Exterior Powers and Pointwise Creation Operators

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Abstract

We develop a theory of pointwise wedge products of vector-valued functions on the circle and the disc, and obtain results which give rise to a new approach to the analysis of the matricial Nehari problem. We investigate properties of pointwise creation operators and pointwise orthogonal complements in the context of operator theory and the study of vector-valued analytic functions on the unit disc.

Keywords Creation operators · Exterior products · Wedge products · Pointwise wedge products

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1 Introduction

The wedge product of Hilbert spaces, though a long established theory (see, for example [3,5,11,12]), deserves in our view to be better exploited in operator theory than it

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has been hitherto. In this paper we put forward a new approach to some aspects of the analysis of E -valued functions on the unit disc $\mathbb{D}$ and on the unit circle $\mathbb{T}$, where E is a separable Hilbert space. Our first application of this idea was to the problem of the superoptimal analytic approximation of a continuous matrix-valued function on the unit circle [2]. The computation of such approximants arises naturally in the context of the classical “Nehari problem”, and also in its application to the “robust stabilization problem” in control engineering. In [2] we give a new algorithm for the construction of the unique superoptimal analytic approximant of a continuous matrix-valued function on the unit circle, making use of exterior powers of operators in preference to spectral or Wiener-Masani factorizations. This algorithm is parallel to the construction of [10], but has the advantage that it requires only the spectral factorisation of *scalar* functions on the circle, together with the calculation of singular-value decompositions, and otherwise requires only rational arithmetic. In the algorithm we make use of pointwise creation operators and pointwise orthogonal complements.

We show how the Hilbert space geometry of the Hardy space $H^2(\mathbb{D}, E)$, where E is a separable Hilbert space, interacts with the pointwise geometry of $H^2(\mathbb{D}, E)$, that is, the geometry of E over each point of $\mathbb{T}$ or $\mathbb{D}$. We work with the completion of the algebraic tensor product of Hilbert spaces, namely the Hilbert tensor product (see [4,6]), and, in particular, with its closed linear subspace of antisymmetric tensors. The space of all antisymmetric tensors $\wedge^p E$, also called a *wedge* or *exterior product*, is a closed linear subspace of the p -fold Hilbert tensor product $\otimes_H^p E$, see Sect. 2.

In Sect. 3, we define the pointwise wedge product of maps that are defined either on the unit disc or on the unit circle and take values in Hilbert spaces. We illustrate the fact that some classical results from function theory extend to the pointwise wedge product. An exemplary result is Proposition 3.7 which asserts that, for any separable Hilbert space E , the pointwise wedge product of two functions in the Hardy space $H^2(\mathbb{D}, E)$ is an element of $H^1(\mathbb{D}, \wedge^2 E)$. We study pointwise creation operators, which are of the form

$$C_\xi: H^2(\mathbb{D}, E) \rightarrow H^2(\mathbb{D}, \wedge^2 E), \quad f \mapsto \xi \dot{\wedge} f,$$

where $\xi: \mathbb{D} \rightarrow E$ is a bounded analytic function, and the symbol $\xi \dot{\wedge} f$ denotes the function

$$(\xi \dot{\wedge} f)(z) = \xi(z) \wedge f(z) \quad \text{for all } z \in \mathbb{D}.$$

Analogous notation applies when ξ, f are E -valued functions defined almost everywhere on $\mathbb{T}$. We connect the kernel of C_ξ to the pointwise linear span and pointwise orthogonal complement of ξ .

Definition 1.1 Let E be a separable Hilbert space. $H^\infty(\mathbb{D}, E)$ denotes the space of bounded analytic E -valued functions Q on the unit disk with supremum norm:

$$\|Q\|_{H^\infty} \stackrel{\text{def}}{=} \|Q\|_\infty \stackrel{\text{def}}{=} \sup_{z \in \mathbb{D}} \|Q(z)\|_E.$$

In Sect. 4 we prove, among other things, the following two statements.

Theorem 4.7 *Let E be a separable Hilbert space, let $\xi_0, \xi_1, \dots, \xi_j \in H^\infty(\mathbb{D}, E)$. Suppose that the set $\{\xi_i(z)\}_{i=0}^j$ is orthonormal in E for almost every $z \in \mathbb{T}$. Then*

$$\xi_0 \wedge \dots \wedge \xi_j \wedge H^2(\mathbb{D}, E)$$

is a closed subspace of $H^2(\mathbb{D}, \wedge^{j+2} E)$.

Let F and E be Hilbert spaces. The Banach space of bounded linear operators from F to E with the operator norm is denoted by $\mathcal{B}(F, E)$.

Definition 1.2 Let E be a separable Hilbert space and let $\xi, \eta \in L^\infty(\mathbb{T}, E)$. We define $\xi\eta^* \in L^\infty(\mathbb{T}, \mathcal{B}(E, E))$ by

$$(\xi\eta^*)(z)x = \langle x, \eta(z) \rangle \xi(z) \text{ for all } x \in E \text{ and for almost every } z \text{ on } \mathbb{T}.$$

Thus $\xi\eta^*(z)$ is the operator of rank one on E that is sometimes denoted by $\xi(z) \otimes \eta(z)$ (for example, in [1, Eq. (1.17)]).

Theorem 4.18 *Let E be a separable Hilbert space. Let $\xi \in H^\infty(\mathbb{D}, E)$ be an inner function. Then, for any $h \in H^2(\mathbb{D}, E)$,*

$$C_\xi^* C_\xi h = P_+ \alpha_\xi,$$

where $\alpha_\xi = h - \xi\xi^*h$ and $P_+ : L^2(\mathbb{T}, E) \rightarrow H^2(\mathbb{D}, E)$ is the orthogonal projection. Moreover

$$C_\xi^* C_\xi h = h - T_{\xi\xi^*} h,$$

where $T_{\xi\xi^*} : H^2(\mathbb{D}, E) \rightarrow H^2(\mathbb{D}, E)$ is the Toeplitz operator with symbol $\xi\xi^*$.

Thus $C_\xi^* C_\xi$ is the Toeplitz operator with symbol $1 - \xi\xi^*$. Finally, we show that although C_ξ is an isometry from the pointwise orthogonal complement of ξ to $H^2(\mathbb{D}, \wedge^2 E)$, a simple example proves that C_ξ fails to be a partial isometry.

2 Exterior Powers of Hilbert Spaces and Operators

In this Section we recall the established notion of the exterior power, or wedge product, of Hilbert spaces. One can find definitions and properties of wedge products in [3, 5, 9, 11–13]. Here we present a concise version of this theory which we need for the development of pointwise wedge products of vector-valued functions on the circle and the disc. We shall assume the notion of the algebraic tensor product of linear spaces as given, for example, in [6].

For E a Hilbert space, we consider the completion of the p -fold algebraic tensor product, denoted by $\otimes_H^p E$, with respect to the norm induced by the inner product (2.1). We define an action of permutation operators on tensor products and examine

various properties. These permutation operators generate two different types of tensors, symmetric and antisymmetric, and we focus on the properties of the antisymmetric tensors. The space of all antisymmetric tensors, also called wedge or exterior products, in $\otimes_H^p E$ is denoted by $\wedge^p E$ and in Theorem 2.9 we prove it is a closed linear subspace of $\otimes_H^p E$.

In the following E denotes a Hilbert space.

Definition 2.1 $\otimes^p E$ is the p -fold algebraic tensor product of E , spanned by tensors of the form $x_1 \otimes x_2 \otimes \cdots \otimes x_p$, where $x_j \in E$ for $j = 1, \dots, p$.

Definition 2.2 An inner product on $\otimes^p E$ is given on elementary tensors by

$$\langle x_1 \otimes x_2 \otimes \cdots \otimes x_p, y_1 \otimes y_2 \otimes \cdots \otimes y_p \rangle_{\otimes^p E} = p! \langle x_1, y_1 \rangle_E \cdots \langle x_p, y_p \rangle_E \quad (2.1)$$

for any $x_1, \dots, x_p, y_1, \dots, y_p \in E$, and is extended to $\otimes^p E$ by sesqui-linearity.

Definition 2.3 $\otimes_H^p E$ is the completion of $\otimes^p E$ with respect to the norm

$$\|u\| = \langle u, u \rangle_{\otimes^p E}^{1/2}, \text{ for } u \in \otimes^p E.$$

Observe that the inner product (2.1), in contrast to the majority of the ones included in the bibliography, invokes a multiple of $p!$. The reason for this choice will be apparent in Theorem 2.11.

In order to introduce antisymmetric tensors, we need to consider the action of the following permutation operators on tensors.

Definition 2.4 Let $\mathfrak{S}_p$ denote the symmetric group on $\{1, \dots, p\}$, with the operation of composition. For $\sigma \in \mathfrak{S}_p$, we define

$$S_\sigma: \otimes^p E \rightarrow \otimes^p E$$

on elementary tensors by

$$S_\sigma(x_1 \otimes x_2 \otimes \cdots \otimes x_p) = x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(p)},$$

and we extend S_σ to $\otimes^p E$ by linearity, that is, for $u = \sum_{i=1}^n \lambda_i x_1^i \otimes \cdots \otimes x_p^i$, we define

$$S_\sigma(u) = \sum_{i=1}^n \lambda_i S_\sigma(x_1^i \otimes \cdots \otimes x_p^i).$$

for any $x_j^i \in E$ and $\lambda_i \in \mathbb{C}$.

Remark 2.5 $(\mathfrak{S}_p, \circ)$ is a group, and so, for every permutation $\sigma \in \mathfrak{S}_p$, there exists $\sigma^{-1} \in \mathfrak{S}_p$ such that

$$\sigma \circ \sigma^{-1} = \text{id} = \sigma^{-1} \circ \sigma,$$

where $\text{id} \in \mathfrak{S}_p$ is the identity map on $\{1, \dots, p\}$.

Elements of $\mathfrak{S}_p$ induce unitary operators on $\otimes_H^p E$ in an obvious way.

Proposition 2.6 *Let E be a Hilbert space, and let p be a positive integer. Then, for any $\sigma \in \mathfrak{S}_p$, S_σ is a linear operator on the normed space $(\otimes^p E, \|\cdot\|)$, which extends to an isometry $\mathbf{S}_\sigma$ on $(\otimes_H^p E, \|\cdot\|)$. Furthermore, $\mathbf{S}_\sigma$ is a unitary operator on $\otimes_H^p E$.*

Proof It is easy to check that S_σ is linear. For any elementary tensors $w = x_1 \otimes x_2 \otimes \cdots \otimes x_p$, $v = y_1 \otimes y_2 \otimes \cdots \otimes y_p$ by the definition of the inner product on $\otimes^p E$,

$$\begin{aligned} \langle S_\sigma^* w, v \rangle_{\otimes^p E} &= \langle w, y_{\sigma(1)} \otimes \cdots \otimes y_{\sigma(p)} \rangle_{\otimes^p E} \\ &= p! \prod_{j=1}^p \langle x_j, y_{\sigma(j)} \rangle_E \\ &= p! \prod_{j=1}^p \langle x_{\sigma^{-1}(j)}, y_j \rangle_E \\ &= \langle S_{\sigma^{-1}} w, v \rangle_{\otimes^p E}. \end{aligned}$$

Hence $S_\sigma^* = S_{\sigma^{-1}}$, thus

$$S_\sigma^* S_\sigma = S_{\sigma^{-1}} S_\sigma = I,$$

the identity operator on $\otimes^p E$, and therefore S_σ is an isometric linear self-map of $\otimes^p E$. Likewise, $S_\sigma S_\sigma^* = I$, and so S_σ is also a surjective self-map of $\otimes^p E$.

Thus one can extend S_σ by continuity to an isometric linear self-map $\mathbf{S}_\sigma$ of the completion $\otimes_H^p E$ of $\otimes^p E$. Since $\mathbf{S}_\sigma$ is isometric, its range is complete, hence closed in $\otimes_H^p E$. Since the range of $\mathbf{S}_\sigma$ contains that of S_σ , $\text{ran } \mathbf{S}_\sigma = \otimes_H^p E$. Being both surjective and isometric, $\mathbf{S}_\sigma$ is a unitary operator on $\otimes_H^p E$. $\square$

Henceforth we shall denote the extended operator $\mathbf{S}_\sigma$ by S_σ .

Definition 2.7 A tensor $u \in \otimes_H^p E$ is said to be *symmetric* if $S_\sigma(u) = u$ for all $\sigma \in \mathfrak{S}_p$. A tensor $u \in \otimes_H^p E$ is said to be *antisymmetric* if $u = \epsilon_\sigma S_\sigma u$ for all $\sigma \in \mathfrak{S}_p$, where ϵ_σ is the signature of σ .

Note that $\epsilon_{\sigma \circ \sigma^{-1}} = \epsilon_\sigma \epsilon_{\sigma^{-1}} = 1$, and hence $\epsilon_\sigma = \epsilon_{\sigma^{-1}}$.

Definition 2.8 The space of all antisymmetric tensors in $\otimes_H^p E$ will be denoted by $\wedge^p E$.

Theorem 2.9 *Let E be a Hilbert space. Then $\wedge^p E$ is a closed linear subspace of the Hilbert space $\otimes_H^p E$ for any $p \geq 2$.*

Proof For $\sigma \in \mathfrak{S}_p$ define the operator

$$f_\sigma \stackrel{\text{def}}{=} S_\sigma - \epsilon_\sigma I: \otimes_H^p E \rightarrow \otimes_H^p E,$$

where I denotes the identity operator on $\otimes_H^p E$. Since S_σ is a continuous linear operator on $\otimes_H^p E$, f_σ is a continuous linear operator. The kernel of the operator f_σ is

$$\begin{aligned}\ker f_\sigma &= \{u \in \otimes_H^p E : (S_\sigma - \epsilon_\sigma I)(u) = 0\} \\ &= \{u \in \otimes_H^p E : S_\sigma(u) = \epsilon_\sigma u\} \\ &= \{u \in \otimes_H^p E : \epsilon_\sigma S_\sigma(u) = u\}.\end{aligned}$$

Since f_σ is a continuous linear operator on $\otimes_H^p E$, $\ker f_\sigma$ is a closed linear subspace of $\otimes_H^p E$. Thus $\wedge^p E$ is a closed linear subspace of $\otimes_H^p E$, since

$$\wedge^p E = \{u \in \otimes_H^p E : \epsilon_\sigma S_\sigma(u) = u \text{ for all } \sigma \in \mathfrak{S}_p\} = \bigcap_{\sigma \in \mathfrak{S}_p} \ker f_\sigma.$$

□

Theorem 2.9 implies that the orthogonal projection from $\otimes_H^p E$ onto $\wedge^p E$ is well-defined.

Definition 2.10 Let E be a Hilbert space. For $x_1, \dots, x_p \in E$, define $x_1 \wedge x_2 \wedge \dots \wedge x_p$ to be the orthogonal projection of the elementary tensor $x_1 \otimes x_2 \otimes \dots \otimes x_p$ onto $\wedge^p E$, that is

$$x_1 \wedge x_2 \wedge \dots \wedge x_p = P_{\wedge^p E}(x_1 \otimes \dots \otimes x_p).$$

Theorem 2.11 For all $u \in \otimes_H^p E$,

$$P_{\wedge^p E}(u) = \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} \epsilon_\sigma S_\sigma(u).$$

Proof Let $u \in \otimes_H^p E$. Then, for any $\sigma \in \mathfrak{S}_p$, $u = \epsilon_\sigma S_\sigma(u) + (u - \epsilon_\sigma S_\sigma(u))$, and so

$$p!u = \sum_{\sigma \in \mathfrak{S}_p} \epsilon_\sigma S_\sigma(u) + \sum_{\sigma \in \mathfrak{S}_p} (u - \epsilon_\sigma S_\sigma(u)).$$

It suffices to show that $\sum_{\sigma \in \mathfrak{S}_p} \epsilon_\sigma S_\sigma(u) \in \wedge^p E$ and

$$\sum_{\sigma \in \mathfrak{S}_p} (u - \epsilon_\sigma S_\sigma(u))$$

is orthogonal to the set of antisymmetric tensors, in other words, that if $v \in \wedge^p E$ then

$$\langle v, \sum_{\sigma \in \mathfrak{S}_p} (u - \epsilon_\sigma S_\sigma(u)) \rangle_{\otimes_H^p E} = 0.$$

Let $w = \sum_{\sigma \in \mathfrak{S}_p} \epsilon_\sigma S_\sigma(u) \in \otimes_H^p E$. For every $\tau \in \mathfrak{S}_p$, we have

$$\begin{aligned} \epsilon_\tau S_\tau(w) &= \epsilon_\tau S_\tau \left(\sum_{\sigma \in \mathfrak{S}_p} \epsilon_\sigma S_\sigma(u) \right) = \sum_{\tau \circ \sigma \in \mathfrak{S}_p} \epsilon_{\tau \circ \sigma} S_{\tau \circ \sigma}(u) \\ &= \sum_{\sigma' \in \mathfrak{S}_p} \epsilon_{\sigma'} S_{\sigma'}(u) = w, \end{aligned}$$

where $\tau \circ \sigma = \sigma'$. Hence $\sum_{\sigma \in \mathfrak{S}_p} \epsilon_\sigma S_\sigma(u) \in \wedge^p E$.

For every $v \in \wedge^p E$, we have $v = \epsilon_\sigma S_\sigma v$ for all $\sigma \in \mathfrak{S}_p$, and

$$\begin{aligned} \langle v, \sum_{\sigma \in \mathfrak{S}_p} (u - \epsilon_\sigma S_\sigma(u)) \rangle_{\otimes_H^p E} &= \sum_{\sigma \in \mathfrak{S}_p} \langle v, u \rangle_{\otimes_H^p E} - \sum_{\sigma \in \mathfrak{S}_p} \epsilon_\sigma \langle v, S_\sigma(u) \rangle_{\otimes_H^p E} \\ &= \sum_{\sigma \in \mathfrak{S}_p} \langle v, u \rangle_{\otimes_H^p E} - \sum_{\sigma \in \mathfrak{S}_p} \epsilon_\sigma \langle S_\sigma^* v, u \rangle_{\otimes_H^p E} \\ &= \sum_{\sigma \in \mathfrak{S}_p} \langle v - \epsilon_{\sigma^{-1}} S_{\sigma^{-1}} v, u \rangle_{\otimes_H^p E} = 0. \end{aligned}$$

□

Remark 2.12 If $p > 1$, then $\mathfrak{S}_p$ contains a transposition, for instance $\sigma = (12)$, and $\epsilon_\sigma = -1$. If $p = 1$, then $\wedge^1 E = E$.

Proposition 2.13 *Let E be a Hilbert space and let $p \geq 2$. The set of antisymmetric tensors and the set of symmetric tensors are orthogonal in $\otimes_H^p E$.*

Proof Suppose that u is a symmetric tensor, that is $S_\sigma u = u$ for all $\sigma \in \mathfrak{S}_p$, and that v is an antisymmetric tensor, that is, $S_\sigma v = \epsilon_\sigma v$ for all $\sigma \in \mathfrak{S}_p$. Since $p \geq 2$ there exists a transposition $\sigma \in \mathfrak{S}_p$, so that $\epsilon_\sigma = -1$, and therefore $S_\sigma v = -v$. By Proposition 2.6, S_σ is a unitary operator on $\otimes_H^p E$. Thus

$$\langle u, v \rangle_{\otimes_H^p E} = \langle S_\sigma u, S_\sigma v \rangle_{\otimes_H^p E} = \langle u, -v \rangle_{\otimes_H^p E}.$$

Thus $\langle u, v \rangle_{\otimes_H^p E} = -\langle u, v \rangle_{\otimes_H^p E}$, and so $\langle u, v \rangle_{\otimes_H^p E} = 0$. □

Proposition 2.14 *Let E be a Hilbert space. The inner product in $\wedge^p E$ is given by*

$$\langle x_1 \wedge \cdots \wedge x_p, y_1 \wedge \cdots \wedge y_p \rangle_{\wedge^p E} = \det \begin{pmatrix} \langle x_1, y_1 \rangle_E & \cdots & \langle x_1, y_p \rangle_E \\ \vdots & \ddots & \vdots \\ \langle x_p, y_1 \rangle_E & \cdots & \langle x_p, y_p \rangle_E \end{pmatrix}$$

for all $x_1, \dots, x_p, y_1, \dots, y_p \in E$.

Proof By Theorem 2.11, we have

$$\begin{aligned}
 & \langle x_1 \wedge \cdots \wedge x_p, y_1 \wedge \cdots \wedge y_p \rangle_{\wedge^p E} \\
 &= \left\langle \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} \epsilon_\sigma S_\sigma(x_1 \otimes x_2 \otimes \cdots \otimes x_p), \frac{1}{p!} \sum_{\tau \in \mathfrak{S}_p} \epsilon_\tau S_\tau(y_1 \otimes y_2 \otimes \cdots \otimes y_p) \right\rangle_{\otimes_H^p E} \\
 &= \frac{1}{p!^2} \sum_{\sigma, \tau \in \mathfrak{S}_p} \langle \epsilon_\sigma S_\sigma(x_1 \otimes x_2 \otimes \cdots \otimes x_p), \epsilon_\tau S_\tau(y_1 \otimes y_2 \otimes \cdots \otimes y_p) \rangle_{\otimes_H^p E} \\
 &= \frac{1}{p!^2} \sum_{\sigma, \tau \in \mathfrak{S}_p} \epsilon_\sigma \epsilon_\tau \langle x_1 \otimes x_2 \otimes \cdots \otimes x_p, S_\sigma^* S_\tau(y_1 \otimes y_2 \otimes \cdots \otimes y_p) \rangle_{\otimes_H^p E} \\
 &= \frac{1}{p!^2} \sum_{\sigma, \tau \in \mathfrak{S}_p} \epsilon_{\sigma^{-1} \tau} \langle x_1 \otimes x_2 \otimes \cdots \otimes x_p, S_{\sigma^{-1} \tau}(y_1 \otimes y_2 \otimes \cdots \otimes y_p) \rangle_{\otimes_H^p E} \\
 &= \frac{1}{p!} \sum_{\sigma' \in \mathfrak{S}_p} \epsilon_{\sigma'} \langle x_1 \otimes x_2 \otimes \cdots \otimes x_p, S_{\sigma'}(y_1 \otimes y_2 \otimes \cdots \otimes y_p) \rangle_{\otimes_H^p E} \\
 &= \sum_{\sigma' \in \mathfrak{S}_p} \epsilon_{\sigma'} \prod_{i=1}^p \langle x_i, y_{\sigma'(i)} \rangle_E \\
 &= \det \begin{pmatrix} \langle x_1, y_1 \rangle_E & \cdots & \langle x_1, y_p \rangle_E \\ \vdots & \ddots & \vdots \\ \langle x_p, y_1 \rangle_E & \cdots & \langle x_p, y_p \rangle_E \end{pmatrix}
 \end{aligned}$$

by Leibniz' formula. $\square$

Since we have already shown that $\wedge^p E$ is a closed linear subspace of the Hilbert space $\otimes_H^p E$, the space $(\wedge^p E, \langle \cdot, \cdot \rangle_{\wedge^p E})$ with inner product given by Proposition 2.14 is itself a Hilbert space.

Lemma 2.15 *Suppose $\{u_1, \dots, u_j\}$ is an orthonormal set in E . Then, for every $x \in E$,*

$$\|u_1 \wedge \cdots \wedge u_j \wedge x\|_{\wedge^{j+1} E} = \|x - \sum_{i=1}^j \langle x, u_i \rangle u_i\|_E.$$

Proof For $x \in E$ we may write

$$x = x - \sum_{i=1}^j \langle x, u_i \rangle u_i + \sum_{i=1}^j \langle x, u_i \rangle u_i.$$

By Proposition 2.14,

$$\begin{aligned}
 & \|u_1 \wedge \cdots \wedge u_j \wedge x\|_{\wedge^{j+1} E}^2 \\
 &= \langle u_1 \wedge \cdots \wedge u_j \wedge x, u_1 \wedge \cdots \wedge u_j \wedge x \rangle_{\wedge^{j+1} E} \\
 &= \det \begin{pmatrix} \langle u_1, u_1 \rangle_E & \langle u_1, u_2 \rangle_E & \cdots & \cdots & \langle u_1, x \rangle_E \\ \langle u_2, u_1 \rangle_E & \langle u_2, u_2 \rangle_E & \cdots & \cdots & \langle u_2, x \rangle_E \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \langle u_j, u_1 \rangle_E & \langle u_j, u_2 \rangle_E & \cdots & \langle u_j, u_j \rangle_E & \langle u_j, x \rangle_E \\ \langle x, u_1 \rangle_E & \langle x, u_2 \rangle_E & \cdots & \cdots & \langle x, x \rangle_E \end{pmatrix}.
 \end{aligned}$$

By assumption,

$$\langle u_i, u_k \rangle = \begin{cases} 0, & \text{if } i \neq k \\ 1, & \text{if } i = k \end{cases},$$

and hence

$$\|u_1 \wedge \cdots \wedge u_j \wedge x\|_{\wedge^{j+1}E}^2 = \det \begin{pmatrix} 1 & 0 & \cdots & \langle u_1, x \rangle_E \\ 0 & 1 & \cdots & \langle u_2, x \rangle_E \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 1 & \langle u_j, x \rangle_E \\ \langle x, u_1 \rangle_E & \langle x, u_2 \rangle_E & \cdots & \langle x, x \rangle_E \end{pmatrix}.$$

If, for $k = 1, \dots, j$, we multiply the k -th column of the determinant by $\langle u_k, x \rangle_E$ and subtract it from the $(j+1)$ -th column, we find that

$$\begin{aligned} & \|u_1 \wedge \cdots \wedge u_j \wedge x\|_{\wedge^{j+1}E}^2 \\ &= \det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \\ \langle x, u_1 \rangle_E & \langle x, u_2 \rangle_E & \cdots & \langle x, x \rangle_E - \sum_{i=1}^j |\langle x, u_i \rangle_E|^2 \end{pmatrix} \\ &= \|x\|_E^2 - \sum_{i=1}^j |\langle x, u_i \rangle_E|^2 \\ &= \|x - \sum_{i=1}^j \langle x, u_i \rangle_E u_i\|_E^2, \end{aligned}$$

the last equality by Pythagoras' theorem. $\square$

Definition 2.16 Let $(E, \|\cdot\|_E)$ be a Hilbert space. The p -fold Cartesian product of E is defined to be the set

$$\underbrace{E \times \cdots \times E}_{p\text{-times}} = \{(x_1, \dots, x_p) : x_i \in E\}.$$

Moreover, we define a norm on $\underbrace{E \times \cdots \times E}_{p\text{-times}}$ by

$$\|(x_1, \dots, x_p)\| = \left\{ \sum_{i=1}^p \|x_i\|_E^2 \right\}^{\frac{1}{2}}.$$

Definition 2.17 Let E be a Hilbert space. We define the multilinear operator

$$\Lambda: \underbrace{E \times \cdots \times E}_{p\text{-times}} \rightarrow \wedge^p E$$

by

$$\Lambda(x_1, \dots, x_p) = x_1 \wedge \dots \wedge x_p \quad \text{for all } x_1, \dots, x_p \in E.$$

Proposition 2.18 [Hadamard's inequality, [7], p. 477] *For any matrix*

$$A = (a_{ij}) \in \mathbb{C}^{n \times n},$$

$$|\det(A)| \leq \prod_{j=1}^n \left(\sum_{i=1}^n |a_{ij}|^2 \right)^{1/2} \quad \text{and} \quad |\det(A)| \leq \prod_{i=1}^n \left(\sum_{j=1}^n |a_{ij}|^2 \right)^{1/2}.$$

Proposition 2.19 *Let E be a Hilbert space. Then the multilinear mapping*

$$\Lambda: \underbrace{E \times \cdots \times E}_{p\text{-times}} \rightarrow \wedge^p E$$

is bounded.

Proof Let $x_i \in E$ for all $i = 1, \dots, p$. Then $\Lambda(x_1, \dots, x_p) = x_1 \wedge \dots \wedge x_p$ and

$$\begin{aligned} \|\Lambda(x_1, \dots, x_p)\|_{\wedge^p E}^2 &= \|x_1 \wedge \dots \wedge x_p\|_{\wedge^p E}^2 \\ &= \langle x_1 \wedge \dots \wedge x_p, x_1 \wedge \dots \wedge x_p \rangle_{\wedge^p E} \\ &= \det \begin{pmatrix} \langle x_1, x_1 \rangle_E & \langle x_1, x_2 \rangle_E & \dots & \langle x_1, x_p \rangle_E \\ \langle x_2, x_1 \rangle_E & \langle x_2, x_2 \rangle_E & \dots & \langle x_2, x_p \rangle_E \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_p, x_1 \rangle_E & \dots & \dots & \langle x_p, x_p \rangle_E \end{pmatrix} \geq 0. \end{aligned}$$

Let

$$X = \begin{pmatrix} \langle x_1, x_1 \rangle_E & \langle x_1, x_2 \rangle_E & \dots & \langle x_1, x_p \rangle_E \\ \langle x_2, x_1 \rangle_E & \langle x_2, x_2 \rangle_E & \dots & \langle x_2, x_p \rangle_E \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_p, x_1 \rangle_E & \dots & \dots & \langle x_p, x_p \rangle_E \end{pmatrix}.$$

By Hadamard's inequality,

$$|\det(X)| \leq \prod_{j=1}^p \left(\sum_{i=1}^p |\langle x_i, x_j \rangle_E|^2 \right)^{1/2}.$$

Moreover, by the Cauchy–Schwarz inequality,

$$|\det(X)| \leq \prod_{j=1}^p \|x_j\|_E \left(\sum_{i=1}^p \|x_i\|_E^2 \right)^{1/2}.$$

Therefore

$$\|\Lambda(x_1, \dots, x_p)\|_{\wedge^p E}^2 \leq \prod_{j=1}^p \|x_j\|_E \left(\sum_{i=1}^p \|x_i\|_E^2 \right)^{1/2}. \quad (2.2)$$

Let $\|(x_1, \dots, x_p)\|_{E^p} \leq 1$. Since $\|x_j\|_E \leq \|(x_1, \dots, x_p)\|_{E^p} \leq 1$ for each j , we have

$$\|\Lambda(x_1, \dots, x_p)\|_{\wedge^p E}^2 \leq 1.$$

Hence the p -linear operator Λ is bounded. $\square$

3 Pointwise Wedge Products

In this section we introduce the notion of pointwise wedge product of vector-valued functions on the unit circle or in the unit disk and explore its features.

Definition 3.1 Let E be a Hilbert space and let $f, g: \mathbb{D} \rightarrow E$ ($f, g: \mathbb{T} \rightarrow E$) be E -valued maps. We define the *pointwise wedge product of f and g* ,

$$f \dot{\wedge} g: \mathbb{D} \rightarrow \wedge^2 E \quad (f \dot{\wedge} g: \mathbb{T} \rightarrow \wedge^2 E)$$

by

$$(f \dot{\wedge} g)(z) = f(z) \wedge g(z) \quad \text{for all } z \in \mathbb{D} \quad (\text{for almost all } z \in \mathbb{T}).$$

Definition 3.2 Let E be a Hilbert space and let $\chi_1, \dots, \chi_n: \mathbb{D} \rightarrow E$ ($\chi_1, \dots, \chi_n: \mathbb{T} \rightarrow E$) be E -valued maps. We call $\chi_1, \dots, \chi_n$ *pointwise linearly dependent on $\mathbb{D}$ (or on $\mathbb{T}$)* if for all $z \in \mathbb{D}$ (for almost all $z \in \mathbb{T}$ respectively) the vectors $\chi_1(z), \dots, \chi_n(z)$ are linearly dependent in E .

Remark 3.3 If $x_1, \dots, x_n$ are pointwise linearly dependent on $\mathbb{T}$, then

$$(x_1 \dot{\wedge} \dots \dot{\wedge} x_n)(z) = 0$$

for almost all $z \in \mathbb{T}$.

3.1 Pointwise Wedge Products on Function Spaces

For vector-valued L^p spaces we use the terminology of [8].

Definition 3.4 Let E be a separable Hilbert space and let $1 \leq p < \infty$. Define

- (i) $L^p(\mathbb{T}, E)$ to be the normed space of measurable (weakly or strongly, which amounts to the same thing, in view of the separability of E) E -valued maps $f: \mathbb{T} \rightarrow E$ such that

$$\|f\|_p = \left(\frac{1}{2\pi} \int_0^{2\pi} \|f(e^{i\theta})\|_E^p d\theta \right)^{1/p} < \infty;$$

- (ii) $H^p(\mathbb{D}, E)$ to be the normed space of analytic E -valued maps $f: \mathbb{D} \rightarrow E$ such that

$$\|f\|_p = \sup_{0 < r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|_E^p d\theta \right)^{1/p} < \infty;$$

- (iii) $L^\infty(\mathbb{T}, E)$ is the space of essentially bounded measurable E -valued functions on the unit circle with essential supremum norm

$$\|f\|_{L^\infty} = \operatorname{ess\,sup}_{|z|=1} \|f(z)\|_E.$$

Proposition 3.5 Let E be a separable Hilbert space and let $\frac{1}{p} + \frac{1}{q} = 1$, where $1 \leq p, q \leq \infty$. Suppose that $x \in L^p(\mathbb{T}, E)$, $y \in L^q(\mathbb{T}, E)$. Then

$$x \wedge y \in L^1(\mathbb{T}, \wedge^2 E)$$

and

$$\|x \wedge y\|_{L^1(\mathbb{T}, \wedge^2 E)} \leq \|x\|_{L^p(\mathbb{T}, E)} \|y\|_{L^q(\mathbb{T}, E)}. \quad (3.1)$$

Proof By Proposition 2.14, for all $z \in \mathbb{T}$,

$$\begin{aligned} \|(x \wedge y)(z)\|_{\wedge^2 E}^2 &= \langle x(z) \wedge y(z), x(z) \wedge y(z) \rangle_{\wedge^2 E} \\ &= \langle x(z), x(z) \rangle_E \cdot \langle y(z), y(z) \rangle_E - |\langle x(z), y(z) \rangle_E|^2 \\ &\leq \|x(z)\|_E^2 \|y(z)\|_E^2. \end{aligned}$$

Thus, for all $z \in \mathbb{T}$,

$$\|(x \wedge y)(z)\|_{\wedge^2 E} \leq \|x(z)\|_E \|y(z)\|_E.$$

By Definition 3.4,

$$\begin{aligned} \|x \dot{\wedge} y\|_{L^1(\mathbb{T}, \wedge^2 E)} &= \frac{1}{2\pi} \int_0^{2\pi} \|(x \dot{\wedge} y)(e^{i\theta})\|_{\wedge^2 E} d\theta \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \|x(e^{i\theta})\|_E \|y(e^{i\theta})\|_E d\theta. \end{aligned} \quad (3.2)$$

Now, by Hölder's inequality,

$$\begin{aligned} &\frac{1}{2\pi} \int_0^{2\pi} \|x(e^{i\theta})\|_E \|y(e^{i\theta})\|_E d\theta \\ &\leq \left(\frac{1}{2\pi} \int_0^{2\pi} \|x(e^{i\theta})\|_E^p d\theta \right)^{1/p} \left(\frac{1}{2\pi} \int_0^{2\pi} \|y(e^{i\theta})\|_E^q d\theta \right)^{1/q}. \end{aligned} \quad (3.3)$$

By inequalities (3.2) and (3.3), $x \dot{\wedge} y \in L^1(\mathbb{T}, \wedge^2 E)$ and the inequality (3.1) holds. $\square$

Proposition 3.6 *Let E be a Hilbert space and $x_1, x_2, \dots, x_n: \mathbb{D} \rightarrow E$ be analytic E -valued maps on $\mathbb{D}$. Then,*

$$x_1 \dot{\wedge} x_2 \dot{\wedge} \dots \dot{\wedge} x_n: \mathbb{D} \rightarrow \wedge^n E$$

is also analytic on $\mathbb{D}$ and

$$\begin{aligned} &(x_1 \dot{\wedge} x_2 \dot{\wedge} \dots \dot{\wedge} x_n)'(z) \\ &= x_1'(z) \wedge x_2(z) \wedge \dots \wedge x_n(z) + \dots + x_1(z) \wedge x_2(z) \wedge \dots \wedge x_n'(z) \end{aligned}$$

for all $z \in \mathbb{D}$.

The proof is straightforward. It follows from Proposition 2.14 and the continuity of $\wedge$, see Hadamard's inequalities (2.2).

Proposition 3.7 *Let E be a separable Hilbert space. Suppose $x, y \in H^2(\mathbb{D}, E)$. Then*

$$x \dot{\wedge} y \in H^1(\mathbb{D}, \wedge^2 E).$$

Proof By Proposition 3.6, $x \dot{\wedge} y$ is analytic on $\mathbb{D}$. By Proposition 2.14, for $0 < r < 1$ and $0 \leq \theta \leq 2\pi$,

$$\|(x \dot{\wedge} y)(re^{i\theta})\|_{\wedge^2 E} \leq \|x(re^{i\theta})\|_E \|y(re^{i\theta})\|_E.$$

By Proposition 2.14 and by Definition 3.4,

$$\begin{aligned} \|x \dot{\wedge} y\|_{H^1(\mathbb{D}, \wedge^2 E)} &= \sup_{0 < r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} \|(x \dot{\wedge} y)(re^{i\theta})\|_{\wedge^2 E} d\theta \right) \\ &\leq \sup_{0 < r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} \|x(re^{i\theta})\|_E \|y(re^{i\theta})\|_E d\theta \right), \end{aligned}$$

for $0 < r < 1$ and $0 \leq \theta \leq 2\pi$. Also, by Hölder's inequality, for $0 < r < 1$ and $0 \leq \theta \leq 2\pi$,

$$\begin{aligned} &\frac{1}{2\pi} \int_0^{2\pi} \|x(re^{i\theta})\|_E \|y(re^{i\theta})\|_E d\theta \\ &\leq \left(\frac{1}{2\pi} \int_0^{2\pi} \|x(re^{i\theta})\|_E^2 d\theta \right)^{1/2} \left(\frac{1}{2\pi} \int_0^{2\pi} \|y(re^{i\theta})\|_E^2 d\theta \right)^{1/2}, \end{aligned}$$

hence

$$\|x \dot{\wedge} y\|_{H^1(\mathbb{D}, \wedge^2 E)} \leq \|x\|_{H^2(\mathbb{D}, E)} \|y\|_{H^2(\mathbb{D}, E)}.$$

Consequently, $x \dot{\wedge} y \in H^1(\mathbb{D}, \wedge^2 E)$. $\square$

Proposition 3.8 *Let E be a separable Hilbert space, let $x \in H^2(\mathbb{D}, E)$ and let $y \in H^\infty(\mathbb{D}, E)$. Then*

$$x \dot{\wedge} y \in H^2(\mathbb{D}, \wedge^2 E).$$

Proof By Proposition 3.6, $x \dot{\wedge} y$ is analytic on $\mathbb{D}$. By Proposition 3.5, for

$$0 < r < 1, \quad 0 \leq \theta \leq 2\pi,$$

we have

$$\|(x \dot{\wedge} y)(re^{i\theta})\|_{\wedge^2 E} \leq \|x(re^{i\theta})\|_E \|y(re^{i\theta})\|_E.$$

Thus,

$$\begin{aligned} \|x \dot{\wedge} y\|_{H^2(\mathbb{D}, \wedge^2 E)} &= \sup_{0 < r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} \|(x \dot{\wedge} y)(re^{i\theta})\|_{\wedge^2 E}^2 d\theta \right)^{1/2} \\ &\leq \sup_{0 < r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} \|x(re^{i\theta})\|_E^2 \|y(re^{i\theta})\|_E^2 d\theta \right)^{1/2} \\ &\leq \|y\|_\infty \sup_{0 < r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} \|x(re^{i\theta})\|_E^2 d\theta \right)^{1/2} < \infty. \end{aligned}$$

$\square$

Definition 3.9 Let E be a Hilbert space. We say that a family of $\{f_\lambda\}_{\lambda \in \Lambda}$ of maps from $\mathbb{T}$ to E is *pointwise orthonormal* on $\mathbb{T}$, if for all z in a set of full measure in $\mathbb{T}$, the set of vectors $\{f_\lambda(z)\}_{\lambda \in \Lambda}$ is orthonormal in E .

Definition 3.10 Let E be a separable Hilbert space. Let F be a subspace of $L^2(\mathbb{T}, E)$ and let X be a subset of $L^2(\mathbb{T}, E)$. We define the *pointwise orthogonal complement* of X in F to be the set

$$\text{POC}(X, F) = \{f \in F : f(z) \perp \{x(z) : x \in X\} \text{ for almost all } z \in \mathbb{T}\}.$$

Proposition 3.11 Let E be a separable Hilbert space, and let $\xi_0, \xi_1, \dots, \xi_j \in L^\infty(\mathbb{T}, E)$ be a pointwise orthonormal set on $\mathbb{T}$, and let $x \in L^2(\mathbb{T}, E)$. Then

$$\xi_0 \wedge \xi_1 \wedge \dots \wedge \xi_j \wedge x \in L^2(\mathbb{T}, \wedge^{j+2} E),$$

and

$$\|\xi_0 \wedge \xi_1 \wedge \dots \wedge \xi_j \wedge x\|_{L^2(\mathbb{T}, \wedge^{j+2} E)} \leq \|x\|_{L^2(\mathbb{T}, E)}.$$

Furthermore,

$$\|\xi_0 \wedge \xi_1 \wedge \dots \wedge \xi_j \wedge x\|_{L^2(\mathbb{T}, \wedge^{j+2} E)} = \|x\|_{L^2(\mathbb{T}, E)}$$

if and only if $x \in \text{POC}(\{\xi_0, \xi_1, \dots, \xi_j\}, L^2(\mathbb{T}, E))$.

Proof By Lemma 2.15, for almost all $z \in \mathbb{T}$,

$$\begin{aligned} & \|\xi_0(z) \wedge \xi_1(z) \wedge \dots \wedge \xi_j(z) \wedge x(z)\|_{\wedge^{j+2} E}^2 \\ &= \|x(z)\|_E^2 - \sum_{i=0}^j |\langle x(z), \xi_i(z) \rangle_E|^2 \leq \|x(z)\|_E^2. \end{aligned}$$

Thus,

$$\begin{aligned} & \|\xi_0 \wedge \xi_1 \wedge \dots \wedge \xi_j \wedge x\|_{L^2(\mathbb{T}, \wedge^{j+2} E)} \\ &= \left(\frac{1}{2\pi} \int_0^{2\pi} \|\xi_0(e^{i\theta}) \wedge \xi_1(e^{i\theta}) \wedge \dots \wedge \xi_j(e^{i\theta}) \wedge x(e^{i\theta})\|_{\wedge^{j+2} E}^2 d\theta \right)^{1/2} \\ &\leq \left(\frac{1}{2\pi} \int_0^{2\pi} \|x(e^{i\theta})\|_E^2 d\theta \right)^{1/2} = \|x\|_{L^2(\mathbb{T}, E)} < \infty. \end{aligned}$$

□

4 Pointwise Creation Operators, Orthogonal Complements and Linear Spans

Definition 4.1 Let E be a separable Hilbert space. Let $\xi \in H^\infty(\mathbb{D}, E)$. We define the *pointwise creation operator*

$$C_\xi : H^2(\mathbb{D}, E) \rightarrow H^2(\mathbb{D}, \wedge^2 E)$$

by

$$C_\xi f = \xi \dot{\wedge} f, \quad \text{for } f \in H^2(\mathbb{D}, E).$$

Remark 4.2 Let E be a separable Hilbert space. Let $\xi \in H^\infty(\mathbb{D}, E)$ and let $f \in H^2(\mathbb{D}, E)$. By the generalized Fatou's Theorem [8, Chapter V], the radial limits

$$\lim_{r \rightarrow 1} \xi(re^{i\theta}) = \tilde{\xi}(e^{i\theta}), \quad \lim_{r \rightarrow 1} f(re^{i\theta}) = \tilde{f}(e^{i\theta}) \quad (0 < r < 1)$$

exist almost everywhere on $\mathbb{T}$ and define functions $\tilde{\xi} \in L^\infty(\mathbb{T}, E)$ and $\tilde{f} \in L^2(\mathbb{T}, E)$ respectively, which satisfy the relations

$$\lim_{r \rightarrow 1} \|\xi(re^{i\theta}) - \tilde{\xi}(e^{i\theta})\|_E = 0, \quad \lim_{r \rightarrow 1} \|f(re^{i\theta}) - \tilde{f}(e^{i\theta})\|_E = 0 \quad (0 < r < 1)$$

for almost all $e^{i\theta} \in \mathbb{T}$.

Lemma 4.3 Let E be a separable Hilbert space. Let $\xi \in H^\infty(\mathbb{D}, E)$ and let $f \in H^2(\mathbb{D}, E)$. Then the radial limits $\lim_{r \rightarrow 1} (\xi(re^{i\theta}) \wedge f(re^{i\theta}))$ exist for almost all $e^{i\theta} \in \mathbb{T}$ and define functions in $L^2(\mathbb{T}, \wedge^2 E)$.

Proof By Proposition 2.19, the bilinear operator $\Lambda : E \times E \rightarrow \wedge^2 E$ is a continuous operator for the norms of E and $\wedge^2 E$. By Remark 4.2, the functions $\xi \in H^\infty(\mathbb{D}, E)$ and $f \in H^2(\mathbb{D}, E)$ have radial limit functions $\tilde{\xi} \in L^\infty(\mathbb{T}, E)$ and $\tilde{f} \in L^2(\mathbb{T}, E)$. Also, by Proposition 3.8, $\xi \dot{\wedge} f \in H^2(\mathbb{D}, \wedge^2 E)$. Hence

$$\lim_{r \rightarrow 1} \|\xi(re^{i\theta}) \wedge f(re^{i\theta}) - \tilde{\xi}(e^{i\theta}) \wedge \tilde{f}(e^{i\theta})\|_{\wedge^2 E} = 0 \quad \text{almost everywhere on } \mathbb{T}$$

and we conclude that

$$\lim_{r \rightarrow 1} (\xi(re^{i\theta}) \wedge f(re^{i\theta})) = \tilde{\xi}(e^{i\theta}) \wedge \tilde{f}(e^{i\theta}) \quad \text{almost everywhere on } \mathbb{T}.$$

This shows that the radial limits

$$\lim_{r \rightarrow 1} (\xi(re^{i\theta}) \wedge f(re^{i\theta}))$$

exist almost everywhere on $\mathbb{T}$ and, by Lemma 3.5, define functions in $L^2(\mathbb{T}, \wedge^2 E)$. Hence one can consider $(C_\xi f)(z) = (\xi \wedge f)(z)$ to be defined for either all $z \in \mathbb{D}$ or for almost all $z \in \mathbb{T}$. $\square$

Remark 4.4 Let E be a separable Hilbert space. By [8, Chapter 5, Sect. 1], for any separable Hilbert space E , the map $f \mapsto \tilde{f}$ is an isometric embedding of $H^2(\mathbb{D}, E)$ in $L^2(\mathbb{T}, E)$, where $\tilde{f}(e^{i\theta}) = \lim_{r \rightarrow 1} f(re^{i\theta})$. Since $H^2(\mathbb{D}, E)$ is complete and the embedding is isometric, the image of the embedding is complete, and therefore is closed in $L^2(\mathbb{T}, E)$. Therefore, the space $H^2(\mathbb{D}, E)$ is identified isometrically with a closed linear subspace of $L^2(\mathbb{T}, E)$. In future we shall use the same notation for f and $\tilde{f}$.

Our next aim is to show that $\text{POC}(X, F)$ is a closed subspace of F . We are going to need the following results.

Lemma 4.5 *Let E be a Hilbert space and let $x \in L^2(\mathbb{T}, E)$. The function $\varphi: L^2(\mathbb{T}, E) \rightarrow \mathbb{C}$ given by*

$$\varphi(g) = \frac{1}{2\pi} \int_0^{2\pi} |\langle g(e^{i\theta}), x(e^{i\theta}) \rangle_E| d\theta$$

is continuous.

Proof Consider $g_0 \in L^2(\mathbb{T}, E)$. For any $\epsilon > 0$, we are looking for a $\delta > 0$ such that

$$\|g - g_0\|_{L^2(\mathbb{T}, E)} = \left(\frac{1}{2\pi} \int_0^{2\pi} \|g(e^{i\theta}) - g_0(e^{i\theta})\|_E^2 d\theta \right)^{1/2} < \delta$$

implies

$$|\varphi(g) - \varphi(g_0)| < \epsilon.$$

Note that

$$\begin{aligned} |\varphi(g) - \varphi(g_0)| &= \left| \frac{1}{2\pi} \int_0^{2\pi} |\langle g(e^{i\theta}), x(e^{i\theta}) \rangle_E| d\theta - \frac{1}{2\pi} \int_0^{2\pi} |\langle g_0(e^{i\theta}), x(e^{i\theta}) \rangle_E| d\theta \right| \\ &= \left| \frac{1}{2\pi} \int_0^{2\pi} \left(|\langle g(e^{i\theta}), x(e^{i\theta}) \rangle_E| - |\langle g_0(e^{i\theta}), x(e^{i\theta}) \rangle_E| \right) d\theta \right|. \end{aligned}$$

For each $e^{i\theta} \in \mathbb{T}$, by the reverse triangle inequality, the integrand satisfies

$$\begin{aligned} |\langle g(e^{i\theta}), x(e^{i\theta}) \rangle_E| - |\langle g_0(e^{i\theta}), x(e^{i\theta}) \rangle_E| &\leq |\langle g(e^{i\theta}), x(e^{i\theta}) \rangle_E - \langle g_0(e^{i\theta}), x(e^{i\theta}) \rangle_E| \\ &= |\langle (g - g_0)(e^{i\theta}), x(e^{i\theta}) \rangle_E|, \end{aligned}$$

$$|\varphi(g) - \varphi(g_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |\langle g(e^{i\theta}) - g_0(e^{i\theta}), x(e^{i\theta}) \rangle_E| d\theta. \quad (4.1)$$

By the Cauchy–Schwarz inequality,

$$\begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} |\langle g(e^{i\theta}) - g_0(e^{i\theta}), x(e^{i\theta}) \rangle_E| d\theta \\ & \leq \left(\frac{1}{2\pi} \int_0^{2\pi} \|g(e^{i\theta}) - g_0(e^{i\theta})\|_E^2 d\theta \right)^{1/2} \left(\frac{1}{2\pi} \int_0^{2\pi} \|x(e^{i\theta})\|_E^2 d\theta \right)^{1/2}. \end{aligned} \quad (4.2)$$

For the given $\epsilon > 0$, let δ be equal to $\frac{\epsilon}{\|x\|_{L^2(\mathbb{T}, E)} + 1}$, and let

$$\left(\frac{1}{2\pi} \int_0^{2\pi} \|g(e^{i\theta}) - g_0(e^{i\theta})\|_E^2 d\theta \right)^{1/2} < \delta.$$

By Eqs. (4.1) and (4.2),

$$\begin{aligned} |\varphi(g) - \varphi(g_0)| & \leq \left(\frac{1}{2\pi} \int_0^{2\pi} \|g(e^{i\theta}) - g_0(e^{i\theta})\|_E^2 d\theta \right)^{1/2} \|x\|_{L^2(\mathbb{T}, E)} \\ & < \frac{\epsilon}{\|x\|_{L^2(\mathbb{T}, E)} + 1} \|x\|_{L^2(\mathbb{T}, E)} < \epsilon. \end{aligned}$$

Hence φ is a continuous function. $\square$

Proposition 4.6 *Let E be a separable Hilbert space. Let $\eta \in L^2(\mathbb{D}, E)$. Then*

- (i) *The space $V = \{f \in H^2(\mathbb{D}, E) : \langle f(z), \eta(z) \rangle_E = 0 \text{ for almost all } z \in \mathbb{T}\}$ is a closed subspace of $H^2(\mathbb{D}, E)$.*
- (ii) *The space $V = \{f \in L^2(\mathbb{T}, E) : \langle f(z), \eta(z) \rangle_E = 0 \text{ for almost all } z \in \mathbb{T}\}$ is a closed subspace of $L^2(\mathbb{T}, E)$.*

Proof (i). V is a linear subspace of $H^2(\mathbb{D}, E)$ since for $\lambda, \mu \in \mathbb{C}$, $\psi, k \in V$ and for almost all $z \in \mathbb{T}$,

$$\langle \lambda\psi(z) + \mu k(z), \eta(z) \rangle_E = \lambda \langle \psi(z), \eta(z) \rangle_E + \mu \langle k(z), \eta(z) \rangle_E = 0,$$

hence $\lambda\psi + \mu k \in V$.

Now suppose that the sequence of functions $(g_n)_{n=1}^\infty$ in V converges to a function g . We need to show that $g \in V$. Since $g_n \in V$ for all $n \in \mathbb{N}$, we have

$$\langle g_n(z), \eta(z) \rangle_E = 0 \text{ for almost all } z \in \mathbb{T}. \quad (4.3)$$

Consider the function $\varphi: H^2(\mathbb{D}, E) \rightarrow \mathbb{C}$ given by

$$\varphi(f) = \frac{1}{2\pi} \int_0^{2\pi} |\langle f(e^{i\theta}), \eta(e^{i\theta}) \rangle_E| d\theta.$$

Then, by Eq. (4.3), we have

$$\varphi(g_n) = \frac{1}{2\pi} \int_0^{2\pi} |\langle g_n(e^{i\theta}), \eta(e^{i\theta}) \rangle_E| d\theta = 0.$$

By Remark 4.4 and Lemma 4.5, φ is a continuous function on $H^2(\mathbb{D}, E)$, thus

$$\lim_{n \rightarrow \infty} \varphi(g_n) = \varphi(g),$$

and so

$$\frac{1}{2\pi} \int_0^{2\pi} |\langle g(e^{i\theta}), \eta(e^{i\theta}) \rangle_E| d\theta = \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_0^{2\pi} |\langle g_n(e^{i\theta}), \eta(e^{i\theta}) \rangle_E| d\theta = 0.$$

Thus $|\langle g(e^{i\theta}), \eta(e^{i\theta}) \rangle_E| = 0$ for almost all $e^{i\theta} \in \mathbb{T}$, and, hence, $g \in V$. We have proved that V is a closed subspace of $H^2(\mathbb{D}, E)$.

(ii). The proof is similar to (i). □

Our motivation for the next theorem is the following. In [2] spaces of the form

$$\xi_0 \wedge \cdots \wedge \xi_j \wedge H^2(\mathbb{D}, E)$$

play a crucial role in the superoptimal Nehari problem. They are the domains of Hankel-type operators whose norms are “superoptimal singular values” of error functions $G - Q$, where G is a given continuous approximand and Q is its analytic approximation.

Theorem 4.7 *Let E be a separable Hilbert space, let $\xi_0, \xi_1, \dots, \xi_j \in H^\infty(\mathbb{D}, E)$. Suppose that the set $\{\xi_i(z)\}_{i=0}^j$ is orthonormal in E for almost every $z \in \mathbb{T}$. Then*

$$\xi_0 \wedge \cdots \wedge \xi_j \wedge H^2(\mathbb{D}, E)$$

is a closed subspace of $H^2(\mathbb{D}, \wedge^{j+2} E)$.

Proof By Proposition 3.6, for every $x \in H^2(\mathbb{D}, E)$,

$$\xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \cdots \dot{\wedge} \xi_j \dot{\wedge} x$$

is analytic on $\mathbb{D}$. By Proposition 3.11, since $\xi_0, \xi_1, \dots, \xi_j$ are pointwise orthogonal on $\mathbb{T}$,

$$\|\xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \cdots \dot{\wedge} \xi_j \dot{\wedge} x\|_{L^2(\mathbb{T}, \wedge^{j+2} E)} < \infty.$$

Thus, for every $x \in H^2(\mathbb{D}, E)$,

$$\xi_0 \dot{\wedge} \xi_1 \dot{\wedge} \cdots \dot{\wedge} \xi_j \dot{\wedge} x \in H^2(\mathbb{D}, \wedge^{j+2} E).$$

Let us first show that $\xi_0 \dot{\wedge} H^2(\mathbb{D}, E)$ is a closed subspace of $H^2(\mathbb{D}, \wedge^2 E)$. Observe that, by Proposition 4.3, $\xi_0 \dot{\wedge} H^2(\mathbb{D}, E) \subset H^2(\mathbb{D}, \wedge^2 E)$. Let

$$\Xi_0 = \{f \in H^2(\mathbb{D}, E) : \langle f(z), \xi_0(z) \rangle_E = 0 \text{ almost everywhere on } \mathbb{T}\}.$$

Consider a vector-valued function $w \in H^2(\mathbb{D}, E)$. For almost every $z \in \mathbb{T}$, we may write w as

$$w(z) = w(z) - \langle w(z), \xi_0(z) \rangle_E \xi_0(z) + \langle w(z), \xi_0(z) \rangle_E \xi_0(z).$$

Then, for all $w \in H^2(\mathbb{D}, E)$ and for almost every $z \in \mathbb{T}$,

$$\begin{aligned} (\xi_0 \dot{\wedge} w)(z) &= \xi_0(z) \wedge (w(z) - \langle w(z), \xi_0(z) \rangle_E \xi_0(z) + \langle w(z), \xi_0(z) \rangle_E \xi_0(z)) \\ &= \xi_0(z) \wedge (w(z) - \langle w(z), \xi_0(z) \rangle_E \xi_0(z)) \end{aligned}$$

due to the pointwise linear dependence of ξ_0 and $z \mapsto \langle w(z), \xi_0(z) \rangle_E \xi_0(z)$ almost everywhere on $\mathbb{T}$. Note that

$$w(z) - \langle w(z), \xi_0(z) \rangle_E \xi_0(z) \in \Xi_0,$$

thus

$$\xi_0 \dot{\wedge} H^2(\mathbb{D}, E) \subset \xi_0 \dot{\wedge} \Xi_0.$$

By Proposition 4.6, Ξ_0 is a closed subspace of $H^2(\mathbb{D}, E)$, hence

$$\xi_0 \dot{\wedge} H^2(\mathbb{D}, E) \supset \xi_0 \dot{\wedge} \Xi_0,$$

and so,

$$\xi_0 \dot{\wedge} H^2(\mathbb{D}, E) = \xi_0 \dot{\wedge} \Xi_0.$$

Consider the mapping

$$C_{\xi_0} : \Xi_0 \rightarrow \xi_0 \wedge \Xi_0$$

given by

$$C_{\xi_0} w = \xi_0 \wedge w$$

for all $w \in \Xi_0$. Notice that, by assumption, $\|\xi_0(e^{i\theta})\|_E^2 = 1$ for almost every $e^{i\theta} \in \mathbb{T}$. Therefore, for any $w \in \Xi_0$, we have

$$\begin{aligned} \|\xi_0 \wedge w\|_{L^2(\mathbb{T}, \wedge^2 E)}^2 &= \frac{1}{2\pi} \int_0^{2\pi} \langle \xi_0 \wedge w, \xi_0 \wedge w \rangle(e^{i\theta}) d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} (\|\xi_0(e^{i\theta})\|_E^2 \|w(e^{i\theta})\|_E^2 - |\langle w(e^{i\theta}), \xi_0(e^{i\theta}) \rangle|^2) d\theta \\ &= \|w\|_{L^2(\mathbb{T}, E)}^2, \end{aligned}$$

since w is pointwise orthogonal to ξ_0 almost everywhere on $\mathbb{T}$. Thus the mapping

$$C_{\xi_0} : \Xi_0 \rightarrow \xi_0 \wedge \Xi_0$$

is an isometry. Furthermore, $C_{\xi_0} : \Xi_0 \rightarrow \xi_0 \wedge \Xi_0$ is a surjective mapping, thus Ξ_0 and $\xi_0 \wedge \Xi_0$ are isometrically isomorphic. Therefore, since Ξ_0 is a closed subspace of $H^2(\mathbb{D}, E)$, the space $\xi_0 \wedge \Xi_0$ is a closed subspace of $H^2(\mathbb{D}, \wedge^2 E)$. Hence $\xi_0 \wedge H^2(\mathbb{D}, E)$ is a closed subspace of $H^2(\mathbb{D}, \wedge^2 E)$.

To prove that $\xi_0 \wedge \cdots \wedge \xi_j \wedge H^2(\mathbb{D}, E)$ is a closed subspace of $H^2(\mathbb{D}, \wedge^{j+2} E)$, let us consider

$$\Xi_j = \{f \in H^2(\mathbb{D}, E) : \langle f(z), \xi_i(z) \rangle_E = 0, \text{ for } i = 0, \dots, j\}$$

to be the pointwise orthogonal complement of $\xi_0, \dots, \xi_j$ in $H^2(\mathbb{D}, E)$. Let $\psi \in H^2(\mathbb{D}, E)$. We may write ψ as

$$\psi(z) = \psi(z) - \sum_{i=0}^j \langle \psi(z), \xi_i(z) \rangle_E \xi_i(z) + \sum_{i=0}^j \langle \psi(z), \xi_i(z) \rangle_E \xi_i(z).$$

Then, for all $\psi \in H^2(\mathbb{D}, E)$ and for almost all $z \in \mathbb{T}$,

$$(\xi_0 \wedge \cdots \wedge \xi_j \wedge \psi)(z) = \xi_0(z) \wedge \cdots \wedge \left(\psi(z) - \sum_{i=0}^j \langle \psi(z), \xi_i(z) \rangle_E \xi_i(z) \right)$$

due to the pointwise linear dependence of ξ_k and $z \mapsto \langle \psi(z), \xi_k(z) \rangle_E \xi_k(z)$ almost everywhere on $\mathbb{T}$.

Notice that $\left(\psi(z) - \sum_{i=0}^j \langle \psi(z), \xi_i(z) \rangle_E \xi_i(z) \right)$ is in Ξ_j , thus

$$\xi_0 \wedge \cdots \wedge \xi_j \wedge H^2(\mathbb{D}, E) \subset \xi_0 \wedge \cdots \wedge \xi_j \wedge \Xi_j.$$

The reverse inclusion holds by the definition of Ξ_j , hence

$$\xi_0 \wedge \cdots \wedge \xi_j \wedge H^2(\mathbb{D}, E) = \xi_0 \wedge \cdots \wedge \xi_j \wedge \Xi_j.$$

Consequently, in order to prove the proposition it suffices to show that $\xi_0 \wedge \cdots \wedge \xi_j \wedge \Xi_j$ is a closed subspace of $H^2(\mathbb{D}, \wedge^{j+2} E)$. By Proposition 4.6, Ξ_j is a closed subspace of $H^2(\mathbb{D}, E)$, being a finite intersection of closed subspaces.

By Lemma 2.15, for any $f \in \Xi_j$, we get

$$\begin{aligned} & \|\xi_0 \wedge \xi_1 \wedge \cdots \wedge \xi_j \wedge f\|_{L^2(\mathbb{T}, \wedge^{j+2} E)}^2 \\ &= \frac{1}{2\pi} \int_0^{2\pi} \|\xi_0(e^{i\theta}) \wedge \xi_1(e^{i\theta}) \wedge \cdots \wedge \xi_j(e^{i\theta}) \wedge f(e^{i\theta})\|_{\wedge^{j+2} E}^2 d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \|f(e^{i\theta}) - \sum_{i=1}^j \langle f(e^{i\theta}), \xi_i(e^{i\theta}) \rangle \xi_i(e^{i\theta})\|_E^2 d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \|f(e^{i\theta})\|_E^2 d\theta = \|f\|_{L^2(\mathbb{T}, E)}^2. \end{aligned}$$

Thus

$$(\xi_0 \wedge \xi_1 \wedge \cdots \wedge \xi_j \wedge \cdot) : \Xi_j \rightarrow \xi_0 \wedge \xi_1 \wedge \cdots \wedge \xi_j \wedge \Xi_j$$

is an isometry. Furthermore

$$(\xi_0 \wedge \xi_1 \wedge \cdots \wedge \xi_j \wedge \cdot) : \Xi_j \rightarrow \xi_0 \wedge \xi_1 \wedge \cdots \wedge \xi_j \wedge \Xi_j$$

is a surjective mapping, thus Ξ_j and $\xi_0 \wedge \cdots \wedge \xi_j \wedge \Xi_j$ are isometrically isomorphic. Therefore, since Ξ_j is a closed subspace of $H^2(\mathbb{D}, E)$, the space $\xi_0 \wedge \cdots \wedge \xi_j \wedge \Xi_j$ is a closed subspace of $H^2(\mathbb{D}, \wedge^{j+2} E)$. Hence

$$\xi_0 \wedge \cdots \wedge \xi_j \wedge H^2(\mathbb{D}, E)$$

is a closed subspace of $H^2(\mathbb{D}, \wedge^{j+2} E)$. □

Proposition 4.8 *Let E be a separable Hilbert space, let F be a subspace of $L^2(\mathbb{T}, E)$ and let X be a subset of $L^2(\mathbb{T}, E)$. The space*

$$\text{POC}(X, F) = \{f \in F : f(z) \perp \{x(z) : x \in X\} \text{ for almost all } z \in \mathbb{T}\}$$

is a closed subspace of F .

Proof It follows from Proposition 4.6, since $\text{POC}(X, F)$ is an intersection of closed subspaces $V_x = \{f \in F : \langle f(z), x(z) \rangle_E = 0 \text{ for almost all } z \in \mathbb{T}\}$ over $x \in X$. $\square$

Remark 4.9 By the generalized Fatou's Theorem, for E a separable Hilbert space, the space $H^\infty(\mathbb{D}, E)$ can be identified with a closed subspace of $L^\infty(\mathbb{T}, E)$.

Definition 4.10 Let E be a separable Hilbert space. Let $f \in H^p(\mathbb{D}, E)$, for $1 \leq p \leq \infty$. By the generalized Fatou's Theorem (see [8], p. 186), the radial limit

$$\lim_{r \rightarrow 1} f(re^{i\theta}) = \tilde{f}(e^{i\theta}) \quad (0 < r < 1)$$

exists almost everywhere on $\mathbb{T}$ and defines a function $\tilde{f} \in L^p(\mathbb{T}, E)$. The set of points on $\mathbb{T}$ at which the above limit does not exist, will be called the *singular set* of the function f and will be denoted by N_f .

Note that the singular sets of functions in $H^p(\mathbb{D}, E)$ for $1 \leq p \leq \infty$ are null sets with respect to Lebesgue measure.

Definition 4.11 Let E be a separable Hilbert space. Let F be a subspace of $L^2(\mathbb{T}, E)$ and let X be a subset of $L^2(\mathbb{T}, E)$. We define the *pointwise linear span* of X in F to be the set

$$\text{PLS}(X, F) = \{f \in F : f(z) \in \text{span}\{x(z) : x \in X\} \text{ for almost all } z \in \mathbb{T}\}.$$

Recall Definition 4.1, for a separable Hilbert space E and for $\xi \in H^\infty(\mathbb{D}, E)$, the pointwise creation operator C_ξ is defined by

$$C_\xi : H^2(\mathbb{D}, E) \rightarrow H^2(\mathbb{D}, \wedge^2 E), \quad f \mapsto \xi \wedge f,$$

where

$$(\xi \wedge f)(z) = \xi(z) \wedge f(z) \text{ for all } z \in \mathbb{D}.$$

Proposition 4.12 Let E be a separable Hilbert space. For $\xi \in H^\infty(\mathbb{D}, E)$,

$$\ker C_\xi \subset \text{PLS}(\{\xi\}, H^2(\mathbb{D}, E)).$$

Proof We have

$$\begin{aligned} \ker C_\xi &= \{f \in H^2(\mathbb{D}, E) : (\xi \wedge f)(z) = 0 \text{ for all } z \in \mathbb{D}\} \\ &= \{f \in H^2(\mathbb{D}, E) : \xi(z) \wedge f(z) = 0 \text{ for all } z \in \mathbb{D}\} \\ &= \{f \in H^2(\mathbb{D}, E) : \xi(z), f(z) \text{ are pointwise linearly dependent for all } z \in \mathbb{D}\} \\ &\subset \text{PLS}(\{\xi\}, H^2(\mathbb{D}, E)). \end{aligned}$$

$\square$

Example 4.13 Let $E = \mathbb{C}^2$. We can find functions $f, g \in H^2(\mathbb{D}, E)$ such that $f \in \text{POC}(\{g\}, H^2(\mathbb{D}, E))$ but it is false that $\langle f(z), g(z) \rangle_E = 0$ for all $z \in \mathbb{D}$. Choose

$$g(z) = \begin{pmatrix} z \\ z^2 \end{pmatrix}, \quad f(z) = \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix} \quad \text{for } z \in \mathbb{D}.$$

Then

$$f \in \text{POC}(\{g\}, H^2(\mathbb{D}, E))$$

is equivalent to

$$\langle f(z), g(z) \rangle_E = 0 \quad \text{for almost all } z \in \mathbb{T}.$$

The later is equivalent to

$$\left\langle \begin{pmatrix} z \\ z^2 \end{pmatrix}, \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix} \right\rangle_E = 0 \quad \text{for almost all } z \in \mathbb{T},$$

which holds if and only if

$$\bar{z}f_1(z) + \bar{z}^2f_2(z) = 0 \quad \text{for almost all } z \in \mathbb{T}.$$

Equivalently

$$f_1(z) = -\bar{z}f_2(z) \quad \text{for almost all } z \in \mathbb{T},$$

which in turn is equivalent to

$$f(z) = \begin{pmatrix} f_1(z) \\ -zf_1(z) \end{pmatrix} \quad \text{for almost all } z \in \mathbb{T}.$$

Now, for $z \in \mathbb{D}$,

$$\begin{aligned} \langle f(z), g(z) \rangle_E &= \left\langle \begin{pmatrix} f_1(z) \\ -zf_1(z) \end{pmatrix}, \begin{pmatrix} z \\ z^2 \end{pmatrix} \right\rangle_E \\ &= \bar{z}f_1(z) - \bar{z}|z|^2f_1(z) \\ &= \bar{z}(1 - |z|^2)f_1(z). \end{aligned}$$

So if we take $f_1(z) = 1$, $f(z) = \begin{pmatrix} 1 \\ -z \end{pmatrix}$ for $z \in \mathbb{D}$, then $f \in \text{POC}(\{g\}, H^2(\mathbb{D}, E))$, but $\langle f(z), g(z) \rangle_E \neq 0$ for all $z \in \mathbb{D} \setminus \{0\}$.

Thus it is not true in general that $\text{POC}(\{g\}, H^2(\mathbb{D}, E)) \subset \{g\}^\perp$.

Lemma 4.14 *Let E be a separable Hilbert space. For $\xi \in H^\infty(\mathbb{D}, E)$,*

$$\text{POC}(\{\xi\}, H^2(\mathbb{D}, E)) \subset H^2(\mathbb{D}, E) \ominus \text{PLS}(\{\xi\}, H^2(\mathbb{D}, E)).$$

Proof Let $f \in \text{POC}(\{\xi\}, H^2(\mathbb{D}, E))$. This is equivalent to $f \in H^2(\mathbb{D}, E)$ and

$$f(z) \perp \xi(z) \text{ for all } z \in \mathbb{T} \setminus (N_f \cup N_\xi),$$

where N_f, N_ξ are the singular sets for the functions f, ξ respectively. This in turn is equivalent to $f \in H^2(\mathbb{D}, E)$ and

$$\langle f(z), \xi(z) \rangle_E = 0 \text{ for all } z \in \mathbb{T} \setminus (N_f \cup N_\xi).$$

The latter implies $f \in H^2(\mathbb{D}, E)$ and

$$\langle f(z), g(z) \rangle_E = 0 \text{ for almost all } z \in \mathbb{T} \text{ for all } g \in \text{PLS}(\{\xi\}, H^2(\mathbb{D}, E)).$$

Thus

$$f \in H^2(\mathbb{D}, E) \ominus \text{PLS}(\{\xi\}, H^2(\mathbb{D}, E)).$$

□

Lemma 4.15 *Let E and F be separable Hilbert spaces, and let $G \in L^\infty(\mathbb{T}, \mathcal{B}(F, E))$. For every $x \in L^2(\mathbb{T}, E)$, the function Gx , defined almost everywhere on $\mathbb{T}$ by*

$$(Gx)(z) = G(z)x(z),$$

belongs to $L^2(\mathbb{T}, E)$.

Proof For almost all $z \in \mathbb{T}$,

$$\|(Gx)(z)\|_E = \|G(z)x(z)\|_E \leq \|G\|_{L^\infty(\mathbb{T}, \mathcal{B}(F, E))} \|x(z)\|_F.$$

Thus

$$\begin{aligned} \|Gx\|_{L^2(\mathbb{T}, E)}^2 &= \frac{1}{2\pi} \int_0^{2\pi} \|Gx(e^{i\theta})\|_E^2 d\theta \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \|G\|_{L^\infty(\mathbb{T}, \mathcal{B}(F, E))}^2 \|x(e^{i\theta})\|_F^2 d\theta \\ &\leq \|G\|_{L^\infty(\mathbb{T}, \mathcal{B}(F, E))}^2 \|x\|_{L^2(\mathbb{T}, F)}^2 < \infty. \end{aligned}$$

□

Definition 4.16 Let E and F be separable Hilbert spaces. Let $P_+ : L^2(\mathbb{T}, E) \rightarrow H^2(\mathbb{D}, E)$ be the orthogonal projection operator. Corresponding to any $G \in L^\infty(\mathbb{T}, \mathcal{B}(F, E))$ we define the *Toeplitz operator* with symbol G to be the operator

$$T_G : H^2(\mathbb{D}, F) \rightarrow H^2(\mathbb{D}, E)$$

given by

$$T_G x = P_+(Gx) \quad \text{for any } x \in H^2(\mathbb{D}, F).$$

Definition 4.17 ([8], p. 190) For a separable Hilbert space E , a function $\xi \in H^\infty(\mathbb{D}, E)$ will be called *inner* if for almost every $z \in \mathbb{T}$,

$$\|\xi(z)\|_E = 1.$$

Theorem 4.18 Let E be a separable Hilbert space, and let $\xi \in H^\infty(\mathbb{D}, E)$. Consider the adjoint operator

$$C_\xi^* : H^2(\mathbb{D}, \wedge^2 E) \rightarrow H^2(\mathbb{D}, E).$$

(i) For every $g \in H^2(\mathbb{D}, E)$ and $f \in H^\infty(\mathbb{D}, E)$,

$$C_\xi^*(f \wedge g) = P_+ \alpha_f,$$

where $\alpha_f \in L^2(\mathbb{T}, E)$ is defined by

$$\alpha_f(e^{i\theta}) = \langle f(e^{i\theta}), \xi(e^{i\theta}) \rangle_E g(e^{i\theta}) - \langle g(e^{i\theta}), \xi(e^{i\theta}) \rangle_E f(e^{i\theta})$$

for all $e^{i\theta} \in \mathbb{T} \setminus (N_f \cup N_\xi \cup N_g)$, and $P_+ : L^2(\mathbb{T}, E) \rightarrow H^2(\mathbb{D}, E)$ is the orthogonal projection. Here N_f, N_g, N_ξ are the singular sets of the functions f, g and ξ respectively.

(ii) Let $\xi \in H^\infty(\mathbb{D}, E)$ be an inner function. Then, for any $g \in H^2(\mathbb{D}, E)$,

$$C_\xi^* C_\xi g = P_+ \alpha_\xi,$$

where $\alpha_\xi = g - \xi \xi^* g$. Moreover

$$C_\xi^* C_\xi g = g - T_{\xi \xi^*} g,$$

where $T_{\xi \xi^*} : H^2(\mathbb{D}, E) \rightarrow H^2(\mathbb{D}, E)$ is the Toeplitz operator with symbol $\xi \xi^*$.

Proof (i) By Proposition 3.8, $f \wedge g \in H^2(\mathbb{D}, \wedge^2 E)$. Now, for all $f \in H^\infty(\mathbb{D}, E)$, and all $g, h \in H^2(\mathbb{D}, E)$, we have

$$\begin{aligned} \langle C_\xi^*(f \wedge g), h \rangle_{H^2(\mathbb{D}, E)} &= \langle f \wedge g, C_\xi h \rangle_{H^2(\mathbb{D}, \wedge^2 E)} \\ &= \langle f \wedge g, \xi \wedge h \rangle_{L^2(\mathbb{T}, \wedge^2 E)} \\ &= \frac{1}{2\pi} \int_0^{2\pi} \langle f(e^{i\theta}) \wedge g(e^{i\theta}), \xi(e^{i\theta}) \wedge h(e^{i\theta}) \rangle_{\wedge^2 E} d\theta, \end{aligned}$$

which, by Proposition 2.14, is equal to

$$\frac{1}{2\pi} \int_0^{2\pi} \det \begin{pmatrix} \langle f(e^{i\theta}), \xi(e^{i\theta}) \rangle_E & \langle f(e^{i\theta}), h(e^{i\theta}) \rangle_E \\ \langle g(e^{i\theta}), \xi(e^{i\theta}) \rangle_E & \langle g(e^{i\theta}), h(e^{i\theta}) \rangle_E \end{pmatrix} d\theta.$$

The latter in turn is equal to

$$\begin{aligned} &\frac{1}{2\pi} \int_0^{2\pi} \left(\langle f(e^{i\theta}), \xi(e^{i\theta}) \rangle_E \langle g(e^{i\theta}), h(e^{i\theta}) \rangle_E \right. \\ &\quad \left. - \langle f(e^{i\theta}), h(e^{i\theta}) \rangle_E \langle g(e^{i\theta}), \xi(e^{i\theta}) \rangle_E \right) d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left\langle \langle f(e^{i\theta}), \xi(e^{i\theta}) \rangle_E g(e^{i\theta}) - \langle g(e^{i\theta}), \xi(e^{i\theta}) \rangle_E f(e^{i\theta}), h(e^{i\theta}) \right\rangle_E d\theta, \end{aligned}$$

which equals

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \langle \alpha_f(e^{i\theta}), h(e^{i\theta}) \rangle_E d\theta &= \langle \alpha_f, h \rangle_{L^2(\mathbb{T}, E)} \\ &= \langle P_+(\alpha_f), h \rangle_{H^2(\mathbb{D}, E)}, \end{aligned}$$

where

$$\alpha_f(e^{i\theta}) = \langle f(e^{i\theta}), \xi(e^{i\theta}) \rangle_E g(e^{i\theta}) - \langle g(e^{i\theta}), \xi(e^{i\theta}) \rangle_E f(e^{i\theta})$$

for all $e^{i\theta} \in \mathbb{T} \setminus (N_\xi \cup N_f \cup N_g)$. Hence $C_\xi^*(f \wedge g) = P_+ \alpha_f$ as required.

Let us show (ii). By Part (i), for all $g, h \in H^2(\mathbb{D}, E)$,

$$\begin{aligned} \langle C_\xi^* C_\xi g, h \rangle_{H^2(\mathbb{D}, E)} &= \langle C_\xi^*(\xi \wedge g), h \rangle_{H^2(\mathbb{D}, E)} \\ &= \langle P_+(\alpha_\xi), h \rangle_{H^2(\mathbb{D}, E)}, \end{aligned}$$

where

$$\alpha_\xi(e^{i\theta}) = \langle \xi(e^{i\theta}), \xi(e^{i\theta}) \rangle_E g(e^{i\theta}) - \langle g(e^{i\theta}), \xi(e^{i\theta}) \rangle_E \xi(e^{i\theta})$$

for all $e^{i\theta} \in \mathbb{T} \setminus (N_\xi \cup N_g \cup N_h)$.

Since ξ is inner, $\|\xi(e^{i\theta})\|_E = 1$ almost everywhere on $\mathbb{T}$, and so, $C_\xi^* C_\xi g = P_+ \alpha_\xi$, where $\alpha_\xi = g - \xi \xi^* g$. Hence

$$C_\xi^* C_\xi g = P_+(g - \xi \xi^* g) = g - T_{\xi \xi^*} g,$$

where $T_{\xi\xi^*}g = P_+(\xi\xi^*g)$ is the Toeplitz operator with symbol $\xi\xi^*$. $\square$

Example 4.19 There exists an inner function $\xi \in H^\infty(\mathbb{D}, \mathbb{C}^2)$ such that, for some $h \in H^2(\mathbb{D}, \mathbb{C}^2)$, $C_\xi^*C_\xi h$ is not in the pointwise orthogonal complement of ξ in E .

Let $\xi \in H^\infty(\mathbb{D}, \mathbb{C}^2)$ be an inner function and let $h \in H^2(\mathbb{D}, \mathbb{C}^2)$. Let ξ, h be given by

$$\xi(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ z \end{pmatrix}, \quad h(z) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{for all } z \in \mathbb{D}.$$

By Theorem 4.18,

$$C_\xi^*C_\xi h = P_+\alpha,$$

where, for all $z \in \mathbb{T}$,

$$\alpha(z) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ z \end{pmatrix} \right\rangle_{\mathbb{C}^2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ z \end{pmatrix}.$$

Calculations yield, for all $z \in \mathbb{T}$,

$$\begin{aligned} \alpha(z) &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{2}}(1 + \bar{z})\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ z \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 + \bar{z} \\ z(1 + \bar{z}) \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 - \bar{z} \\ 1 - z \end{pmatrix}. \end{aligned}$$

Thus

$$(P_+\alpha)(z) = \frac{1}{2} \begin{pmatrix} 1 \\ 1 - z \end{pmatrix} \quad \text{for all } z \in \mathbb{T}.$$

The latter expression is not in the pointwise orthogonal complement of ξ in $\mathbb{C}^2$, since for all $z \in \mathbb{T}$,

$$\begin{aligned} \left\langle \frac{1}{2} \begin{pmatrix} 1 \\ 1 - z \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ z \end{pmatrix} \right\rangle_{\mathbb{C}^2} &= \frac{1}{2\sqrt{2}}(1 + \bar{z}) \begin{pmatrix} 1 \\ 1 - z \end{pmatrix} \\ &= \frac{1}{2\sqrt{2}}(1 + \bar{z} - |z|^2) = \frac{1}{2\sqrt{2}}\bar{z} \neq 0. \end{aligned}$$

Lemma 4.20 Let E be a separable Hilbert space. For every inner function $\xi \in H^\infty(\mathbb{D}, E)$,

$$\{x \in H^2(\mathbb{D}, E) : \|C_\xi x\|_{H^2(\mathbb{D}, \wedge^2 E)} = \|x\|_{H^2(\mathbb{D}, E)}\} = \text{POC}(\{\xi\}, H^2(\mathbb{D}, E)).$$

Proof By Remark 4.4, for every $x \in H^2(\mathbb{D}, E)$, $\|x\|_{H^2(\mathbb{D}, E)} = \|x\|_{L^2(\mathbb{T}, E)}$. Hence

$$\begin{aligned} & \{x \in H^2(\mathbb{D}, E) : \|C_\xi x\|_{H^2(\mathbb{D}, \wedge^2 E)}^2 = \|x\|_{H^2(\mathbb{D}, E)}^2\} \\ &= \{x \in H^2(\mathbb{D}, E) : \|C_\xi x\|_{L^2(\mathbb{T}, \wedge^2 E)}^2 = \|x\|_{L^2(\mathbb{T}, E)}^2\}. \end{aligned}$$

By Proposition 2.14, the latter set is equal to

$$\begin{aligned} & \left\{ x \in H^2(\mathbb{D}, E) : \frac{1}{2\pi} \int_0^{2\pi} \det \begin{pmatrix} \langle \xi(e^{i\theta}), \xi(e^{i\theta}) \rangle_E & \langle \xi(e^{i\theta}), x(e^{i\theta}) \rangle_E \\ \langle x(e^{i\theta}), \xi(e^{i\theta}) \rangle_E & \langle x(e^{i\theta}), x(e^{i\theta}) \rangle_E \end{pmatrix} d\theta \right. \\ & \quad \left. = \frac{1}{2\pi} \int_0^{2\pi} \|x(e^{i\theta})\|_E^2 d\theta \right\}. \end{aligned}$$

Since ξ is inner, $\|\xi(e^{i\theta})\|_E = 1$ almost everywhere on $\mathbb{T}$, hence the last set is equal to

$$\begin{aligned} & \left\{ x \in H^2(\mathbb{D}, E) : \frac{1}{2\pi} \int_0^{2\pi} (\|x(e^{i\theta})\|_E^2 - |\langle \xi(e^{i\theta}), x(e^{i\theta}) \rangle_E|^2) d\theta \right. \\ & \quad \left. = \frac{1}{2\pi} \int_0^{2\pi} \|x(e^{i\theta})\|_E^2 d\theta \right\} \\ &= \left\{ x \in H^2(\mathbb{D}, E) : \frac{1}{2\pi} \int_0^{2\pi} |\langle \xi(e^{i\theta}), x(e^{i\theta}) \rangle_E|^2 d\theta = 0 \right\} \\ &= \{x \in H^2(\mathbb{D}, E) : \xi(e^{i\theta}) \perp x(e^{i\theta}) \text{ almost everywhere on } \mathbb{T}\} \\ &= \text{POC}(\{\xi\}, H^2(\mathbb{D}, E)). \end{aligned}$$

□

The following example illustrates the fact that, although C_ξ is an isometry on $\text{POC}(\xi, H^2(\mathbb{D}, \mathbb{C}^2))$, it is not a partial isometry on $H^2(\mathbb{D}, \mathbb{C}^2)$.

Example 4.21 $C_\xi^* C_\xi$ fails to be a projection for some inner function $\xi \in H^\infty(\mathbb{D}, \mathbb{C}^2)$.

Let us calculate $C_\xi^* C_\xi$ for $\xi(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ z \end{pmatrix}$, $z \in \mathbb{D}$. By Theorem 4.18, for $h \in H^2(\mathbb{D}, \mathbb{C}^2)$

$$C_\xi^* C_\xi h = P_+ \alpha,$$

where, for all $z \in \mathbb{T}$, $\alpha(z)$ is given by

$$\begin{aligned} \alpha(z) &= h(z) - \langle h(z), \xi(z) \rangle_E \xi(z) \\ &= \begin{pmatrix} h_1(z) \\ h_2(z) \end{pmatrix} - \frac{1}{\sqrt{2}} (h_1(z) + \bar{z} h_2(z)) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ z \end{pmatrix} \\ &= \begin{pmatrix} h_1(z) - \frac{1}{2} (h_1(z) + \bar{z} h_2(z)) \\ h_2(z) - \frac{1}{2} (z h_1(z) + h_2(z)) \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} h_1(z) - \bar{z} h_2(z) \\ h_2(z) - z h_1(z) \end{pmatrix}. \end{aligned}$$

Thus

$$P_{+\alpha} = \frac{1}{2} \begin{pmatrix} h_1 - S^* h_2 \\ -S h_1 + h_2 \end{pmatrix},$$

where S, S^* denote the shift and the backward shift operators on $H^2(\mathbb{D}, \mathbb{C})$ respectively. Hence

$$C_\xi^* C_\xi = \frac{1}{2} \begin{pmatrix} 1 & -S^* \\ -S & 1 \end{pmatrix}. \quad (4.4)$$

By Eq. (4.4),

$$\begin{aligned} (C_\xi^* C_\xi)^2 &= \frac{1}{4} \begin{pmatrix} 1 + S^* S & -2S^* \\ -2S & SS^* + 1 \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 2 & -2S^* \\ -2S & 2 - P_0 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & -S^* \\ -S & 1 - \frac{1}{2}P_0 \end{pmatrix} \neq C_\xi^* C_\xi, \end{aligned}$$

since $SS^* = 1 - P_0$, where $P_0(\sum_{n=0}^{\infty} a_n z^n) = a_0$.

Consequently $C_\xi^* C_\xi$ is not a projection and hence C_ξ is not a partial isometry on $H^2(\mathbb{D}, \mathbb{C}^2)$.

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