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Complex Analysis and Operator Theory

Products of Positive Operators

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Abstract
On finite dimensional spaces, it is apparent that an operator is the product of two positive operators if and only if it is similar to a positive operator. Here, the class $\mathcal{L}^{+2}$ of bounded operators on separable infinite dimensional Hilbert spaces which can be written as the product of two bounded positive operators is studied. The structure is much richer, and connects (but is not equivalent to) quasi-similarity and quasi-affinity to a positive operator. The spectral properties of operators in $\mathcal{L}^{+2}$ are developed, and membership in $\mathcal{L}^{+2}$ among special classes, including algebraic and compact operators, is examined.

Keywords Products of positive operators · Schur complements · Quasi-similarity · Quasi-affinity · Local spectral theory · Generalized scalar operators

Mathematics Subject Classification 47A05 · 47A65

1 Introduction

This work aims to shed light on two questions, "Which bounded Hilbert space operators are products of two bounded positive operators?", and "What properties do such operators share?" Here, positive means selfadjoint with non-negative spectrum. This class is denoted throughout by $\mathcal{L}^{+2}$. The answer is easily given on finite dimensional spaces: an operator will be in $\mathcal{L}^{+2}$ if and only if it is similar to a positive operator [22], and this in turn is equivalent to the operator being diagonalizable with positive

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To Henk de Snoo, on his 75th birthday.

Extended author information available on the last page of the article

spectrum. Answering the questions on infinite dimensional Hilbert spaces is a much more delicate matter. Similarity no longer suffices.

Apostol [1] studied the question as to which operators are quasi-similar to normal operators, and his work readily adapts to this setting, making it possible to construct operators which are quasi-similar to positive operators. Another difficulty then arises, since not every operator which is quasi-similar to a positive operator will be the product of two *bounded* positive operators. For this, something extra is needed.

This is not the end of the story though, since the quasi-similar operators which are in $\mathcal{L}^{+2}$ only form a part of the whole class. One can relax the quasi-similarity condition to quasi-affinity. Here again, the class of operators which are in $\mathcal{L}^{+2}$ and which are quasi-affine to a positive operator can be characterized. However, even this falls short of giving the entire class. Nevertheless, it comes close, and in general $T \in \mathcal{L}^{+2}$ has the property that it has both a restriction and extension in $\mathcal{L}^{+2}$ which are quasi-affine to a positive operator.

Despite the fact that similarity to a positive operator fails to capture the whole of $\mathcal{L}^{+2}$, a surprising number of the spectral properties of operators similar to positive operators do carry over. It is an elementary observation that the spectrum of an operator in $\mathcal{L}^{+2}$ is contained in $\mathbb{R}^+$, the non-negative reals. Also, it was observed by Wu [22] that the only quasi-nilpotent operator in the class is 0. It happens that operators which are similar to positive operators are spectral operators, and so decompose as the sum of a scalar operator (having a spectral decomposition) and a quasi-nilpotent operator. Moreover, in this case the quasi-nilpotent part is 0. Using local spectral theory, it is possible to define an invariant linear manifold (so not necessarily closed) on which an operator is quasi-nilpotent [14]. In case the operator has the single valued extension property, which enables the definition of a unique local resolvent, this manifold is closed. Since the operators in $\mathcal{L}^{+2}$ have thin spectrum, they also have the single valued extension property. It then follows that for any operator in $\mathcal{L}^{+2}$, the quasi-nilpotent part is the restriction to the kernel. In addition, for non-zero point spectra, there is no non-trivial (generalized) Jordan structure. These ideas enable the study operators in $\mathcal{L}^{+2}$ which are either algebraic or compact. While the only operators in $\mathcal{L}^{+2}$ which are similar to positive operators are scalar, all are *generalized scalar operators* (having a C^∞ functional calculus). Furthermore, the algebraic spectral subspaces for operators in $\mathcal{L}^{+2}$ have the same form as that exhibited by normal operators.

Elements of $\mathcal{L}^{+2}$ with closed range are the ones which behave most similarly to the finite dimensional case, since they are similar to positive operators. In this case it is possible to explicitly describe the Moore-Penrose inverse of the operator, and to find a generalized inverse which is also in $\mathcal{L}^{+2}$.

A good deal of the paper hinges on a theorem due to Sebestyén [18]. Sebestyén's theorem states that for fixed operators A and T, the equation $T = AX$ has a positive solution if and only if $TT^* \leq \lambda AT^*$ for some $\lambda > 0$. A proof of a refined version is given in Sect. 2 using Schur complement techniques (see also [2]), enabling $T \in \mathcal{L}^{+2}$ to be written as AB, $A, B \geq 0$, where $\overline{\operatorname{ran}}\, A = \overline{\operatorname{ran}}\, T$ and $\overline{\operatorname{ran}}\, B = \overline{\operatorname{ran}}\, T^*$. Such a pair (A, B) is called *optimal* for T. Optimal pairs happen to be extremely useful.

Section 3 looks at those operators (not necessarily in $\mathcal{L}^{+2}$) which are either quasi-affine or quasi-similar to positive operators. Rigged Hilbert spaces are used to show that for an operator T quasi-affine to a positive operator, $\operatorname{ran} T \cap \ker T = \{0\}$ and

$\overline{\text{ran}\, T \dotplus \ker T} = \mathcal{H}$. In the quasi-similar case, since this will hold for both T and T^*, one has instead that $\overline{\text{ran}}\, T \dotplus \ker T = \mathcal{H}$. Work of Hassi, Sebestyén, and de Snoo [10] plays a key role in describing those operators quasi-affine to a positive operator.

The paper then turns to describing general properties of the class $\mathcal{L}^{+2}$ in Sect. 4. Central here are optimal pairs, the properties of which are explored in detail. Examples are given which show that operators in $\mathcal{L}^{+2}$ which are similar to a positive operator, quasi-similar to a positive operator, and quasi-affine to a positive operator form strictly increasingly larger subclasses when $\dim \mathcal{H} = \infty$, and that there are operators in $\mathcal{L}^{+2}$ which do not fall into any of these, further hinting at the complexities of the class.

Similarity to a positive operator completely characterizes $\mathcal{L}^{+2}$ on finite dimensional spaces, and this is examined in Sect. 6. The closed range operators are considered as a special sub-category. In Sect. 7, attention turns to those operators in $\mathcal{L}^{+2}$ which are either quasi-affine or quasi-similar to a positive operator, where there are characterizations given which are analogous to those found for operators similar to a positive operator. Generally, there is only a weak connection between the spectra of quasi-similar operators. However, for an operator in $\mathcal{L}^{+2}$, quasi-affinity to a positive operator preserves the spectrum. It is also proved that not every operator which is quasi-similar to a positive operator is in $\mathcal{L}^{+2}$, by proving that any operator in $\mathcal{L}^{+2}$ which is quasi-similar to a positive operator has a square root which is quasi-similar to a positive operator, but not all have square roots in $\mathcal{L}^{+2}$. In Sect. 8, general operators in $\mathcal{L}^{+2}$ are considered, and the main point is that for any $T \in \mathcal{L}^{+2}$, there exist both restrictions and extensions (on the same Hilbert space) which are also in $\mathcal{L}^{+2}$ and which are quasi-affine to a positive operator.

A constant refrain throughout is that operators in $\mathcal{L}^{+2}$ have many of the properties of positive operators. Section 5 examines this resemblance with regards to local spectral properties. This is applied in the final section to algebraic operators and compact operators in $\mathcal{L}^{+2}$.

2 Preliminaries

Throughout, all spaces are complex and separable Hilbert spaces. The domain, range, closure of the range, null space or kernel, spectrum and resolvent of any given operator A are denoted by $\text{dom}\,(A)$, $\text{ran}\, A$, $\overline{\text{ran}}\, A$, $\ker A$, $\sigma(A)$, and $\rho(A)$, respectively, and $\sigma(A) \subseteq [0, \infty)$ is indicated as $\sigma(A) \geq 0$.

The space of everywhere defined bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ is written as $L(\mathcal{H}, \mathcal{K})$, or $L(\mathcal{H})$ when $\mathcal{H} = \mathcal{K}$, while $CR(\mathcal{H})$ denotes the subset in $L(\mathcal{H})$ of closed range operators. The identity operator on $\mathcal{H}$ is written as 1, or $1_{\mathcal{H}}$ if it is necessary to disambiguate.

As usual, the direct sum of two subspaces $\mathcal{M}$ and $\mathcal{N}$ with $\mathcal{M} \cap \mathcal{N} = \{0\}$ is indicated by $\mathcal{M} \dotplus \mathcal{N}$, and the orthogonal direct sum by $\mathcal{M} \oplus \mathcal{N}$. The orthogonal complement of a space $\mathcal{M}$ is written $\mathcal{M}^{\perp}$. The symbol $\mathcal{P}$ denotes the class of all Hilbert space orthogonal projections, while $P_{\mathcal{M}}$ is the orthogonal projection with range $\mathcal{M}$.

Write $GL(\mathcal{H})$ for the group of invertible operators in $L(\mathcal{H})$, $\mathcal{L}^+ = L(\mathcal{H})^+$, the class of positive semidefinite operators, $GL(\mathcal{H})^+ := GL(\mathcal{H}) \cap \mathcal{L}^+$ and $CR(\mathcal{H})^+ :=$

$CR(\mathcal{H}) \cap \mathcal{L}^+$. The paper focuses on the operators in

$$\mathcal{L}^{+2} := \{T \in L(\mathcal{H}) : T = AB \text{ where } A, B \in \mathcal{L}^+\}.$$

Occasionally, this will be written as $\mathcal{L}^{+2}(\mathcal{H})$ if it is necessary to clarify on which space the operators are acting.

Given two operators $S, T \in L(\mathcal{H})$, the notation $T \leq S$ signifies that $S - T \in \mathcal{L}^+$ (the *Löwner order*). Given any $T \in L(\mathcal{H})$, $|T| := (T^*T)^{1/2}$ is the modulus of T and $T = U|T|$ is the polar decomposition of T, with U the partial isometry such that $\ker U = \ker T$ and $\operatorname{ran} U = \overline{\operatorname{ran}}\, T$.

For $B \in \mathcal{L}^+$, the *Schur complement* $B_{/\mathcal{S}}$ of B to a closed subspace $\mathcal{S} \subseteq \mathcal{H}$ is the maximal element of $\{X \in L(\mathcal{H}) : 0 \leq X \leq B \text{ and } \operatorname{ran} X \subseteq \mathcal{S}^\perp\}$. It always exists. The $\mathcal{S}$-*compression* of B is defined as $B_{\mathcal{S}} := B - B_{/\mathcal{S}}$.

Let $B \in L(\mathcal{H})$ be selfadjoint, $\mathcal{S} \subseteq \mathcal{H}$ a closed subspace, relative to $\mathcal{S} \oplus \mathcal{S}^\perp$,

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix}.$$

Suppose that $B \geq 0$. Write $B^{1/2} = \begin{pmatrix} R_1^* \\ R_2^* \end{pmatrix}$, where R_1^*, R_2^* are the rows of $B^{1/2}$. Then for $j = 1, 2$, $R_j^* R_j = B_{jj}$, and so by Douglas' lemma, there are isometries $V_j : \overline{\operatorname{ran}}\, B_{jj}^{1/2} \to \overline{\operatorname{ran}}\, R_j$ such that $R_j = V_j B_{jj}^{1/2}$. Then $B_{12} = R_1^* R_2 = B_{11}^{1/2} F B_{22}^{1/2}$, where $F = V_1^* V_2 : \overline{\operatorname{ran}}\, B_{22}^{1/2} \to \overline{\operatorname{ran}}\, B_{11}^{1/2}$ is a contraction.

On the other hand, if $B_{11}, B_{22} \geq 0$ and B_{12} has this form, then

$$\begin{aligned} B &= \begin{pmatrix} B_{11}^{1/2} & 0 \\ B_{22}^{1/2} F^* & B_{22}^{1/2} D_F \end{pmatrix} \begin{pmatrix} B_{11}^{1/2} & F B_{22}^{1/2} \\ 0 & D_F B_{22}^{1/2} \end{pmatrix} \\ &= \begin{pmatrix} B_{11}^{1/2} \\ B_{22}^{1/2} F^* \end{pmatrix} \begin{pmatrix} B_{11}^{1/2} & F B_{22}^{1/2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & B_{22}^{1/2}(1 - F^*F) B_{22}^{1/2} \end{pmatrix}, \end{aligned} \tag{2.1}$$

where $D_F = (1_{\overline{\operatorname{ran}}\, B_{22}} - F^*F)^{1/2}$ on $\overline{\operatorname{ran}}\, B_{22}$. Therefore, positivity of B is equivalent to $B_{11}, B_{22} \geq 0$ and the existence of such a contraction F. The second term in the sum in (2.1) is the Schur complement $B_{/\mathcal{S}}$, while the first term is the $\mathcal{S}$-compression of B. In general, it is not difficult to verify that whenever $B = C^*C$, where $C : \mathcal{S} \oplus \mathcal{S}^\perp \to \mathcal{S}$, then $B_{/\mathcal{S}} = 0$.

The next theorem is a slightly strengthened form of one due to Sebestyén ([18], see also [2, Corollary 2.4]). It plays a central role in what follows.

Theorem 2.1 (Sebestyén) *Let $A, T \in L(\mathcal{H})$. The equation $AX = T$ has a positive solution if and only if*

$$TT^* \leq \lambda AT^*$$

for some $\lambda \geq 0$, in which case X can be chosen so that $\ker X = \ker T$, $X_{/\overline{\operatorname{ran}}\, T} = 0$. *Furthermore, if $A \geq 0$ with $\overline{\operatorname{ran}}\, A = \overline{\operatorname{ran}}\, T$, then $P_{\overline{\operatorname{ran}}\,(T)} X|_{\overline{\operatorname{ran}}\,(T)}$ will be injective with dense range.*

Proof If $AX = T$ has a positive solution, then $\lambda = \|X\|$ suffices. On the other hand, if for some $\lambda \geq 0$, $0 \leq TT^* \leq \lambda AT^*$, then $AT^* \geq 0$ and by Douglas' lemma, there exists G with $\|G\| \leq \lambda^{1/2}$ and $\overline{\operatorname{ran}}\, G \subseteq \overline{\operatorname{ran}}\,(TA^*)^{1/2}$ satisfying $T = (TA^*)^{1/2}G$. Clearly then, $\ker T = \ker G$ and $\overline{\operatorname{ran}}\, T \subseteq \overline{\operatorname{ran}}\,(TA^*)^{1/2}$. Also $\overline{\operatorname{ran}}\,(TA^*)^{1/2} = \overline{\operatorname{ran}}\,(TA^*) \subseteq \overline{\operatorname{ran}}\, T$, so equality holds. The equality $TA^* = (TA^*)^{1/2}GA^*$ then implies $(TA^*)^{1/2} = GA^* = AG^*$. Thus $T = AG^*G$, and so $X = G^*G \geq 0$ with $\ker X = \ker T$ (equivalently, $\overline{\operatorname{ran}}\, X = \overline{\operatorname{ran}}\, T^*$). Also, $\overline{\operatorname{ran}}\, G = \overline{\operatorname{ran}}\,(TA^*)^{1/2} = \overline{\operatorname{ran}}\, T$. Decomposing $\mathcal{H} = \overline{\operatorname{ran}}\, T \oplus \ker T^*$, the operator G has the form $G = \begin{pmatrix} G_1 & G_2 \\ 0 & 0 \end{pmatrix}$, and by (2.1), $X_{/\overline{\operatorname{ran}}\, T} = 0$.

Finally, if $A \geq 0$ with $\overline{\operatorname{ran}}\, A = \overline{\operatorname{ran}}\, T$, then $(TA)^{1/2} = GA = G_1 A = AG_1^*$. Hence $\ker(G_1^* G_1) = \{0\}$, and so $P_{\overline{\operatorname{ran}}\,(T)} X|_{\overline{\operatorname{ran}}\,(T)}$ is injective with dense range. □

3 Similarity and Quasi-Similarity to a Positive Operator

Recall that two operators $S, T \in L(\mathcal{H})$ are *similar* if there exists $G \in GL(\mathcal{H})$ such that $TG = GS$.

Mimicking the spectral theory for normal operators, an operator T is *spectral* if for all $\omega \subseteq \mathbb{C}$ Borel, there are (not necessarily orthogonal), uniformly bounded, countably additive projections $E(\omega)$ commuting with T such that $\sigma(T|_{\operatorname{ran} E(\omega)}) = \overline{\omega}$. If in addition, $T = \int_{\sigma(T)} \lambda dE(\lambda)$, T is termed a *scalar operator*, in which case it is similar to a normal operator A and $\sigma(T) = \sigma(A)$. More generally, any spectral operator T has a unique decomposition $T = S + N$, where S is scalar, N is quasi-nilpotent, and $SN = NS$. See, for example, [7].

Various papers, including [17, Theorem 2], have considered operators similar to selfadjoint operators. See also [20] for the connection with scalar operators. The following collects conditions for an operator to be similar to a positive operator.

Theorem 3.1 *Let $T \in L(\mathcal{H})$. The following statements are equivalent:*

(i) *$TG = GS$ for some $G \in GL(\mathcal{H})$ and $S \in \mathcal{L}^+$;*
(ii) *$TX = XT^*$ with $X \in GL(\mathcal{H})^+$ and $\sigma(T) \geq 0$;*
(iii) *$T = AB$, with $A, B \in \mathcal{L}^+$, where B, respectively A, is invertible;*
(iv) *There exist $W, Z \in GL(\mathcal{H})^+$ such that $TW \in \mathcal{L}^+$, respectively $ZT \in \mathcal{L}^+$;*
(v) *T is a scalar operator and $\sigma(T) \geq 0$.*

If any of these hold, then

$$\overline{\operatorname{ran}}\, T \dotplus \ker T = \mathcal{H}.$$

Proof *(i)* $\Rightarrow$ *(ii)*: If $0 \leq S = G^{-1}TG$, $G \in GL(\mathcal{H})$, then $\sigma(T) = \sigma(S) \geq 0$. Also, since $G^{-1}TG = G^*T^*G^{*-1}$, it follows that $(GG^*)^{-1}T(GG^*) = T^*$, or equivalently, $T(GG^*) = (GG^*)T^*$.

$(ii) \Rightarrow (iii)$: Let $T = XT^*X^{-1}$, $X \in GL(\mathcal{H})^+$, and assume that $\sigma(T) \geq 0$. Then

$$X^{1/2}T^*X^{-1/2} = X^{-1/2}TX^{1/2} = (X^{-1/2}TX^{1/2})^* \in \mathcal{L}^+,$$

and so $A := X^{1/2}(X^{-1/2}TX^{1/2})X^{1/2} = TX \geq 0$. Consequently, $T = AB$, where $B = X^{-1} > 0$. Work instead with $T^* = X^{-1}TX$ to obtain $T^* = BA$, $B \geq 0$ and $A > 0$.

$(iii) \Rightarrow (iv)$: Suppose that $T = AB$, with $A, B \in \mathcal{L}^+$ and B invertible. Let $W := B^{-1} \in GL(\mathcal{H})^+$. Then $TW = A \in \mathcal{L}^+$. If on the other hand A is invertible, $Z = A^{-1}$ yields $ZT \geq 0$.

$(iv) \Rightarrow (i)$: Suppose $W \in GL(\mathcal{H})^+$ and $TW \in \mathcal{L}^+$. Then

$$W^{-1/2}(TW)W^{-1/2} = W^{-1/2}TW^{1/2} \geq 0.$$

Similarly if $ZT \in \mathcal{L}^+$.

$(v) \Leftrightarrow (i)$: If $G \in GL(\mathcal{H})$ is such that $S = G^{-1}TG \in \mathcal{L}^+$, then $\sigma(T) \geq 0$. Let E^S be the spectral measure of S, so that $S = \int_{\sigma(S)} \lambda \, dE^S(\lambda)$. Then $E^T(\cdot) := G^{-1}E^S(\cdot)G$ is a resolution of the identity for T and $T = \int_{\sigma(S)} \lambda \, dE^T(\lambda)$. Thus T is scalar.

Conversely, if T is scalar and $\sigma(T) \geq 0$, then T is similar to S normal with $\sigma(S) = \sigma(T) \geq 0$, and thus $S \in \mathcal{L}^+$.

To prove the last statement, assume (i). Since $S \geq 0$, $\overline{\operatorname{ran}}\, S \oplus \ker S = \mathcal{H}$. Also, $\overline{\operatorname{ran}}\, T = G\overline{\operatorname{ran}}\, S$ and $\ker T = G \ker S$. Since G is injective, $\overline{\operatorname{ran}}\, T \cap \ker T = \{0\}$, and since G is surjective $\overline{\operatorname{ran}}\, T + \ker T = \mathcal{H}$. Hence $\overline{\operatorname{ran}}\, T \dotplus \ker T = \mathcal{H}$. □

If $T \in \mathcal{L}^{+2}$ is similar to $S \geq 0$, with $TG = GS$ (where without loss of generality in (i) in the last theorem, G can be taken to be positive), then as previously noted, $\Omega = \sigma(T) = \sigma(S) \subset \mathbb{R}^+$. Moreover, since it is possible to define a Borel functional calculus for S on Ω, the same then holds for T (see Theorem 3.1, where this is essentially what is implied by T being a scalar operator). In particular, if f is a Borel function, then $f(T) = Gf(S)G^{-1}$ is well-defined.

If $f(\Omega) \subset \mathbb{R}^+$, then $f(S) \geq 0$ and

$$f(T) = (Gf(S)G)(G^{-2}) \in \mathcal{L}^{+2}.$$

A case of particular interest is $f(x) = x^{1/2}$. Since $T = AB, A = (GSG), B = (G^{-2})$, it follows that $T^{1/2} = A'B$ when

$$A'BA' = A, \qquad A' \geq 0.$$

This is an example of a *Ricatti equation*, and more generally, an operator $T \in \mathcal{L}^{+2}$ will have a square root if for some factorization $T = AB$, $A, B \geq 0$, there exists $A' \geq 0$ satisfying this equality. This is examined more closely later in the section.

There is also a close connection with the *geometric mean*, defined for two positive operators E and F with E invertible as

$$E \# F = E^{1/2}(E^{-1/2}FE^{-1/2})^{1/2}E^{1/2},$$

and so satisfies $(E\,\#\,F)E^{-1}(E\,\#\,F) = F$.

Lemma 3.2 *If T is similar to a positive operator and $T = AB$ with $B \in GL(\mathcal{H})^+$, respectively, $A \in GL(\mathcal{H})^+$, then*

$$T^{1/2} = (B^{-1}\,\#\,A)B,$$

respectively, $T^{1/2} = A(A^{-1}\,\#\,B)$.

Proof By Theorem 3.1, B can be chosen invertible in $T = AB$, and then with $G = B^{-1/2}$ and $S = B^{1/2}AB^{1/2}$, $TG = GS$. Setting $E = B^{-1} = G^2$ and $F = A = GSG$, it follows that $E\,\#\,F = GS^{1/2}G$, and hence $T^{1/2} = (B^{-1}\,\#\,A)B$ since $(B^{-1}\,\#\,A)B(B^{-1}\,\#\,A) = A$. If instead A is chosen to be invertible, then working with $G^{-1}T = SG^{-1}$, one obtains $T^{1/2} = A(A^{-1}\,\#\,B)$. □

An operator which is injective with dense range is termed a *quasi-affinity*. An operator T is *quasi-affine* to C if there is a quasi-affinity X such that

$$TX = XC.$$

The operators T and C are said to be *quasi-similar* if there exist quasi-affinities $X, Y \in L(\mathcal{H})$ such that

$$TX = XC \quad \text{and} \quad YT = CY.$$

A finite or countable system $\{\mathcal{S}_n\}_{1\le n<m}$ of subspaces of $\mathcal{H}$ is called *basic* if $\mathcal{S}_n \dotplus \overline{\bigvee}_{k\neq n}\mathcal{S}_k = \mathcal{H}$ for every n ($\overline{\bigvee}$ indicating the closed span), and $\bigcap_{n\ge1}(\overline{\bigvee}_{k\ge n}\mathcal{S}_k) = \{0\}$ if $m = \infty$. In [1], Apostol uses basic systems to characterize those operators which are quasi-similar to normal operators. With only minor modification, his proof works to characterize quasi-similarity to positive operators.

Theorem 3.3 (Apostol) *The operator $T \in L(\mathcal{H})$ is quasi-similar to a positive operator if and only if there exists a basic system $\{\mathcal{S}_n\}_{n\ge1}$ of invariant subspaces of T such that $T|_{\mathcal{S}_n}$ is similar to a positive operator.*

It is sometimes useful to relax the conditions in the definition of quasi-similarity so that instead, $TX = XC$ and $YT = DY$. The next lemma shows that if C and D are positive, this is no more general.

Lemma 3.4 *Let $T \in L(\mathcal{H})$ such that $TX = XC$ and $YT = DY$, with X, Y quasi-affinities and $C, D \in \mathcal{L}^+$. Then*

(i) *C is quasi-similar to D, and*
(ii) *T is quasi-similar to C.*

Proof (i): Since $(YX)C = YTX = D(YX)$, $C(YX)^* = (YX)^*D$, and the claim follows since YX and $(YX)^*$ are quasi-affinities.

(ii): Set $Y' := (YX)^*Y$. Then Y' is a quasi-affinity and

$$\begin{aligned} Y'T &= (YX)^*YT = (YX)^*DY = X^*Y^*DY \\ &= X^*T^*Y^*Y = (TX)^*Y^*Y = CX^*Y^*Y = CY'. \end{aligned}$$

By assumption $TX = XC$, so it follows that T is quasi-similar to C. □

Definition A *rigged Hilbert space* is a triple $(\mathcal{S}, \mathcal{H}, \mathcal{S}^*)$ with $\mathcal{H}$ a Hilbert space and $\mathcal{S} \subseteq \mathcal{H}$ a dense subspace such that the inclusion $\iota : \mathcal{S} \to \mathcal{H}$ is continuous. The space $\mathcal{S}^*$ is the dual of $\mathcal{S}$, and $\mathcal{H}^* = \mathcal{H}$ is mapped into $\mathcal{S}^*$ via the adjoint map ι^*. The spaces $\mathcal{S}$ and $\mathcal{S}^*$ are identified as Hilbert spaces, with $\iota^*\iota(\mathcal{H}) = \mathcal{H}^*$.

Let $X \in L(\mathcal{H})$. Define an inner product on $\operatorname{ran} X$ by

$$\langle x, y \rangle_X := \left\langle X^{-1}x, X^{-1}y \right\rangle, \qquad x, y \in \operatorname{ran} X.$$

Then $\mathcal{H}_X := (\operatorname{ran} X, \langle \cdot, \cdot \rangle_X)$ is a Hilbert space.

The primary case of interest is when X is a quasi-affinity, in which case $\mathcal{H}_X$ can be viewed as a rigged Hilbert space.

Proposition 3.5 *Let $T \in L(\mathcal{H})$ such that $TX = XC$ with X a quasi-affinity and $C \in \mathcal{L}^+$. Then $\mathcal{H}_X$ can be identified with a rigged Hilbert space and $\hat{T} := T|_{\operatorname{ran} X} \in L(\mathcal{H}_X)^+$. Furthermore,* $\operatorname{ran} T \cap \ker T = \{0\}$ *and*

$$\overline{\operatorname{ran} T \dotplus \ker T} = \mathcal{H}.$$

Proof Let $y = XX^{-1}y \in \operatorname{ran} X$. Then $\|y\| \le \|X\|\|X^{-1}y\| = \|X\|\|y\|_X$. Therefore the inclusion map $\iota : \mathcal{H}_X \hookrightarrow \mathcal{H}$ is continuous. Thus $\mathcal{H}_X$ (or more properly, the triple $(\mathcal{H}_X, \mathcal{H}, \mathcal{H}_X^*)$) is a rigged Hilbert space. This space is simply denoted as $\mathcal{H}_X$. Note that for any set $\mathcal{S} \subseteq \operatorname{ran} X$, $\overline{\mathcal{S}}^{\mathcal{H}_X} \subseteq \overline{\mathcal{S}}$.

Since $TX = XC$, $T(\operatorname{ran} X) \subseteq \operatorname{ran} X$ and $\hat{T}$ is well defined. Also, if $y = Xx$, $v = Xw$ for some $x, w \in \mathcal{H}$,

$$\left\langle \hat{T}y, v \right\rangle_X = \left\langle X^{-1}Ty, X^{-1}v \right\rangle = \left\langle X^{-1}TXx, w \right\rangle = \left\langle X^{-1}XCx, w \right\rangle = \langle Cx, w \rangle .$$

Since $\|y\|_X = \|x\|$ and $\|v\|_X = \|w\|$, taking the supremum over y and v with norm 1 gives that $\|\hat{T}\|_X = \|C\|$ and so $\hat{T}$ is bounded in $\mathcal{H}_X$. Taking $v = y$ then yields $\hat{T} \in L(\mathcal{H}_X)^+$. By positivity $\mathcal{H} = \overline{\operatorname{ran}} C \oplus \ker C$, and $\overline{\operatorname{ran}} T = \overline{X \operatorname{ran} C}$, $\ker T = \overline{X \ker C}$, so $\overline{\operatorname{ran}}^{\mathcal{H}_X}(\hat{T}) \oplus_{\mathcal{H}_X} \ker \hat{T} = \mathcal{H}_X = \operatorname{ran} X \subseteq \overline{\operatorname{ran}} T + \ker T$, and thus $\overline{\operatorname{ran} T + \ker T} = \mathcal{H}$.

Now suppose that $0 \neq z = Tx \in \ker T$. There is a sequence $\{h_n\}$ in $\mathcal{H}$ such that $XX^*h_n \to x$, and so $T^2XX^*h_n = XC^2X^*h_n \to 0$. Let $g_n = X^*h_n$ for all n. Then $Xg_n \to x$ and for all $y \in \mathcal{H}$,

$$\left\langle Cg_n, CX^*y \right\rangle = \left\langle XC^2X^*h_n, y \right\rangle \to 0.$$

Since $\overline{\text{ran}}\,(CX) = \overline{\text{ran}}\,C$, it follows that $Cg_n \to 0$. Hence $TXX^*h_n = XCX^*h_n = XCg_n \to 0$, which implies that $Tx = 0$, a contradiction. □

Corollary 3.6 *Let $T \in L(\mathcal{H})$ be quasi-similar to a positive operator. Then $\overline{\text{ran}}\,T \cap \ker T = \{0\}$ and $\overline{\text{ran}}\,T \dotplus \ker T$ is dense in $\mathcal{H}$.*

Proof If $T \in L(\mathcal{H})$ is quasi-similar to a positive operator C, by Proposition 3.5, $\overline{\text{ran}\,T \dotplus \ker T} = \mathcal{H}$ and $\overline{\text{ran}\,T^* \dotplus \ker T^*} = \mathcal{H}$. Hence $\overline{\text{ran}}\,T \cap \ker T = \{0\}$, and so $\overline{\text{ran}}\,T \dotplus \ker T$ is dense in $\mathcal{H}$. □

The following is a special case of more general results found in [8, Corollary 2.12] and [19, Theorem 2].

Lemma 3.7 *If $T \in L(\mathcal{H})$ is quasi-affine to $C \in \mathcal{L}^+$, then $\sigma(T) \supseteq \sigma(C)$.*

If $T \in \mathcal{L}^{+2}$, then it will be shown that these spectra are equal (Proposition 7.2).

Proposition 3.8 *Let $T \in L(\mathcal{H})$. The following statements are equivalent:*

(i) *T is quasi-affine to a positive operator;*
(ii) *$T^* = BA$, with B a closed surjective positive operator and $A \in \mathcal{L}^+$;*
(iii) *There exists a quasi-affinity $X \in \mathcal{L}^+$ such that $TX \in \mathcal{L}^+$.*

Proof $(i) \Rightarrow (ii)$: Assume $TG = GS$, G a quasi-affinity, $S \geq 0$. Then $GG^*T^* = GSG^*$ and

$$T^* = (GG^*)^{-1}(GSG^*).$$

The operator GG^* is a quasi-affinity, hence $(GG^*)^{-1}$ maps $\text{ran}(GG^*)$ onto $\mathcal{H}$, and it is thus surjective, closed, and so selfadjoint. Since for all x, $\left\langle (GG^*)^{-1}GG^*x, GG^*x \right\rangle = \langle x, GG^*x \rangle \geq 0$, $(GG^*)^{-1}$ is positive.

$(ii) \Rightarrow (iii)$: Assume (ii). Since B is surjective, by the closed graph theorem, $B^{-1} : \mathcal{H} \to \text{dom}\,(B)$ is bounded, and since B is positive, B^{-1} is a quasi-affinity, and is also positive. Then $B^{-1}T^* = A \geq 0$, and the claim follows with X = B^{-1}.

$(iii) \Rightarrow (i)$: Suppose there exists a quasi-affinity $X \in \mathcal{L}^+$ such that $TX = XT^* \geq 0$. According to [10, Theorem 5.1], if A, B, and C are bounded operators with $A \geq 0$ and $AB = C^*A$, there exists a unique bounded S with $\ker A \subseteq \ker S$ such that $A^{1/2}B = SA^{1/2}$ and $C^*A^{1/2} = A^{1/2}S$. Translating to the present context, take $A = X$ and $B = C = T^*$. Then there exists a bounded S so that $X^{1/2}T^* = SX^{1/2}$, equivalently, $TX^{1/2} = X^{1/2}S^*$. Thus $S^* = X^{-1/2}TX^{1/2}$.

For all $x \in \mathcal{H}$ and $y = X^{1/2}x$,

$$\left\langle S^*y, y \right\rangle = \left\langle X^{-1/2}TXx, X^{1/2}x \right\rangle = \langle TXx, x \rangle \geq 0.$$

It follows by polarization that S is selfadjoint, and so $S \geq 0$. □

Corollary 3.9 *Let $T \in L(\mathcal{H})$. The following statements are equivalent:*

(i) *T is quasi-similar to a positive operator;*
(ii) *$T = AB$, with A a closed surjective positive operator and $B \in \mathcal{L}^+$, and $T^* = B'A'$, with B' a closed surjective positive operator and $A' \in \mathcal{L}^+$;*
(iii) *There exist quasi-affinities $W, Z \in \mathcal{L}^+$ such that TW and $ZT \in \mathcal{L}^+$;*
(iv) *There exists a basic system $\{\mathcal{S}_n\}_{n\geq 1}$ of invariant subspaces of T such that for all n, $T|_{\mathcal{S}_n}$ is scalar and $\sigma(T|_{\mathcal{S}_n}) \geq 0$.*

Proof The equivalence of (i)—(iii) is a direct consequence of Proposition 3.8. The last item is equivalent to (i) by Theorem 3.3. □

The last result resembles Theorem 3.1, though under the weaker condition of quasi-similarity it appears not to be possible to say much about the spectrum of T without some extra conditions. See Sect. 7.

Coming back to square roots, suppose that $TG = GS$, where $G \geq 0$ is a quasi-affinity and $S \geq 0$. Then there exists a densely defined linear operator R mapping $\operatorname{ran} X$ to itself such that $RX = XC^{1/2}$. However it may not be the case that R is bounded. Circumstances when it is will be addressed further on.

4 The Set $\mathcal{L}^{+2}$

The remainder of the paper is devoted to the study of the set of products of two positive bounded operators,

$$\mathcal{L}^{+2} := \{T \in L(\mathcal{H}) : T = AB \text{ with } A, B \in \mathcal{L}^+\}.$$

The subclasses $\mathcal{P}\cdot\mathcal{P}$ and $\mathcal{P}\cdot\mathcal{L}^+$ were considered in [2] and [4].

If $T \in \mathcal{L}^{+2}$, it is straightforward to check that $T^* \in \mathcal{L}^{+2}$ and $GTG^{-1} \in \mathcal{L}^{+2}$ for all $G \in GL(\mathcal{H})$. Then the *similarity orbit* of T, $\mathbb{O}_T := \{GTG^{-1} : G \in GL(\mathcal{H})\} \subseteq \mathcal{L}^{+2}$. Also, it can easily be verified that $\{T^n : n \in \mathbb{N}\} \subseteq \mathcal{L}^{+2}$.

From the basic fact that for two operators C and D, $\sigma(CD) \cup \{0\} = \sigma(DC) \cup \{0\}$, the following is immediate.

Lemma 4.1 *Let $T = AB \in \mathcal{L}^{+2}$, $A, B \in \mathcal{L}^+$. Then $\sigma(T) = \sigma(A^{1/2}BA^{1/2}) \geq 0$.*

Proof As already observed, $\sigma(T) \cup \{0\} = \sigma(A^{1/2}BA^{1/2}) \cup \{0\} \geq 0$. If $0 \notin \sigma(T)$, then A and B are invertible, so $0 \notin \sigma(A^{1/2}BA^{1/2})$. Likewise, $0 \notin \sigma(A^{1/2}BA^{1/2})$ implies $0 \notin \sigma(T)$, and so the stated equality holds. □

Example 1 Lemma, 4.1 implies that a normal operator in $\mathcal{L}^{+2}$ is positive. Suppose now that $T \in \mathcal{L}^{+2}$ is subnormal. Let N be the minimal normal extension of T. Then $\sigma(N) = \sigma(T) \geq 0$, and so N is positive. Since T is the restriction of N to an invariant subspace, it too is then positive.

It will be proved in Proposition 6.3 that an operator in $\mathcal{L}^{+2}$ with closed range is similar to a positive operator. This will imply then that any partial isometry V in $\mathcal{L}^{+2}$ is similar to an orthogonal projection, and so is itself a projection. Since V is a

contraction, this means that ran V is orthogonal to ker V, and so $V \geq 0$ is an orthogonal projection.

Proposition 4.2 *Let $T \in \mathcal{L}^{+2}$. Then there exist $A, B \in \mathcal{L}^+$ such that $T = AB$, $\overline{\text{ran}}\, A = \overline{\text{ran}}\, T$ and $\ker B = \ker T$. For this pair, $\text{ran}\, B \cap \ker A = \text{ran}\, A \cap \ker B = \{0\}$, and it follows then that*

$$\text{ran}\, T \cap \ker T = \{0\}.$$

Proof Let $T = A_0 B_0 \in \mathcal{L}^{+2}$. Then, by Theorem 2.1, there exists $B \in \mathcal{L}^+$ such that $T = A_0 B$ and $\ker B = \ker T$. On the other hand, $T^* = B A_0 \in \mathcal{L}^{+2}$ and again by Theorem 2.1, there exists $A \in \mathcal{L}^+$ such that $T^* = BA$ and $\ker A = \ker T^*$. If $x \in \text{ran}\, B \cap \ker A$ then $x = By$ for some $y \in \mathcal{H}$ and $0 = Ax = ABy = Ty$. Hence $y \in \ker T = \ker B$, and so $x = 0$. The other equality follows in a similar way. It is then immediate from this that $\text{ran}\, T \cap \ker T = \{0\}$. □

Corollary 4.3 *If $T \in \mathcal{L}^{+2}$, then $\overline{\text{ran}}\,(T|_{\overline{\text{ran}}\, T}) = \overline{\text{ran}}\, T$.*

Proof If $T \in \mathcal{L}^{+2}$, then $T = AB$ with $\overline{\text{ran}}\, T = \overline{\text{ran}}\, A$ by Proposition 4.2. Therefore, $\overline{\text{ran}}\,(T|_{\overline{\text{ran}}\, T}) = \overline{\text{ran}}\,(T P_{\overline{\text{ran}}\, T}) = (\ker(P_{\overline{\text{ran}}\, T} T^*))^\perp$. But $\ker(P_{\overline{\text{ran}}\, T} T^*) = (\text{ran}\, T^* \cap \ker T^*) + \ker T^* = \ker T^*$, again by Proposition 4.2. □

Definition For $A, B \in \mathcal{L}^+$, the pair (A, B) is called *optimal* for $T = AB$ if $\overline{\text{ran}}\, T = \overline{\text{ran}}\, A$ and $\ker B = \ker T$.

According to Proposition 4.2, whenever $T \in \mathcal{L}^{+2}$, it can be written as a product involving an optimal pair. Clearly, the pair (A, B) is optimal for T if and only if the pair (B, A) is optimal for T^*.

Example 2 Any oblique projection Q is in $\mathcal{L}^{+2}$. For suppose that $\mathcal{M} = \text{ran}\, Q$. Then $Q P_{\mathcal{M}} = P_{\mathcal{M}} = P_{\mathcal{M}} Q^*$ and $P_{\mathcal{M}} Q = Q$. Therefore $Q = P_{\mathcal{M}}(Q^* Q)$. Obviously, $(P_{\mathcal{M}}, Q^* Q)$ is an optimal pair for Q.

If $T \in \mathcal{P}^2$ then $\overline{\text{ran}}\, T \cap \ker T = \{0\}$ (see [4, Theorem 3.2]). This no longer need be the case if $T \in \mathcal{L}^{+2}$, as the following example shows.

Example 3 [2, Lemma 3.1] Let $A \in \mathcal{L}^+$ with non-closed dense range and $x \in \overline{\text{ran}}\, A \setminus \text{ran}\, A$. Define $\mathcal{S} = \text{span}\{x\}^\perp$ and $T = A P_{\mathcal{S}} \in \mathcal{L}^{+2}$. Then $\ker T = \text{span}\{x\}$, $\overline{\text{ran}}\, T^* = \mathcal{S}$, $\ker T^* = \{y : Ay \in \ker P_{\mathcal{S}}\} = \{0\}$, and $\overline{\text{ran}}\, T = \mathcal{H}$. Hence $\overline{\text{ran}}\, T \cap \ker T = \text{span}\{x\}$, $\overline{\text{ran}}\, T^* \cap \ker T^* = \{0\}$, and $\overline{\overline{\text{ran}}\, T^* \dotplus \ker T^*} = \mathcal{S}$. By Proposition 3.4, T^* is not quasi-affine to any positive operator, though T is.

If instead $T = (A \oplus B)(B \oplus A)$ on $\mathcal{H} \oplus \mathcal{H}$, then for neither T nor T^* is the closure of the range intersected with the kernel nontrivial, nor the sum of the range with the kernel dense in $\mathcal{H} \oplus \mathcal{H}$. As a consequence of Proposition 3.4, neither is quasi-affine to a positive operator. Clearly, in these examples T is not quasi-similar to a positive operator.

Operators $T \in \mathcal{L}^{+2}$ with a factorization $T = AB$ where one of A or B has closed range have special properties (see, for example, Theorem 5.4 and Corollary 7.4).

Proposition 4.4 *Let $T \in \mathcal{L}^{+2}$. If T is similar to a positive operator, then there exists an optimal pair with* ran A, *respectively* ran B, *closed.*

Proof Suppose that $T \in \mathcal{L}^{+2}$ is similar to a positive operator, so $T = GCG^{-1}$, $C \geq 0$. Let P be the projection onto the closure of the range of C. Then GPG^* and $G^{*-1}PG^{-1}$ have closed range, and $T = (GPG^*)(G^{*-1}CG^{-1}) = (GCG^*)(G^{*-1}PG^{-1})$. It is readily seen that $(GPG^*, G^{*-1}CG^{-1})$ and $(GCG^*, G^{*-1}PG^{-1})$ are optimal pairs. □

It is natural to wonder at this point if the class of operators in $\mathcal{L}^{+2}$ which are quasi-similar to a positive operator is strictly larger than the class of those which are similar to a positive operator.

Example 4 As noted, if $T \in \mathcal{P}^2$ then $\overline{\text{ran}}\, T \cap \ker T = \{0\}$. Furthermore, $\overline{\text{ran}}\, T \dotplus \ker T = \mathcal{H}$ if and only if ran T is closed. An operator $T \in \mathcal{P}^2$ without closed range is constructed as follows. Assuming $\dim \mathcal{H} = \infty$, there exist two closed subspaces $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{M} \cap \mathcal{N} = \{0\}$ and $\mathcal{M} \dotplus \mathcal{N}$ is dense in, but not equal to $\mathcal{H}$. Take $T = AB$, A and B be orthogonal projections onto $\mathcal{M}$ and $\mathcal{N}^\perp$, respectively. Then $\overline{\text{ran}}\, T = \mathcal{M}$ and $\ker T = \mathcal{N}$, so $\overline{\text{ran}}\, T \dotplus \ker T$ is dense in, but not equal to $\mathcal{H}$.

Let $W = A + P_{\mathcal{N}}$ and $Z = B + P_{\mathcal{M}^\perp}$. Clearly, both are positive. Also, $\ker W = \mathcal{M}^\perp \cap \mathcal{N}^\perp = \{0\}$, and similarly, $\ker Z = \{0\}$, so both are quasi-affinities. Since $TW = ABA$ and $ZT = BAB$ are both positive, it follows from Corollary 3.9 that T is quasi-similar to a positive operator. By Theorem 3.1, T cannot be similar to a positive operator, since $\overline{\text{ran}}\, T \dotplus \ker T \neq \mathcal{H}$.

The above example can also be used to construct $T \in \mathcal{L}^{+2}$ which again is quasi-similar, but not similar to a positive operator, but now with $\ker T = \ker T^* = \{0\}$. Let $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$, where $\dim \mathcal{K} = \infty$. Define $\mathcal{M} := \mathcal{K} \oplus \{0\}$, and choose $\mathcal{N}$ as above. Notice that $\dim \mathcal{N} = \dim \mathcal{M}$, so there is a unitary V on $\mathcal{H}$ mapping $\mathcal{N}$ to $\mathcal{M}$ and $\mathcal{N}^\perp$ to $\mathcal{M}^\perp$.

Let $A_1 = P_{\mathcal{M}}$, $B_1 = P_{\mathcal{N}^\perp}$, $A_2 = P_{\mathcal{N}}$, $B_2 = P_{\mathcal{M}^\perp}$. So $A_2 = V^*A_1V$ and $B_2 = V^*B_1V$. Set $A = \frac{1}{\sqrt{2}}(A_1 + A_2)$, $B = \frac{1}{\sqrt{2}}(B_1 + B_2)$. These are both positive and injective, but since neither $\mathcal{M} \dotplus \mathcal{N}$ nor $\mathcal{N}^\perp \dotplus \mathcal{M}^\perp$ equals $\mathcal{H}$, the ranges of A and B are not closed.

Let $W = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ V \end{pmatrix}$, and set

$$T = AB = A_1B_1 + A_2B_2 = W^*(A_1B_1 \otimes 1_2)W,$$

where $A_1B_1 \otimes 1_2$ is the 2×2 diagonal operator matrix with diagonal entries A_1B_1. The operator W is an isometry, and T is injective with dense range.

Suppose that T is similar to a positive operator, $T = GCG^{-1}$. The operators

$$W' := \frac{1}{\sqrt{2}} \begin{pmatrix} -V^* \\ 1 \end{pmatrix}, \qquad W'' := \frac{1}{\sqrt{2}} \begin{pmatrix} V^* \\ 1 \end{pmatrix}$$

are also isometric and $U = \begin{pmatrix} W & W' \end{pmatrix}$ is unitary. Furthermore,

$$W'^*(A_1B_1 \otimes 1_2)W' = W''^* \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} (A_1B_1 \otimes 1_2) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} W''$$
$$= W''^*(A_1B_1 \otimes 1_2)W'' = VW^*(A_1B_1 \otimes 1_2)WV^* = (VG)C(VG)^{-1}.$$

Hence

$$A_1B_1 \otimes 1_2 = U \begin{pmatrix} G & 0 \\ 0 & VG \end{pmatrix} (C \otimes 1_2) \begin{pmatrix} G & 0 \\ 0 & VG \end{pmatrix}^{-1} U^{-1};$$

that is, $A_1B_1 \otimes 1_2$ is similar to a positive operator. But by the same reasoning employed in showing that A_1B_1 is quasi-similar, but not similar to a positive operator, the same holds for $A_1B_1 \otimes 1_2$, giving a contradiction. Hence, T is also quasi-similar, but not similar to a positive operator.

It is noteworthy that if the operator T just constructed were to have a factorization $T = AB$, where one of A or B has closed range, then by Theorem 3.1, T would be similar to a positive operator. Hence there can be no such factorization for this T.

The following characterization of the elements of $\mathcal{L}^{+2}$ is immediate from Theorem 2.1.

Theorem 4.5 *Let $T \in L(\mathcal{H})$. Then $T \in \mathcal{L}^{+2}$ if and only if the inequality $TT^* \leq XT^*$ admits a positive solution.*

Proof If $T \in \mathcal{L}^{+2}$ then there exist $A, B \in \mathcal{L}^+$ such that $T = AB$. Since $B^2 \leq \|B\|B$ then $TT^* = AB^2A \leq \|B\|ABA = \|B\|AT^*$. Therefore, $\|B\|A$ is a positive solution of $TT^* \leq XT^*$. Conversely, if $A \in \mathcal{L}^+$ satisfies $TT^* \leq AT^*$ then, by Theorem 2.1, the equation $T = AX$ admits a positive solution. Therefore $T \in \mathcal{L}^{+2}$. □

Corollary 4.6 *Let $T \in \mathcal{L}^{+2}$ and $A \in \mathcal{L}^+$. Then T can be factored as $T = AB$, with $B \in \mathcal{L}^+$ if and only if λA is a solution of $TT^* \leq XT^*$, for some $\lambda \geq 0$.*

Corollary 4.7 *The operator $T \in \mathcal{L}^+ \cdot \mathcal{P}$ if and only if $TT^* = XT^*$ admits a positive solution. Moreover, $T \in \mathcal{P}^2$ if and only if $TT^* = XT^*$ admits a solution in $\mathcal{P}$.*

Proof If $T = AP$, $A \geq 0$ and P an orthogonal projection, then $TT^* = AP^2A = APA = AT^*$. Conversely, if $TT^* = XT^*$ admits a positive solution $X = A \geq 0$, then $|T^*|^2 = AU|T^*|$, where U is a partial isometry from $\overline{\text{ran}}\, T$ onto $\overline{\text{ran}}\, T^*$. Thus $|T^*| = AU = U^*A$, and so $T^* = UU^*A$, where UU^* is an orthogonal projection. □

The next result will be particularly useful for describing spectral properties of elements of $\mathcal{L}^{+2}$ in Sect. 5. It was proved for invariant subspaces in the finite dimensional case in [22]. Recall that a subspace $\mathcal{M}$ is *invariant* for an operator T if $T\mathcal{M} \subseteq \mathcal{M}$.

Proposition 4.8 *Let $T \in \mathcal{L}^{+2}$ and suppose $\mathcal{M}$ is invariant for T. Then $TP_{\mathcal{M}} \in \mathcal{L}^{+2}$.*

Proof Write $T = AB$, $A, B \in \mathcal{L}^+$. Then $T^*T \leq \lambda BT$ for $\lambda = \|A\|$. Assume that $\mathcal{M}$ is invariant. Then

$$P_{\mathcal{M}} T^* T P_{\mathcal{M}} \leq \lambda P_{\mathcal{M}} BT P_{\mathcal{M}} = \lambda P_{\mathcal{M}} B P_{\mathcal{M}} T P_{\mathcal{M}}.$$

Since $\lambda P_{\mathcal{M}} B P_{\mathcal{M}} \geq 0$, by Theorem 4.5, $T P_{\mathcal{M}} \in \mathcal{L}^{+2}$. □

From the proof of Theorem 4.5, $T P_{\mathcal{M}}$ above has the form $C(P_{\mathcal{M}} B P_{\mathcal{M}})$ for some $C \in \mathcal{L}^+$.

In fact, it is not difficult to see that since $T^* \in \mathcal{L}^{+2}$, the above proposition is true more generally for *semi-invariant* subspaces; that is, subspaces of the form $\mathcal{M} = \mathcal{M}_1 \ominus \mathcal{M}_2$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are invariant for T.

Definition Given $T \in \mathcal{L}^{+2}$ and $A \geq 0$ with $\overline{\operatorname{ran}}\, A = \overline{\operatorname{ran}}\, T$, define

$$\mathcal{B}_T^A = \{X \geq 0 : T = AX\}.$$

Note that even if $\overline{\operatorname{ran}}\, A = \overline{\operatorname{ran}}\, T$ and $\operatorname{ran} A \supseteq \operatorname{ran} T$, the set $\mathcal{B}_T^A$ may be empty. As just seen in Corollary 4.6, A must also satisfy $TT^* \leq \lambda AT^*$ for some $\lambda > 0$.

Theorem 4.9 *Let $T \in \mathcal{L}^{+2}$ and $A \geq 0$ be such that $\mathcal{B}_T^A \neq \emptyset$. Then $\mathcal{B}_T^A$ has a minimum B_0. The pair (A, B_0) is optimal and the set $\mathcal{B}_T^A$ is the cone*

$$\mathcal{B}_T^A = \{B_0 + Z : Z \in \mathcal{L}^+ \text{ and } \overline{\operatorname{ran}}\, Z \subseteq \ker T^*\}.$$

Moreover, for every $B \in \mathcal{B}_T^A$, $B_{\overline{\operatorname{ran}}\, T} = B_0$, and the pair (A, B) is optimal if and only if $\operatorname{ran} Z \subseteq \overline{\operatorname{ran}}\, T^* \cap \ker T^*$.

Proof Let $B \in \mathcal{B}_T^A$ and $B_0 = G^*G$ the solution of $T = AX$ constructed in the proof of Theorem 2.1. With respect to the decomposition $\mathcal{H} = \overline{\operatorname{ran}}\, T \oplus \ker T^*$, B has an LU-decomposition,

$$B = F^*F = \begin{pmatrix} F_1^* & 0 \\ F_2^* & F_3^* \end{pmatrix} \begin{pmatrix} F_1 & F_2 \\ 0 & F_3 \end{pmatrix}.$$

Also by Theorem 2.1,

$$B_0 = \begin{pmatrix} G_1^* & 0 \\ G_2^* & 0 \end{pmatrix} \begin{pmatrix} G_1 & G_2 \\ 0 & 0 \end{pmatrix}.$$

Since the theorem also gives in this circumstance that $G_1^* G_1$ is a quasi-affinity, there is no loss in generality in taking $G_1 \geq 0$ with dense range.

Now

$$TA = ABA = AF_1^* F_1 A = AB_0 A = AG_1^2 A,$$

and since $\overline{\operatorname{ran}} A = \overline{\operatorname{ran}} T$, $F_1^* F_1 = G_1^2$. Without loss of generality, take $F_1 = G_1$ (adjusting F_2 and F_3 as necessary). So

$$T = AG^*G = A\begin{pmatrix} G_1^2 & G_1G_2 \end{pmatrix} = AF^*F = A\begin{pmatrix} G_1^2 & G_1F_2 \end{pmatrix}.$$

Therefore,

$$G_2^* G_1 A = F_2^* G_1 A.$$

Since both G_1 and A are positive with dense ranges in $\overline{\operatorname{ran}} T$, $\overline{\operatorname{ran}}(G_1 A) = \overline{\operatorname{ran}} T$. Hence by continuity, $F_2 = G_2$. Therefore $F = \begin{pmatrix} G_1 & G_2 \\ 0 & F_3 \end{pmatrix}$ and

$$Z := B_{/\overline{\operatorname{ran}} T} = \begin{pmatrix} 0 & 0 \\ 0 & F_3^* F_3 \end{pmatrix} \geq 0,$$

giving $B = B_0 + Z$, $Z \geq 0$, $\overline{\operatorname{ran}} Z \subseteq \ker T^*$, and $B_{\overline{\operatorname{ran}} T} = B_0$.

Finally, suppose that the pair (A, B) is optimal. So $\overline{\operatorname{ran}} B = \overline{\operatorname{ran}} T^*$, where $B = B_0 + Z$, and since $\overline{\operatorname{ran}} B_0 = \overline{\operatorname{ran}} T^*$, it must be that $\overline{\operatorname{ran}} Z \subseteq \overline{\operatorname{ran}} T^*$. Hence, $\overline{\operatorname{ran}} Z \subseteq \overline{\operatorname{ran}} T^* \cap \ker T^*$. On the other hand, if $B = B_0 + Z$, $Z \geq 0$, and $\overline{\operatorname{ran}} Z \subseteq \overline{\operatorname{ran}} T^* \cap \ker T^*$, then $\overline{\operatorname{ran}} B = \overline{\operatorname{ran}} T^*$, and so (A, B) is optimal. □

Theorem 4.9 states that if $T \in \mathcal{L}^{+2}$ admits an optimal pair (A, B), then there is an optimal pair (A, B_0) where B_0 has minimal norm among the operators in the set $\mathcal{B}_T^A$. Furthermore, B_0 is the minimal positive completion of the operator matrix $\begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & * \end{pmatrix}$. However, (A, B_0) need not be the unique optimal pair for T with A as the first factor.

Example 5 Consider $T = P_{\mathcal{S}} A$, where $A \geq 0$ and $\mathcal{S}$ are defined as in Example 3. Then by Theorem 4.9, for any $\lambda > 0$, $(P_{\mathcal{S}}, A + \lambda(1 - P_{\mathcal{S}}))$ is an optimal pair for T.

The next result gives a condition for the optimal pair (A, B) to be unique when one of the terms is fixed.

Corollary 4.10 *Let $T \in \mathcal{L}^{+2}$ and A such that $\mathcal{B}_T^A \neq \emptyset$ with minimal element B_0. Then (A, B_0) is the unique optimal pair for T with A as the first factor if and only if $\overline{\operatorname{ran}} T \dotplus \ker T = \mathcal{H}$. Additionally, for fixed A, (A, B_0) and (B_0, A) are unique optimal pairs for T and T^*, respectively, if and only if $\overline{\operatorname{ran}} T \dotplus \ker T = \mathcal{H}$.*

Proof This follows directly from Theorem 4.9, since there can be more than one optimal pair (A, B) for fixed A if and only if $\overline{\operatorname{ran}} T^* \cap \ker T^* \neq \{0\}$. The condition $\overline{\operatorname{ran}} T \dotplus \ker T = \mathcal{H}$ implies that $\overline{\operatorname{ran}} T^* \cap \ker T^* = \{0\}$, and by a similar argument as at the end of the proof of Theorem 4.9, this condition is necessary and sufficient for there to be a unique optimal pair (B, A) for T^* with A fixed. □

There is a dilation theory for elements of $\mathcal{L}^{+2}$ which mimics that of contractions on Hilbert spaces.

Proposition 4.11 *Let $T \in \mathcal{L}^{+2}$. Then there is a Hilbert space $\mathcal{H}' \supseteq \mathcal{H}$, and an operator $T' \in \mathcal{L}^+ \cdot \mathcal{P}$ on $\mathcal{H}'$ such that T is the restriction of T' to an invariant subspace, $\overline{\operatorname{ran}}\, T' = \overline{\operatorname{ran}}\, T$, and $\ker T' \supseteq \ker T$. There is also a Hilbert space $\mathcal{H}'' \supseteq \mathcal{H}'$ and $T'' \in \mathcal{P}^2$ on $\mathcal{H}''$, such that $\mathcal{H}'$ is invariant for T''^*, $\mathcal{H}$ is semi-invariant for T'', and T is the compression of cT'' for some $c > 0$.*

Proof Suppose that $T = AB$, where (A, B) is optimal and by scaling if necessary, that $\|B\| \leq 1$. On $\mathcal{H}' = \mathcal{H} \oplus \mathcal{H}$, the operator

$$\tilde{B} := \begin{pmatrix} B & B^{1/2}(1-B)^{1/2} \\ (1-B)^{1/2}B^{1/2} & 1-B \end{pmatrix} = \begin{pmatrix} B^{1/2} \\ (1-B)^{1/2} \end{pmatrix} \begin{pmatrix} B^{1/2} & (1-B)^{1/2} \end{pmatrix}$$

is seen to be a projection since the column operator is an isometry. Extend A to $\tilde{A}$ by padding with 0s. Then $T' := \tilde{A}\tilde{B} \in \mathcal{L}^+ \cdot \mathcal{P}$ and

$$T' = \begin{pmatrix} AB & AB^{1/2}(1-B)^{1/2} \\ 0 & 0 \end{pmatrix}.$$

Clearly, $\mathcal{H}$ is invariant for T' and $T = P_{\mathcal{H}} T'|_{\mathcal{H}}$. Also, $\overline{\operatorname{ran}}\, T \subseteq \overline{\operatorname{ran}}\, T' \subseteq \overline{\operatorname{ran}}\, A = \overline{\operatorname{ran}}\, T$, so equality holds throughout. It is obvious that if $f \in \mathcal{H}$ is in $\ker T$, it is in $\ker T'$.

The operator T'' is constructed by applying the same method to cT'^*, where c is chosen so that $\|cT'^*\| \leq 1$. □

Let $T \in \mathcal{L}^{+2}$. Using the Löwner order, define a partial order on the set of optimal pairs for T by

$$(A_\alpha, B_\alpha) \prec (A_\beta, B_\beta)$$

if $A_\alpha \leq A_\beta$ and $B_\alpha \leq B_\beta$.

Definition Let $T \in \mathcal{L}^{+2}$. An optimal pair for T, (A_{min}, B_{min}), is said to be *minimal* if for an optimal pair (A, B), $(A, B) \prec (A_{min}, B_{min})$ implies that $(A, B) = (A_{min}, B_{min})$.

Proposition 4.12 *Let $T \in \mathcal{L}^{+2}$. For every optimal pair (A, B) for T, there exists a minimal optimal pair $(A_{min}, B_{min}) \prec (A, B)$.*

Proof Suppose that with respect to the partial order $\prec$, $(A_\lambda, B_\lambda)_{\lambda \in \Lambda}$ is a chain in the collection of optimal pairs for T. Then the decreasing nets of positive operators $(A_\lambda)_\lambda$, $(B_\lambda)_\lambda$ converge strongly to some $A, B \in \mathcal{L}^+$, respectively, and $T = AB$. Since $\ker B_\lambda = \ker T$, $\ker T \subseteq \ker B$, and since $T = AB$, equality holds. Likewise, $\ker T^* = \ker A$. Hence (A, B) is optimal. Thus every chain has a lower bound, and so minimal optimal pairs exist by Zorn's lemma. □

Remark For any minimal optimal pair (A, B), $A = A_{\overline{\text{ran}}\, T^*}$ and $B = B_{\overline{\text{ran}}\, T}$. So $A = F^*F$, $B = G^*G$, where

$$F = \begin{pmatrix} F_1 & F_2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} G_1 & G_2 \\ 0 & 0 \end{pmatrix}$$

on $\overline{\text{ran}}\, T^* \oplus \ker T$ and $\overline{\text{ran}}\, T \oplus \ker T^*$, respectively.

Minimal optimal pairs need not be unique. As a simple example, let $R > 1$ on $\mathcal{H}$, and $T = R \oplus R^{-1}$ on $\mathcal{H} \oplus \mathcal{H}$. Then for $A = R \oplus 1$, $B = 1 \oplus R^{-1}$, both (A, B) and (B, A) are minimal optimal pairs for T.

Lemma 4.1 already hints that operators in $\mathcal{L}^{+2}$ share certain properties with positive operators, many more of which will be explored in the next section. It is reasonable to wonder if an operator in $\mathcal{L}^{+2}$ has a square root in $\mathcal{L}^{+2}$. Partial results in this direction are given next. First, recall the following result of Pedersen and Takesaki [15] (slightly rephrased).

Proposition 4.13 (Pedersen-Takesaki) *Let $H, K \in \mathcal{L}^+$, and write $\mathcal{K}$ for $\overline{\text{ran}}\, H$. A necessary and sufficient condition for the existence of $X \in \mathcal{L}^+$ such that $P_{\mathcal{K}} K P_{\mathcal{K}} = XHX$ is that $(H^{1/2} K H^{1/2})^{1/2} \leq aH$ for some $a \geq 0$.*

Though they do not show it, under the conditions of the proposition, with respect to the decomposition $\mathcal{H} = \mathcal{S}_1 \oplus \mathcal{S}_2$, $\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{K}$, X can be chosen as the $(2, 2)$ entry of the $\mathcal{S}_1$-compression of

$$\begin{pmatrix} aH & (H^{1/2} K H^{1/2})^{1/4} \\ (H^{1/2} K H^{1/2})^{1/4} & 1 \end{pmatrix} \geq 0.$$

Proposition 4.14 *Let $T \in \mathcal{L}^{+2}$ and suppose that T is quasi-affine to a positive operator, $TX = XC$. If $C^{1/2} \leq aX^*X$ for some $a \geq 0$, then T has a square root in $\mathcal{L}^{+2}$. Otherwise, if $\overline{\text{ran}}\, T = \overline{\text{ran}}\, T^*$ and T has a factorization satisfying the conditions of Proposition 4.13, then T admits a square root in $\mathcal{L}^{+2}$.*

Proof If $C^{1/2} \leq aX^*X$, by Douglas' lemma, $C^{1/4} = X^*F$. Hence $C^{1/2} = X^*GX$, where $G \geq 0$, and so $TX = (XX^*)G(XX^*)GX$. Since $\text{ran}\, X$ is dense, $T = ((XX^*)G)^2$.

Now suppose instead that $\overline{\text{ran}}\, T = \overline{\text{ran}}\, T^*$ and T has a factorization satisfying the conditions of Proposition 4.13. Choose $H = B$ and $K = A$ in Proposition 4.13. Then $A = XBX$ for some $X \in \mathcal{L}^+$, and so $T = (XB)^2$. □

It was already noted in Theorem 3.1 that if T is similar to a positive operator, it has a square root in $\mathcal{L}^{+2}$. The next example shows that this fails more generally. The explanation requires a lemma showing that an injective positive operator has a unique square root in $\mathcal{L}^{+2}$—namely the positive square root.

Lemma 4.15 *If $A \in \mathcal{L}^+$ is injective, $R^2 = A^2$, and $RX = XA$ for some quasi-affinity X, then $R = A$. Consequently, if $R \in \mathcal{L}^{+2}$ satisfies $R^2 = A^2$, then $R = A$.*

Proof Suppose that $A \in \mathcal{L}^+$ is injective, $R^2 = A^2$, and $RX = XA$, where X is a quasi-affinity. Then $A^2X = R^2X = XA^2$, and so $AX = XA$ since A is in the commutative C^*-algebra generated by A^2. Thus $RX = AX$, and therefore $R = A$.

Write $R = XY$, $X, Y \in \mathcal{L}^+$ and injective. Then $(XY)^2 = R^2 = A^2 = R^{*2} = (YX)^2$. Hence $A^2X^{1/2} = X^{1/2}(X^{1/2}YX^{1/2})^2$, and by [6, Lemma 4.1], there is a unitary U such that $A^2 = U(X^{1/2}YX^{1/2})^2U^*$, and so $A = U(X^{1/2}YX^{1/2})U^*$. Thus

$$R(X^{1/2}U^*) = (XY)(X^{1/2}U^*) = (X^{1/2}U^*)A.$$

Since $X^{1/2}U^*$ is a quasi-affinity, $R = A$. □

Example 6 Following ideas from [4], let $T \in \mathcal{P}^2$ be the product of A and B, non-trivial projections, given as follows. With respect to the decomposition $\mathcal{H} = \operatorname{ran} B \oplus (\operatorname{ran} B)^\perp$,

$$T = AB = \begin{pmatrix} S & S^{1/2}(1-S)^{1/2} \\ (1-S)^{1/2}S^{1/2} & 1-S \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} S & 0 \\ (1-S)^{1/2}S^{1/2} & 0 \end{pmatrix},$$

where $S \geq 0$ is injective but not invertible on $\operatorname{ran} B$ with $\|S\| < 1$, so that $(1-S)^{1/2}$ is invertible. Then $\ker T = \ker B$, and $\overline{\operatorname{ran}}\, T = \operatorname{ran} G$, $G = \begin{pmatrix} S^{1/2} & 0 \\ (1-S)^{1/2} & 0 \end{pmatrix}$, and so equals $\operatorname{ran} A$ since $A = GG^*$. This has zero intersection with $\ker T$ and $\overline{\operatorname{ran}}\, T \dotplus \ker T = \operatorname{ran} S \oplus \ker B$, which is dense in $\mathcal{H}$. As an aside, in Corollary 7.3 it will be shown that this condition implies that T is quasi-similar to a positive operator.

Suppose that $T = (XY)^2$ for some $X, Y \in \mathcal{L}^+$; that is, T has a square root in $\mathcal{L}^{+2}$. Without loss of generality, (X, Y) can be chosen to be an optimal pair for $R := XY$. Clearly, $\ker R \subseteq \ker T$, and since $\operatorname{ran} R \cap \ker R = \{0\}$, the reverse containment also holds. Hence $\overline{\operatorname{ran}}\, Y = \operatorname{ran} B$. A similar calculation gives $\overline{\operatorname{ran}}\, T = \operatorname{ran} A$.

Now

$$R^2 = \left(\begin{pmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{pmatrix} \begin{pmatrix} Y_{11} & 0 \\ 0 & 0 \end{pmatrix} \right)^2 = \begin{pmatrix} (X_{11}Y_{11})^2 & 0 \\ (X_{12}^*Y_{11})(X_{11}Y_{11}) & 0 \end{pmatrix}.$$

Thus $(X_{11}Y_{11})^2 = S$, and so by Lemma 4.15, $X_{11}Y_{11} = S^{1/2}$. Since $(1-S)^{1/2} = X_{12}^*Y_{11} = Y_{11}X_{12}$ is invertible and Y_{11} is injective, Y_{11} is invertible by the open mapping theorem, and hence the same is true for X_{12}. The operator S is not invertible, so X_{11} is not invertible. However, since $X \geq 0$, $X_{12} = X_{11}^{1/2}G$, and so $\operatorname{ran} X_{11}^{1/2}$ is closed. This implies that $X_{11}^{1/2}$ is invertible, giving a contradiction.

5 Spectral Properties of $\mathcal{L}^{+2}$

Recall by Theorem 3.1, any operator which is similar to a positive operator (and so in $\mathcal{L}^{+2}$) is necessarily scalar (that is, it is spectral and has no quasi-nilpotent part). It will be shown further that finite rank operators in $\mathcal{L}^{+2}$ are completely characterized by the property that the spectrum is positive and the operator is diagonalizable (Corollary 6.6).

It has already been noted that operators in $\mathcal{L}^{+2}$ need not be quasi-affine to a positive operator, much less similar to one, and as a result they are in general not spectral. Despite this, the spectral properties of operators in $\mathcal{L}^{+2}$ are found to reflect what is observed in these special cases.

The spectrum $\sigma(T)$ of an operator T can be divided into two, potentially overlapping parts; the *compression spectrum* $\sigma_c(T)$, points λ of which have the property that $T - \lambda 1$ is not surjective, and the approximate point spectrum $\sigma_a(T)$, in which $T - \lambda 1$ is not bounded below. The subset of $\sigma_a(T)$ of points λ for which $T - \lambda 1$ is not injective constitute the point spectrum $\sigma_p(T)$. Standard results in operator theory are that $\lambda \in \sigma_p(T)$ is equivalent to $\overline{\lambda} \in \sigma_c(T^*)$, and that the topological boundary of the spectrum is contained in $\sigma_a(T)$. In the case of $T \in \mathcal{L}^{+2}$, where the spectrum lacks interior, this means that $\sigma(T) = \sigma_a(T)$.

The parts of the spectrum already mentioned are for the most part enough when studying normal operators on Hilbert spaces. Outside of this class, it helps to refine these by looking at *local spectral properties*. This is ordinarily developed for (potentially unbounded) Banach space operators, though here bounded Hilbert space operators are solely considered.

Let $T \in L(\mathcal{H})$. If a point μ is in $\rho(T)$, the resolvent of T, $T - \mu 1$ is invertible. Equivalently, for all $x \in \mathcal{H}$ and $\lambda \in U$, an open neighborhood of μ, $f(\lambda) = (T - \lambda 1)^{-1}x$ is an analytic function from U into $\mathcal{H}$ and satisfies $(T - \lambda 1) f(\lambda) = x$. Even if $\mu \notin \rho(T)$, it may happen that for some $x \in \mathcal{H}$ and neighborhood U of μ, there is an analytic $f : U \to \mathcal{H}$ such that $(T - \lambda 1) f(\lambda) = x$. In this case, $\mu \in \rho_T(x)$, the *local resolvent of T at x*. For fixed x, the complement in $\mathbb{C}$ of $\rho_T(x)$ is called the local spectrum of T at x, and is denoted by $\sigma_T(x)$.

An operator T is said to have the *single valued extension property* (abbreviated *SVEP*) if whenever $U \subseteq \mathbb{C}$ is open and $f : U \to \mathcal{H}$ is an analytic function satisfying $(T - \lambda 1) f(\lambda) = 0$ for all $\lambda \in U$, then $f = 0$. The point of SVEP is that if T has this property, any solution f to $(T - \lambda 1) f(\lambda) = x$ in a neighborhood of a point μ is unique. Operators like those in $\mathcal{L}^{+2}$ with thin spectrum have SVEP.

For $F \subseteq \mathbb{C}$ closed, an *(analytic) local spectral subspace for $T \in L(\mathcal{H})$* is defined as

$$\mathcal{H}_T(F) := \{x \in \mathcal{H} : \sigma_T(x) \subseteq F\}.$$

This is a (not necessarily closed) linear manifold. Properties include that $\mathcal{H}_T(F) = \mathcal{H}_T(\sigma(T) \cap F)$, and if T has SVEP, $\mathcal{H}_T(\emptyset) = \{0\}$. Hence for operators in $\mathcal{L}^{+2}$, it will suffice to consider $\mathcal{H}_T(F)$ for closed subsets of $\sigma(T)$. It is also the case that for $\lambda \notin F$, $(T - \lambda 1)\mathcal{H}_T(F) = \mathcal{H}_T(F)$, $\mathcal{H}_T(F)$ is invariant for all operators commuting with T (in other words, it is *hyperinvariant*). Also, for all $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$, $\ker(T - \lambda 1)^n \subseteq \mathcal{H}_T(\{\lambda\})$, and more generally, if for $x \in \mathcal{H}$ and $\lambda \in F$, $(T - \lambda 1)x \in \mathcal{H}_T(F)$, then $x \in \mathcal{H}_T(F)$. See [12, Proposition 1.2.16]. By [14, Proposition 1.3], when T has SVEP, $\mathcal{H}_T(\{\lambda\}) = \{x : \lim_n \|(T - \lambda 1)^n x\|^{1/n} = 0\}$.

The following is a special case of a result due to Putnam, and Pták and Vrbová (see [12, Theorem 1.5.7]). The proof in this case is elementary and is included for completeness. The more general result is discussed below.

Lemma 5.1 *Let $T \in L(\mathcal{H})$ be normal and $\lambda \in \mathbb{C}$. Then $\mathcal{H}_T(\{\lambda\}) = \ker(T - \lambda 1)$.*

Proof Recall that for a normal operator T, the norm equals the spectral radius:

$$\|T\| = \lim_{n\to\infty} \|T^n\|^{1/n}.$$

Also, T is spectral so has SVEP. Let $\mathcal{H}_T(\{\lambda\}) = \{x : \lim_{n\to\infty} \|(T-\lambda 1)^n x\|^{1/n} \to 0\}$, and $\mathcal{E} = \overline{\mathcal{H}_T(\{\lambda\})}$. Then $(T-\lambda 1)\mathcal{E} \subseteq \mathcal{E}$, so $T\mathcal{E} \subseteq \mathcal{E}$. Since for all y, $\|(T-\lambda 1)y)\| = \|(T^* - \lambda 1)y)\|$, $T^*\mathcal{E} \subseteq \mathcal{E}$. Thus $\mathcal{E}$ reduces T, and $T_0 := P_{\mathcal{E}} T|_{\mathcal{E}}$ is normal.

Let $x \in \mathcal{H}_T(\{\lambda\})$, $\|x\| = 1$. Then for all $\epsilon > 0$, for sufficiently large n, $\|(T_0 - \lambda 1_{\mathcal{E}})^n x\| < \epsilon^n$. So if $y \in \mathcal{E}$ with $\|y\| = 1$,

$$\epsilon^n > \left\langle (T_0 - \lambda 1_{\mathcal{E}})^n x, y \right\rangle = \left\langle x, (T_0 - \lambda 1_{\mathcal{E}})^{*n} y \right\rangle.$$

Since $\mathcal{H}_T(\{\lambda\})$ is dense in $\mathcal{E}$ and ϵ is arbitrary, $\|(T_0 - \lambda 1_{\mathcal{E}})^{*n} y\|^{1/n} \to 0$ for all $y \in \mathcal{E}$. Thus $\sigma(T_0 - \lambda 1_{\mathcal{E}}) = \sigma((T_0 - \lambda 1_{\mathcal{E}})^*) = \{0\}$. Hence by normality, $T_0 - \lambda 1_{\mathcal{E}} = 0$, and so $\mathcal{H}_T(\{\lambda\}) = \ker(T - \lambda 1)$. □

Proposition 5.2 *For $T \in \mathcal{L}^{+2}$ and $\lambda \in \mathbb{C}$, $\mathcal{H}_T(\{\lambda\}) = \ker(T - \lambda 1)$.*

Proof By definition, $\mathcal{H}_T(\{\lambda\}) = \{x : \sigma_T(x) = \{\lambda\}\} = \{x : \sigma_{T-\lambda 1}(x) = \{0\}\} \supseteq \ker(T-\lambda 1)$. If $\lambda \in \rho(T)$, then $T-\lambda 1$ is invertible, and so for all $x \neq 0$, $\rho_T(x) \supseteq \rho(T)$, or equivalently, $\sigma_T(x) \subseteq \sigma(T)$. In particular then, if $\lambda \in \rho(T)$, $\mathcal{H}_T(\{\lambda\}) = \{0\} = \ker(T - \lambda 1)$.

So suppose that $\lambda \geq 0$ is in $\sigma(T)$. Write $T = AB$ for some optimal pair (A, B), and set $C = B^{1/2}AB^{1/2}$. Then $B^{1/2}(T - \lambda 1) = (C - \lambda 1)B^{1/2}$, and by induction, $B^{1/2}(T - \lambda 1)^n = (C - \lambda 1)^n B^{1/2}$ for $n \in \mathbb{N}$. Let $x \in \mathcal{H}_T(\{\lambda\}) = \{y : \lim_n \|(T - \lambda 1)^n y\|^{1/n} = 0\}$. Then

$$\|B^{1/2}(T - \lambda 1)^n x\|^{1/n} \leq \|B^{1/2}\|^{1/n} \|(T - \lambda 1)^n x\|^{1/n} \to 1 \cdot 0 = 0,$$

and so

$$\|(C - \lambda 1)^n B^{1/2} x\|^{1/n} \to 0.$$

Thus $B^{1/2}x \in \mathcal{H}_C(\{\lambda\})$. By the previous lemma $\mathcal{H}_C(\{\lambda\}) = \ker(C - \lambda 1)$, hence

$$B^{1/2}(T - \lambda 1)x = (C - \lambda 1)B^{1/2}x = 0.$$

If $\lambda = 0$, then either $x \in \ker T$ or $Tx \in \ker B = \ker T$. But by Proposition 4.2, $\operatorname{ran} T \cap \ker T = \{0\}$, and so this also implies that $x \in \ker T$. If $\lambda > 0$, then similar reasoning gives either $(T-\lambda 1)x = 0$ or $(T-\lambda 1)x \in \ker B = \ker T$. Suppose the latter. Since $(T - \lambda 1)Tx = T(T - \lambda 1)x = 0$, it follows that $(T - \lambda 1)^2 x = -\lambda(T - \lambda 1)x$, and in general, by induction for all n,

$$(T - \lambda 1)^n x = (-1)^{n-1}\lambda^{n-1}(T - \lambda 1)x.$$

Hence

$$\lambda^{(n-1)/n}\|(T-\lambda 1)x\|^{1/n} = \|\lambda^{n-1}(T-\lambda 1)x\|^{1/n} = \|(T-\lambda 1)^n x\|^{1/n}.$$

Since as $n \to \infty$, the right hand term goes to 0, $\lambda^{(n-1)/n} \to \lambda > 0$, and $\|(T-\lambda 1)x\|^{1/n} \to 1$ if $\|(T-\lambda 1)x\| > 0$, the conclusion is that $x \in \ker(T-\lambda 1)$. □

A simplified version of the above argument can be used to show the following.

Proposition 5.3 *If $T \in L(\mathcal{H})$ is quasi-affine to a normal operator, then*

$$\mathcal{H}_T(\{\lambda\}) = \ker(T-\lambda 1).$$

There are further ways in which operators in $\mathcal{L}^{+2}$ resemble positive operators. To explain this requires the introduction of some additional ideas from local spectral theory, details for which can be found in [12] and [3].

Recall that a scalar operator is one which is similar to a normal operator, and so has a Borel functional calculus. By Theorem 3.1, if $T \in \mathcal{L}^{+2}$ and is similar to a positive operator, then it is scalar, and so also has a Borel functional calculus. An operator T is termed a *generalized scalar operator* if it has a C^∞ functional calculus; that is, there is a continuous homomorphism $\Phi : C^\infty(\mathbb{C}) \to L(\mathcal{H})$ with $\Phi(1) = 1$ and $\Phi(z) = T$. An operator which is the restriction of a generalized scalar operator to an invariant subspace is said to be *subscalar*. Obviously, the classes of generalized scalar and subscalar operators include that of scalar operators.

Theorem 5.4 *Let $T \in \mathcal{L}^{+2}$. Then T is a generalized scalar operator and T has a $C^2([0, \|T\|])$ functional calculus.*

Proof To begin with, claim that if either A or B has closed range, then T is generalized scalar. So suppose $T = AB$, $A, B \in \mathcal{L}^+$, where $\operatorname{ran} A$ is closed (the case where $\operatorname{ran} B$ is closed can be handled identically by working with T^*). Decompose $\mathcal{H} = \operatorname{ran} A \oplus (\operatorname{ran} A)^\perp$, and write

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}$$

with respect to this decomposition. By the assumption that $\operatorname{ran} A$ is closed, T_1 is similar to a positive operator, and so is scalar by Theorem 3.1. Hence there is a constant $\kappa \geq 1$ such that for $\lambda \in \mathbb{C}\backslash\mathbb{R}$,

$$\|(T_1-\lambda 1)^{-1}\| \leq \kappa(1 + |\operatorname{Im}\lambda|^{-1}).$$

Since

$$(T-\lambda 1)^{-1} = \begin{pmatrix} (T_1-\lambda 1)^{-1} & \frac{1}{\lambda}(T_1-\lambda 1)^{-1}T_2 \\ 0 & -\frac{1}{\lambda} \end{pmatrix},$$

it follows that for sufficiently large κ',

$$\begin{aligned}\|(T-\lambda 1)^{-1}\| &\leq \kappa(1+|\operatorname{Im}\lambda|^{-1})\left(1+|\operatorname{Im}\lambda|^{-1}\left\|\begin{pmatrix}0 & T_2\\ 0 & 0\end{pmatrix}\right\|\right)\\ &\leq \kappa'|1+|\operatorname{Im}\lambda|^{-2}|.\end{aligned}$$

From [12, Theorem 1.5.19], T is a generalized scalar operator.

For the general case, let $T \in \mathcal{L}^{+2}$ and let $T' \in \mathcal{L}^+ \cdot \mathcal{P}$ on $\mathcal{H}'$ be the dilation of T from Proposition 4.11. So $\mathcal{H}$ is an invariant subspace for T' and T is the restriction of T' to $\mathcal{H}$. Hence T is subscalar.

Subscalar operators need not be generalized scalar. However, in this case any $\lambda \in \mathbb{C}\backslash\mathbb{R}$ is in the resolvents of both T and T'. So writing $T' = \begin{pmatrix}T & T_2\\ 0 & 0\end{pmatrix}$ with respect to the decomposition $\mathcal{H}' = \mathcal{H} \oplus \mathcal{H}^\perp$, for $\lambda \in \mathbb{C}\backslash\mathbb{R}$,

$$(T'-\lambda 1_{\mathcal{H}'})^{-1} = \begin{pmatrix}(T-\lambda 1_{\mathcal{H}})^{-1} & \frac{1}{\lambda}(T-\lambda 1_{\mathcal{H}})^{-1}T_2\\ 0 & -\frac{1}{\lambda}\end{pmatrix}.$$

Consequently, for all $\lambda \in \mathbb{C}\backslash\mathbb{R}$, there is a $\kappa' > 0$ such that

$$\|(T-\lambda 1_{\mathcal{H}})^{-1}\| \leq \|(T'-\lambda 1_{\mathcal{H}'})^{-1}\| \leq \kappa'(1+|\operatorname{Im}\lambda|^{-2})$$

by the first part of the proof. Thus [12, Theorem 1.5.19] gives that T is a generalized scalar operator. The fact that T has a $C^2([0, \|T\|])$ functional calculus follows from the proof of that theorem. □

Let $F \subset \mathbb{C}$. For an operator T, the *algebraic spectral subspace* $\mathcal{E}_T(F)$ is the largest linear manifold such that $(T-\lambda 1)\mathcal{E}_T(F) = \mathcal{E}_T(F)$ for all $\lambda \notin F$. Moreover, for every positive integer p,

$$\mathcal{H}_T(F) \subseteq \mathcal{E}_T(F) \subseteq \bigcap_{\lambda\notin F} \operatorname{ran}(T-\lambda 1)^p.$$

When T is normal and $E_T(F)$ is the spectral projection for the set F, it turns out that $\mathcal{H}_T(F) = \mathcal{E}_T(F) = \bigcap_{\lambda\notin F} \operatorname{ran}(T-\lambda 1) = \operatorname{ran} E_T(F)$ [12, Theorem 1.5.7]. The next result states that the operators in $\mathcal{L}^{+2}$ behave in this respect like normal operators.

Proposition 5.5 *Let $T \in \mathcal{L}^{+2}$. Then for closed $F \subset \mathbb{C}$,*

$$\mathcal{H}_T(F) = \mathcal{E}_T(F) = \bigcap_{\lambda\notin F} \operatorname{ran}(T-\lambda 1).$$

Proof By Theorem 5.4, $T \in \mathcal{L}^{+2}$ is a generalized operator, so by [12, Theorem 1.5.4], there exists an integer p such that for any closed set F, $\mathcal{H}_T(F) = \mathcal{E}_T(F) = \bigcap_{\lambda\notin F} \operatorname{ran}(T-\lambda 1)^p$. Fix $\lambda \notin F$. Since $T^* \in \mathcal{L}^{+2}$, by Proposition 5.2, $\ker(T^*-\bar{\lambda}1)^p = \ker(T-\lambda 1)^{*p} = \ker(T^*-\bar{\lambda}1)$ for all $p \in \mathbb{N}$, and so $\operatorname{ran}(T-\lambda 1)^p = \operatorname{ran}(T-\lambda 1)$ for all p. □

6 $\mathcal{L}^{+2}$ and Similarity; the Set $\mathcal{L}^{+2}_{cr}$

In Proposition 4.2, it was proved that if $T \in \mathcal{L}^{+2}$ then $\operatorname{ran} T \cap \ker T = \{0\}$. It is always the case then that

$$\mathcal{H} = \overline{\operatorname{ran} T \dotplus \ker T} \oplus (\ker T^* \cap \overline{\operatorname{ran}} T^*).$$

This section considers the case where $\operatorname{ran} T \dotplus \ker T$ is dense in $\mathcal{H}$. In Sect. 8, the general case will be taken up.

Recall from Proposition 3.1, $T \in \mathcal{L}^{+2}$ and T admits a factorization $T = AB$ where $A, B \in \mathcal{L}^+$ and either A or B is invertible is equivalent to T being similar to a positive operator.

Proposition 6.1 *Let $T \in \mathcal{L}^{+2}$ and $A \in \mathcal{L}^+$ such that $\overline{\operatorname{ran}} A = \overline{\operatorname{ran}} T$. Then the following are equivalent:*

(i) *There exists $B \in GL(\mathcal{H})^+$ such that $T = AB$;*
(ii) *There exists $B \in \mathcal{L}^+$ such that (A, B) is optimal for T and $\operatorname{ran} B \dotplus \ker A = \mathcal{H}$;*
(iii) *$\mathcal{B}_T^A \neq \emptyset$ and $\operatorname{ran} T = \operatorname{ran} A$.*

As a result, for this choice of A, there is a unique optimal pair (A, B_0) and B_0 has closed range.

Proof $(i) \Rightarrow (ii)$: Suppose that $T = AB$ with $B \in GL(\mathcal{H})^+$ then $\operatorname{ran} T = \operatorname{ran} A = \operatorname{ran}(AB')$ for any optimal pair (A, B'). Then $\mathcal{H} = A^{-1} \operatorname{ran}(AB') = \operatorname{ran} B' \dotplus \ker A$, where the sum is direct by Proposition 4.2.

$(ii) \Rightarrow (iii)$: Suppose that there exists $B \in \mathcal{L}^+$ such that (A, B) is optimal for T and $\mathcal{H} = \operatorname{ran} B \dotplus \ker A$. Applying A to both sides gives $\operatorname{ran} A = \operatorname{ran}(AB) = \operatorname{ran} T$.

$(iii) \Rightarrow (i)$: Let (A, B') be an optimal pair for T. Such a pair exists by Proposition 4.2. Since $\operatorname{ran} T = \operatorname{ran} A$, by the same calculation as above, $\mathcal{H} = \operatorname{ran} B' \dotplus \ker A$. Then by [9, Theorem 2.3], which states that if an operator range is complemented, then it is closed, $\operatorname{ran} B'$ is closed.

Now define the positive operator $B = B' + P_{\ker A}$. By [9, Theorem 2.2], $\operatorname{ran} B^{1/2} = \operatorname{ran} B' + \operatorname{ran} P_{\ker A} = \mathcal{H}$, and so B is invertible.

The last statement follows from Corollary 4.10. □

Theorem 3.1 indicates a number of ways of finding operators which are similar to positive operators. In addition, it combines with the last result to give yet another.

Corollary 6.2 *Let $T \in L(\mathcal{H})$. Then T is similar to a positive operator if and only if $T \in \mathcal{L}^{+2}$, $\overline{\operatorname{ran}} T \dotplus \ker T = \mathcal{H}$ and there exists and optimal pair (A, B) such that either A or B has closed range.*

It is not true in general that if T is similar to a positive operator, then every optimal pair for T is such that one of its factors has closed range.

Example 7 Let $A \in \mathcal{L}^+$ be such that $\operatorname{ran} A$ is not closed. Then clearly A is similar to a positive operator and $(A^{1/2}, A^{1/2})$ is an optimal pair for A. But, since $\operatorname{ran} A \subsetneq \operatorname{ran} A^{1/2}$, then none of the factors of this optimal pair has closed range. However, since $A = AP_{\overline{\operatorname{ran}} A}$, the optimal pair $(A, P_{\overline{\operatorname{ran}} A})$ is as in Corollary 6.2.

The situation when the range of $T \in \mathcal{L}^{+2}$ is closed happens to be special as well. Write

$$\mathcal{L}^{+2}_{cr} := \{T \in \mathcal{L}^{+2} : T \text{ has closed range}\}.$$

Proposition 6.3 *Let $T \in \mathcal{L}^{+2}$. Then the following are equivalent:*

(i) $T \in CR(\mathcal{H})$;
(ii) $\operatorname{ran} T \dotplus \ker T = \mathcal{H}$;
(iii) *For any optimal pair (A, B), $A, B \in CR(\mathcal{H})$ and $\operatorname{ran} A \dotplus \ker B$ is closed.*

In this case, T is similar to a positive operator.

Proof Let $T \in \mathcal{L}^{+2}$ and suppose that $T \in CR(\mathcal{H})$. Then $T^* \in CR(\mathcal{H})$. Let (A, B) be an optimal pair. Then $\operatorname{ran} A \supseteq \operatorname{ran} T = \overline{\operatorname{ran}}\, T = \overline{\operatorname{ran}}\, A$, and similarly, $\operatorname{ran} B = \overline{\operatorname{ran}}\, B$. Thus $A, B \in CR(\mathcal{H})$ and $\mathcal{H} = B^{-1} \operatorname{ran} T^* = \operatorname{ran} A \dotplus \ker B = \operatorname{ran} T \dotplus \ker T$. Conversely, if $\operatorname{ran} T \dotplus \ker T = \mathcal{H}$, by [9, Theorem 2.3], $\operatorname{ran} T$ is closed.

Finally, suppose that for an optimal pair (A, B), $A, B \in CR(\mathcal{H})$ and $\operatorname{ran} A \dotplus \ker B$ is closed. By [11, Corollary 2.5], $\operatorname{ran} T = \operatorname{ran}(AB)$ is closed. On the other hand, if $\operatorname{ran} T \dotplus \ker T = \mathcal{H}$, then arguing as above, $\operatorname{ran} A \dotplus \ker B = \mathcal{H}$, and so is closed. Hence all of the items are equivalent.

The statement that T is similar to a positive operator follows from Corollary 6.2. □

Proposition 6.3 and Theorem 3.1 together imply the following.

Corollary 6.4 *Let $T \in L(\mathcal{H})$. The following are equivalent:*

(i) $T \in \mathcal{L}^{+2}_{cr}$;
(ii) $T = ST^*S^{-1}$ *with* $S \in GL(\mathcal{H})^+$ *and* $\sigma(T) \subseteq \{0\} \cup [c, \infty)$ *for* $c > 0$;
(iii) *There exists* $G \in GL(\mathcal{H})$ *such that* $GTG^{-1} \in CR(\mathcal{H})^+$;
(iv) T *is a scalar operator and* $\sigma(T) \subseteq \{0\} \cup [c, \infty)$ *for* $c > 0$.

If $T \in \mathcal{L}^{+2}_{cr}$, then by Proposition 6.3, T is similar to a positive operator C. From this it is not difficult to check that $\sigma(T) = \sigma(C)$, C also has closed range, and consequently the spectrum of both operators have the form indicated in the corollary.

Corollary 6.5

$$\mathcal{L}^{+2}_{cr} = \bigcup_{W \in CR(\mathcal{H})^+} \mathbb{O}_W.$$

Corollary 6.6 *Suppose that $T \in L(\mathcal{H})$ is finite rank. Then $T \in \mathcal{L}^{+2}$ if and only if T is diagonalizable and $\sigma(T) \geq 0$.*

Remark If T on $\mathcal{H}$ with $dim(\mathcal{H}) < \infty$ is diagonalizable with positive spectrum, it is in principle straightforward to write T as a product of two positive operators. Let C be the diagonal matrix of eigenvalues of T, V the matrix with columns consisting of the eigenvectors of T, arranged in the same order as the diagonal entries of C. The matrix V is invertible, and $TV = VC$. Therefore, $T = (VV^*)(V^{*-1}CV^{-1})$.

Example 8 The situation for the product of three or more positive operators is more complicated. In particular, such products need not be diagonalizable. As a simple example,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Hence the class $\mathcal{L}^{+3}$ of products of three positive operators strictly contains $\mathcal{L}^{+2}$.

Maganja showed in [13] that every bounded operator on a Hilbert space is the sum of at most three operators which are similar to positive operators (and so by Theorem 3.1, three operators in $\mathcal{L}^{+2}$). On finite dimensional spaces, Wu proved that if $\det T \geq 0$ (which includes those T with non-negative spectrum), T is the product of at most 5 positive matrices [22], and in [5], an algorithm is given for determining the number of matrices between 1 and 5 needed. In the setting of separable Hilbert spaces, Wu also showed that any operator which is the norm limit of a sequence of invertible operators is the product of at most 18 positive operators [21]. For invertible operators, this was improved by Phillips to at most 7 [16].

It is also possible to give an explicit formula for the Moore-Penrose inverse $T^\dagger$ of an operator $T \in \mathcal{L}^{+2}_{cr}$. In this case, if $Q := P_{\operatorname{ran} T^* // \ker T^*}$ is the oblique projection onto $\operatorname{ran} T^*$ along $\ker T^*$, Q is bounded. Recall that an operator T' is called a $(1,2)$-inverse of T if $TT'T = T$ and $T'TT' = T'$. Generally, there will be infinitely many $(1,2)$-inverses for an operator T. The Moore-Penrose inverse is the $(1,2)$-inverse for which $TT^\dagger$ is the orthogonal projection onto $\operatorname{ran} T$ and $T^\dagger T$ is the orthogonal projection onto $\operatorname{ran} T^*$.

Proposition 6.7 *Let $T \in \mathcal{L}^{+2}_{cr}$ with optimal pair (A, B). Then*

$$T^\dagger = B^\dagger Q A^\dagger.$$

Furthermore, $T' := Q^* T^\dagger Q^*$ is a $(1,2)$-inverse of T in $\mathcal{L}^{+2}_{cr}$.

Proof Let $T \in \mathcal{L}^{+2}_{cr}$. The fact that A and B have closed range follows from Proposition 6.3. Hence $A^\dagger$ and $B^\dagger$ are bounded positive operators. Also, $\operatorname{ran} T^*$ is closed. For $Q = P_{\operatorname{ran} T^* // \ker T^*}$, $P_{\operatorname{ran} T^*} Q P_{\operatorname{ran} T} = Q P_{\operatorname{ran} T} = Q$. For $W = B^\dagger Q A^\dagger$,

$$TWT = AB(B^\dagger Q A^\dagger)AB = A P_{\operatorname{ran} T^*} Q P_{\operatorname{ran} T} B = AQB = T.$$

Therefore TW is a projection. Furthermore,

$$TW = ABB^\dagger Q A^\dagger = A P_{\operatorname{ran} T^*} Q A^\dagger = AQA^\dagger.$$

Also, $\operatorname{ran}(TW) = \operatorname{ran} T$ and $\ker(TW) = \ker T^*$ since

$$\begin{aligned} \operatorname{ran} T = \operatorname{ran}(TWT) \subseteq \operatorname{ran}(TW) \subseteq \operatorname{ran} T, \quad \text{and} \\ \ker W \subseteq \ker(TW) \subseteq \ker(WTW) = \ker W = \ker T^*. \end{aligned}$$

The last equality holds since if $x \in \ker W$, $QA^\dagger x \in \ker T \cap \operatorname{ran} Q = \ker T \cap \operatorname{ran} T^* = \{0\}$, so $A^\dagger x \in \ker Q \cap \operatorname{ran} A^\dagger = \ker T^* \cap \operatorname{ran} T = \{0\}$. Thus $x \in \ker A^\dagger = \ker T^*$. Hence $\operatorname{ran}(TW)$ and $\ker(TW)$ are orthogonal, and so $TW \in \mathcal{P}$.

Similar calculations show that $WTW = W$, hence WT is a projection, and by identical reasoning, it is an orthogonal projection. Thus, $T^\dagger = B^\dagger QA^\dagger$, as claimed.

Since $Q^*T = TQ^* = T$, it is easy to see that for $T' = Q^*T^\dagger Q^*$, $TT'T = T$ and $T'TT' = T'$. Also,

$$\operatorname{ran} T' = Q^*T^\dagger \operatorname{ran} T = Q^*T^\dagger\mathcal{H} = Q^* \operatorname{ran} T^* = Q^*\mathcal{H} = \operatorname{ran} T.$$

Finally,

$$T' = (Q^*B^\dagger Q)(QA^\dagger Q^*) \in \mathcal{L}^{+2}_{cr}. \qquad \square$$

Remark If $T \in \mathcal{P}^2$ with closed range, the formula $T^\dagger = P_{\operatorname{ran} T^* /\!/ \ker T^*}$ from [4] is recovered.

7 $\mathcal{L}^{+2}$, Quasi-Affinity and Quasi-Similarity

In Proposition 3.8 it was seen that the statement that T being quasi-affine to a positive operator is equivalent to, among other things, being able to write $T^* = BA$ where B and A are positive, but where B may be unbounded. The situation for quasi-similarity is no better (Corollary 3.9). Conditions equivalent to $T = AB$ where A and B are bounded and positive require something extra, and this will then imply $\sigma(T) \geq 0$ by Lemma 4.1.

Theorem 7.1 *For $T \in L(\mathcal{H})$, the following are equivalent:*

(i) $T \in \mathcal{L}^{+2}$ *and is quasi-affine to a positive operator;*
(ii) $T \in \mathcal{L}^{+2}$ *and* $\overline{\operatorname{ran} T \dotplus \ker T} = \mathcal{H}$*;*
(iii) *There exists a quasi-affinity $X \in \mathcal{L}^+$ such that* $\operatorname{ran} T \subseteq \operatorname{ran} X$ *and $TX \geq 0$;*
(iv) $T = AB$*, $A, B \in \mathcal{L}^+$ and A injective;*
(v) $\sigma(T) \cap (-\infty, 0) = \emptyset$*, and there exists a quasi-affinity $X \in \mathcal{L}^+$ such that $TX = XT^*$ and* $\operatorname{ran} T \subseteq \operatorname{ran} X$*;*
(vi) *There exists $C \in \mathcal{L}^+$ and a quasi-affinity $G \in L(\mathcal{H})$ such that $TG = GC$ and* $\operatorname{ran} T \subseteq \operatorname{ran}(GG^*)$.

Proof $(i) \Rightarrow (ii)$: This follows from Proposition 3.5.

$(ii) \Rightarrow (iii)$: Let $T = AB$, where (A, B) is optimal. Define $X := A + P_{\ker B} \in \mathcal{L}^+$. Then $\ker X = \ker T^* \cap \overline{\operatorname{ran}}\, T^* = (\overline{\operatorname{ran} T \dotplus \ker T})^\perp = \{0\}$, and so X is a quasi-affinity. Consequently, $TX = ABA \geq 0$ and $XB = AB = T$. Hence $\operatorname{ran} T = \operatorname{ran}(XB) \subseteq \operatorname{ran} X$.

$(iii) \Rightarrow (iv)$: Since $TX \geq 0$, X is a quasi-affinity, and $\operatorname{ran} T \subseteq \operatorname{ran} X$, it follows from Douglas' lemma that $T = XP$ and $TX = XT^* = XPX \geq 0$, where $P \geq 0$. So $T = XP \in \mathcal{L}^{+2}$, and by Lemma 4.1, $\sigma(T) \geq 0$.

$(iv) \Rightarrow (v)$: If $T = AB$, $A, B \in \mathcal{L}^+$ and A injective, then $X = A$ is a quasi-affinity and $TX \geq 0$. By Lemma 4.1, $\sigma(T) \geq 0$.

$(v) \Rightarrow (vi)$: By Douglas' lemma, $T = XP$, and since $TX = XPX$ is selfadjoint and X is a quasi-affinity, P is selfadjoint. By the assumption $\sigma(T) \cap (-\infty, 0) = \emptyset$, it follows from [10, Corollary 4.2] that $TX \geq 0$, and hence that $P \geq 0$. Thus $T \in \mathcal{L}^{+2}$. Set $G = X^{1/2}$, which is also a quasi-affinity, and define $C = GPG \geq 0$. Then $TG = GC$. The last condition in (vi) then follows since $T = XP$.

$(vi) \Rightarrow (i)$: Since $TGG^* = GCG^* \geq 0$, and since $\operatorname{ran} T \subseteq \operatorname{ran}(GG^*)$, by Douglas' lemma $T = GG^*P$. Moreover, $TG = G(G^*PG)$, and since G is a quasi-affinity, $G^*PG = C$. Hence $P \geq 0$ and so $T \in \mathcal{L}^{+2}$. □

Remark A simple example shows that even if $T \in \mathcal{L}^{+2}$ and there is a quasi-affinity $X \in \mathcal{L}^{+}$ such that $TX = XT^* \geq 0$, it need not be true that $\operatorname{ran} T \subseteq \operatorname{ran} X$. For example, take $T = 1$ on an infinite dimensional Hilbert space, and $X \in \mathcal{L}^{+}$, but without closed range. Also, $T = C = 1$ and G any quasi-affinity without closed range together satisfy $TG = GC$, but obviously, $\operatorname{ran} T$ is not contained in $\operatorname{ran}(GG^*)$.

In [19, Corollary 3], Stampfli showed that quasi-similar operators with Dunford's property C have equal spectra. Since by Theorem 5.4, any $T \in \mathcal{L}^{+2}$ has property C, and positive operators, being scalar, also have this property, it follows that if $T \in \mathcal{L}^{+2}$ is quasi-similar to a positive operator C, then $\sigma(T) = \sigma(C)$. As the next result shows, this continues to be true with the weaker assumption of quasi-affinity, and as a bonus, the proof does not use any of the material from Sect. 5.

Proposition 7.2 *If $T \in \mathcal{L}^{+2}$ is quasi-affine to $C \in \mathcal{L}^{+}$, then $\sigma(T) = \sigma(C)$.*

Proof Suppose to begin with that T is quasi-similar to C. Write, using Theorem 7.1, $T = AB$, $A, B \in \mathcal{L}^{+}$ and A injective. Then T is quasi-affine to $C_A := A^{1/2}BA^{1/2}$ (with quasi-affinity $A^{1/2}$). Applying Lemma 3.7, C is quasi-affine to C_A. From [6, Lemma 4.1], C and C_A are unitarily equivalent, and so have equal spectra. As noted in Lemma 4.1, T and C_A also have equal spectra, so the result follows in this case.

Now suppose that T is just quasi-affine to C. If $\mathcal{N} = \overline{\operatorname{ran}}\, T^*$, $\mathcal{N}$ is invariant for T^*, and $T^*P_{\mathcal{N}} \in \mathcal{L}^{+2}$ by Proposition 4.8. As observed in Lemma 4.1, $\sigma(T) \subseteq \mathbb{R}^+$, so $\sigma(T^*) = \sigma(T)$, and consequently

$$\sigma(T) = \sigma(T^*) = \sigma(T^*P_{\mathcal{N}}) = \sigma(P_{\mathcal{N}}T).$$

The middle equality follows by the same argument as in the proof of Lemma 4.1.

Define $\tilde{T} : \mathcal{N} \to \mathcal{N}$ as the compression of T to $\mathcal{N}$. If $0 \notin \sigma(\tilde{T})$, so that $\tilde{T}$ is invertible, then $\operatorname{ran} T^* = \mathcal{N}$. By Proposition 6.3, T is similar to a positive operator, and by Lemma 3.4, T is quasi-similar to C, and this has already been dealt with.

If $0 \in \sigma(\tilde{T})$, then $\sigma(P_{\mathcal{N}}T) = \sigma(\tilde{T})$. Suppose that $TX = XC$, X a quasi-affinity. Then $\mathcal{R} := \overline{X^*\mathcal{N}} = \overline{\operatorname{ran}}\, C$. Since $X^*P_{\mathcal{N}} = P_{\mathcal{R}}X^*P_{\mathcal{N}}$, for $\tilde{C} = P_{\mathcal{R}}C|_{\mathcal{R}}$ and $\tilde{X} = P_{\mathcal{N}}X|_{\mathcal{R}}$, $\tilde{X}$ is a quasi-affinity and $\tilde{T}\tilde{X} = \tilde{X}\tilde{C}$. Note that $\sigma(\tilde{C}) \cup \{0\} = \sigma(C) \cup \{0\}$, and if $0 \in \sigma(C)$, then $0 \in \sigma(\tilde{C})$. By Theorem 7.1, $\overline{\operatorname{ran}\, \tilde{T} \dotplus \ker \tilde{T}} = \mathcal{N}$. By definition, $\overline{\operatorname{ran} \tilde{T}^* \dotplus \ker \tilde{T}^*} = \overline{\operatorname{ran} \tilde{T}^* \dotplus \{0\}} = \mathcal{N}$. Applying Theorem 7.1 to $\tilde{T}$ and $\tilde{T}^*$, $\tilde{T}$ is quasi-similar to some positive operator, C'. It then follows from Lemma 3.4 that $\tilde{T}$ is quasi-similar to $\tilde{C}$. Hence $\sigma(\tilde{T}) = \sigma(\tilde{C})$. Finally,

$$\sigma(\tilde{T}) = \sigma(T) \supseteq \sigma(C) = \sigma(\tilde{C}),$$

where the containment is by Lemma 3.7, and the second equality follows since $0 \in \sigma(\tilde{C})$. Consequently, equality holds throughout. □

The next is a corollary of Theorem 7.1.

Corollary 7.3 *For $T \in L(\mathcal{H})$, the following are equivalent:*

(i) *$T \in \mathcal{L}^{+2}$ and T is quasi-similar to a positive operator;*
(ii) *$T \in \mathcal{L}^{+2}$ and $\overline{\operatorname{ran}} T \dotplus \ker T = \mathcal{H}$;*
(iii) *There exist quasi-affinities $X, Y \in \mathcal{L}^{+}$ such that $\operatorname{ran} T \subseteq \operatorname{ran} X$, $\operatorname{ran} T^* \subseteq \operatorname{ran} Y$, $TX \geq 0$, and $TY \geq 0$;*
(iv) *$\sigma(T) \cap (-\infty, 0) = \emptyset$, and there exists quasi-affinities $X, Y \in \mathcal{L}^{+}$ such that $TX = XT^*$, $YT = T^*Y$, and either $\operatorname{ran} T \subseteq \operatorname{ran} X$ or $\operatorname{ran} T^* \subseteq \operatorname{ran} Y$;*
(v) *There exists $C \in \mathcal{L}^{+}$ and quasi-affinities $G, F \in L(\mathcal{H})$ such that $TG = GC$, $FT = CF$, and either $\operatorname{ran} T \subseteq \operatorname{ran}(GG^*)$ or $\operatorname{ran} T^* \subseteq \operatorname{ran}(F^*F)$.*

Proof The equivalence of (i) and (ii) in Theorem 7.1 gives the equivalence of the first two items here. Assuming $T \in \mathcal{L}^{+2}$ and T quasi-similar to a positive operator, one has T^* quasi-affine to a positive operator, and from this $\overline{\operatorname{ran}} T^* \dotplus \ker T^* = \mathcal{H}$. Taking orthogonal complements gives $\overline{\operatorname{ran}} T \cap \ker T = \{0\}$ and so $\overline{\operatorname{ran}} T \dotplus \ker T = \mathcal{H}$. On the other hand, if $\overline{\operatorname{ran}} T \dotplus \ker T = \mathcal{H}$, then $\overline{\operatorname{ran}} T \cap \ker T = \{0\}$, and so taking orthogonal complements, $\overline{\operatorname{ran}} T^* \dotplus \ker T^* = \mathcal{H}$.

Consequently, Theorem 7.1 applies to both T and T^*. Since $\sigma(T^*) = \{\bar{\lambda} : \lambda \in \sigma(T)\}$, $\sigma(T^*) \cap (-\infty, 0) = \emptyset$ as well. The rest of the equivalences then easily follow. □

Corollary 7.4 *If $T \in \mathcal{L}^{+2}$ and $T = AB$ where (A, B) is an optimal pair and either* $\operatorname{ran} B$, *respectively* $\operatorname{ran} A$, *is closed, then T, respectively T^* is quasi-affine to a positive operator. If there is such a pair with both* $\operatorname{ran} A$ *and* $\operatorname{ran} B$ *closed, then T is quasi-similar to a positive operator.*

Proof Suppose $T \in \mathcal{L}^{+2}$ and $T = AB$ where (A, B) is an optimal pair and $\operatorname{ran} B$ is closed. From Proposition 4.2, $\operatorname{ran} B \cap \ker A = \{0\}$, and taking orthogonal complements gives that $\ker T \dotplus \operatorname{ran} T$ is dense in $\mathcal{H}$. Therefore, by Theorem 7.1, T is quasi-affine to a positive operator. The other case is handled identically. If both A and B have closed range, T and T^* are both quasi-affine to positive operators. By Lemma 3.4, T is quasi-similar to a positive operator. □

Remark As was noted in Example 3 in Sect. 4, there exists an operator $T \in \mathcal{L}^{+2}$ for which neither $\operatorname{ran} T \dotplus \ker T$ nor $\operatorname{ran} T^* \dotplus \ker T^*$ are dense. Hence by the results of this section, in this particular example neither T nor T^* is quasi-affine to a positive operator, and in particular, T will not be quasi-similar to a positive operator.

On the other hand, Example 4 gives an operator $T \in \mathcal{L}^{+2}$ which is quasi-similar to a positive where there is no optimal pair (A, B) with $\operatorname{ran} A$ or $\operatorname{ran} B$ closed, so there is no converse to Corollary 7.4.

While the operators which are similar to a positive operator are in $\mathcal{L}^{+2}$, this is no longer necessarily true for those which are quasi-similar to a positive operator.

Proposition 7.5 *For $T \in \mathcal{L}^{+2}$, T is quasi-affine, respectively quasi-similar, to a positive operator if and only if T has a square root which is quasi-affine, respectively, quasi-similar to a positive operator. Consequently, there exists an operator which is quasi-similar to a positive operator which is not in $\mathcal{L}^{+2}$.*

Proof Suppose that T is quasi-affine to a positive operator. By Theorem 7.1, $T = AB$ with $A, B \in \mathcal{L}^+$ and A injective. Set $C = A^{1/2}BA^{1/2}$ and $X = A^{1/2}$. Then $TX = XC$. By Douglas' lemma, $C^{1/2} = A^{1/2}B^{1/2}G$, so in particular, $\operatorname{ran} C^{1/2}X^* \subseteq \operatorname{ran} X^*$, and so another application of Douglas' lemma gives $R \in L(\mathcal{H})$ such that $RX = XC^{1/2}$. Thus R is quasi-affine to $C^{1/2}$. Since $R^2X = XC = TX$ and $\operatorname{ran} X$ is dense, $R^2 = T$. Conversely, if $R^2 = T$ and $RX = XD$, $D \geq 0$ and X a quasi-affinity, then $R^2X = XD^2$.

If in addition, T is quasi-similar to C, $YR^2 = CY$. Hence $C(YX) = YR^2X = (YX)C$, and by Lemma 4.15, $C^{1/2}(YX) = (YX)C^{1/2}$. Therefore,

$$YR(XC^{1/2}) = YR^2X = C(YX) = C^{1/2}Y(XC^{1/2}).$$

It is straightforward to see that $\overline{\operatorname{ran}}\,(XC^{1/2}) = \overline{\operatorname{ran}}\,T$, $\overline{\operatorname{ran}}\,(R^*Y^*) = \overline{\operatorname{ran}}\,T^*$, and $\overline{\operatorname{ran}}\,(Y^*C^{1/2}) = \overline{\operatorname{ran}}\,T^*$. Also, $\ker(R^*Y^*) = \ker(Y^*C^{1/2}) = \ker T$. By Corollary 7.3, $\overline{\operatorname{ran}}\,T \dotplus \ker T = \mathcal{H}$, and so $YR = C^{1/2}Y$ on a dense subset. By continuity, $YR = C^{1/2}Y$ on all of $\mathcal{H}$, and thus R is quasi-similar to $C^{1/2}$. The converse is as in the quasi-affine case.

By Example 6, there is a $T \in \mathcal{L}^{+2}$ which is quasi-similar to a positive operator, yet does not have a square root in $\mathcal{L}^{+2}$. Hence for this operator, the square root constructed above is quasi-similar to a positive operator but is not in $\mathcal{L}^{+2}$. □

Remark In the last proposition, the operator R constructed there has $\sigma(R) \subseteq \sigma(C^{1/2}) \cup \sigma(-C^{1/2})$, which is a slight strengthening of Lemma 3.7 in this setting.

There is no obvious way to rule out negative values in $\sigma(R)$ if R is not in $\mathcal{L}^{+2}$. Nevertheless, for the operator in Example 6, the square root is $R = \begin{pmatrix} S^{1/2} & 0 \\ (1-S)^{1/2} & 0 \end{pmatrix}$, which happens to be a partial isometry with $\sigma(R) = \sigma(S^{1/2}) \cup \{0\} = \sigma(S^{1/2})$. For

$$C = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} S^{1/2} & 0 \\ (1-S)^{1/2} & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 \\ -(1-S)^{1/2} & S^{1/2} \end{pmatrix},$$

$C \geq 0$, X and Y are quasi-affinities, $RX = XC^{1/2}$ and $YR = C^{1/2}Y$. So even though $R \notin \mathcal{L}^{+2}$, $\sigma(R) = \sigma(C^{1/2})$.

8 $\mathcal{L}^{+2}$: The General Case

The sole remaining case to consider are those operators $T \in \mathcal{L}^{+2}$ for which neither $\mathcal{M} := \overline{\operatorname{ran}}\,T \dotplus \ker T$ nor $\mathcal{N} := \overline{\operatorname{ran}}\,T^* \dotplus \ker T^*$ equals $\mathcal{H}$. Decompose

$$\mathcal{H} = \mathcal{M} \oplus (\overline{\operatorname{ran}}\,T^* \cap \ker T^*) = \mathcal{N} \oplus (\overline{\operatorname{ran}}\,T \cap \ker T).$$

The spaces $\mathcal{M}$ and $\mathcal{N}^\perp$ are invariant for T, while $\mathcal{N}$ and $\mathcal{M}^\perp$ are invariant for T^*. In what follows, statements involving only the spaces $\mathcal{M}$ and $\mathcal{M}^\perp$ are given, since it is obvious what the equivalent statements for $\mathcal{N}$ and $\mathcal{N}^\perp$ should be.

Lemma 8.1 *Let $T \in \mathcal{L}^{+2}$. Then $T_\mathcal{M} := TP_\mathcal{M} \in \mathcal{L}^{+2}$, $\overline{\operatorname{ran}}\, T_\mathcal{M} = \overline{\operatorname{ran}}\, T$, $\ker T_\mathcal{M} = \ker T \oplus \mathcal{M}^\perp$, and $T_\mathcal{M}$ is quasi-affine to a positive operator. Also, if (A, B) is an optimal pair for T, then $\operatorname{ran} T \subseteq \operatorname{ran}(A(P_\mathcal{M} B P_\mathcal{M})^{1/2})$ and $(A, P_\mathcal{M} B P_\mathcal{M})$ is an optimal pair for $T_\mathcal{M}$.*

Proof Applying Proposition 4.8 and Corollary 4.3, $T_\mathcal{M} \in \mathcal{L}^{+2}$ and $\overline{\operatorname{ran}}\, T_\mathcal{M} = \overline{\operatorname{ran}}\, T$.

Write $T = AB$, where (A, B) is optimal. Since $\ker B = \ker T \subseteq \mathcal{M}$ and $P_\mathcal{M} A = A = AP_\mathcal{M}$,

$$T_\mathcal{M} = (A + P_{\ker T} + P_{\mathcal{M}^\perp})(P_\mathcal{M} B P_\mathcal{M}),$$

where $A + P_{\ker T} + P_{\mathcal{M}^\perp}$ is positive and injective since by now standard calculations, $A + P_{\ker T}$ has this property on $\mathcal{M}$. It then follows from Theorem 7.1 that $T_\mathcal{M}$ is quasi-affine to a positive operator. Also $\ker T_\mathcal{M} = \ker(P_\mathcal{M} B P_\mathcal{M}) = \ker T \oplus \mathcal{M}^\perp$.

Finally, since $B \geq 0$, $\operatorname{ran}(P_\mathcal{M} B P_{\mathcal{M}^\perp}) \subseteq \operatorname{ran}(P_\mathcal{M} B P_\mathcal{M})^{1/2}$. Then from

$$T = \begin{pmatrix} T_\mathcal{M} & TP_{\mathcal{M}^\perp} \end{pmatrix} = A \begin{pmatrix} P_\mathcal{M} B P_\mathcal{M} & P_\mathcal{M} B P_{\mathcal{M}^\perp} \end{pmatrix},$$

the last claim follows. □

It is also true that T is the restriction of an operator in $\mathcal{L}^{+2}$ which is quasi-affine to a positive operator in the following sense.

Lemma 8.2 *Let $T \in \mathcal{L}^{+2}$. Then there is an operator $T^\mathcal{M} \in \mathcal{L}^{+2}$ with the properties that $T^\mathcal{M}$ is quasi-affine to a positive operator, $T = P_\mathcal{M} T^\mathcal{M}$, $\overline{\operatorname{ran}}\, T^\mathcal{M} = \overline{\operatorname{ran}}\, T \oplus \mathcal{M}^\perp$ and $\ker T^\mathcal{M} = \ker T$.*

Proof Write $T = AB$ with (A, B) optimal. Set

$$T^\mathcal{M} = T + P_{\mathcal{M}^\perp} B = (A + P_{\mathcal{M}^\perp})B = (A + P_{\ker T} + P_{\mathcal{M}^\perp})B.$$

Then $T^\mathcal{M} \in \mathcal{L}^{+2}$, $A + P_{\ker T} + P_{\mathcal{M}^\perp} \geq 0$ is injective, and $\ker T^\mathcal{M} = \ker B$. By Theorem 7.1, $T^\mathcal{M}$ is quasi-affine to a positive operator.

Since $T^{\mathcal{M}*} = B(A + P_{\mathcal{M}^\perp})$, $\ker T^{\mathcal{M}*} \supseteq \ker(A + P_{\mathcal{M}^\perp}) = \ker A \cap \mathcal{M}$, and if $T^{\mathcal{M}*} x = 0$, then $(A + P_{\mathcal{M}^\perp})x \in \ker B \cap \operatorname{ran}(A + P_{\mathcal{M}^\perp}) = \{0\}$. The last equality follows since if $x \in \ker B \subseteq \mathcal{M}$ and $x = x_1 + x_2$, $x_1 \in \operatorname{ran} A \subseteq \mathcal{M}$ and $x_2 \in \mathcal{M}^\perp$, then $x_2 = 0$, and since $\operatorname{ran} A \cap \ker B = \{0\}$ by Proposition 4.2, $x = 0$. Hence $\ker T^{\mathcal{M}*} = \ker(A + P_{\mathcal{M}^\perp})$, and so $\overline{\operatorname{ran}}\, T^\mathcal{M} = \overline{\operatorname{ran}}\, T \oplus \mathcal{M}^\perp$. □

Theorem 8.3 *Let $T \in L(\mathcal{H})$ and $\mathcal{M} = \overline{\operatorname{ran} T + \ker T}$. The following are equivalent:*

(i) $T \in \mathcal{L}^{+2}$;
(ii) $T_\mathcal{M} := TP_\mathcal{M} \in \mathcal{L}^{+2}$ *and there exists an optimal pair* (A, B) *for* $T_\mathcal{M}$ *such that* $\operatorname{ran} T \subseteq AB^{1/2}$;

(iii) *There exists* $T^{\mathcal{M}} \in \mathcal{L}^{+2}$ *satisfying* $T = P_{\mathcal{M}} T^{\mathcal{M}}$ *and an optimal pair* (A, B) *for* $T^{\mathcal{M}}$ *such that* $A\mathcal{M}^{\perp} = \mathcal{M}^{\perp}$.

In this case, both $T_{\mathcal{M}}$ *and* $T^{\mathcal{M}}$ *are quasi-affine to positive operators.*

Proof $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$ follow from the last two lemmas.

Assume (ii) holds and that $T_{\mathcal{M}} = AB$ for an optimal pair (A, B) such that $\operatorname{ran}(TP_{\mathcal{M}^{\perp}}) \subseteq \operatorname{ran} T \subseteq \operatorname{ran}(AB^{1/2})$. By Douglas' lemma, there is an operator $Z \in L(\mathcal{H})$ with $\ker Z = \ker(TP_{\mathcal{M}^{\perp}}) = \mathcal{M}$ and such that $TP_{\mathcal{M}^{\perp}} = AB^{1/2}Z$. Hence,

$$T = T_{\mathcal{M}} + TP_{\mathcal{M}^{\perp}} = A(B + B^{1/2}Z) = AB^{1/2}(B^{1/2} + Z) = A(B^{1/2} + Z^*)(B^{1/2} + Z)$$

is in $\mathcal{L}^{+2}$, the last equality following since $\overline{\operatorname{ran}}\, Z^* \subseteq \mathcal{M}^{\perp} \subseteq \ker A$. Thus (i) holds.

Now assume (iii) is true. Then for the optimal pair (A, B) there, $P_{\mathcal{M}} A = P_{\mathcal{M}}(AP_{\mathcal{M}} + AP_{\mathcal{M}^{\perp}}) = P_{\mathcal{M}} A P_{\mathcal{M}}$. Hence $T = P_{\mathcal{M}} A P_{\mathcal{M}} B \in \mathcal{L}^{+2}$, which is (i). □

Remark Since $T_{\mathcal{M}} = TP_{\mathcal{M}}$, $\sigma(T_{\mathcal{M}}) \cup \{0\} = \sigma(P_{\mathcal{M}} T) \cup \{0\} = \sigma(T) \cup \{0\}$. If $0 \notin \sigma(T)$, $P_{\mathcal{M}} = 1$, and likewise, if $0 \notin \sigma(T_{\mathcal{M}})$, $\operatorname{ran} P_{\mathcal{M}} = \mathcal{H}$, so again $P_{\mathcal{M}} = 1$. Thus, $\sigma(T_{\mathcal{M}}) = \sigma(T)$. Unfortunately, there does not seem to be any similar relation between $\sigma(T^{\mathcal{M}})$ and $\sigma(T)$.

There is also the dilation result for the class $\mathcal{L}^{+2}$ in Proposition 4.11, though there does not appear to be such a close connection for the spectra of these with that of T. These dilations are in a sense extremal for the family $\mathcal{L}^{+2}$, in that any further dilations are direct sums.

Theorem 5.4 indicates that all operators in $\mathcal{L}^{+2}$ are generalized scalar, so it is natural to wonder if there is some characterization of the class $\mathcal{L}^{+2}$ in terms of this property and the spectrum of the operator being in $\mathbb{R}^+$.

9 Examples

Recall that an operator T is *algebraic* if there is a polynomial p such that $p(T) = 0$. By the spectral mapping theorem, $\sigma(T)$ is then contained in the set of roots of the polynomial.

Proposition 9.1 *Suppose that* $T \in \mathcal{L}^{+2}$ *is algebraic. Then* T *has the form*

$$T = \sum_j \lambda_j Q_j,$$

where each $\lambda_j \geq 0$ *is an eigenvalue for* T *and* Q_j *is an oblique projection. In this case,* $\operatorname{ran} T$ *is closed and* T *is similar to* $C = \sum_j \lambda_j P_j \geq 0$, *where each* P_j *is an orthogonal projection and* $\bigoplus_j P_j = 1$. *Conversely, if* T *has this form, then* $T \in \mathcal{L}^{+2}$ *and is algebraic.*

Proof As noted above, if $p(T) = 0$ for a polynomial p, the spectrum of T is a finite set of points taken from the non-negative roots of p. For each $\lambda_j \in \sigma(T)$, let Q_j be the Riesz projection for $\lambda_j \in \sigma(T)$. Then $\mathcal{H}_j = \operatorname{ran} Q_j$ is invariant for T and $\sigma(T|_{\mathcal{H}_j}) = \lambda_j$. Furthermore, $Q_i Q_j = 0$ for $i \neq j$. By Proposition 4.8 and Proposition 5.2, $T|_{\mathcal{H}_j} = GAG^{-1}$, with $A \geq 0$, G invertible in $\mathcal{H}_j$, and $\sigma(A) = \{\lambda_j\}$. Thus $A = \lambda_j 1_{\mathcal{H}_j}$, and so $T|_{\mathcal{H}_j} = \lambda_j 1_{\mathcal{H}_j}$. Therefore, $TQ_j = \lambda_j Q_j$ for some oblique projection Q_j and if $\lambda_j \neq 0$, $T|_{\mathcal{H}_j}$ is invertible. Since $Q_i Q_j = 0$ when $i \neq j$, $\sum_j Q_j$ is a projection, and moreover $\sum_j Q_j = 1$. So T has the claimed form.

Since $Q_i Q_j = 0$ when $i \neq j$, $\operatorname{ran} Q_i \dotplus \operatorname{ran} Q_j$ is closed, and consequently, $\operatorname{ran} T = \bigvee_{j \neq 0} \mathcal{H}_j$ is closed. So by Corollary 6.6, T is similar to a positive operator C. In this case, C must be as in the statement of the proposition.

For the converse, the statement that T is algebraic follows from the spectral mapping theorem, using a polynomial with roots equal to the set of eigenvalues. Furthermore, T is a scalar operator and its spectrum is in a set of the form $\{0\} \cup [c, \infty)$, $c > 0$. Therefore by Corollary 6.4, T is in $\mathcal{L}^{+2}$. □

Remark Using Example 2, it is possible to write down an optimal pair for any T in $\mathcal{L}^{+2}$ which is algebraic. Let (A_j, B_j) be the optimal pair for the oblique projection Q_j, as constructed in that example. Claim that for $A = \sum_j \lambda_j A_j$, $B = \sum_j B_j$, (A, B) is an optimal pair for T. Since $Q_j Q_k = 0$ if $k \neq j$, $B_j A_k = 0$, or equivalently, $A_k B_j = 0$. Hence $T = AB$. It is immediate that $\operatorname{ran} A = \operatorname{ran} T$ and $\operatorname{ran} B = \operatorname{ran} T^*$, so the pair is optimal.

Next consider the class of compact operators in $\mathcal{L}^{+2}$.

Corollary 9.2 *Let T be a compact operator in $\mathcal{L}^{+2}$ and let $\sigma(T) = \{\lambda_j\}$. Then restricted to the range $\mathcal{H}_j$ of the Riesz projection corresponding to $\lambda_j \neq 0$, $T|_{\mathcal{H}_j} = \lambda_j 1_{\mathcal{H}_j}$. Furthermore, T has no quasi-nilpotent part other than $T|\ker T$.*

Proof The first part is obtained in the same way as in the proof of the last proposition, The last part follows directly from Proposition 5.2. □

Remark Despite the simple form of the eigenspaces for a compact operator in $\mathcal{L}^{+2}$, a compact operator is generally not as nice as an algebraic operator. Indeed, it need not be quasi-affine to a positive operator, even if it is in a Schatten class. It suffices to verify this with the trace class operators.

For example, let $(e_n)_{n=1}^{\infty}$ be an orthonormal basis on $\mathcal{H}$, and $\{\lambda_j\} \subset \mathbb{R}^+$ non-zero and absolutely summable. Also let P_n be the orthonormal projection onto the span of e_n. Define $A = \sum_n \lambda_n P_n$, a positive trace class operator. If $x = \sum_n \lambda_n e_n$, then $x \in \mathcal{H}$, and there is obviously no vector $y \in \mathcal{H}$ such that $Ay = x$. As in Example 3, define B to be the orthogonal projection onto $(\bigvee x)^{\perp}$. Then $T = AB$ is trace class, and is not quasi-similar to a positive operator. With minor modifications, T can be chosen to be trace class and not even quasi-affine to a positive operator.

It follows from Apostol's theorem (Theorem 3.3) that the eigenspaces of T do not form a basic system of subspaces. Moreover, it is not clear that a compact operator with eigenvalues and eigenspaces as in Corollary 9.2 will necessarily be in $\mathcal{L}^{+2}$, even if the eigenspaces do form a basic system.

Suppose that T is compact and of the form given in Corollary 9.2. Suppose furthermore that $\overline{\bigvee_n \mathcal{H}_n} = \mathcal{H}$. In this case the eigenspaces form a basic system. For $\{\alpha_n\} \subset \mathbb{R}^+ \backslash \{0\}$ with $\sum_n \alpha_n < \infty$, define $X : \bigoplus_n \mathcal{H}_n \to \mathcal{H}$ by

$$X(\oplus_n x_n) = \sum_n \alpha_n x_n, \qquad x_n \in \mathcal{H}_n.$$

Notice that $\bigoplus_n \mathcal{H}_n$ is a sort of "straightened" version of $\mathcal{H}$ and is isomorphic to $\mathcal{H}$. By the arguments in [1], X is bounded. Let $Q_n : \bigoplus_n \mathcal{H}_n \to \mathcal{H}$ be the oblique projection defined by

$$Q_n x = \begin{cases} x, & x \in \mathcal{H}_n; \\ 0, & x \in \bigoplus_{k \neq n} \mathcal{H}_n. \end{cases}$$

Lemma 9.3 *For X defined as above,*

$$X^* y = \sum_n \alpha_n Q_n^* y, \qquad y \in \mathcal{H}.$$

Proof As defined, Q_n has the properties that $\overline{\operatorname{ran}}\, Q_n^* = (\ker Q_n)^\perp = \mathcal{H}_n$ and $\ker Q_n^* = (\operatorname{ran} Q_n)^\perp = \mathcal{H}_n^\perp$. Thus, for $y \in \mathcal{H}$ and $x = \oplus_n x_n \in \bigoplus_n \mathcal{H}_n$,

$$\begin{aligned} \left\langle \sum_n \alpha_n Q_n^* y, x \right\rangle &= \sum_n \alpha_n \langle y, Q_n x \rangle = \sum_n \alpha_n \langle y, x_n \rangle \\ &= \left\langle y, \sum_n \alpha_n x_n \right\rangle = \langle y, Xx \rangle . \end{aligned}$$

□

Proposition 9.4 *Let T be a compact operator in $L(\mathcal{H})$ with $\sigma(T) = \{\lambda_j\} \geq 0$, and suppose that when restricted to the the range $\mathcal{H}_j$ of the Riesz projection corresponding to $\lambda_j \neq 0$, $T|_{\mathcal{H}_j} = \lambda_j 1_{\mathcal{H}_j}$, and that the quasi-nilpotent part of T is $T|\ker T$. If $\sum_j \lambda_j^{1/2} < \infty$, then $T \in \mathcal{L}^{+2}$.*

Proof Take X defined as above, but with $\alpha_n = \lambda_n^{1/2}$ if $\lambda_n > 0$ and 1 otherwise. Then $\sum_n \alpha_n < \infty$ and by Lemma 9.3, for $x = \oplus_n x_n \in \bigoplus_n \mathcal{H}_n$,

$$\begin{aligned} \|Xx\|^2 &= \left\langle X^*(\sum_n \alpha_n x_n), \oplus_n x_n \right\rangle = \sum_n \alpha_n \left\langle X^* x_n, \oplus_n x_n \right\rangle \\ &= \sum_n \alpha_n^2 \langle \oplus_n x_n, \oplus_n x_n \rangle = \sum_n \alpha_n^2 \|x_n\|^2. \end{aligned}$$

Define $C \geq 0$ on $\bigoplus_n \mathcal{H}_n$ by $\langle Cx, x \rangle = \sum_n \lambda_n \|x_n\|^2$. Then X is a quasi-affinity and $X^*X \geq C$ (in fact, it will be equal if $\ker T = \{0\}$). By Douglas' lemma, $C^{1/2} = X^* Z$

for bounded Z. By Apostol's theorem (or rather, the proof of it),

$$TX = XC = XX^*ZZ^*X,$$

and so since ran X is dense, $T \in \mathcal{L}^{+2}$. □

Next consider Fredholm operators in $\mathcal{L}^{+2}$. Recall that T is *left-semi-Fredholm* if there exists a bounded operator R and a compact operator K such that $RT = 1 + K$. On the other hand, it is *right semi-Fredholm* if there exist such R and K such that $TR = 1 + K$. Finally, T is *Fredholm* if it is both left and right semi-Fredholm.

Proposition 9.5 *Let $T \in \mathcal{L}^{+2}$. Then T is left / right semi-Fredholm if and only if T is Fredholm and similar to a positive operator with closed range and finite dimensional kernel. In this case,*

$$\operatorname{ind} T := \dim \ker T - \dim \ker T^* = 0.$$

Proof Suppose that $T \in \mathcal{L}^{+2}$ and that it is left semi-Fredholm (the other case is handled identically). Then by Atkinson's theorem, ran T is closed and $\dim \ker T < \infty$. Hence by Proposition 6.3, T is similar to a positive operator. If $T = LCL^{-1}$ where $C \geq 0$ and L is invertible, ran T closed implies that ran C is closed, and $\dim \ker T < \infty$ gives $\dim \ker C < \infty$. Since $T^* = L^{*-1}CL^*$, $\dim \ker T^* = \dim \ker T$. □

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References

1. Apostol, C.: Operators quasisimilar to a normal operator. Proc. Amer. Math. Soc. **53**(1), 104–106 (1975)
2. Arias, M.L., Corach, G., Gonzalez, C.M.: Products of projections and positive operators. Linear Algebra Appl. **439**(7), 1730–1741 (2013)
3. Colojoară, I., Foiaş, C.: *Theory of generalized spectral operators*, Gordon and Breach, Science Publishers, New York-London-Paris, 1968, Mathematics and its Applications, Vol. 9
4. Corach, G., Maestripieri, A.: Products of orthogonal projections and polar decompositions. Linear Algebra Appl. **434**(6), 1594–1609 (2011)

5. Cui, J., Li, C.-K., Sze, N.-S.: Products of positive semi-definite matrices. Linear Algebra Appl. **528**, 17–24 (2017)
6. Douglas, R.G.: On the operator equation $S^*XT = X$ and related topics. Acta Sci. Math. (Szeged) **30**, 19–32 (1969)
7. Dunford, N.: Spectral operators. Pacific J. Math. **4**, 321–354 (1954)
8. Fialkow, L.A.: A note on quasisimilarity of operators. Acta Sci. Math. (Szeged) **39**(1–2), 67–85 (1977)
9. Fillmore, P.A., Williams, J.P.: On operator ranges. Adv. Math. **7**, 254–281 (1971)
10. Hassi, S., Sebestyén, Z., de Snoo, H.S.V.: On the nonnegativity of operator products. Acta Math. Hungar. **109**(1–2), 1–14 (2005)
11. Izumino, S.: The product of operators with closed range and an extension of the reverse order law. Tohoku Math. J. **34**(1), 43–52 (1982)
12. Laursen, K.B., Neumann, M.M.:*An introduction to local spectral theory*, London Mathematical Society Monographs. New Series, vol. 20, The Clarendon Press, Oxford University Press, New York (2000)
13. Magajna, B.: Sums of products of positive operators and spectra of Lüders operators. Proc. Amer. Math. Soc. **141**(4), 1349–1360 (2013)
14. Mbekhta, M.: Sur la théorie spectrale locale et limite des nilpotents. Proc. Amer. Math. Soc. **110**(3), 621–631 (1990)
15. Pedersen, G.K., Takesaki, M.: The operator equation $THT = K$. Proc. Amer. Math. Soc. Proc. Amer. Math. Soc. **36**, 311–312 (1972)
16. Phillips, N.C.: Every invertible Hilbert space operator is a product of seven positive operators. Canad. Math. Bull. **38**(2), 230–236 (1995)
17. Radjavi, H., Williams, J.P.: Products of self-adjoint operators. Michigan Math. J. **16**, 177–185 (1969)
18. Sebestyén, Z.: Restrictions of positive operators. Acta Sci. Math. (Szeged) **46**(1–4), 299–301 (1983)
19. Stampfli, J.G.: Quasisimilarity of operators. Proc. Roy. Irish Acad. Sect. A **81**(1), 109–119 (1981)
20. Tzafriri, L.: The connection between normalizable and spectral operators. Israel J. Math. **3**, 75–80 (1965)
21. Wu, P.Y.: Products of normal operators. Canad. J. Math. **40**(6), 1322–1330 (1988)
22. Wu, P.Y.: Products of positive semidefinite matrices. Linear Algebra Appl. **111**, 53–61 (1988)

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