{N_k} \int_{\partial\mathbb{D}} \mathcal{R}^p(P_{k,j})(\omega z) d\lambda(\omega) \\
&\overset{(p6)}{\leq} C^2 \sum_{k=1}^{\infty} \Big(\int_{\partial\mathbb{D}} \mathcal{R}^p(Q_k)(\omega z) d\lambda(\omega) \\
&\quad + \left\|\Phi - \mathcal{R}^p\Big(\sum_{m=0}^{k-1} Q_m\Big)\right\|_{\partial\Omega}^{k}\Big) \\
&\overset{(15)}{<} C^2 \sum_{k=1}^{\infty} \Big(\Big(\frac{3}{4}\Big)^k ||\Phi||_{\partial\Omega} + \Big(\frac{3}{4}\Big)^{(k-1)k}\Big) < \infty.
\end{aligned}$$

Hence $\mathcal{S}^p(f) \in L_2(z\partial\mathbb{D})$ for $p \in (1, 2]$, $z \in \partial\Omega$. The proof is now complete. □

Observe that for $p = 2$ due to orthogonality of $p_{k,j,n}$ in (17) each slice function of $\mathcal{S}^2(f)$ additionally has divergent series of Taylor coefficients with every exponent $s < 2$.

Data Availibility Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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