Solvability Criterion for Convolution Equations on a Half-Strip

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Abstract

We obtain the criterion of solvability of homogeneous convolution equation in a half-strip. Proof is based on a new decomposition property of the weighted Hardy space. This result has relations to the spectral analysis-synthesis problem, cyclicity problem, information theory. All data generated or analysed during this study are included in this published article.

Keywords Convolution equation · Hardy space · Analytic continuation · Spectral analysis · Spectral synthesis · Signal processing

Mathematics Subject Classification Primary 45E10; Secondary 30H10 · 42A84

1 Introduction

Convolution operations are very important in mathematical community, as well as in signal processing, sampling, filter design and applications. Lax et al. (see [1]–[3]) considered the equation

$$\int_{-\infty}^0 f(u + \tau)g(u)du = 0, \quad \tau \leq 0, \quad g \in L^2(-\infty; 0), \quad (1.1)$$

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where unknown function f belongs to $L^2(-\infty; 0)$. For the analysis of this equation the following characteristic functions

$$G(z) = \frac{1}{i\sqrt{2\pi}} \int_{-\infty}^0 g(u) e^{uz} du, \quad (1.2)$$

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 f(u) e^{-zu} du \quad (1.3)$$

was used. Equation (1.1) has a number of direct applications in theory of translation invariant subspaces, cyclicity problem, spectral analysis-synthesis problem; also this equation is of interest as a model case for the study of locally compact abelian groups (see [3, 19]). In [32] the author obtained the criterion of the Bedrosian identity in terms of equation (1.1). Therefore generalizations of equation (1.1) have a good perspective in applications.

Let $H^p(\mathbb{C}_+)$, $1 \leq p < +\infty$, be the Hardy space of analytic in the half-plane $\mathbb{C}_+ := \{z : \Re z > 0\}$ functions for which

$$\|f\|_* = \sup_{x>0} \left\{ \int_{-\infty}^{+\infty} |f(x+iy)|^p dy \right\}^{1/p} < +\infty.$$

Properties of these spaces are described in details in [18]. Each function $\psi \in H^p(\mathbb{C}_+)$, $1 \leq p < +\infty$, has nontangential boundary values almost everywhere on the imaginary axis $i\mathbb{R} = \partial\mathbb{C}_+$. These values we also define by ψ and $\psi \in L^p(i\mathbb{R})$.

N. Wiener and R. Paley [23] proved that (1.2) is a bijection of $H^2(\mathbb{C}_+) \ni G$ onto $L^2(-\infty; 0) \ni g$. It is clear, equality (1.3) is an bijection of $L^2(-\infty; 0) \ni f$ onto $H^2(\mathbb{C}_-) \ni F$, where $H^2(\mathbb{C}_-)$ is the Hardy space on $\mathbb{C}_- := \{z : \Re z < 0\}$.

Let us define the function $\Phi(iy) = F(iy)G(iy)$ almost everywhere on $i\mathbb{R}$, where $F(iy)$ and $G(iy)$ are nontangential boundary values of the functions F and G . The following solvability criterion (see, for example [3], p. 89) is the fundamental fact for the analysis of Eq. (1.1).

Theorem 1.1 [1] $f \in L^2(-\infty; 0)$ is a solution of Eq. (1.1) if and only if there exists a function $P \in H^1(\mathbb{C}_+)$ such that nontangential boundary values of P equal to Φ almost everywhere on $i\mathbb{R}$.

While often applied to real-valued signals, convolution can be used on complex signals. Many types of signals, such as seismic data or electrical signals, contain significant information in the phase of the signal [10, 11].

Some analogues of Theorem 1.1 for Hardy-type spaces on angular domain is proved in [9]. But for the case of a half-strip domain another methods is required: characteristic functions of functions in the Hardy (Smirnov) spaces on an angular domain belong to Hardy spaces on (generally other) angular domain; characteristic functions of functions

in the Hardy (Smirnov) spaces on an half-strip belong to weighted Hardy spaces on half-plane.

The aim of this paper is to obtain an equivalent result to Theorem 1.1 for the convolution equation on half-strip. This equation has direct applications to signal theory [24], to the studies of cyclic functions [26]. But for the studies of translation invariant subspaces [19] and some integral operators the full analogue of Theorem 1.1 is needed. In [25] only some partial results was obtained. The case of half-strip is interest as a model case for the study of equations of convolution type, for study of nonselfadjoint operators, Toeplitz operators, locally compact groups (see [4]–[8,13,27,28]).

2 The Main Result

First we need some definitions of analogues of the space $L^2(-\infty; 0)$. Let $D_\sigma = \{z : \Re z < 0, |\Im z| < \sigma\}$, $D_\sigma^* = \mathbb{C} \setminus \overline{D}_\sigma$, $\sigma > 0$. Let $E^p[D_\sigma]$, $1 \leq p < +\infty$, be the space of analytic functions on D_σ for which

$$\sup \left\{ \int_{\gamma} |f(z)|^p |dz| \right\}^{1/p} < +\infty, \quad (2.1)$$

where the supremum is over all segments γ which lay in D_σ and let $E_*^p[D_\sigma]$ be the space of analytic functions f on D_σ^* for which condition (2.1) holds, where the supremum is over all segments γ which lay in D_σ^* . Functions f belonging to these spaces have almost everywhere on ∂D_σ nontangential boundary values, which we denote by $f(z)$ and $f \in L^p[\partial D_\sigma]$. In fact, $E^p[D_\sigma]$ and $E_*^p[D_\sigma]$ are the Hardy spaces in their domains.

Sedletsii [17] proved that the space $H^p(\mathbb{C}_+)$ can be defined as the class of analytic on $\mathbb{C}_+$ functions for which

$$\|f\| := \sup_{-\frac{\pi}{2} < \varphi < \frac{\pi}{2}} \left\{ \int_0^{+\infty} |f(re^{i\varphi})|^p dr \right\}^{1/p} < +\infty.$$

Therefore Vynnytskiy [20] has considered the following generalization of the Hardy space. Let $H_\sigma^p(\mathbb{C}_+)$, $\sigma \geq 0$, $1 \leq p < +\infty$, be the space of analytic in $\mathbb{C}_+$ functions for which

$$\|f\| := \sup_{-\frac{\pi}{2} < \varphi < \frac{\pi}{2}} \left\{ \int_0^{+\infty} |f(re^{i\varphi})|^p e^{-pr\sigma|\sin \varphi|} dr \right\}^{1/p} < +\infty.$$

In [21] the following generalization of equation (1.1)

$$\int_{\partial D_\sigma} f(w + \tau)g(w)dw = 0, \quad \tau \leq 0, \quad g \in E_*^2[D_\sigma], \quad (2.2)$$

is considered, where an unknown function f belongs to $E^2[D_\sigma]$. The inclusion $f \in E^2[D_\sigma]$, $\sigma > 0$, implies that f is defined almost everywhere on $\overline{D}_\sigma = \{z : \Re z \leq 0, |\Im z| \leq \sigma\}$ and $f \in L^2(\partial D_\sigma)$. Therefore it is natural to define $D_0 := (-\infty; 0)$, $E^2[D_0] = L^2(-\infty; 0)$ and $E_*^2[D_0] = L^2(-\infty; 0)$ as well. Hence for the limit case $\sigma = 0$ the above equation is equal to (1.1).

The set of solutions of Eq. (1.1) is significantly different [26] from (1.2). By [21], each of the equalities

$$G(z) = \frac{1}{i\sqrt{2\pi}} \int_{\partial D_\sigma} g(w)e^{zw}dw \quad (2.3)$$

and

$$g(w) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} G(x)e^{-xw}dx, \quad \Re w > 0. \quad (2.4)$$

is the bijection of $H_\sigma^2(\mathbb{C}_+)$ onto $E_*^2[D_\sigma]$.

Let $\sigma > 0$, $p > 0$. We define the Paley - Wiener space W_σ^p as the space of entire functions, whose restriction on $\mathbb{R}$ belongs to $L^p(\mathbb{R})$ and which satisfy the inequality $|f(z)| < ce^{\sigma|z|}$ for a positive constant C .

Let $T_\sigma^2(\mathbb{C}_-)$ be a set of points $F = (F_1, F_2, F_3)$, where $F_1(z)e^{-i\sigma z} \in H^2(\mathbb{C}_-)$, $F_3(z)e^{i\sigma z} \in H^2(\mathbb{C}_-)$, $F_2 \in W_\sigma^2$, and $F_1(z) + F_2(z) + F_3(z) \equiv 0$ for $z \in \mathbb{C}_- = \{\Re z < 0\}$.

Suppose $l_1 = \{w = u - i\sigma : -\infty < u \leq 0\}$, $l_2 = [-i\sigma; i\sigma]$, and $l_3 = \{w = u + i\sigma : -\infty < u \leq 0\}$; l_j are oriented anticlockwise with respect to D_σ .

Lemma 2.1 [21] *Equalities*

$$F_j(z) = \frac{1}{\sqrt{2\pi}} \int_{l_j} f(w)e^{-zw}dw, \quad f \in E^2[D_\sigma], \quad j \in \{1, 2, 3\}, \quad (2.5)$$

are the bijection of $T_\sigma^2(\mathbb{C}_-)$ onto $E^2[D_\sigma]$.

Each function $f \in W_\sigma^2$ can be represented as cardinal series by Whittaker-Kotelnikov-Shannon-Nyquist formula:

Lemma 2.2 ([12, 14, 15]) *The space W_σ^2 coincides with the space of functions represented as*

$$f(t) = \sigma \sqrt{\frac{2}{\pi}} \sum_{k=-\infty}^{+\infty} c_k \frac{\sin \sigma t}{\sigma t - \pi k}, \quad (c_k) \in l^2. \quad (2.6)$$

The space W_σ^1 coincides with the space of functions represented as (2.6) such that $(c_k) \in l^1$,

$$\sum_{m=-\infty}^{+\infty} \left| \sum_{k=-\infty}^{+\infty} (-1)^{k+m} c_{k+m} \frac{k}{1+k^2} \right| < +\infty. \quad (2.7)$$

The space W_σ^p can be defined also [16] as the space of entire functions satisfying the condition

$$\sup_{\varphi \in (0; 2\pi)} \left\{ \int_0^{+\infty} |(f r e^{i\varphi})|^p e^{-p\sigma r |\sin \varphi|} dr \right\}^{1/p} < +\infty.$$

It is clear that $H_\sigma^p(\mathbb{C}_+)$ is a weighted Hardy space and is an analogue of the Paley-Wiener space for the half-plane as well.

Let $\Phi_j(it) = F_j(it)G(it)$, $j \in \overline{1; 3}$, $t \in \mathbb{R}$. The main result of this paper is the following.

Theorem 2.1 *A function $f \in E^2[D_\sigma]$ is a solution of equation (2.2) if and only if there exists a function P_1 , $P_1 e^{-i\sigma z} \in H_\sigma^1(\mathbb{C}_+)$, such that nontangential boundary values of P_1 equal to $\Phi_1(it)$, $t \in \mathbb{R}$, almost everywhere on the imaginary axis.*

Clearly, Theorem 1.1 is a partial case of Theorem 2.1.

Otherwise, Theorem 2.1 can be reformulate as

Theorem 2.2 *A function $f \in E^2[D_\sigma]$ is a solution of equation (2.2) if and only if there exists a function P_3 , $P_3 e^{-i\sigma z} \in H_\sigma^1(\mathbb{C}_+)$, such that nontangential boundary values of P_3 equal to $\Phi_3(it)$, $t \in \mathbb{R}$, almost everywhere on the imaginary axis.*

3 Proof of the Main Result

Sufficiency is proved in [25]. The proof of necessity is based on the following convolution theorem.

Lemma 3.1 [21] *If $g \in E_*^2[D_\sigma]$, $f \in E^2[D_\sigma]$, then for all $\tau \leq 0$ the following equality*

$$\int_{\partial D_\sigma} f(w + \tau) g(w) dw = \int_0^{+\infty} \Phi_1(z) e^{\tau z} dz + \int_{-\infty}^0 \Phi_3(z) e^{\tau z} dz + \int_0^{+\infty} \Phi_2(z) e^{\tau z} dz, \quad (3.1)$$

is valid.

The main idea of the following proof is the reduction of properties of the weighted Hardy space to the classical Hardy space.

Proof of Theorem 2.1 Note that $F_2 \in W_\sigma^2$ by the Paley-Wiener Theorem. Hence $\Phi_2 \in H_{2\sigma}^1(\mathbb{C}_+)$ by Hölder's inequality. Suppose

$$S(z) = -\frac{1}{2\pi i} \int_0^{+i\infty} \Phi_1(w) \frac{1}{w-z} dw + \frac{1}{2\pi i} \int_{-i\infty}^0 \Phi_3(w) \frac{1}{w-z} dw \\ - \frac{1}{2\pi i} \int_0^{+\infty} \Phi_2(w) \frac{1}{w-z} dw.$$

Let $E^p[\mathbb{C}(\alpha; \gamma)]$, $0 < \gamma - \alpha < 2\pi$, $1 \leq p < +\infty$, be the space of analytic functions f in $\mathbb{C}(\alpha; \gamma) = \{z : \gamma < \arg z < \beta\}$ for which

$$\sup_{\alpha < \varphi < \gamma} \left\{ \int_0^{+\infty} |f(re^{i\varphi})| dr \right\} < +\infty.$$

Functions $f \in E^p[\mathbb{C}(\alpha; \gamma)]$ have almost everywhere on $\partial\mathbb{C}(\alpha; \gamma)$ [9] the angular boundary values, which we denote by f and $f \in L^p[\mathbb{C}(\alpha; \gamma)]$.

The first step. We claim the function

$$P_1(z) = \begin{cases} S(z), & z \in \mathbb{C}(0; \pi/2), \\ S(z) - \Phi_2(z), & z \in \mathbb{C}(-\pi/2; 0) \end{cases}$$

is the analytic continuation of Φ_1 from $i\mathbb{R}$ to $\mathbb{C}_+$.

We designate by $\eta(\tau)$ the right hand side of (3.1). Since

$$\int_{-\infty}^0 \eta(\tau) e^{\tau\zeta} d\tau = 0, \quad \Re\zeta < 0,$$

by Fubini theorem we have

$$S(z) = 0 \text{ for all } z \in \mathbb{C}_-. \quad (3.2)$$

Hence for $z \in \mathbb{C}_+$

$$S(z) = S(z) - S(-\bar{z}) = \frac{1}{\pi} \int_0^\infty \frac{x\Phi_1(iv)}{(v-y)^2 + x^2} dv - \frac{1}{\pi} \int_{-\infty}^0 \frac{x\Phi_3(iv)}{(v-y)^2 + x^2} dv \\ - \frac{1}{\pi i} \int_0^{+\infty} \frac{\Phi_2(u)x}{(u-x-iy)(u+x-iy)} du.$$

The first integral in the left hand side of the above equality is the Poisson integral for the function

$$\psi_1(y) := \begin{cases} \Phi_1(iy), & y > 0, \\ 0, & y < 0, \end{cases}$$

therefore the nontangential boundary values from $\mathbb{C}_+$ of this integral are equal to $\psi_1(y)$ almost everywhere on $i\mathbb{R}$. Analogously the nontangential boundary values from $\mathbb{C}_+$ of the second integral are equal to

$$\psi_3(y) := \begin{cases} \Phi_3(iy), & y < 0, \\ 0, & y > 0, \end{cases}$$

almost everywhere on $i\mathbb{R}$. The third integral tends to zero as x tends to zero. Hence the nontangential boundary values of $S(z)$ are equal to

$$\psi(y) := \begin{cases} \Phi_1(iy), & y > 0, \\ -\Phi_3(iy), & y < 0. \end{cases}$$

Since $\Phi_1 \equiv -\Phi_2 - \Phi_3$, we have that nontangential boundary values of P_1 equal to $\Phi_1(it)$, $t \in \mathbb{R}$, almost everywhere on $i\mathbb{R}$.

It remains to prove $P_1(z)e^{-i\sigma z} \in H_\sigma^1(\mathbb{C}_+)$.

The second step. There exist analytic in $\mathbb{C}_+$ functions β_1, β_2 such that $\Phi_2 = \beta_1 - \beta_2$, $\beta_1 \in E^1[\mathbb{C}(0; \pi/2)]$, $\beta_2 \in E^1[\mathbb{C}(-\pi/2; 0)]$.

It is easy to see

$$\begin{aligned} \Phi_2(z) &= F_2(z)G(z) = -\frac{1}{\sqrt{2\pi}}F_2(z) \left(\int_{l_1} g(w)e^{zw}dw + \int_{l_3} g(w)e^{zw}dw \right. \\ &\quad \left. + i \int_{-\sigma}^{\sigma} g(iv)e^{izv}dv \right) = -\frac{1}{\sqrt{2\pi}}F_2(z) \left(e^{-i\sigma z} \int_{-\infty}^0 g(u-i\sigma)e^{uz}du \right. \\ &\quad \left. - e^{i\sigma z} \int_{-\infty}^0 g(u+i\sigma)e^{uz}du + i \int_{-\sigma}^{\sigma} g(iv)e^{izv}dv \right) \\ &= F_2(z) \left(e^{-i\sigma z}h_1(z) + e^{i\sigma z}h_3(z) + h_2(z) \right). \end{aligned}$$

By Paley-Wiener Theorem [18] $h_1 \in H^2(\mathbb{C}_+)$, $h_3 \in H^2(\mathbb{C}_+)$. Therefore $F_2(z)h_3(z)e^{i\sigma z} \in E^1[\mathbb{C}(0; \pi/2)]$, $F_2(z)h_1(z)e^{-i\sigma z} \in E^1[\mathbb{C}(-\pi/2; 0)]$. By another Paley-Wiener Theorem [18] $h_2 \in W_\sigma^2$. It remains to show that $\Phi_{22} := F_2h_2$ has an appropriate decomposition as Φ_2 .

The third step. There exist analytic in $\mathbb{C}_+$ functions μ_1, μ_2 such that $\Phi_{22} = \mu_1 - \mu_2$, $\mu_1 \in E^1[\mathbb{C}(0; \pi/2)]$, $\mu_2 \in E^1[\mathbb{C}(-\pi/2; 0)]$.

Using Hölder's inequality, it is easy to see that $\Phi_{22} \in W_{2\sigma}^1$. By Lemma 2.2 we have

$$\Phi_{22}(t) = \sum_{k=-\infty}^{+\infty} b_k \frac{\sin 2\sigma t}{2\sigma t - \pi k}, \quad (b_k) \in l^1$$

and

$$\sum_{m=-\infty}^{+\infty} \left| \sum_{k=-\infty}^{+\infty} (-1)^{k+m} b_{k+m} \frac{k}{1+k^2} \right| < +\infty.$$

Therefore

$$\Phi_{22}(z) = \sum_{n=-\infty}^{+\infty} b_{2n} \frac{\sin 2\sigma z}{2\sigma z - 2\pi n} + b_{2n+1} \frac{\sin 2\sigma z}{2\sigma z - 2\pi n - \pi} = \psi(z) + \eta(z),$$

where

$$\psi(z) = \frac{1}{\pi} \sum_{n=-\infty}^{+\infty} \frac{-b_{2n} \sin 2\sigma z}{(2\sigma z - 2\pi n)(2\sigma z - 2\pi n - \pi)}, \quad (3.3)$$

$$\eta(z) = \frac{1}{\pi} \sum_{n=-\infty}^{+\infty} \frac{(b_{2n} + b_{2n+1}) \sin 2\sigma z}{2\sigma z - 2\pi n - \pi}. \quad (3.4)$$

First of all we prove that $\psi \in W_{2\sigma}^1$. Since series in (3.3) converges absolutely and uniformly on each compact set in $\mathbb{C}_+$, we have

$$\begin{aligned} \int_{-\infty}^{+\infty} |\psi(x+i)| dx &\leq d_1 \sum_{k=-\infty}^{+\infty} |b_{2k}| \left(\int_{-\infty}^{-2\pi k-2\pi} \frac{dx}{|2\sigma x + 2\sigma i - 2\pi k| |2\sigma x + 2\sigma i - 2\pi k - \pi|} \right. \\ &+ \int_{-2\pi k-2\pi}^{-2\pi k+\pi} \frac{dx}{|2\sigma x + 2\sigma i - 2\pi k| |2\sigma x + 2\sigma i - 2\pi k - \pi|} \\ &\left. + \int_{-2\pi k+\pi}^{+\infty} \frac{dx}{|2\sigma x + 2\sigma i - 2\pi k| |2\sigma x + 2\sigma i - 2\pi k - \pi|} \right) \\ &\leq d_1 \sum_{k=-\infty}^{+\infty} |b_{2k}| \left(\int_{-\infty}^{-2\pi k-2\pi} \frac{1}{\sqrt{(2\sigma x - 2\pi k)^2 + 4\sigma^2} \sqrt{(2\sigma x - 2\pi k)^2 + 4\sigma^2}} dx \right. \end{aligned}$$

$$\begin{aligned}
& + \int_{-2\pi k - 2\pi}^{-2\pi k + 2\pi} \frac{1}{4\sigma} dx + \int_{-2\pi k + 2\pi}^{+\infty} \frac{1}{\sqrt{(2\sigma x - 2\pi k - \pi)^2 + 4\sigma^2} \sqrt{(2\sigma x - 2\pi k - \pi)^2 + 4\sigma^2}} dx \Bigg) \\
& \leq d_2 \sum_{k=-\infty}^{+\infty} |b_{2k}| < +\infty,
\end{aligned}$$

where $d_1 > 0$, $d_2 > 0$.

Since $\psi(x+i) \in L^1(\mathbb{R})$, it follows from definition that $\psi(z+i) \in W_{2\sigma}^1$. Therefore $\psi(z) \in W_{2\sigma}^1$ and $\eta = f - \psi \in W_{2\sigma}^1$. It follows from (3.4) that

$$\eta(z) = \frac{1}{\pi} \sum_{n=-\infty}^{+\infty} \frac{(b_{2n} + b_{2n+1}) \sin 2\sigma z}{2\sigma z - 2\pi n - \pi} = \mu_{11} - \mu_{21},$$

where

$$\begin{aligned}
\mu_{11}(z) &= \sum_{k=-\infty}^{+\infty} \frac{(e^{2i\sigma z} + 1)(b_{2n} + b_{2n+1})}{2\pi i(2\sigma z - 2\pi n - \pi)}, \\
\mu_{21}(z) &= \sum_{k=-\infty}^{+\infty} \frac{(e^{-2i\sigma z} + 1)(b_{2n} + b_{2n+1})}{2\pi i(2\sigma z - 2\pi n - \pi)}.
\end{aligned}$$

Each series above converges absolutely and uniformly on each compact set in $\mathbb{C}$ because sequence $(b_{2n} + b_{2n+1})$ belongs to l_1 . Therefore both the functions μ_{11} and μ_{21} are entire functions of exponential type $\leq \sigma$. Let us remark that

$$\begin{aligned}
\mu_{11}(z) &= \sum_{k=-\infty}^{+\infty} \frac{e^{i\sigma z}(e^{i\sigma z} + e^{-i\sigma z})(b_{2n} + b_{2n+1})}{2\pi i(2\sigma z - 2\pi n - \pi)} \\
&= \sum_{k=-\infty}^{+\infty} \frac{(b_{2n} + b_{2n+1}) \sin 2\sigma z}{\pi(2\sigma z - 2\pi n - \pi)} \times \frac{e^{i\sigma z}(e^{i\sigma z} + e^{-i\sigma z})}{2i \sin 2\sigma z} = \eta(z) \frac{e^{i\sigma z}}{i \sin \sigma z}.
\end{aligned}$$

Since on the line $z = x + i$ the function

$$\frac{e^{i\sigma z}}{i \sin \sigma z}$$

is bounded and $\eta \in L^1$, we have $\eta(z) \frac{e^{i\sigma z}}{i \sin \sigma z} \in L^1(\mathbb{R} + i)$. Therefore $\mu_{11}(z)e^{-i\sigma z} \in W_{\sigma}^1$. Similarly, we can prove that $\mu_{21}(z)e^{i\sigma z} \in W_{\sigma}^1$. As a consequence $\mu_{11} \in E^1[\mathbb{C}(0; \pi/2)]$ and $\mu_{21} \in E^1[\mathbb{C}(-\pi/2; 0)]$.

Hishchak proves [22] that for $f \in W_{\sigma}^1$ defined by (2.6) the decomposition

$$f = \mu_1 - \mu_2, \quad \mu_1 \in E^1[\mathbb{C}(0; \pi/2)], \quad \mu_2 \in E^1[\mathbb{C}(-\pi/2; 0)]$$

is valid if and only if both of the following conditions are fulfilled

$$\sum_{m=1}^{+\infty} \left| \sum_{k=-\infty}^{+\infty} c_k \frac{k}{(m - \frac{i}{2} - k)(m - \frac{i}{2} - ik)} \right| < +\infty, \quad (3.5)$$

$$\sum_{m=1}^{+\infty} \left| \sum_{k=-\infty}^{+\infty} c_k \frac{k}{(m + \frac{i}{2} + ik)(m + \frac{i}{2} - k)} \right| < +\infty. \quad (3.6)$$

Note that ψ has representation (2.6). Let us consider condition (3.5) for ψ and denote by L the left hand side.

$$\begin{aligned} L = & \frac{1}{\sqrt{2}} \sum_{m=1}^{+\infty} \left| \sum_{n=-\infty}^{+\infty} \frac{c_{2n}}{m - \frac{i}{2} - 2n} - \frac{c_{2n}}{m - \frac{i}{2} - 2ni} \right. \\ & \left. + \sum_{n=-\infty}^{+\infty} \frac{c_{2n+1}}{m - \frac{i}{2} - 2n - 1} - \frac{c_{2n+1}}{m - \frac{i}{2} - 2ni - i} \right|. \end{aligned}$$

Since $c_{2n} = -c_{2n+1}$, we obtain

$$\begin{aligned} L = & \frac{1}{\sqrt{2}} \sum_{m=1}^{+\infty} \left| \sum_{n=-\infty}^{+\infty} \frac{c_{2n}}{(m - \frac{i}{2} - 2n)(m - \frac{i}{2} - 2n - 1)} + \frac{c_{2n}i}{(m - \frac{i}{2} - 2ni)(m - \frac{i}{2} - 2ni - i)} \right| \\ \leq & \frac{1}{\sqrt{2}} \sum_{m=1}^{+\infty} \left(\sum_{n=-\infty}^{+\infty} \frac{|c_{2n}|}{|m - \frac{i}{2} - 2n||m - \frac{i}{2} - 2n - 1|} + \sum_{n=-\infty}^{+\infty} \frac{|c_{2n}|}{(m - \frac{i}{2} - 2ni)(m - \frac{i}{2} - 2ni - i)} \right). \end{aligned}$$

Replacing $l = m - 2n$, we get

$$\begin{aligned} L \leq & \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{+\infty} |c_{2n}| \sum_{m=-\infty}^{+\infty} \frac{1}{\sqrt{(m - 2n)^2 + \frac{1}{4}} \sqrt{(m - 2n - 1)^2 + \frac{1}{4}}} \\ & + \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{+\infty} |c_{2n}| \sum_{m=-\infty}^{+\infty} \frac{1}{\sqrt{m^2 + (\frac{1}{2} + 2n)^2} \sqrt{m^2 + (\frac{1}{2} + 2n + 1)^2}} \\ \leq & \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{+\infty} |c_{2n}| \sum_{l=-\infty}^{+\infty} \frac{1}{\sqrt{l^2 + \frac{1}{4}} \sqrt{(l - 1)^2 + \frac{1}{4}}} \\ & + \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{+\infty} |c_{2n}| \sum_{m=-\infty}^{+\infty} \frac{1}{m^2} < +\infty \end{aligned}$$

Similarly we can prove (3.6) for ψ .

The last step. We claim $P_1 \in E^1[\mathbb{C}(0; \pi/2)]$, $P_1(z)e^{-2i\sigma z} \in E^1[\mathbb{C}(-\pi/2; 0)]$.

It is sufficient to prove $S \in E^1[\mathbb{C}(0; \pi/2)]$, $S \in E^1[\mathbb{C}(-\pi/2; 0)]$. Since $\Phi_2 = \beta_1 - \beta_2$, $\beta_1 \in E^1[\mathbb{C}(0; \pi/2)]$, $\beta_2 \in E^1[\mathbb{C}(-\pi/2; 0)]$, we get

$$\begin{aligned} \frac{1}{2\pi i} \int_0^{+\infty} \Phi_2(w) \frac{1}{w-z} dw &= \frac{1}{2\pi i} \int_0^{+\infty} (\beta_1(w) - \beta_2(w)) \frac{1}{w-z} dw, \\ \frac{1}{2\pi i} \int_{\partial\mathbb{C}(0; \pi/2)} \beta_1(w) \frac{1}{w-z} dw &= \begin{cases} \beta_1(z), & z \in \mathbb{C}(0; \pi/2), \\ 0, & z \in \mathbb{C} \setminus \overline{\mathbb{C}(0; \pi/2)}, \end{cases} \\ \frac{1}{2\pi i} \int_{\partial\mathbb{C}(-\pi/2; 0)} \beta_2(w) \frac{1}{w-z} dw &= \begin{cases} \beta_2(z), & z \in \mathbb{C}(-\pi/2; 0), \\ 0, & z \in \mathbb{C} \setminus \overline{\mathbb{C}(-\pi/2; 0)}. \end{cases} \end{aligned}$$

Hence

$$S(z) = -\frac{1}{2\pi i} \int_0^{+\infty} \frac{\Phi_1(w) + \beta_1(w)}{w-z} dw + \frac{1}{2\pi i} \int_{-\infty}^0 \frac{\Phi_3(w) + \beta_1(w)}{w-z} dw.$$

Then for $z \in \mathbb{C}_+$ by (3.2) we have

$$S(z) = S(z) - S(-\bar{z}) = \frac{1}{\pi} \int_0^{\infty} \frac{x(\Phi_1(iv) + \beta_1(iv))}{(v-y)^2 + x^2} dv - \frac{1}{\pi} \int_{-\infty}^0 \frac{x(\Phi_3(iv) + \beta_2(iv))}{(v-y)^2 + x^2} dv.$$

Using Fubini's theorem, this gives

$$\begin{aligned} \int_{-\infty}^{+\infty} |S(x+iy)| dy &\leq \frac{1}{\pi} \int_0^{\infty} |\Phi_1(iv) + \beta_1(iv)| dv \int_{-\infty}^{+\infty} \frac{x}{(v-y)^2 + x^2} dy \\ &+ \frac{1}{\pi} \int_{-\infty}^0 |\Phi_3(iv) + \beta_2(iv)| dv \int_{-\infty}^{+\infty} \frac{x}{(v-y)^2 + x^2} dy \leq \alpha_1 < +\infty, \end{aligned}$$

where α_1 does not depend on x . Similarly, $\int_0^{+\infty} |S(x+iy)| dx \leq \alpha_2 < +\infty$,

where α_2 does not depend on y . Therefore (see [29]) $S \in E^1[\mathbb{C}(0; \pi/2)]$, $S \in E^1[\mathbb{C}(-\pi/2; 0)]$. $\square$

The above proof motivates the following open problem.

Problem. Let $\widehat{H}_{2\sigma}^1 = \{f = f_1 f_2 : f_1 \in H_{\sigma}^2, f_2 \in H_{\sigma}^2\}$. It is clear that $\widehat{H}_{2\sigma}^1 \subset H_{2\sigma}^1$. But what about $\widehat{H}_{2\sigma}^1 = H_{2\sigma}^1$?

This problem is similar in some sense to the Ehrenpreis factorization problem (see [30,31]).

If the answer is negative, the following question about the so-called weak factorization is interesting: is it possible for every function from $H_{2\sigma}^1$ a representation as a sum (finite or infinite norm-controlled) of products of pairs of functions from H_σ^2 ?

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