The Zak Transform on Gelfand–Shilov and Modulation Spaces with Applications to Operator Theory

Joachim Toft¹

Received: 3 April 2020 / Accepted: 18 September 2020 / Published online: 13 November 2020
© The Author(s) 2020

Abstract

We characterize Gelfand–Shilov spaces, their distribution spaces and modulation spaces in terms of estimates of their Zak transforms. We use these results for general investigations of quasi-periodic functions and distributions. We also establish necessary and sufficient conditions for linear operators in order for these operators should be conjugations by Zak transforms.

Keywords Zak transform · Quasi-periodic · Wiener spaces · Modulation spaces · Gelfand–Shilov spaces · Quasi-Banach spaces · Characterizations

Mathematics Subject Classification Primary: 42C20 · 43A32 · 42B35 · 46E10; Secondary: 46A16 · 35A22 · 37A05 · 46E35

0 Introduction

In the paper we characterise Gelfand–Shilov spaces of functions and distributions, modulation spaces and Gevrey classes in background of mapping properties of the Zak transforms. We apply these results to deduce duality properties of spaces of quasi-periodic functions and distributions and for investigating transitions of linear operators under the Zak transform.

The Zak transforms are unpredictable and exciting in several ways. They appear in natural ways when dealing with Gabor frame operators in the cases of *critical sampling*, where the Gabor theory cease to work properly. This ought to be the reason

Communicated by Serap Oztop.

This article is part of the topical collection “Harmonic Analysis and Operator Theory” edited by H. Turgay Kaptanoglu, Aurelian Gheondea and Serap Oztop.

✉ Joachim Toft
joachim.toft@lnu.se

¹ Department of Mathematics, Linnaeus University, Växjö, Sweden

why the transform possess several exciting and almost magical properties, useful in Gabor theory.

For example, in critical sampling cases, the Zak transform Z , adapted to the sampling parameters, takes the Gabor frame operator $S_{\phi, \psi}$ into the multiplication operator

$$F \mapsto c \cdot \overline{Z\phi} \cdot Z\psi \cdot F$$

for some constant c which depends on the sampling parameters. (See [21,40] and Sect. 1 for notations.) We remark that this property is heavily used when showing that Gabor atoms and their canonical dual atoms often belong to the same function classes. (See [4,5,18].)

An other example concerns the fact that if Zf is continuous, then it has zeros. This property is important when deducing various kinds of Balian–Low theorems, which are essential when finding limitations for bases and Gabor frames in Gabor analysis (see Theorem 8.4.1 and its consequences in [18]).

Before entering the Gabor theory, Zak transforms were first introduced and used in a problem in differential equation by Gelfand [16]. Subsequently, the transforms were applied in various contexts, especially in solid state physics by Zak [41] and in differential equations by Brezin [3].

In these considerations it is essential to understand various kinds of mapping properties of the Zak transforms. The transforms take suitable functions, defined on the configuration space $\mathbf{R}^d$ into quasi-periodic functions, defined on the phase space $\mathbf{R}^{2d}$. Hence, in similar ways as for periodic functions, Zak transformed functions are completely described by their behaviour on suitable rectangles.

For example, the (standard) Zak transform is given by

$$(Z_1 f)(x, \xi) \equiv \sum_{j \in \mathbf{Z}^d} f(x - j) e^{i \langle j, \xi \rangle},$$

when f is a suitable function or distribution (see (1.19) for the general definition of the Zak transform). By the definition it follows that if $F = Z_1 f$ and $Q_{d,r}$ is the cube $[0, r]^d$, then F is quasi-periodic (with respect to $Q_{d,1} \times Q_{d,2\pi}$). That is,

$$F(x + k, \xi) = e^{i \langle k, \xi \rangle} F(x, \xi) \quad \text{and} \quad F(x, \xi + 2\pi \kappa) = F(x, \xi), \quad k, \kappa \in \mathbf{Z}^d.$$

It follows from these equalities that F is completely reconstructable from its data on $Q_{d,1} \times Q_{d,2\pi}$.

It is well-known that Z_1 is bijective from $L^2(\mathbf{R}^d)$ to the set of quasi-periodic elements in $L^2(Q_{d,1} \times Q_{d,2\pi})$ and that

$$\|Z_1 f\|_{L^2(Q_{d,1} \times Q_{d,2\pi})} = (2\pi)^{\frac{d}{2}} \|f\|_{L^2}, \quad f \in L^2(\mathbf{R}^d). \quad (0.1)$$

(Cf. e.g. [18, Theorem 8.2.3].) Consequently, $L^2(\mathbf{R}^d)$ can be characterized in a convenient way by its image under the Zak transform.

An other space that can be characterized by related mapping properties concerns the Schwartz space $\mathcal{S}(\mathbf{R}^d)$. In fact, it is proved in [22] by Janssen that Z_1 is continuous and bijective from $\mathcal{S}(\mathbf{R}^d)$ to the set of quasi-periodic elements in $C^\infty(\mathbf{R}^{2d})$.

In [32,33], Heil and Tinaztepe deduce some important mapping properties for the Zak transform on modulation spaces, and apply these results to deduce Balian–Low properties in the framework of such spaces. These mapping properties on modulation spaces seems not to be (complete) characterizations, because of absence of bijectivity. In fact, apart from the spaces $L^2(\mathbf{R}^d)$ and $\mathcal{S}(\mathbf{R}^d)$, the whole theory seems to lack characterizations of essential function and distribution spaces via the Zak transform as remarked in Subsection 8.2 (f) in [18].

In Sect. 2 we make this part more complete and furnish the theory with various kinds of characterizations. Especially we characterize modulation and Lebesgue spaces by suitable Lebesgue estimates of short-time Fourier transforms of the Zak transforms of the involved functions. We also characterize the dual $\mathcal{S}'(\mathbf{R}^d)$ of $\mathcal{S}(\mathbf{R}^d)$, the (standard) Gelfand–Shilov spaces and their distribution spaces by their images under the Zak transform.

For example we prove that Z_1 is continuous and bijective from $\mathcal{S}'(\mathbf{R}^d)$ to the set of all quasi-periodic distributions on $Q_{d,1} \times Q_{d,2\pi}$. (See Theorem 2.1.) In Theorems 2.2 and 2.5 we deduce similar characterizations for Gelfand–Shilov spaces and their distribution spaces. As a consequence of Theorem 2.2 we have that the Zak transform Z_1 maps the Gelfand–Shilov space $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ bijectively to $\mathcal{E}_{Z,1}^{\sigma,s}(\mathbf{R}^{2d})$, the set of all quasi-periodic functions on $Q_{d,1} \times Q_{d,2\pi}$ in the Gevrey class $\mathcal{E}^{\sigma,s}(\mathbf{R}^{2d})$. In the same way it follows from Theorem 2.5 that the Zak transform Z_1 maps the Gelfand–Shilov distribution space $(\mathcal{S}_s^\sigma)'(\mathbf{R}^d)$ bijectively to $(\mathcal{E}_{Z,1}^{\sigma,s})'(\mathbf{R}^{2d})$, the set of all quasi-periodic distributions on $Q_{d,1} \times Q_{d,2\pi}$ in $(\mathcal{S}_{s,\sigma}^{\sigma,s})'(\mathbf{R}^{2d})$. As a consequence, if $s + \sigma < 1$, then there are no non-trivial quasi-periodic functions in $\mathcal{E}^{\sigma,s}(\mathbf{R}^{2d})$ (cf. Corollary 2.3).

An other consequence of our results is that Z_1 maps the modulation space $M^p(\mathbf{R}^d)$ continuously and bijectively to the set of all elements in $W^{\infty,p}(\mathbf{R}^{2d})$ which are quasi-periodic on $Q_{d,1} \times Q_{d,2\pi}$. Furthermore,

$$\|Z_1 f\|_{W^{\infty,p}} \asymp \|f\|_{M^p} \quad (0.2)$$

(see Theorem 2.13 and Corollary 2.14).

We also use some recent results in [39] on Wiener estimates to deduce different versions of the latter characterization. For example we show that (0.2) in combination with results in [39, Section 2] give

$$f \in M^p(\mathbf{R}^d) \Leftrightarrow V_\Phi(Z_1 f) \in L^p(Q_{d,1} \times Q_{d,2\pi} \times \mathbf{R}^d \times \mathbf{R}^d). \quad (0.3)$$

If $p = 2$, then an application of Parseval’s formula implies that (0.3) is the same as

$$f \in M^2(\mathbf{R}^d) \Leftrightarrow Z_1 f \in L^2(Q_{d,1} \times Q_{d,2\pi}),$$

which is a slightly weaker form of (0.1).

In Sect. 3 we apply the mapping results of the Zak transform to deduce duality properties for spaces of quasi-periodic functions and distributions. For example, if

$p \in [1, \infty)$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then we prove that the dual of quasi-periodic elements in $\mathcal{E}^{\sigma,s}$ and in $W^{\infty,p}$ can be identified with the set of quasi-periodic elements in the Gelfand–Shilov distribution space $(\mathcal{S}_{s,\sigma}^{\sigma,s})'$, respective in $W^{\infty,p'}$. An essential part of these investigations concerns characterizations of quasi-periodic elements in terms of estimates of their short-time Fourier transforms, given in the end of Sect. 2 and the beginning of Sect. 3.

Finally, in Sect. 4 we show how linear operators, T are transformed under conjugation of the Zak transform, $T_Z = Z_1 \circ T \circ Z_1^{-1}$. It follows from our investigations that the map $T \mapsto T_Z$ is a bijection between the set of all continuous linear mappings

$$T : \mathcal{S}_s^\sigma \rightarrow (\mathcal{S}_s^\sigma)'$$

and the set of all continuous linear mappings

$$T_Z : \mathcal{E}_{Z,1}^{\sigma,s} \rightarrow (\mathcal{E}_{Z,1}^{\sigma,s})'$$

(cf. Theorems 4.1 and 4.2). At the same time we prove that a map T_Z maps quasi-periodic functions or distributions into quasi-periodic functions or distributions, if and only if T_Z commutes with each operator $U_{y,\eta}$, $y, \eta \in \mathbf{R}^d$, where

$$(U_{y,\eta}F)(x, \xi) = e^{-i\langle y, \xi + \eta \rangle} F(x + y, \xi + \eta), \quad y, \eta \in \mathbf{R}^d.$$

1 Preliminaries

In this section we recall some basic facts. We start by discussing Gelfand–Shilov spaces and their properties. Thereafter we recall some properties of modulation spaces and discuss different aspects of periodic distributions.

1.1 Gelfand–Shilov Spaces and Gevrey Classes

We shall consider certain extended classes of the standard Gelfand–Shilov spaces of functions and distributions compared to [17]. Let $n \in \mathbf{Z}_+$, $s_j, \sigma_j, h \in \mathbf{R}_+$ and $d_j \in \mathbf{Z}_+$, $j = 1, \dots, n$ be fixed, $d = d_1 + \dots + d_n$,

$$s = (s_1, \dots, s_n) \quad \text{and} \quad \sigma = (\sigma_1, \dots, \sigma_n). \quad (1.1)$$

Also let

$$V = V_{d_1, \dots, d_n} = \mathbf{R}^{d_1} \times \dots \times \mathbf{R}^{d_n}, \quad (1.2)$$

which we often identify with $\mathbf{R}^d$. Then

$$\mathcal{S}_{s,h}^\sigma(V) = \mathcal{S}_{s,h}^\sigma(\mathbf{R}^d) = \mathcal{S}_{s_1, \dots, s_n; h}^{\sigma_1, \dots, \sigma_n}(V)$$

is the Banach space which consists of all $f \in C^\infty(\mathbf{R}^d)$ such that

$$\|f\|_{\mathcal{S}_{s;h}^\sigma} \equiv \sup \left(\frac{|x_1^{\alpha_1} \cdots x_n^{\alpha_n} \partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n} f(x_1, \dots, x_n)|}{h^{|\alpha_1+\beta_1|+\cdots+|\alpha_n+\beta_n|} \alpha_1!^{s_1} \cdots \alpha_n!^{s_n} \beta_1!^{\sigma_1} \cdots \beta_n!^{\sigma_n}} \right) \quad (1.3)$$

is finite. Here the supremum should be taken over all $\alpha_j, \beta_j \in \mathbf{N}^{d_j}$ and $x_j \in \mathbf{R}^{d_j}$, $j = 1, \dots, n$.

The Gelfand–Shilov space

$$\mathcal{S}_s^\sigma(V) = \mathcal{S}_s^\sigma(\mathbf{R}^d) = \mathcal{S}_{s_1, \dots, s_n}^{\sigma_1, \dots, \sigma_n}(V) \quad (\Sigma_s^\sigma(V) = \Sigma_s^\sigma(\mathbf{R}^d) = \Sigma_{s_1, \dots, s_n}^{\sigma_1, \dots, \sigma_n}(V))$$

of Roumieu type (Beurling type) with parameters $s_1, \dots, s_n, \sigma_1, \dots, \sigma_n$ is the inductive limit (projective limit) of $\mathcal{S}_{s;h}^\sigma(V)$ with respect to $h > 0$. In particular,

$$\mathcal{S}_s^\sigma(V) = \bigcup_{h>0} \mathcal{S}_{s;h}^\sigma(V) \quad \text{and} \quad \Sigma_s^\sigma(V) = \bigcap_{h>0} \mathcal{S}_{s;h}^\sigma(V)$$

It follows that $\Sigma_s^\sigma(V)$ is a Fréchet space with topology induced by the seminorms $\|\cdot\|_{\mathcal{S}_{s;h}^\sigma}$, $h > 0$. The space $\mathcal{S}_s^\sigma(V)$ is complete and the topology is the largest one in order for the inclusion map $i : \mathcal{S}_{s;h}^\sigma(V) \rightarrow \mathcal{S}_s^\sigma(V)$ to be continuous for every $h > 0$.

We have $\Sigma_s^\sigma(V) \neq \{0\}$, if and only if $s_j + \sigma_j \geq 1$ and $(s_j, \sigma_j) \neq (\frac{1}{2}, \frac{1}{2})$, $j = 1, \dots, n$, and $\mathcal{S}_s^\sigma(V) \neq \{0\}$, if and only if $s_j + \sigma_j \geq 1$, $j = 1, \dots, n$. By $(j+k)! \leq 2^{j+k} j!k!$ when $j, k \geq 0$ are integers we get $\mathcal{S}_{s, \dots, s}^{\sigma, \dots, \sigma} = \mathcal{S}_s^\sigma$ and $\Sigma_{s, \dots, s}^{\sigma, \dots, \sigma} = \Sigma_s^\sigma$.

The Gelfand–Shilov distribution spaces

$$(\mathcal{S}_s^\sigma)'(V) = (\mathcal{S}_s^\sigma)'(\mathbf{R}^d) = (\mathcal{S}_{s_1, \dots, s_n}^{\sigma_1, \dots, \sigma_n})'(V)$$

and

$$(\Sigma_s^\sigma)'(V) = (\Sigma_s^\sigma)'(\mathbf{R}^d) = (\Sigma_{s_1, \dots, s_n}^{\sigma_1, \dots, \sigma_n})'(V)$$

are the (strong) dual spaces of $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ and $\Sigma_s^\sigma(\mathbf{R}^d)$, respectively. Let $(\mathcal{S}_{s;h}^\sigma)'(V)$ be the dual of $\mathcal{S}_{s;h}^\sigma(V)$ with respect to the form $(\cdot, \cdot)_{L^2(\mathbf{R}^d)}$. Then $(\mathcal{S}_s^\sigma)'(V)$ and $(\Sigma_s^\sigma)'(V)$ can be identified with the projective limit respectively inductive limit of $(\mathcal{S}_{s;h}^\sigma)'(V)$ with respect to $h > 0$, when $\mathcal{S}_s^\sigma(V)$ respective $\Sigma_s^\sigma(V)$ are non-trivial. (See e.g. [2, Theorem 4.16]. See also [17, 24, 26].) In particular,

$$(\mathcal{S}_s^\sigma)'(V) = \bigcap_{h>0} (\mathcal{S}_{s;h}^\sigma)'(V) \quad \text{and} \quad (\Sigma_s^\sigma)'(V) = \bigcup_{h>0} (\mathcal{S}_{s;h}^\sigma)'(V)$$

in such situations. For conveniency we set

$$\mathcal{S}_s = \mathcal{S}_s^s, \quad \mathcal{S}'_s = (\mathcal{S}_s^s)', \quad \Sigma_s = \Sigma_s^s \quad \text{and} \quad \Sigma'_s = (\Sigma_s^s)'.$$

For any $s_j, \sigma_j, s_{0,j}, \sigma_{0,j} > 0$ such that $s_j > s_{0,j}, \sigma_j > \sigma_{0,j}$ and $s_{0,j} + \sigma_{0,j} \geq 1$, $j = 1, \dots, n$, we have

$$\begin{aligned} \mathcal{S}_{s_0}^{\sigma_0}(V) &\hookrightarrow \Sigma_s^\sigma(V) \hookrightarrow \mathcal{S}_s^\sigma(V) \hookrightarrow \mathcal{S}(\mathbf{R}^d) \\ &\hookrightarrow \mathcal{S}'(\mathbf{R}^d) \hookrightarrow (\mathcal{S}_s^\sigma)'(V) \hookrightarrow (\Sigma_s^\sigma)'(V) \hookrightarrow (\mathcal{S}_{s_0}^{\sigma_0})'(V), \end{aligned} \quad (1.4)$$

with dense embeddings, where s and σ are the same as above,

$$s_0 = (s_{0,1}, \dots, s_{0,n}) \quad \text{and} \quad \sigma_0 = (\sigma_{0,1}, \dots, \sigma_{0,n}).$$

Here and in what follows we use the notation $A \hookrightarrow B$ when the topological spaces A and B satisfy $A \subseteq B$ with continuous embeddings.

Gelfand–Shilov spaces and their distribution spaces possess convenient mapping properties under (partial) Fourier transformations. In fact, let $\mathcal{F}$ be the Fourier transform which takes the form

$$(\mathcal{F}f)(\xi) = \widehat{f}(\xi) \equiv (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} f(x) e^{-i\langle x, \xi \rangle} dx$$

when $f \in L^1(\mathbf{R}^d)$. Let $\mathcal{F}_j f$ denote the partial Fourier transform of $f(x_1, \dots, x_n) \in \mathcal{S}(\mathbf{R}^d)$ with respect to x_j , $j = 1, \dots, n$. Then $\mathcal{F}_j$ restricts to homeomorphisms

$$\mathcal{F}_j : \mathcal{S}_s^\sigma(\mathbf{R}^d) \rightarrow \mathcal{S}_t^\tau(\mathbf{R}^d) \quad \text{and} \quad \mathcal{F}_j : \Sigma_s^\sigma(\mathbf{R}^d) \rightarrow \Sigma_t^\tau(\mathbf{R}^d), \quad (1.5)$$

and extends uniquely to homeomorphisms

$$\mathcal{F}_j : (\mathcal{S}_s^\sigma)'(\mathbf{R}^d) \rightarrow (\mathcal{S}_t^\tau)'(\mathbf{R}^d) \quad \text{and} \quad \mathcal{F}_j : (\Sigma_s^\sigma)'(\mathbf{R}^d) \rightarrow (\Sigma_t^\tau)'(\mathbf{R}^d), \quad (1.6)$$

where

$$t = (t_1, \dots, t_n) \quad \text{and} \quad \tau = (\tau_1, \dots, \tau_n)$$

are given by

$$t_k = \begin{cases} s_k, & k \neq j \\ \sigma_j, & k = j \end{cases} \quad \text{and} \quad \tau_k = \begin{cases} \sigma_k, & k \neq j \\ s_j, & k = j. \end{cases}$$

(See [17] and the analysis therein.) Consequently,

$$\begin{aligned} \mathcal{F} : \mathcal{S}_s^\sigma(\mathbf{R}^d) &\rightarrow \mathcal{S}_\sigma^s(\mathbf{R}^d), & \mathcal{F} : \Sigma_s^\sigma(\mathbf{R}^d) &\rightarrow \Sigma_\sigma^s(\mathbf{R}^d), \\ \mathcal{F} : (\mathcal{S}_s^\sigma)'(\mathbf{R}^d) &\rightarrow (\mathcal{S}_\sigma^s)'(\mathbf{R}^d) & \text{and} & \mathcal{F} : (\Sigma_s^\sigma)'(\mathbf{R}^d) \rightarrow (\Sigma_\sigma^s)'(\mathbf{R}^d) \end{aligned} \quad (1.7)$$

are homeomorphisms.

Gelfand–Shilov spaces can in convenient ways be characterized in terms of estimates of the involved functions and their Fourier transforms. For example we have the following. Here $f(\theta) \lesssim g(\theta)$ means that $f(\theta) \leq cg(\theta)$ for some constant $c > 0$ which is independent of θ in the domain of f and g . We also set $f(\theta) \asymp g(\theta)$ when $f(\theta) \lesssim g(\theta)$ and $g(\theta) \lesssim f(\theta)$.

Proposition 1.1 *Let $p \in [1, \infty]$, $f \in \mathcal{S}'(\mathbf{R}^d)$, $n \in \mathbf{Z}_+$, $d_j \in \mathbf{Z}_+$, $j = 1, \dots, n$, $s, \sigma \in \mathbf{R}_+^n$, V be as in (1.2) and let*

$$v_{r,s}(x_1, \dots, x_n) = e^{r(|x_1|^{\frac{1}{s_1}} + \dots + |x_n|^{\frac{1}{s_n}})}, \quad r \geq 0, \quad x_j \in \mathbf{R}^{d_j}.$$

Then the following conditions are equivalent:

- (1) $f \in \mathcal{S}_s^\sigma(V)$ ($f \in \Sigma_s^\sigma(V)$);
- (2) f and $\widehat{f}$ are measurable and satisfy

$$\|f \cdot v_{r,s}\|_{L^p(\mathbf{R}^d)} < \infty \quad \text{and} \quad \|\widehat{f} \cdot v_{r,\sigma}\|_{L^p(\mathbf{R}^d)} < \infty \quad (1.8)$$

for some $r > 0$ (for every $r > 0$);

- (3) f is smooth and satisfies

$$\|(\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} f) \cdot v_{r,s}\|_{L^p(\mathbf{R}^d)} \lesssim \prod_{j=1}^n h^{|\alpha_j|} \alpha_j!^{\sigma_j},$$

for some $h, r > 0$ (for every $h, r > 0$).

We refer to [6,8] and their analyses for the proof of the equivalence between (1) and (2), and to [17] and their analyses for the proof of the equivalence between (1) and (3) in Proposition 1.1.

Gelfand–Shilov spaces and their distribution spaces can also be characterized by estimates of short-time Fourier transforms, (see e.g. [20,30,36]). More precisely, let $\phi \in \mathcal{S}(\mathbf{R}^d)$ be fixed. Then the *short-time Fourier transform* $V_\phi f$ of $f \in \mathcal{S}'(\mathbf{R}^d)$ with respect to the *window function* ϕ is the Schwartz distribution on $\mathbf{R}^{2d}$, defined by

$$V_\phi f(x, \xi) = \mathcal{F}(f \overline{\phi(\cdot - x)})(\xi).$$

If $f, \phi \in \mathcal{S}(\mathbf{R}^d)$, then it follows that

$$V_\phi f(x, \xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbf{R}^d} f(y) \overline{\phi(y - x)} e^{-i\langle y, \xi \rangle} dy.$$

By [31, Theorem 3.2] or [34, Theorem 2.3] and their analyses, it follows that the map $(f, \phi) \mapsto V_\phi f$ from $\mathcal{S}(\mathbf{R}^d) \times \mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}(\mathbf{R}^{2d})$ is uniquely extendable to a continuous map from $(\mathcal{S}_s^\sigma)'(V) \times (\mathcal{S}_s^\sigma)'(V)$ to $(\mathcal{S}_{s,\sigma}^{\sigma,s})'(V \times V)$, and restricts to a continuous map from $\mathcal{S}_s^\sigma(V) \times \mathcal{S}_s^\sigma(V)$ to $\mathcal{S}_{s,\sigma}^{\sigma,s}(V \times V)$. The same conclusion holds

with Σ_s^σ and $\Sigma_{s,\sigma}^{\sigma,s}$ in place of $\mathcal{S}_s^\sigma$ and $\mathcal{S}_{s,\sigma}^{\sigma,s}$, respectively, at each occurrence. Here $\mathcal{S}_{s,\sigma}^{\sigma,s}$ and $\Sigma_{s,\sigma}^{\sigma,s}$ are given by

$$\mathcal{S}_{s,\sigma}^{\sigma,s} = \mathcal{S}_{s_1, \dots, s_n, \sigma_1, \dots, \sigma_n}^{\sigma_1, \dots, \sigma_n, s_1, \dots, s_n} \quad \text{and} \quad \Sigma_{s,\sigma}^{\sigma,s} = \Sigma_{s_1, \dots, s_n, \sigma_1, \dots, \sigma_n}^{\sigma_1, \dots, \sigma_n, s_1, \dots, s_n},$$

when s and σ are given by (1.1).

The following results characterize Gelfand–Shilov spaces and their distribution spaces in terms of estimates of short-time Fourier transform.

Proposition 1.2 *Let $s \in \mathbf{R}_+^n$, $\sigma \in \mathbf{R}_+^n$, V be as in (1.1) and (1.2), $p \in [1, \infty]$, f be a Gelfand–Shilov distribution on V ,*

$$\phi \in \mathcal{S}_s^\sigma(V) \setminus 0 \quad (\phi \in \Sigma_s^\sigma(V) \setminus 0)$$

and let

$$v_{r,s,\sigma}(x_1, \dots, x_n) = e^{r(|x_1|^{\frac{1}{s_1}} + \dots + |x_n|^{\frac{1}{s_n}} + |\xi_1|^{\frac{1}{\sigma_1}} + \dots + |\xi_n|^{\frac{1}{\sigma_n}})}, \quad r \geq 0, \quad x_j, \xi_j \in \mathbf{R}^{d_j}.$$

Then the following is true:

- (1) $f \in \mathcal{S}_s^\sigma(V)$ ($f \in \Sigma_s^\sigma(V)$), if and only if

$$\|V_\phi f \cdot v_{r,s,\sigma}\|_{L^p(\mathbf{R}^{2d})} < \infty \quad (1.9)$$

for some $r > 0$ (for every $r > 0$);

- (2) $f \in (\mathcal{S}_s^\sigma)'(V)$ ($f \in (\Sigma_s^\sigma)'(V)$), if and only if

$$\|V_\phi f \cdot (v_{r,s,\sigma})^{-1}\|_{L^p(\mathbf{R}^{2d})} < \infty \quad (1.10)$$

for every $r > 0$ (for some $r > 0$).

A proof of Proposition 1.2 (1) can be found in e.g. [20] (cf. [20, Theorem 2.7]) and a proof of Proposition 1.2 (2) in the case $n = 1$ can be found in [36]. The general case of Proposition 1.2 (2) follows by similar arguments as in [36] and is left for the reader. See also [7] for related results.

Next we consider Gevrey classes on $\mathbf{R}^d$. Let $\sigma \geq 0$. For any compact set $K \subseteq \mathbf{R}^d$, $h > 0$ and $f \in C^\infty(K)$ let

$$\|f\|_{K,h,\sigma} \equiv \sup_{\alpha \in \mathbf{N}^d} \left(\frac{\|\partial^\alpha f\|_{L^\infty(K)}}{h^{|\alpha|} \alpha!^\sigma} \right). \quad (1.11)$$

The Gevrey class $\mathcal{E}^\sigma(K)$ ($\mathcal{E}^{\sigma;0}(K)$) of order σ and of Roumieu type (of Beurling type) is the set of all $f \in C^\infty(K)$ such that (1.11) is finite for some (for every) $h > 0$. We equip $\mathcal{E}^\sigma(K)$ ($\mathcal{E}^{\sigma;0}(K)$) by the inductive (projective) limit topology with respect to

$h > 0$, supplied by the seminorms in (1.11). Finally if $\{K_j\}_{j \geq 1}$ is an exhaustion by compact sets of $\mathbf{R}^d$, then let

$$\mathcal{E}^\sigma(\mathbf{R}^d) = \text{proj} \lim_j \mathcal{E}^\sigma(K_j) \quad \text{and} \quad \mathcal{E}^{\sigma;0}(\mathbf{R}^d) = \text{proj} \lim_j \mathcal{E}^{\sigma;0}(K_j).$$

In particular,

$$\mathcal{E}^\sigma(\mathbf{R}^d) = \bigcap_{j \geq 1} \mathcal{E}^\sigma(K_j) \quad \text{and} \quad \mathcal{E}^{\sigma;0}(\mathbf{R}^d) = \bigcap_{j \geq 1} \mathcal{E}^{\sigma;0}(K_j).$$

It is clear that $\mathcal{E}^{0;0}(\mathbf{R}^d)$ contains all constant functions on $\mathbf{R}^d$, and that $\mathcal{E}^0(\mathbf{R}^d)$ contains all non-constant trigonometric polynomials.

1.2 Ordered, Dual and Phase Split Bases

Our discussions involving Zak transforms, periodicity, modulation spaces and Wiener spaces are done in terms of suitable bases.

Definition 1.3 Let $E = \{e_1, \dots, e_d\}$ be an *ordered* basis of $\mathbf{R}^d$. Then E' denotes the basis of $e'_1, \dots, e'_d$ in $\mathbf{R}^d$ which satisfies

$$\langle e_j, e'_k \rangle = 2\pi \delta_{jk} \quad \text{for every } j, k = 1, \dots, d.$$

The corresponding lattices are given by

$$\Lambda_E = \{n_1 e_1 + \dots + n_d e_d; (n_1, \dots, n_d) \in \mathbf{Z}^d\},$$

and

$$\Lambda'_E = \Lambda_{E'} = \{v_1 e'_1 + \dots + v_d e'_d; (v_1, \dots, v_d) \in \mathbf{Z}^d\}.$$

The sets E' and Λ'_E are called the dual basis and dual lattice of E and Λ_E , respectively. If E_1, E_2 are ordered bases of $\mathbf{R}^d$ such that a permutation of E_2 is the dual basis for E_1 , then the pair (E_1, E_2) are called *permuted dual bases* (to each others on $\mathbf{R}^d$).

Remark 1.4 Evidently, if E is the same as in Definition 1.3, then there is an invertible matrix T_E with E as the image of the standard basis in $\mathbf{R}^d$. Then E' is the image of the standard basis under the map $T_{E'} = 2\pi(T_E^{-1})^t$.

Definition 1.5 Let E_1, E_2 be ordered bases of $\mathbf{R}^d$,

$$V_1 = \{(x, 0) \in \mathbf{R}^{2d}; x \in \mathbf{R}^d\}, \quad V_2 = \{(0, \xi) \in \mathbf{R}^{2d}; \xi \in \mathbf{R}^d\}$$

and let π_j from $\mathbf{R}^{2d}$ to $\mathbf{R}^d$, $j = 1, 2$, be the projections

$$\pi_1(x, \xi) = x \quad \text{and} \quad \pi_2(x, \xi) = \xi.$$

Then $E_1 \times E_2$ is the ordered basis $\{e_1, \dots, e_{2d}\}$ of $\mathbf{R}^{2d}$ such that

$$\begin{aligned} \{e_1, \dots, e_d\} &\subseteq V_1, & E_1 &= \{\pi_1(e_1), \dots, \pi_1(e_d)\}, \\ \{e_{d+1}, \dots, e_{2d}\} &\subseteq V_2 & \text{and } E_2 &= \{\pi_2(e_{d+1}), \dots, \pi_2(e_{2d})\}. \end{aligned}$$

In the phase space it is convenient to consider phase split bases, which are defined as follows.

Definition 1.6 Let V_1, V_2, π_1 and π_2 be as in Definition 1.5, E be an ordered basis of the phase space $\mathbf{R}^{2d}$ and let $E_0 \subseteq E$. Then E is called *phase split* (with respect to E_0), if the following is true:

- (1) the span of E_0 and $E \setminus E_0$ equal V_1 and V_2 , respectively;
- (2) if $E_1 = \pi_1(E_0)$ and $E_2 = \pi_2(E \setminus E_0)$, then (E_1, E_2) are permuted dual bases.

If E is a phase split basis with respect to E_0 and that E_0 consists of the first d vectors in E , then E is called *strongly phase split* (with respect to E_0).

In Definition 1.6 it is understood that the orderings of E_0 and $E \setminus E_0$ are inherited from the ordering in E .

Remark 1.7 Let E and $E_j, j = 0, 1, 2$ be the same as in Definition 1.6. It is evident that E_0 and $E \setminus E_0$ consist of d elements, and that E_1 and E_2 are uniquely defined if the orders of E_0 and $E \setminus E_0$ are preserved. The pair (E_1, E_2) is called *the pair of permuted dual bases, induced by E and E_0* .

On the other hand, suppose that (E_1, E_2) is a pair of permuted dual bases to each others on $\mathbf{R}^d$. Then it is clear that for $E_1 \times E_2 = \{e_1, \dots, e_{2d}\}$ in Definition 1.5 and $E_0 = \{e_1, \dots, e_d\}$, we have that E_0 and E fullfils all properties in Definition 1.6. In this case, $E_1 \times E_2$ is called the phase split basis (of $\mathbf{R}^{2d}$) induced by (E_1, E_2) .

It follows that if E', E'_1 and E'_2 are the dual bases of E, E_1 and E_2 , respectively, then $E' = E'_1 \times E'_2$.

1.3 Quasi-Banach Spaces and Spaces of Mixed Quasi-Normed Spaces of Lebesgue Types

We recall that a quasi-norm $\|\cdot\|_{\mathcal{B}}$ of order $r \in (0, 1]$ on the vector-space $\mathcal{B}$ over $\mathbf{C}$ is a nonnegative functional on $\mathcal{B}$ which satisfies

$$\begin{aligned} \|f + g\|_{\mathcal{B}} &\leq 2^{\frac{1}{r}-1}(\|f\|_{\mathcal{B}} + \|g\|_{\mathcal{B}}), & f, g &\in \mathcal{B}, \\ \|\alpha \cdot f\|_{\mathcal{B}} &= |\alpha| \cdot \|f\|_{\mathcal{B}}, & \alpha &\in \mathbf{C}, \quad f \in \mathcal{B} \end{aligned} \quad (1.12)$$

and

$$\|f\|_{\mathcal{B}} = 0 \quad \Leftrightarrow \quad f = 0.$$

The space $\mathcal{B}$ is then called a quasi-normed space. A complete quasi-normed space is called a quasi-Banach space. If $\mathcal{B}$ is a quasi-Banach space with quasi-norm satisfying

(1.12) then by [1,29] there is an equivalent quasi-norm to $\|\cdot\|_{\mathcal{B}}$ which additionally satisfies

$$\|f + g\|_{\mathcal{B}}^r \leq \|f\|_{\mathcal{B}}^r + \|g\|_{\mathcal{B}}^r, \quad f, g \in \mathcal{B}. \quad (1.13)$$

From now on we always assume that the quasi-norm of the quasi-Banach space $\mathcal{B}$ is chosen in such way that both (1.12) and (1.13) hold.

Next we recall some facts on weight functions. A *weight* or *weight function* on $\mathbf{R}^d$ is a positive function $\omega \in L_{loc}^\infty(\mathbf{R}^d)$ such that $1/\omega \in L_{loc}^\infty(\mathbf{R}^d)$. The weight ω is called *moderate*, if there is a positive weight v on $\mathbf{R}^d$ such that

$$\omega(x + y) \lesssim \omega(x)v(y), \quad x, y \in \mathbf{R}^d. \quad (1.14)$$

If ω and v are weights on $\mathbf{R}^d$ such that (1.14) holds, then ω is also called *v-moderate*. We note that (1.14) implies that ω fulfills the estimates

$$v(-x)^{-1} \lesssim \omega(x) \lesssim v(x), \quad x \in \mathbf{R}^d. \quad (1.15)$$

We let $\mathcal{P}_E(\mathbf{R}^d)$ be the set of all moderate weights on $\mathbf{R}^d$.

By [19] it follows that if $\omega \in \mathcal{P}_E(\mathbf{R}^d)$, then

$$e^{-r|x|} \lesssim \omega(x) \lesssim e^{r|x|} \quad \text{and} \quad \omega(x + y) \lesssim \omega(x)e^{r|y|}, \quad x, y \in \mathbf{R}^d.$$

for some $r > 0$.

We say that v is *submultiplicative* if v is even and (1.14) holds with $\omega = v$. In the sequel, v and v_j for $j \geq 0$, always stand for submultiplicative weights if nothing else is stated.

In the following we define a broad family of mixed quasi-normed Lebesgue spaces. Here Q_E denotes the closed parallelepiped (or the *E-cube*) which is spanned by the ordered basis E of $\mathbf{R}^d$.

Definition 1.8 Let $E = \{e_1, \dots, e_d\}$ be an ordered basis of $\mathbf{R}^d$, Q_E be the parallelepiped spanned by E , $\omega \in \mathcal{P}_E(\mathbf{R}^d)$, $\mathbf{q} = (q_1, \dots, q_d) \in (0, \infty]^d$ and $r = \min(1, \mathbf{q})$. If $f \in L_{loc}^r(\mathbf{R}^d)$, then

$$\|f\|_{L_{E,(\omega)}^{\mathbf{q}}} \equiv \|g_{d-1}\|_{L^{q_d}(\mathbf{R})}$$

where $g_k : \mathbf{R}^{d-k} \rightarrow \mathbf{R}$, $k = 0, \dots, d-1$, are inductively defined as

$$g_0(x_1, \dots, x_d) \equiv |f(x_1 e_1 + \dots + x_d e_d) \omega(x_1 e_1 + \dots + x_d e_d)|,$$

and

$$g_k(\mathbf{z}_k) \equiv \|g_{k-1}(\cdot, \mathbf{z}_k)\|_{L^{q_k}(\mathbf{R})}, \quad \mathbf{z}_k \in \mathbf{R}^{d-k}, \quad k = 1, \dots, d-1.$$

If $\Omega \subseteq \mathbf{R}^d$ is measurable, then $L_{E,(\omega)}^q(\Omega)$ consists of all $f \in L_{loc}^r(\Omega)$ with finite quasi-norm

$$\|f\|_{L_{E,(\omega)}^q(\Omega)} \equiv \|f_\Omega\|_{L_{E,(\omega)}^q(\mathbf{R}^d)}, \quad f_\Omega(x) \equiv \begin{cases} f(x), & \text{when } x \in \Omega, \\ 0, & \text{when } x \notin \Omega. \end{cases}$$

The space $L_{E,(\omega)}^q(\Omega)$ is called *E-split Lebesgue space (with respect to ω , q and Ω)*.

We let $\ell'_0(\Lambda'_E)$ be the set of all formal sequences $a = \{a(j)\}_{j \in \Lambda_E} \subseteq \mathbf{C}$ on Λ_E and $\ell_{E,(\omega)}^p(\Lambda_E)$ be the *discrete version* of $L_{E,(\omega)}^p(\mathbf{R}^d)$ when $p \in (0, \infty]^d$. That is, $\ell_{E,(\omega)}^p(\Lambda_E)$ consists of all $a \in \ell'_0(\Lambda_E)$ such that

$$\|a\|_{\ell_{E,(\omega)}^p} = \|a\|_{\ell_{E,(\omega)}^p(\Lambda_E)} \equiv \left\| \sum_{j \in \Lambda_E} a(j) \chi_{j+Q_E} \right\|_{L_{E,(\omega)}^p(\mathbf{R}^d)}$$

is finite. Here χ_Ω is the characteristic function of the set Ω .

Remark 1.9 Evidently, $L_{E,(\omega)}^q(\Omega)$ and $\ell_{E,(\omega)}^q(\Lambda)$ in Definition 1.8 are quasi-Banach spaces of order $\min(q, 1)$. We set

$$L_E^q = L_{E,(\omega)}^q \quad \text{and} \quad \ell_E^q = \ell_{E,(\omega)}^q$$

when $\omega = 1$. For conveniency we identify $\mathbf{q} = (q, \dots, q) \in (0, \infty]^d$ with $q \in (0, \infty]$ when considering spaces involving Lebesgue exponents. In particular,

$$L_{E,(\omega)}^q = L_{E,(\omega)}^q, \quad L_E^q = L_E^q, \quad \ell_{E,(\omega)}^q = \ell_{E,(\omega)}^q \quad \text{and} \quad \ell_E^q = \ell_E^q$$

for such $\mathbf{q}$, and notice that these spaces agree with

$$L_{(\omega)}^q, \quad L^q, \quad \ell_{(\omega)}^q \quad \text{and} \quad \ell^q,$$

respectively, with equivalent quasi-norms.

1.4 Modulation and Wiener Spaces

We consider the following broad family of modulation spaces which contains the classical modulation spaces, introduced by Feichtinger [10].

Definition 1.10 Let $\mathbf{p}, \mathbf{q} \in (0, \infty]^d$, E_1 and E_2 be ordered bases of $\mathbf{R}^d$, $E = E_1 \times E_2$, $\phi \in \Sigma_1(\mathbf{R}^d) \setminus 0$ and let $\omega \in \mathcal{P}_E(\mathbf{R}^{2d})$. For any $f \in \Sigma'_1(\mathbf{R}^d)$ set

$$\|f\|_{M_{E,(\omega)}^{\mathbf{p},\mathbf{q}}} \equiv \|H_{1,f,E_1,\mathbf{p},\omega}\|_{L_{E_2}^q},$$

where $H_{1,f,E_1,\mathbf{p},\omega}(\xi) \equiv \|V_\phi f(\cdot, \xi)\omega(\cdot, \xi)\|_{L_{E_1}^p}$

and

$$\|f\|_{W_{E,(\omega)}^{p,q}} \equiv \|H_{2,f,E_2,q,\omega}\|_{L_{E_1}^p},$$

$$\text{where } H_{2,f,E_2,q,\omega}(x) \equiv \|V_\phi f(x, \cdot)\omega(x, \cdot)\|_{L_{E_2}^q}.$$

The modulation space $M_{E,(\omega)}^{p,q}(\mathbf{R}^d)$ ($W_{E,(\omega)}^{p,q}(\mathbf{R}^d)$) consist of all $f \in \Sigma'_1(\mathbf{R}^d)$ such that $\|f\|_{M_{E,(\omega)}^{p,q}}$ ($\|f\|_{W_{E,(\omega)}^{p,q}}$) is finite.

The theory of modulation spaces has developed in different ways since they were introduced in [10] by Feichtinger. (Cf. e.g. [11,15,18,35].) For example, let p, q, E and ω be the same as in Definition 1.10, and let $L_E^{p,q}(\mathbf{R}^{2d})$, $r = \min(1, p, q)$, $v \in \mathcal{P}_E(\mathbf{R}^{2d})$ be such that ω is v -moderate, and let $\phi \in M_{(v)}^r(\mathbf{R}^d) \setminus 0$. Then $M_{E,(\omega)}^{p,q}(\mathbf{R}^d)$ is a quasi-Banach space of order r . Moreover, $f \in M_{E,(\omega)}^{p,q}(\mathbf{R}^d)$ if and only if $V_\phi f \cdot \omega \in L_E^{p,q}(\mathbf{R}^{2d})$, and different choices of ϕ give rise to equivalent quasi-norms in Definition 1.10. We also note that

$$\Sigma_1(\mathbf{R}^d) \subseteq M_{E,(\omega)}^{p,q}(\mathbf{R}^d) \subseteq \Sigma'_1(\mathbf{R}^d).$$

Similar facts hold for the space $W_{E,(\omega)}^{p,q}(\mathbf{R}^d)$. (Cf. [15,35].)

Definition 1.11 Let $p, r \in (0, \infty]^d$, $\omega_0 \in \mathcal{P}_E(\mathbf{R}^d)$, $\omega \in \mathcal{P}_E(\mathbf{R}^{2d})$, $\phi \in \Sigma_1(\mathbf{R}^d) \setminus 0$, $E \subseteq \mathbf{R}^d$ be an ordered basis, and let Q_E be the closed parallelepiped spanned by E . Also let f and F be measurable on $\mathbf{R}^d$ respective $\mathbf{R}^{2d}$, and let $F_\omega = F \cdot \omega$. Then $\|f\|_{W_{E,(\omega_0, \ell_E^p)}^r}$ is given by

$$\|f\|_{W_{E,(\omega_0, \ell_E^p)}^r} \equiv \|h_{E, \omega_0, r, f}\|_{\ell_E^p(\Lambda_E)},$$

$$h_{E, \omega_0, r, f}(j) = \|f\|_{L_E^r(j+Q_E)} \omega_0(j), \quad j \in \Lambda_E.$$

The set $W_{E,(\omega, \ell_E^p)}^r$ consists of all measurable f on $\mathbf{R}^d$ such that $\|f\|_{W_{E,(\omega_0, \ell_E^p)}^r} < \infty$.

We observe that $W_{E,(\omega_0, \ell_E^p)}^r$ is equal to $W(L^r, L_{(\omega_0)}^p)$ in [12,15,18,27,28] when E is the standard basis. In particular, $W_{E,(\omega_0, \ell_E^p)}^r$ is related to so-called coorbit spaces. (See [9,12–14,27,28].)

Remark 1.12 Let $p, q, \omega_0, \omega, E, f$ and F be the same as in Definition 1.11. Evidently, by using the fact that ω_0 is v_0 -moderate for some v_0 , it follows that

$$\|f \cdot \omega_0\|_{W_{E,(1, \ell_E^p)}^q} \asymp \|f\|_{W_{E,(\omega_0, \ell_E^p)}^q}.$$

Remark 1.13 For the spaces in Definition 1.11 we set $W^{q_0, r_0} = W^r$, when

$$r_0 = (r_1, \dots, r_d) \in (0, \infty]^d, \quad \text{and } r = (q_0, \dots, q_0, r_1, \dots, r_d) \in (0, \infty]^{2d},$$

and similarly for other types of exponents and for the spaces in Definition 1.10. (See also Remark 1.9.) We also set

$$M_{E,(\omega)}^{\infty,q} = M_{E_2,(\omega)}^{\infty,q} \quad \text{and} \quad W_{E,(\omega)}^{\infty,q} = W_{E_2,(\omega)}^{\infty,q}$$

when E_1, E_2 are ordered bases of $\mathbf{R}^d$ and $E = E_1 \times E_2$, for spaces in Definition 1.10, since these spaces are independent of E_1 .

The following result is essential when characterizing elements in modulation spaces in terms of estimates of their Zak transforms. We omit the proof since the result is a consequence of [39, Proposition 2.6].

Proposition 1.14 *Let E_0 be a basis for $\mathbf{R}^d$, E'_0 be its dual basis, $E = E_0 \times E'_0$, $\mathbf{q}, \mathbf{r} \in (0, \infty]^d$, $\omega_0 \in \mathcal{P}_E(\mathbf{R}^d)$ and $\omega(x, \xi) = \omega_0(\xi)$, $x, \xi \in \mathbf{R}^d$. Then*

$$\|f\|_{M_{E,(\omega)}^{\infty,q}} \asymp \|\varphi_{f,\omega,E,\mathbf{r}}\|_{L_{E'_0}^q} \quad \text{and} \quad \|f\|_{W_{E,(\omega)}^{\infty,q}} \asymp \sup_{j \in \Lambda_E} (\|\psi_{f,\omega,\mathbf{q}}\|_{L^r(j+Q_E)}),$$

where

$$\varphi_{f,\omega,E,\mathbf{r}}(\xi) \equiv \sup_{j \in \Lambda_E} (\|V_\phi f(\cdot, \xi)\omega(\cdot, \xi)\|_{L_E^r(j+Q_E)})$$

and

$$\psi_{f,\omega,\mathbf{q}}(x) \equiv \|V_\phi f(x, \cdot)\omega(x, \cdot)\|_{L_{E'_0}^q}.$$

The next result is a restatement of Proposition 1.15' in [39]. The proof is therefore omitted. Here

$$(\Theta_\rho v)(x, \xi) = v(x, \xi)\langle x, \xi \rangle^\rho, \quad \text{where} \quad \rho \geq 2d \left(\frac{1}{r} - 1 \right). \quad (1.16)$$

Proposition 1.15 *Let E be a phase split basis for $\mathbf{R}^{2d}$, $\mathbf{p}, \mathbf{r} \in (0, \infty]^{2d}$, $r \in (0, \min(1, \mathbf{p})]$, $\omega, v \in \mathcal{P}_E(\mathbf{R}^{2d})$ be such that ω is v -moderate, ρ and $\Theta_\rho v$ ρ be as in (1.16) with strict inequality when $r < 1$, and let $\phi_1, \phi_2 \in M_{(\Theta_\rho v)}^1(\mathbf{R}^d) \setminus 0$. Then*

$$\|f\|_{M_{E,(\omega)}^{\mathbf{p}}} \asymp \|V_{\phi_1} f\|_{L_{E,(\omega)}^{\mathbf{p}}} \asymp \|V_{\phi_2} f\|_{W_E^{\mathbf{r}}(\omega, \ell_E^{\mathbf{p}})}, \quad f \in \mathcal{S}'_{1/2}(\mathbf{R}^d).$$

In particular, if $f \in \mathcal{S}'_{1/2}(\mathbf{R}^d)$, then

$$f \in M_{E,(\omega)}^{\mathbf{p}}(\mathbf{R}^{2d}) \Leftrightarrow V_{\phi_1} f \in L_{E,(\omega)}^{\mathbf{p}}(\mathbf{R}^{2d}) \Leftrightarrow V_{\phi_2} f \in W_E^{\mathbf{r}}(\omega, \ell_E^{\mathbf{p}}(\Lambda_E)).$$

1.5 Classes of Periodic Elements

Let $s, \sigma \in \mathbf{R}_+$ be such that $s + \sigma \geq 1$, $f \in (\mathcal{S}_s^\sigma)'(\mathbf{R}^d)$, E be a basis of $\mathbf{R}^d$ and let $E_0 \subseteq E$. Then f is called E_0 -periodic if $f(x + y) = f(x)$ for every $x \in \mathbf{R}^d$ and $y \in E_0$.

We note that for any E -periodic function $f \in C^\infty(\mathbf{R}^d)$, we have

$$f = \sum_{\alpha \in \Lambda'_E} c(f, \alpha) e^{i(\cdot, \alpha)}, \quad (1.17)$$

where $c(f, \alpha)$ are the Fourier coefficients given by

$$c(f, \alpha) \equiv |Q_E|^{-1} (f, e^{i(\cdot, \alpha)})_{L^2(E)}, \quad \alpha \in \Lambda'_E.$$

For any $\sigma \geq 0$ and basis $E \subseteq \mathbf{R}^d$ we let $\mathcal{E}_E^{\sigma;0}(\mathbf{R}^d)$ and $\mathcal{E}_E^\sigma(\mathbf{R}^d)$ be the sets of all E -periodic elements in $\mathcal{E}^{\sigma;0}(\mathbf{R}^d)$ and in $\mathcal{E}^\sigma(\mathbf{R}^d)$, respectively.

Let $s, s_0, \sigma, \sigma_0 > 0$ be such that $s + \sigma \geq 1, s_0 + \sigma_0 \geq 1$ and $(s_0, \sigma_0) \neq (\frac{1}{2}, \frac{1}{2})$. Then we recall that the duals $(\mathcal{E}_E^\sigma)'(\mathbf{R}^d)$ and $(\mathcal{E}_E^{\sigma_0;0})'(\mathbf{R}^d)$ of $\mathcal{E}_E^\sigma(\mathbf{R}^d)$ and $\mathcal{E}_E^{\sigma_0;0}(\mathbf{R}^d)$, respectively, can be identified with the E -periodic elements in $(\mathcal{S}_s^\sigma)'(\mathbf{R}^d)$ and $(\Sigma_{s_0}^{\sigma_0})'(\mathbf{R}^d)$ respectively via unique extensions of the form

$$(f, \phi)_E = \sum_{\alpha \in \Lambda'_E} c(f, \alpha) \overline{c(\phi, \alpha)}$$

on $\mathcal{E}_E^{\sigma_0;0}(\mathbf{R}^d) \times \mathcal{E}_E^{\sigma;0}(\mathbf{R}^d)$. We also let $(\mathcal{E}_E^0)'(\mathbf{R}^d)$ be the set of all formal expansions in (1.17) and $\mathcal{E}_E^0(\mathbf{R}^d)$ be the set of all formal expansions in (1.17) such that at most finite numbers of $c(f, \alpha)$ are non-zero (cf. [40]). We refer to [25,40] for more characterizations of $\mathcal{E}_E^\sigma, \mathcal{E}_E^{\sigma;0}$ and their duals.

The following definition takes care of spaces of formal expansions (1.17) with coefficients obeying specific quasi-norm estimates.

Definition 1.16 Let E be a basis of $\mathbf{R}^d$, $\mathcal{B}$ be a quasi-Banach space continuously embedded in $\ell'_0(\Lambda'_E)$ and let ω_0 be a weight on $\mathbf{R}^d$. Then $\mathcal{E}_E(\omega_0, \mathcal{B})$ consists of all $f \in (\mathcal{E}_E^0)'(\mathbf{R}^d)$ such that

$$\|f\|_{\mathcal{E}_E(\omega_0, \mathcal{B})} \equiv \|\{c(f, \alpha)\omega_0(\alpha)\}_{\alpha \in \Lambda'_E}\|_{\mathcal{B}}$$

is finite.

The next result is a reformulation of Proposition 3' in [39]. The proof is therefore omitted.

Proposition 1.17 Let E be an ordered basis of $\mathbf{R}^d$, $E = E_0 \times E'_0$, $\mathbf{q}, \mathbf{r} \in (0, \infty]^d$, $\omega_0 \in \mathcal{P}_E(\mathbf{R}^d)$ and let $\omega(x, \xi) = \omega_0(\xi)$, $x, \xi \in \mathbf{R}^d$. Then

$$\mathcal{E}_E(\omega_0, \ell_{E'_0}^{\mathbf{q}}(\Lambda_{E'_0})) = M_{E,(\omega)}^{\infty,\mathbf{q}}(\mathbf{R}^d) \bigcap (\mathcal{E}_E^0)'(\mathbf{R}^d) = W_{E,(\omega)}^{\infty,\mathbf{q}}(\mathbf{R}^d) \bigcap (\mathcal{E}_E^0)'(\mathbf{R}^d).$$

Remark 1.18 Let E_0 , q , r and ω be the same as in Proposition 1.17. Also let $v_0 \in \mathcal{P}_E(\mathbf{R}^d)$ be such that ω_0 is v_0 -moderate, $v(x, \xi) = v_0(\xi)$, $\phi \in \Sigma_1(\mathbf{R}^d) \setminus 0$, $r_0 \leq \min(r)$, $f \in (\mathcal{E}_E^0)'(\mathbf{R}^d)$ with Fourier series expansion (1.17),

$$E_1 = E_0 \times E'_0 = \{e_1, \dots, e_{2d}\} \quad \text{and} \quad E_2 = \{e_{d+1}, \dots, e_{2d}, e_1, \dots, e_d\}.$$

Then it follows from Proposition 2.7 and Remark 2.8 in [39] that

$$\|V_\phi f \cdot \omega\|_{L_{E_1}^{r,q}(Q(E_0) \times \mathbf{R}^d)} \asymp \|V_\phi f \cdot \omega\|_{L_{E_2}^{q,r}(\mathbf{R}^d \times Q(E_0))} \asymp \|c(f, \cdot)\|_{\ell_{E'_0, (\omega_0)}^q}. \quad (1.18)$$

1.6 The Zak Transform

For any ordered basis E of $\mathbf{R}^d$ and $f \in \mathcal{S}(\mathbf{R}^d)$, the *Zak transform* is defined by

$$(Z_E f)(x, \xi) \equiv \sum_{j \in \Lambda_E} f(x - j) e^{i\langle j, \xi \rangle}. \quad (1.19)$$

Several properties for the Zak transform can be found in [18]. For example, by the definition it follows that Z_E is continuous from $\mathcal{S}(\mathbf{R}^d)$ to the set of all smooth functions on $\mathbf{R}^{2d}$ which are bounded together with all their derivatives. It is also clear that $Z_E f$ is quasi-periodic of order E . Here, if F is a function or an ultra-distribution, then F is called quasi-periodic of order E , when

$$F(x + k, \xi) = e^{i\langle k, \xi \rangle} F(x, \xi) \quad \text{and} \quad F(x, \xi + \kappa) = F(x, \xi), \quad (1.20)$$

$$k \in \Lambda_E, \quad \kappa \in \Lambda'_E.$$

By interpreting (1.19) as a Fourier series in the ξ variable, we regain $f(x)$ as the zero order Fourier coefficient, which is evaluated by

$$f(x) = (Z_E^{-1} F)(x) = |Q_{E'}|^{-1} \int_{Q_{E'}} F(x, \xi) d\xi. \quad (1.21)$$

For conveniency we set $Z_1 = Z_E$ when E is the standard basis of $\mathbf{R}^d$, and recall the following important mapping properties on $L^2(\mathbf{R}^d)$.

Proposition 1.19 *Let E be an ordered basis of $\mathbf{R}^d$. Then the operator Z_E is homeomorphic from $L^2(\mathbf{R}^d)$ to the set of all quasi-periodic elements of order E in $L_{loc}^2(\mathbf{R}^{2d})$, and*

$$\|Z_E f\|_{L^2(Q_{E \times E'})} = |Q_{E'}|^{\frac{1}{2}} \|f\|_{L^2}, \quad f \in L^2(\mathbf{R}^d). \quad (0.1)'$$

Proof Let T_E be as in Remark 1.4. By straight-forward computations it follows that

$$Z_E f(x, \xi) = (Z_1 f_E)(T_E^{-1} x, T_E^* \xi), \quad f_E = f \circ T_E. \quad (1.22)$$

The assertion now follows from (0.1), (1.22) and suitable changes of variables in the involved integrals. The details are left for the reader. $\square$

2 Zak Transform on Gelfand–Shilov Spaces, Lebesgue Spaces and Modulation Spaces

In this section we deduce characterizations of Lebesgue spaces, modulation spaces, and Gelfand–Shilov spaces and their distribution spaces in terms of suitable estimates of the Zak transforms of the involved elements. The characterizations on modulation spaces are related to results given in [32,33].

2.1 Spaces of Quasi-Periodic Functions and Distributions

Since quasi-periodic functions depend on the phase space variable $(x, \xi) \in \mathbf{R}^{2d}$, it is suitable that the Gevrey regularity with respect to $x \in \mathbf{R}^d$ for such functions might be different to the Gevrey regularity with respect to $\xi \in \mathbf{R}^d$. We therefore consider two parameters analogies of $\mathcal{E}^\sigma$ and $\mathcal{E}^{\sigma;0}$, where the parameter σ is replaced by the pair σ, s . More precisely, for any compact $K \subseteq \mathbf{R}^{2d}$ and $s, \sigma \geq 0$, $\mathcal{E}^{\sigma,s}(K)$ ($\mathcal{E}^{\sigma,s;0}(K)$) is the set of all $F \in C^\infty(K)$ such that

$$\sup_{\alpha, \beta \in \mathbf{N}^d} \sup_{x, \xi \in \mathbf{R}^d} \left(\frac{|\partial_x^\alpha \partial_\xi^\beta F(x, \xi)|}{h^{|\alpha+\beta|} \alpha!^\sigma \beta!^s} \right)$$

is finite for some $h > 0$ (for every $h > 0$). The two parameter Gevrey classes, $\mathcal{E}^{\sigma,s}(\mathbf{R}^d)$ and $\mathcal{E}^{\sigma,s;0}(\mathbf{R}^d)$, are the projective limits of $\mathcal{E}^{\sigma,s}(K_j)$ respective $\mathcal{E}^{\sigma,s;0}(K_j)$, when $\{K_j\}_{j \geq 1}$ is an exhaustion by compact sets of $\mathbf{R}^{2d}$. Furthermore we let

$$\begin{array}{lll} \mathcal{E}_{Z,E}^{\sigma,s;0}(\mathbf{R}^{2d}), & \mathcal{E}_{Z,E}^{\sigma,s}(\mathbf{R}^{2d}), & C_{Z,E}^\infty(\mathbf{R}^{2d}), \\ \mathcal{S}'_{Z,E}(\mathbf{R}^{2d}), & (\mathcal{E}_{Z,E}^{\sigma,s})'(\mathbf{R}^{2d}), & (\mathcal{E}_{Z,E}^{\sigma,s;0})'(\mathbf{R}^{2d}) \end{array}$$

be the set of all quasi-periodic elements in

$$\begin{array}{lll} \mathcal{E}^{\sigma,s;0}(\mathbf{R}^{2d}), & \mathcal{E}^{\sigma,s}(\mathbf{R}^{2d}), & C^\infty(\mathbf{R}^{2d}), \\ \mathcal{S}'(\mathbf{R}^{2d}), & (\mathcal{S}_{s,\sigma}^{\sigma,s})'(\mathbf{R}^{2d}), & (\Sigma_{s,\sigma}^{\sigma,s})'(\mathbf{R}^{2d}), \end{array}$$

respectively, with respect to the ordered basis E on $\mathbf{R}^d$. For conveniency we also set

$$\mathcal{E}_Z^{\sigma,s;0} = \mathcal{E}_{Z,E}^{\sigma,s;0} \quad \text{and} \quad \mathcal{E}_Z^{\sigma,s} = \mathcal{E}_{Z,E}^{\sigma,s},$$

when E is the standard basis of $\mathbf{R}^d$.

Next we introduce spaces of quasi-periodic functions and distributions which correspond to Lebesgue spaces and modulation spaces. We let $L_{Z,E}^p(\mathbf{R}^{2d})$ for $p \in (0, \infty]$

be the set of all quasi-periodic measurable functions F on $\mathbf{R}^{2d}$ with respect to the ordered basis E such that

$$\|F\|_{L_{Z,E}^p} \equiv |Q_{E'}|^{-\frac{1}{p}} \|F\|_{L^p(Q_{E \times E'})}$$

is finite. Evidently, we may identify $L_{Z,E}^p(\mathbf{R}^{2d})$ by $L^p(Q_{E \times E'})$, and the scalar product on $L_{Z,E}^2(\mathbf{R}^{2d})$ is given by

$$(F, G)_{Z,E} \equiv (F, G)_{L_{Z,E}^2(\mathbf{R}^{2d})} = |Q_{E'}|^{-1} (F, G)_{L^2(Q_{E \times E'})} \quad (2.1)$$

when $F, G \in L_{Z,E}^2(\mathbf{R}^{2d})$.

Let $\mathbf{p}, \mathbf{r} \in (0, \infty]^{2d}$, $\omega \in \mathcal{P}_E(\mathbf{R}^{4d})$ and $\Phi \in \Sigma_1(\mathbf{R}^{2d}) \setminus 0$ be fixed, and let E_0 be an ordered basis in $\mathbf{R}^d$. Then set

$$H_{1,F,\omega}(x, \xi, y, \eta) = |V_\Phi F(x, \xi, \eta, y) \omega(x, \xi, \eta, y)|, \quad (2.2)$$

and

$$H_{2,F,\omega,E,E_0,\mathbf{p}}(x, \xi) \equiv \|H_{1,F,\omega}(x, \xi, \cdot, \cdot)\|_{L_{E \times E_0'}^p(\mathbf{R}^{2d})}, \quad (2.3)$$

when F is a quasi-periodic Gelfand–Shilov distribution with respect to the ordered basis E . We let $W_{Z,E,E_0,(\omega)}^{r,\mathbf{p}}(\mathbf{R}^{2d})$ be the set of all quasi-periodic Gelfand–Shilov distributions F with respect to the ordered basis E such that

$$\|F\|_{W_{Z,E,E_0,(\omega)}^{r,\mathbf{p}}} \equiv \|H_{2,F,\omega,E,E_0,\mathbf{p}}\|_{L_{E \times E'}^r(Q_{E \times E'})} \quad (2.4)$$

is finite. We also let the topology of $W_{Z,E,E_0,(\omega)}^{r,\mathbf{p}}(\mathbf{R}^{2d})$ be induced by the quasi-norm $\|\cdot\|_{W_{Z,E,E_0,(\omega)}^{r,\mathbf{p}}}$. Usually we assume that ω is given by

$$\omega(x, \xi, \eta, y) = \omega_0(x - y, \eta), \quad (2.5)$$

for some $\omega_0 \in \mathcal{P}_E(\mathbf{R}^{2d})$.

2.2 The Zak Transform on Test Function Spaces and Their Distribution Spaces

For the classical spaces $\mathcal{S}(\mathbf{R}^d)$ and its distribution space $\mathcal{S}'(\mathbf{R}^d)$ we have the following.

Theorem 2.1 *Let E be an ordered basis of $\mathbf{R}^d$. Then the following is true:*

- (1) *The operator Z_E is a homeomorphism from $\mathcal{S}(\mathbf{R}^d)$ to $C_{Z,E}^\infty(\mathbf{R}^{2d})$;*
- (2) *The operator Z_E from $\mathcal{S}(\mathbf{R}^d)$ to $C^\infty(\mathbf{R}^{2d})$ is uniquely extendable to a homeomorphism from $\mathcal{S}'(\mathbf{R}^d)$ to $\mathcal{S}'_{Z,E}(\mathbf{R}^{2d})$.*

The assertion (1) in Theorem 2.1 follows from (1.22) and [18, Theorem 8.2.5], and (2) in the same theorem follows by similar arguments as in the proof of Theorem 2.5 below. The verifications of Theorem 2.1 are therefore left for the reader.

The analogous result of the previous theorem for Gelfand–Shilov functions and their distributions, are given in Theorems 2.2 and 2.5 below.

Theorem 2.2 *Let $s, \sigma > 0$ and E be an ordered basis. Then the operator Z_E from $\mathcal{S}(\mathbf{R}^d)$ to $C^\infty(\mathbf{R}^{2d})$ restricts to a homeomorphism from $S_s^\sigma(\mathbf{R}^d)$ to $\mathcal{E}_{Z,E}^{\sigma,s}(\mathbf{R}^{2d})$.*

The same holds true with $\Sigma_s^\sigma(\mathbf{R}^d)$ and $\mathcal{E}_{Z,E}^{\sigma,s;0}(\mathbf{R}^{2d})$ in place of $S_s^\sigma(\mathbf{R}^d)$ and $\mathcal{E}_{Z,E}^{\sigma,s}(\mathbf{R}^{2d})$, respectively at each occurrence.

By the previous result and the facts that $S_s^\sigma(\mathbf{R}^d)$ is trivially equal to $\{0\}$ when $s + \sigma < 1$, and $\Sigma_s^\sigma(\mathbf{R}^d)$ is trivial when $s + \sigma < 1$ or $s = \sigma = \frac{1}{2}$, we get the following.

Corollary 2.3 *Let $s, \sigma > 0$ and E be an ordered basis. Then the following is true:*

- (1) *if $s + \sigma < 1$ and $F \in \mathcal{E}_{Z,E}^{\sigma,s}(\mathbf{R}^{2d})$, then $F = 0$;*
- (2) *if $s + \sigma < 1$ or $s = \sigma = \frac{1}{2}$ and $F \in \mathcal{E}_{Z,E}^{\sigma,s;0}(\mathbf{R}^{2d})$, then $F = 0$.*

Remark 2.4 Let $\sigma \geq 0$, E be a basis of $\mathbf{R}^d$ and let f be an E -periodic distribution on $\mathbf{R}^d$. Then $f \in \mathcal{E}^\sigma(\mathbf{R}^d)$, if and only if its Fourier coefficients $c(f, \alpha)$ in (1.17) satisfies

$$|c(f, \alpha)| \lesssim e^{-r|\alpha|^{\frac{1}{\sigma}}}$$

for some $r > 0$. In particular, $\mathcal{E}_E^\sigma(\mathbf{R}^d) \neq \{0\}$ for every $\sigma \geq 0$ (cf. [25,40]). Consequently, the conclusions in Corollary 2.3 are not true for periodic functions in place of quasi-periodic functions.

In similar ways we may characterize Gelfand–Shilov distributions through their Zak transforms as in the following result.

Theorem 2.5 *Let $s, \sigma > 0$ be such that $s + \sigma \geq 1$ and E be an ordered basis of $\mathbf{R}^d$. Then the operator Z_E from $\mathcal{S}(\mathbf{R}^d)$ to $C^\infty(\mathbf{R}^{2d})$ extends uniquely to a homeomorphism from $(S_s^\sigma)'(\mathbf{R}^d)$ to $(\mathcal{E}_{Z,E}^{\sigma,s})'(\mathbf{R}^{2d})$.*

If in addition $(s, \sigma) \neq (\frac{1}{2}, \frac{1}{2})$, then the same holds true with $(\Sigma_s^\sigma)'(\mathbf{R}^d)$ and $(\mathcal{E}_{Z,E}^{\sigma,s;0})'(\mathbf{R}^{2d})$ in place of $(S_s^\sigma)'(\mathbf{R}^d)$ and $(\mathcal{E}_{Z,E}^{\sigma,s})'(\mathbf{R}^{2d})$, respectively at each occurrence.

We shall first prove Theorem 2.2 before proving Theorem 2.5. We need the following lemma for the proofs.

Lemma 2.6 *Let $r, s > 0$. Then there is a constant $h > 0$ such that*

$$|t|^\beta e^{-r|t|^{\frac{1}{s}}} \leq h^\beta \beta!^s, \quad t \in \mathbf{R}, \quad \beta \in \mathbf{N}, \quad h = \left(\frac{s}{r}\right)^s.$$

Lemma 2.6 is a straight-forward consequence of the inequality

$$\frac{t^\beta}{\beta!} \cdot e^{-t} \leq 1, \quad t \geq 0, \beta \in \mathbf{N},$$

due to the Taylor expansion of e^t . The details are left for the reader.

Proof of Theorem 2.2 Let T_E be the same as in (1.22). Then the map $F(x, \xi) \mapsto F(T_E^{-1}x, T_E^*\xi)$ maps quasi-periodic elements of order E to quasi-periodic elements with respect to the standard basis. Since $f \mapsto f \circ T$ maps E -periodic elements to 1-periodic functions, it follows from these observations and (1.22) that it suffices to prove the result when E is the standard basis.

We only prove the result in the Roumieu case, i.e. we prove that Z_E restricts to a homeomorphism from $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ to $\mathcal{E}_{Z,E}^{\sigma,s}(\mathbf{R}^{2d})$. The other case follows by similar arguments and is left for the reader.

We shall follow the proof of Theorem 8.2.5 in [18]. In fact, assume first that $f \in \mathcal{S}_s^\sigma(\mathbf{R}^d)$, $x \in k_0 + Q_{d,1}$ for some fixed $k_0 \in \mathbf{Z}^d$, and let $F = Z_1 f$. Then it follows by straight-forward applications of Proposition 1.1 that

$$|\partial_x^\alpha f(x - k)| \leq Ch^{|\alpha|} e^{-2r(|k_1|^{\frac{1}{s}} + \dots + |k_d|^{\frac{1}{s}})} \alpha!^\sigma, \quad x \in k_0 + Q_{d,1},$$

for some positive constant C which only depends on the positive constants h and r . In particular, C is independent of x , k_0 , k and α . (See also [23, Proposition 6.1.7].)

The series in (1.19) is absolutely convergent together with all its derivatives. This gives

$$\begin{aligned} |(\partial_x^\alpha \partial_\xi^\beta F)(x, \xi)| &\leq \sum_{k \in \mathbf{Z}^d} |f^{(\alpha)}(x - k)| |k^\beta| \\ &\lesssim h_1^{|\alpha+\beta|} \sum_{k \in \mathbf{Z}^d} e^{-2r(|k_1|^{\frac{1}{s}} + \dots + |k_d|^{\frac{1}{s}})} |k^\beta| \\ &= h_1^{|\alpha+\beta|} \prod_{j=1}^d \left(\sum_{k_j \in \mathbf{Z}} e^{-r|k_j|^{\frac{1}{s}}} \left(e^{-r|k_j|^{\frac{1}{s}}} |k_j|^{\beta_j} \right) \right), \end{aligned}$$

for some constant $h_1 > 0$. By Lemma 2.6 it follows that

$$e^{-r|k_j|^{\frac{1}{s}}} |k_j|^{\beta_j} \lesssim h^{\beta_j} \beta_j!^s$$

for some constant $h > 0$. A combination of these estimates gives

$$|(\partial_x^\alpha \partial_\xi^\beta F)(x, \xi)| \lesssim h^{|\alpha+\beta|} \alpha!^\sigma \beta!^s,$$

and it follows that $F \in \mathcal{E}^{\sigma,s}(\mathbf{R}^{2d})$. This shows that Z_1 is continuous from $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ to $\mathcal{E}_Z^{\sigma,s}(\mathbf{R}^{2d})$.

Next we show that any F in $\mathcal{E}_Z^{\sigma,s}(\mathbf{R}^{2d})$ is the Zak transform of an element in $\mathcal{S}_s^\sigma(\mathbf{R}^d)$. By Theorem 8.2.5 in [18] it follows that $F = Z_1 f$ when

$$f(x) = \int_{Q_{d,2\pi}} F(x, \xi) d\xi.$$

We need to prove that $f \in \mathcal{S}_s^\sigma(\mathbf{R}^d)$.

Since $k \mapsto f(x - k)$ is the Fourier coefficient of order k for the function $\xi \mapsto F(x, \xi)$, we have

$$f(x - k) = \int_{Q_{d,2\pi}} F(x, \xi) e^{-i\langle k, \xi \rangle} d\xi.$$

By applying the operator $k^\alpha \partial_x^\beta$ and integrating by parts we get

$$\begin{aligned} k^\alpha (\partial_x^\beta f)(x - k) &= \int_{Q_{d,2\pi}} (\partial_x^\beta F)(x, \xi) k^\alpha e^{-i\langle k, \xi \rangle} d\xi \\ &= (-1)^{|\beta|} i^{|\alpha|} \int_{Q_{d,2\pi}} (\partial_x^\beta \partial_\xi^\alpha F)(x, \xi) e^{-i\langle k, \xi \rangle} d\xi. \end{aligned}$$

This gives

$$\begin{aligned} \sup_{x \in \mathbf{R}^d} |x^\alpha f^{(\beta)}(x)| &\asymp \sup_{k \in \mathbf{Z}^d} \sup_{x \in Q_{d,\rho}} |k^\alpha f^{(\beta)}(x - k)| \\ &\lesssim \|\partial_x^\beta \partial_\xi^\alpha F\|_{L^\infty(Q_{d,\rho} \times Q_{d,2\pi})} \lesssim h^{|\alpha|+|\beta|} \alpha!^s \beta!^\sigma, \end{aligned}$$

which is the same as $f \in \mathcal{S}_s^\sigma(\mathbf{R}^d)$. $\square$

For the proof of Theorem 2.5 we need the following lemma on tensor products of Gelfand–Shilov distributions.

Lemma 2.7 *Let $s_j, \sigma_j > 0$ and $f_j \in (\mathcal{S}_{s_j}^{\sigma_j})'(\mathbf{R}^{d_j})$, $j = 1, 2$. Then there is a unique $f \in (\mathcal{S}_{s_1, s_2}^{\sigma_1, \sigma_2})'(\mathbf{R}^{d_1+d_2})$ such that*

$$\langle f, \varphi_1 \otimes \varphi_2 \rangle = \langle f_1, \varphi_1 \rangle \langle f_2, \varphi_2 \rangle, \quad \varphi_j \in \mathcal{S}_{s_j}^{\sigma_j}(\mathbf{R}^{d_j}), \quad j = 1, 2.$$

Moreover, if $\varphi \in \mathcal{S}_{s_1, s_2}^{\sigma_1, \sigma_2}(\mathbf{R}^{d_1+d_2})$,

$$\varphi_1(x_1) = \langle f_2, \varphi(x_1, \cdot) \rangle \quad \text{and} \quad \varphi_2(x_2) = \langle f_1, \varphi(\cdot, x_2) \rangle,$$

then

$$\langle f, \varphi \rangle = \langle f_1, \varphi_1 \rangle = \langle f_2, \varphi_2 \rangle.$$

The same holds true with $\Sigma_{s_j}^{\sigma_j}$, $\Sigma_{s_1, s_2}^{\sigma_1, \sigma_2}$, $(\Sigma_{s_j}^{\sigma_j})'$ and $(\Sigma_{s_1, s_2}^{\sigma_1, \sigma_2})'$ in place of $\mathcal{S}_{s_j}^{\sigma_j}$, $\mathcal{S}_{s_1, s_2}^{\sigma_1, \sigma_2}$, $(\mathcal{S}_{s_j}^{\sigma_j})'$ and $(\mathcal{S}_{s_1, s_2}^{\sigma_1, \sigma_2})'$, respectively, $j = 1, 2$.

Lemma 2.7 is essentially a restatement of Theorem 2.4 in [38]. The proof is therefore omitted.

Remark 2.8 We notice that the uniqueness assertions in Lemma 2.7 is an immediate consequence of [38, Lemma 2.3] which asserts that if $f \in (\mathcal{S}_{s_1, s_2}^{\sigma_1, \sigma_2})'(\mathbf{R}^{d_1+d_2})$ ($f \in (\Sigma_{s_1, s_2}^{\sigma_1, \sigma_2})'(\mathbf{R}^{d_1+d_2})$) satisfies

$$\langle f, \varphi_1 \otimes \varphi_2 \rangle = 0$$

for every $\varphi_j \in \mathcal{S}_{s_j}^{\sigma_j}(\mathbf{R}^{d_j})$ ($\varphi_j \in \Sigma_{s_j}^{\sigma_j}(\mathbf{R}^{d_j})$), then $f = 0$ (as an element in $(\mathcal{S}_{s_1, s_2}^{\sigma_1, \sigma_2})'(\mathbf{R}^{d_1+d_2})$ ($(\Sigma_{s_1, s_2}^{\sigma_1, \sigma_2})'(\mathbf{R}^{d_1+d_2})$)).

Proof of Theorem 2.5 By similar arguments as in the proof of Theorem 2.2 we may assume that E is the standard basis for $\mathbf{R}^d$.

Let $\Phi \in \mathcal{S}_{s, \sigma}^{\sigma}(\mathbf{R}^{2d})$ and $\Phi_2 = \mathcal{F}_2^{-1}\Phi$. Then

$$\begin{aligned} \Phi_2 &\in \mathcal{S}_s^{\sigma}(\mathbf{R}^{2d}), & \mathcal{F}_1\Phi_2 &\in \mathcal{S}_{\sigma, s}^{\sigma}(\mathbf{R}^{2d}), \\ |\Phi_2(x, y)| &\lesssim e^{-r_0(|x|^{\frac{1}{s}} + |y|^{\frac{1}{s}})} & \text{and} & |\mathcal{F}_1\Phi_2(\xi, y)| \lesssim e^{-r_0(|\xi|^{\frac{1}{\sigma}} + |y|^{\frac{1}{s}})}, \end{aligned} \quad (2.6)$$

for some $r_0 > 0$.

If $f \in \mathcal{S}'(\mathbf{R}^d)$, then

$$\begin{aligned} \langle Z_1 f, \Phi \rangle &= \iint_{\mathbf{R}^{2d}} \left(\sum_{j \in \mathbf{Z}^d} f(x - j) e^{i\langle j, \xi \rangle} \right) \Phi(x, \xi) dx d\xi \\ &= \sum_{j \in \mathbf{Z}^d} \left(\int_{\mathbf{R}^d} f(x) \Phi_2(x + j, j) dx \right) \\ &= \sum_{j \in \mathbf{Z}^d} (\check{f} * \Phi_2(\cdot, j))(j), \end{aligned} \quad (2.7)$$

where $\check{f}(x) = f(-x)$.

Assume instead that $f \in (\mathcal{S}_s^{\sigma})'(\mathbf{R}^d)$ is arbitrary. We claim that the series on the right-hand side of (2.7) converges absolutely for every Φ as above.

In fact, let $\phi \in \mathcal{S}_s^{\sigma}(\mathbf{R}^d)$ and $r > 0$. Since $f \in (\mathcal{S}_s^{\sigma})'(\mathbf{R}^d)$, we have

$$|\langle f, \phi \rangle| \lesssim \|e^{r|\cdot|^{\frac{1}{s}}} \phi\|_{L^\infty} + \|e^{r|\cdot|^{\frac{1}{\sigma}}} \widehat{\phi}\|_{L^\infty}.$$

giving that

$$|(\check{f} * \Phi_2(\cdot, y))(x)| \lesssim \|e^{r|\cdot|^{\frac{1}{s}}} \Phi_2(x - \cdot, y)\|_{L^\infty} + \|e^{r|\cdot|^{\frac{1}{\sigma}}} \widehat{\Phi}_2(\cdot, y)\|_{L^\infty}.$$

By (2.6) we obtain for some $r_0 > 0$ and $c \geq 1$ that

$$|e^{r|z|^{\frac{1}{s}}} \Phi_2(x - z, y)| \lesssim e^{r|z|^{\frac{1}{s}} - r_0(|x-z|^{\frac{1}{s}} + |y|^{\frac{1}{s}})} \lesssim e^{cr|x|^{\frac{1}{s}} - r_0|y|^{\frac{1}{s}}}$$

and

$$|e^{r|\xi|^{\frac{1}{s}}}(\mathcal{F}_1\Phi_2)(\xi, y)| \lesssim e^{r|\xi|^{\frac{1}{s}} - r_0(|\xi|^{\frac{1}{\sigma}} + |y|^{\frac{1}{s}})} \lesssim e^{-r_0|y|^{\frac{1}{s}}},$$

for some constant $c \geq 1$ which only depends on s , provided r is chosen strictly smaller than r_0/c . A combination of these estimates gives

$$|(\check{f} * \Phi_2(\cdot, y))(x)| \lesssim e^{cr|x|^{\frac{1}{s}} - r_0|y|^{\frac{1}{s}}/c}.$$

Hence, if r is chosen smaller than $r_0/(2c)$ and letting $x = y = j$, we obtain

$$|(\check{f} * \Phi_2(\cdot, j))(j)| \lesssim e^{-r_0|j|^{\frac{1}{s}}/2}, \quad (2.8)$$

when $f \in (\mathcal{S}_s^\sigma)'(\mathbf{R}^d)$. The absolutely convergence of the series of the right-hand side of (2.7) now follows from (2.8).

If $f \in (\mathcal{S}_s^\sigma)'(\mathbf{R}^d)$, then $Z_1 f$ is defined as the element in $(\mathcal{S}_{s,\sigma}^{\sigma,s})'(\mathbf{R}^{2d})$, given by the right-hand side of (2.7). The previous estimates show that this definition makes sense, and that the map $f \mapsto Z_1 f$ is continuous from $(\mathcal{S}_s^\sigma)'(\mathbf{R}^d)$ to $(\mathcal{E}_Z^{\sigma,s})'(\mathbf{R}^{2d})$. By approximating elements in $(\mathcal{S}_s^\sigma)'(\mathbf{R}^d)$ by sequences of elements in $\mathcal{S}'(\mathbf{R}^d)$, it also follows that the continuous extension of Z_1 to such distributions is unique.

We need to prove that any element in $(\mathcal{E}_Z^{\sigma,s})'(\mathbf{R}^{2d})$ is the Zak transform of an element in $(\mathcal{S}_s^\sigma)'(\mathbf{R}^d)$. Therefore, let $\varphi \in \mathcal{S}_s^\sigma(\mathbf{R}^d)$, $F \in (\mathcal{E}_Z^{\sigma,s})'(\mathbf{R}^{2d})$, and let $g_\varphi \in (\mathcal{S}_\sigma^s)'(\mathbf{R}^d)$ be defined by

$$\langle g_\varphi, \psi \rangle = \langle F, \varphi \otimes \psi \rangle, \quad \psi \in \mathcal{S}_\sigma^s(\mathbf{R}^d).$$

Then g_φ is 2π -periodic, and it follows from Remark 2.3 and Proposition 2.5 in [40] that if $\phi \in \mathcal{S}_\sigma^s(\mathbf{R}^d) \setminus 0$, then

$$g_\varphi = \sum_{k \in \mathbf{Z}^d} c(g_\varphi, k) e^{i\langle k, \xi \rangle},$$

where the series converges in $(\mathcal{S}_\sigma^s)'(\mathbf{R}^d)$, and

$$c(g_\varphi, 0) = \frac{1}{(2\pi)^d \|\phi\|_{L^2}^2} \int_{Q_{d,2\pi}} \left(\int_{\mathbf{R}^d} (V_\phi g_\varphi)(\eta, y) \widehat{\phi}(-y) e^{i\langle y, \eta \rangle} dy \right) d\eta$$

and

$$c(g_\varphi, k) = c(g_\varphi e^{-i\langle k, \cdot \rangle}, 0)$$

By straight-forward computations we get

$$c(g_\varphi, 0) = \|\phi\|_{L^2}^{-2} \int_{Q_{d,2\pi}} h_\varphi(\eta) d\eta, \quad (2.9)$$

where

$$h_\varphi(\eta) = (2\pi)^{-\frac{3d}{2}} \int_{\mathbf{R}^d} \langle F, \varphi \otimes (\overline{\phi(\cdot - \eta)} e^{-i\langle y, \cdot \rangle}) \rangle \widehat{\phi}(-y) e^{i\langle y, \eta \rangle} dy,$$

and it is clear that the map which takes φ into the right-hand side in (2.9) defines a continuous linear form on $\mathcal{S}_s^\sigma(\mathbf{R}^d)$. Hence

$$c(g_\varphi, 0) = \langle f, \varphi \rangle, \quad \varphi \in \mathcal{S}_s^\sigma(\mathbf{R}^d),$$

for some $f \in (\mathcal{S}_s^\sigma)'(\mathbf{R}^d)$. Furthermore, by the quasi-periodicity of F we obtain

$$\begin{aligned} \langle g_\varphi \cdot e^{-i\langle k, \cdot \rangle}, \psi \rangle &= \langle g_\varphi, \psi \cdot e^{-i\langle k, \cdot \rangle} \rangle = \langle F, \varphi \otimes (\psi e^{-i\langle k, \cdot \rangle}) \rangle \\ &= \langle F(\cdot - (k, 0)), \varphi \otimes \psi \rangle = \langle F, \varphi(\cdot + k) \otimes \psi \rangle = \langle g_{\varphi(\cdot + k)}, \psi \rangle, \end{aligned}$$

that is,

$$g_\varphi e^{-i\langle k, \cdot \rangle} = g_{\varphi(\cdot + k)}.$$

A combination of these facts now gives

$$\begin{aligned} c(g_\varphi, k) &= c(g_\varphi e^{-i\langle k, \cdot \rangle}, 0) \\ &= c(g_{\varphi(\cdot + k)}, 0) = \langle f, \varphi(\cdot + k) \rangle = \langle f(\cdot - k), \varphi \rangle, \end{aligned}$$

giving that

$$\begin{aligned} \langle Z_1 f, \varphi \otimes \psi \rangle &= \sum_{k \in \mathbf{Z}^d} \langle f(\cdot - k), \varphi \rangle \langle e^{i\langle k, \cdot \rangle}, \psi \rangle \\ &= \sum_{k \in \mathbf{Z}^d} c(g_\varphi, k) \langle e^{i\langle k, \cdot \rangle}, \psi \rangle = \langle g_\varphi, \psi \rangle = \langle F, \varphi \otimes \psi \rangle. \end{aligned}$$

Hence, if $F_0 = F - Z_1 f$, then $\langle F_0, \varphi \otimes \psi \rangle = 0$ when $\varphi \in \mathcal{S}_s^\sigma(\mathbf{R}^d)$ and $\psi \in \mathcal{S}_\sigma^s(\mathbf{R}^d)$. By Remark 2.8 it now follows that $F = Z_1 f$. This gives the result in the Roumieu case, i.e. we have proved that Z_E is homeomorphic between $(\mathcal{S}_s^\sigma)'(\mathbf{R}^d)$ and $(\mathcal{E}_{Z,E}^{\sigma,s})'(\mathbf{R}^{2d})$. By similar arguments it follows that Z_E is homeomorphic between $(\Sigma_s^\sigma)'(\mathbf{R}^d)$ and $(\mathcal{E}_{Z,E}^{\sigma,s;0})'(\mathbf{R}^{2d})$. The details are left for the reader. $\square$

For completeness we also show that all quasi-periodic distributions are tempered or Gelfand–Shilov distributions. (Cf. [21, Section 7.2].) Here $(\mathcal{D}^{\sigma,s})'(\mathbf{R}^{2d})$ and $(\mathcal{D}^{\sigma,s;0})'(\mathbf{R}^{2d})$ are the duals of $\mathcal{E}^{\sigma,s}(\mathbf{R}^{2d}) \cap C_0^\infty(\mathbf{R}^{2d})$ and $\mathcal{E}^{\sigma,s;0}(\mathbf{R}^{2d}) \cap C_0^\infty(\mathbf{R}^{2d})$, respectively.

Proposition 2.9 *Let $s, \sigma > 1$ and E be an ordered basis of $\mathbf{R}^d$. Then the following is true:*

- (1) *The set of all quasi-periodic elements of order E in $\mathcal{D}'(\mathbf{R}^{2d})$ is equal to $\mathcal{S}'_{Z,E}(\mathbf{R}^{2d})$;*
- (2) *The set of all quasi-periodic elements of order E in $(\mathcal{D}^{\sigma,s})'(\mathbf{R}^{2d})$ is equal to $(\mathcal{E}^{\sigma,s}_{Z,E})'(\mathbf{R}^{2d})$;*
- (3) *The set of all quasi-periodic elements of order E in $(\mathcal{D}^{\sigma,s;0})'(\mathbf{R}^{2d})$ is equal to $(\mathcal{E}^{\sigma,s;0}_{Z,E})'(\mathbf{R}^{2d})$.*

Proof We only prove (2). The other assertions follow by similar arguments and are left for the reader.

Let $F \in C^\infty_{Z,E}(\mathbf{R}^{2d})$, $\Phi \in \mathcal{S}^{\sigma,s}_{s,\sigma}(\mathbf{R}^{2d})$ and let $\chi \in C^\infty_0(\mathbf{R}^{2d}) \cap \mathcal{E}^{\sigma,s}(\mathbf{R}^{2d})$ be such that

$$\sum_{k \in \Lambda_E} \sum_{\kappa \in \Lambda'_E} \chi(x + k, \xi + \kappa) = 1.$$

If $\Phi \in \mathcal{S}^{\sigma,s}_{s,\sigma}(\mathbf{R}^{2d})$, then it follows by the quasi-periodicity of F and some straightforward computations that

$$\langle F, \Phi \rangle = \langle F, T_\chi \Phi \rangle, \quad (2.10)$$

where

$$(T_\chi \Phi)(x, \xi) = \sum_{k \in \Lambda_E} \sum_{\kappa \in \Lambda'_E} e^{-i\langle k, \xi \rangle} \Phi(x - k, \xi - \kappa) \chi(x, \xi), \quad (2.11)$$

and that T_χ in (2.11) is continuous from $\mathcal{S}^{\sigma,s}_{s,\sigma}(\mathbf{R}^{2d})$ to $C^\infty_0(\mathbf{R}^{2d}) \cap \mathcal{E}^{\sigma,s}(\mathbf{R}^{2d})$.

By letting $\langle F, \Phi \rangle$ be defined by the right-hand side of (2.10) when $F \in (\mathcal{D}^{\sigma,s})'(\mathbf{R}^{2d})$ and $\Phi \in \mathcal{S}^{\sigma,s}_{s,\sigma}(\mathbf{R}^{2d})$, it follows that $\Phi \mapsto \langle F, T_\chi \Phi \rangle$ in (2.10) defines a linear and continuous form on $\mathcal{S}^{\sigma,s}_{s,\sigma}(\mathbf{R}^{2d})$ which agrees with the usual distribution action, $\Phi \mapsto \langle F, \Phi \rangle$ when $\Phi \in C^\infty_0(\mathbf{R}^{2d}) \cap \mathcal{E}^{\sigma,s}(\mathbf{R}^{2d})$. $\square$

The mapping properties of Gelfand–Shilov distributions also lead to some queries concerning the inversion formula (1.21) for the Zak transform. Evidently, if F is a general quasi-periodic distribution or even Gelfand–Shilov distribution, then the integral on the right-hand side of (1.21) might not be defined. On the other hand, since $Z_E^{-1}F(x)$ is the zero order Fourier coefficient of the expansion (1.19), it follows from [40, Remark 2.3] that the following is true. The details are left for the reader.

Proposition 2.10 *Let $s, \sigma > 0$ and $\phi \in \mathcal{S}^\sigma_s(\mathbf{R}^d) \setminus 0$ ($\phi \in \Sigma^\sigma_s(\mathbf{R}^d) \setminus 0$) be fixed, $f \in (\mathcal{S}^\sigma_s)'(\mathbf{R}^d)$ ($f \in (\Sigma^\sigma_s)'(\mathbf{R}^d)$), and let $F = Z_E f$. Then*

$$f(x) = (Z_E^{-1}F)(x) = (\|\phi\|_{L^2}^2 |\mathcal{Q}_{E'}|)^{-1} \int_{\mathcal{Q}_{E'}} \left(\int_{\mathbf{R}^d} (V_\phi F(x, \cdot))(\xi, y) \widehat{\phi}(-y) e^{i\langle y, \xi \rangle} dy \right) d\xi. \quad (1.21)'$$

2.3 The Zak Transform on Lebesgue and Modulation Spaces

For completeness we begin the subsection by making a review of the Zak transform when acting on Lebesgue spaces. Here we let

$$\theta_1(p) = \min\left(0, 1 - \frac{2}{p}\right) \quad \text{and} \quad \theta_2(p) = \max\left(0, 1 - \frac{2}{p}\right), \quad p \in (0, \infty].$$

Proposition 2.11 *Let E be an ordered basis of $\mathbf{R}^d$. Then the following is true:*

- (1) *if $p \in (0, 2]$, then Z_E from $\mathcal{S}(\mathbf{R}^d)$ to $C_{Z,E}^\infty(\mathbf{R}^{2d})$ is uniquely extendable to a continuous map from $L^p(\mathbf{R}^d)$ to $L_{Z,E}^p(\mathbf{R}^{2d})$, and*

$$\|Z_E f\|_{L_{Z,E}^p(\mathbf{R}^{2d})} \leq \|f\|_{L^p(\mathbf{R}^d)}, \quad f \in L^p(\mathbf{R}^d);$$

- (2) *if $p \in [2, \infty]$, then Z_E^{-1} from $C_{Z,E}^\infty(\mathbf{R}^{2d})$ to $\mathcal{S}(\mathbf{R}^d)$ is uniquely extendable to a continuous map from $L_{Z,E}^p(\mathbf{R}^{2d})$ to $L^p(\mathbf{R}^d)$, and*

$$\|Z_E^{-1} F\|_{L^p(\mathbf{R}^d)} \leq \|F\|_{L_{Z,E}^p(\mathbf{R}^{2d})}, \quad F \in L_{Z,E}^p(\mathbf{R}^{2d}).$$

- (3) *if $f \in \mathcal{S}'(\mathbf{R}^d)$, $p \in [1, \infty]$ and $v \in \mathcal{P}_E(\mathbf{R}^d)$ satisfy $1/v \in L^1(\mathbf{R}^d)$, then*

$$\|f \cdot v^{\theta_1(p)}\|_{L^p(\mathbf{R}^d)} \lesssim \|Z_E f\|_{L_{Z,E}^p(\mathbf{R}^{2d})} \lesssim \|f \cdot v^{\theta_2(p)}\|_{L^p(\mathbf{R}^d)}.$$

Proposition 2.11 (1) and (2) are evidently true for $p = 2$, in view of Proposition 1.19, and is presented in [32, Lemma 3.1.2], without any proof in the case $p = 1$. In order to be self-contained, we give a proof in Appendix A. We also observe that by choosing $p = 2$ in Proposition 2.11 (1) and (2) we regain Proposition 1.19.

When investigation mapping properties of the Zak transform on modulation spaces, we need to deduce various kinds of estimates on short-time Fourier transforms and partial short-time Fourier transforms of Zak transforms. Especially we search suitable estimates on $V_\Phi(Z_E f)$, and on

$$(ZV_{E,\phi_1}^{(1)} f)(x, \xi, \eta) \equiv (V_{\phi_1}(Z_E f(\cdot, \xi)))(x, \eta) \quad (2.12)$$

and

$$(ZV_{E,\phi_2}^{(2)} f)(x, \xi, y) \equiv (V_{\phi_2}(Z_E f(x, \cdot)))(\xi, y), \quad (2.13)$$

which are compositions of the Zak transform and the partial short-time Fourier transforms with respect to the first and second variable, respectively.

From the previous subsection it is clear that there is a one-to-one correspondence between quasi-periodic functions and distributions, and Zak transforms of functions

and distributions. For a quasi-periodic function or distribution F on $\mathbf{R}^{2d}$ which satisfies (1.20), and a suitable function or distribution Φ on $\mathbf{R}^{2d}$, we have

$$\begin{aligned}(V_\Phi F)(x+k, \xi, \eta, y) &= e^{-i\langle k, \eta \rangle} (V_\Phi F)(x, \xi, \eta, y-k), & k \in \Lambda_E, \\ (V_\Phi F)(x, \xi+\kappa, \eta, y) &= e^{-i\langle y, \kappa \rangle} (V_\Phi F)(x, \xi, \eta, y), & \kappa \in \Lambda'_E,\end{aligned}\quad (2.14)$$

which follows by straight-forward computations. We remark that functions and distributions which satisfy conditions given in (2.14) are examples of so-called *echo-periodic functions and distributions*, considered in [37].

First we have the following result concerning identifying Lebesgue spaces via estimates of corresponding Zak transforms.

Theorem 2.12 *Let E be an ordered basis of $\mathbf{R}^d$, $p, r \in (0, \infty]$, $\omega \in \mathcal{P}_E(\mathbf{R}^d)$, $\phi \in \Sigma_1(\mathbf{R}^d) \setminus 0$, and let f be a Gelfand–Shilov distribution on $\mathbf{R}^d$. Then*

$$\|f\|_{L^p_{(\omega)}} \asymp \|G_{E,r,\omega,f}\|_{L^p(Q_E \times \mathbf{R}^d)}, \quad (2.15)$$

where

$$G_{E,r,\omega,f}(x, y) \equiv \|(ZV_{E,\phi}^{(2)} f)(x, \cdot, y)\omega(-y)\|_{L^r(Q_{E'})}. \quad (2.16)$$

In particular,

$$\|f\|_{L^p} \asymp \|ZV_{E,\phi}^{(2)} f\|_{L^p(Q_{E \times E'} \times \mathbf{R}^d)}. \quad (2.17)$$

Proof We only prove the result for $p < \infty$. The case $p = \infty$ follows by similar arguments and is left for the reader.

The distribution $\xi \mapsto Z_E f(x, \xi)$ is E' -periodic, and it follows from (1.18) that

$$\begin{aligned}\left(\sum_{j \in \Lambda_E} |f(x-j)\omega(x-j)|^p\right)^{\frac{1}{p}} &\asymp \left(\sum_{j \in \Lambda_E} |f(x-j)\omega(-j)|^p\right)^{\frac{1}{p}} \\ &\asymp \|G_{E,r,\omega,f}(x, \cdot)\|_{L^p(\mathbf{R}^d)}, \quad x \in Q_E.\end{aligned}$$

The result now follows by applying the $L^p(Q_E)$ quasi-norm with respect to the x -variable. $\square$

In the same way we may identify modulation spaces by using the Zak transform as in the next result. Here recall Definition 1.10, Remark 1.13 and (2.2)–(2.4) for definitions of the Wiener amalgam spaces $W_{E,(\omega)}^p(\mathbf{R}^d)$ and $W_{Z,E,E_0,(\omega)}^{r,p}(\mathbf{R}^{2d})$.

Theorem 2.13 *Let E, E_0 be an ordered bases of $\mathbf{R}^d$, $p, r \in (0, \infty]^{2d}$, $\omega_0 \in \mathcal{P}_E(\mathbf{R}^{2d})$ and $\omega \in \mathcal{P}_E(\mathbf{R}^{4d})$ be such that (2.5) holds. Then Z_E from $\Sigma_1(\mathbf{R}^d)$ to $C^\infty(\mathbf{R}^{2d})$ is*

uniquely extendable to a homeomorphism from $M_{E \times E'_0, (\omega_0)}^p(\mathbf{R}^d)$ to $W_{Z, E, E_0, (\omega)}^{r, p}(\mathbf{R}^{2d})$, and

$$\|f\|_{M_{E \times E'_0, (\omega_0)}^p} \asymp \|Z_E f\|_{W_{Z, E, E_0, (\omega)}^{r, p}}, \quad f \in \Sigma'_1(\mathbf{R}^d). \quad (2.18)$$

Proof First we prove the result for $r = \infty$. Let $\Phi = \phi_1 \otimes \phi_2$ with $\phi_1, \phi_2 \in \Sigma_1(\mathbf{R}^d) \setminus 0$. By straight-forward computations we get

$$(\mathbf{ZV}_{1, \phi_1}^{(1)} f)(x, \xi, \eta) = \sum_{j \in \Lambda_E} ((V_{\phi_1} f)(x - j, \eta) e^{-i \langle j, \eta \rangle}) e^{i \langle j, \xi \rangle}. \quad (2.19)$$

Let

$$\mathbf{p}_1 = (p_1, \dots, p_d) \quad \text{and} \quad \mathbf{p}_2 = (p_{d+1}, \dots, p_{2d})$$

when

$$\mathbf{p} = (p_1, \dots, p_{2d}),$$

and consider the functions

$$F(x, \eta) = \|(V_{\phi_1} f)(x - \cdot, \eta) \omega_0(x - \cdot, \eta)\|_{\ell_E^{p_1}(\Lambda_E)},$$

$$g_0(x) = \|F(x, \cdot)\|_{L_{E'_0}^{p_2}(\mathbf{R}^d)},$$

$$G_0(x, \xi, \eta, y) \equiv |V_{\Phi}(Z_E f)(x, \xi, \eta, y) \omega_0(x - y, \eta)|,$$

$$G(x, \xi, \eta) \equiv \|G_0(x, \xi, \eta, \cdot)\|_{L_E^{p_1}(\mathbf{R}^d)},$$

$$H(x, \xi) = \|G(x, \xi, \cdot)\|_{L_{E'_0}^{p_2}(\mathbf{R}^d)},$$

and

$$h_0(x) = h_{0, r_0}(x) = \|H(x, \cdot)\|_{L_{E'_0}^{r_0}(Q_{E'_0})}, \quad r_0 \in (0, \infty].$$

Since $\xi \mapsto (\mathbf{ZV}_{1, \phi_1}^{(1)} f)(x, \xi, \eta)$ is E' -periodic with Fourier coefficients

$$j \mapsto (V_{\phi_1} f)(x - j, \eta) e^{-i \langle j, \eta \rangle}$$

(cf. (2.19)), and the (partial) short-time Fourier transform of that distribution equals $V_{\Phi}(Z_1 f)$, it follows from (1.18) that

$$F(x, \eta) \asymp \|G(x, \cdot, \eta)\|_{L_{E'}^r(Q_{E'})}, \quad r \in (0, \infty]. \quad (2.20)$$

Let $r_0 \leq \min(\mathbf{p})$. If we apply the $L_{E'_0}^{p_2}$ norm on (2.20) with respect to the η variable and using Hölder's inequality we get

$$\begin{aligned} g_0(x) &= \|F(x, \cdot)\|_{L_{E'_0}^{p_2}(\mathbf{R}^d)} \asymp \|G(x, \cdot)\|_{L_{E' \times E'_0}^{r_0, p_2}(Q_{E'} \times \mathbf{R}^d)} \\ &\lesssim \|H(x, \cdot)\|_{L_{E'}^{r_0}(Q_{E'})} = h_{0, r_0}(x). \end{aligned} \quad (2.21)$$

If

$$g_1(x) = \|F_1(x, \cdot)\|_{\ell_{E'_0}^{p_2}} \quad \text{with} \quad F_1(x, \iota) = \|F(x, \iota)\|_{L_{E'_0}^{r_0}(\iota + Q_{E'_0})},$$

then the fact that $r_0 \leq \min(\mathbf{p})$ and Jensen's inequality give $g_1 \lesssim g_0$.

By applying the $L_E^{r_0}(Q_E)$ norm on the latter inequality, using the fact that

$$\omega_0(x - y, \eta) \asymp \omega_0(-y, \eta), \quad x \in Q_E$$

and Jensen's inequality again we obtain

$$\|a_{1, f}\|_{\ell_{E_1}^p} \lesssim \|g_0\|_{L^{r_0}(Q_E)}, \quad \text{where} \quad a_{1, f}(j, \iota) = \|V_{\phi_1} f \cdot \omega_0\|_{L_{E_1}^{r_0}((j, \iota) + Q_{E_1})}.$$

Here $E_1 = E \times E'_0$. That is,

$$\|V_{\phi_1} f\|_{W_{E_1}^{r_0}(\omega_0, \ell_{E_1}^p(\Lambda_{E_1}))} \lesssim \|g_0\|_{L^{r_0}(Q_E)},$$

which is the same as

$$\|f\|_{M_{E \times E'_0(\omega_0)}^p} \lesssim \|g_0\|_{L^{r_0}(Q_E)} \quad (2.22)$$

in view of Proposition 1.15.

In order to estimate h_{0, r_0} we apply (2.14) to get

$$\begin{aligned} &|V_{\Phi}(Z_E f)(x + j, \xi + \iota, \eta, y)\omega_0(x + j - y, \eta)| \\ &= |V_{\Phi}(Z_E f)(x, \xi, \eta, y - j)\omega_0(x - y + j, \eta)|, \quad (j, \iota) \in \Lambda_{E \times E'}. \end{aligned}$$

By first applying the $L_E^{p_1}(\mathbf{R}^d)$ norm with respect to the y variable and then the $L_{E'_0}^{p_2}(\mathbf{R}^d)$ norm with respect to the η variable we get

$$H(x + j, \xi + \iota) = H(x, \xi), \quad (j, \iota) \in \Lambda_{E \times E'}.$$

Hence, by applying the $L_E^{r_0}(Q_E)$ norm on h_{0,r_0} and using Hölder's and Jensen's inequalities we get

$$\begin{aligned} \|h_{0,r_0}\|_{L_E^{r_0}(Q_E)} &= \|H\|_{L_{E \times E'}^{r_0}(Q_{E \times E'})} \lesssim \|H\|_{L_{E \times E'}^\infty(Q_{E \times E'})} \\ &= \sup_{(j,\iota) \in \Lambda_{E \times E'}} \left(\|H\|_{L_{E \times E'}^\infty((j,\iota)+Q_{E \times E'})} \right). \end{aligned} \quad (2.23)$$

A combination of (2.21)–(2.23) and Proposition 1.15 now gives

$$\|f\|_{M_{E \times E'_0,(\omega_0)}^p} \lesssim \|Z_E f\|_{W_{Z,E,E_0,(\omega)}^{\infty,p}}, \quad f \in \Sigma'_1(\mathbf{R}^d). \quad (2.24)$$

In order to get the reversed estimate we again apply the $L_{E'_0}^{p_2}$ norm on (2.20) with respect to the η variable and use Hölder's inequality to get

$$\begin{aligned} g_0(x) &= \|F(x, \cdot)\|_{L_{E'_0}^{p_2}(\mathbf{R}^d)} \asymp \|G(x, \cdot)\|_{L_{E' \times E'_0}^{\infty,p_2}(Q_{E'} \times \mathbf{R}^d)} \\ &\gtrsim \|H(x, \cdot)\|_{L_{E'}^\infty(Q_{E'})} = h_{0,\infty}(x). \end{aligned} \quad (2.25)$$

If

$$g_2(x) = \|F_2(x, \cdot)\|_{\ell_{E'_0}^{p_2}} \quad \text{with} \quad F_2(x, \iota) = \|F(x, \cdot)\|_{L_{E'_0}^\infty(\iota+Q_{E'_0})},$$

then Jensen's inequality give $g_0 \lesssim g_2$.

By applying the $L_E^\infty(Q_E)$ norm on the latter inequality and using Jensen's inequality again we obtain

$$\|a_{2,f}\|_{\ell_{E_1}^p} \gtrsim \|g_0\|_{L^\infty(Q_E)}, \quad \text{where} \quad a_{2,f}(j, \iota) = \|V_{\phi_1} f \cdot \omega_0\|_{L^\infty((j,\iota)+Q_{E_1})}.$$

That is,

$$\|V_{\phi_1} f\|_{W_{E_1}^\infty(\omega_0, \ell_{E_1}^p(\Lambda_{E_1}))} \gtrsim \|g_0\|_{L^{r_0}(Q_E)},$$

which is the same as

$$\|f\|_{M_{E \times E'_0,(\omega_0)}^p} \gtrsim \|g_0\|_{L^\infty(Q_E)} \quad (2.26)$$

in view of Proposition 1.15.

By applying the $L_E^\infty(Q_E)$ norm on $h_{0,\infty}$ and using (2.23) we get

$$\|h_{0,\infty}\|_{L_E^\infty(Q_E)} = \|H\|_{L_{E \times E'}^\infty(Q_{E \times E'})} \asymp \|Z_E f\|_{W_{Z,E,E_0,(\omega)}^{\infty,p}}, \quad (2.27)$$

where the last relation follows from Proposition 1.15. A combination of (2.25), (2.26) and (2.27) now gives

$$\|f\|_{M_{E \times E'_0, (\omega_0)}^p} \gtrsim \|Z_E f\|_{W_{Z, E, E_0, (\omega)}^{\infty, p}}, \quad f \in \Sigma'_1(\mathbf{R}^d), \quad (2.28)$$

and the result in the case $r = \infty$ follows by combining (2.28) with (2.24).

For general $r \in (0, \infty]^{2d}$, we notice that the echo-periodicity (2.14) implies that $H_{2, F, \omega, E, E_0, p}$ in (2.3) is $E \times E'$ periodic. The general case now follows from Proposition 1.15, the previous observation and that we have already proved the result for $r = \infty$. The details are left for the reader. $\square$

A consequence of the previous result is that $W_{Z, E, E_0, (\omega)}^{r, p}(\mathbf{R}^{2d})$ is independent of r (also in topological sense), and for this reason we set

$$W_{Z, E, E_0, (\omega)}^p = W_{Z, E, E_0, (\omega)}^{r, p}.$$

As a special case of the previous result we have the following.

Corollary 2.14 *Let E be an ordered basis in $\mathbf{R}^d$, $\Phi \in \Sigma_1(\mathbf{R}^{2d}) \setminus 0$ and let $p \in (0, \infty]$. Then*

$$\|f\|_{M^p} \asymp \|(V_\Phi(Z_E f))\|_{L^p(Q_{E \times E'} \times \mathbf{R}^{2d})}, \quad f \in \Sigma'_1(\mathbf{R}^d).$$

3 Duality Properties and Some Further Characterizations of Quasi-Periodic Elements

In this section we discuss various aspects concerning duality and characterizations for quasi-periodic elements, as well as transitions of linear operators under the Zak transform. In Sect. 3.1 we show that the elements in $\mathcal{E}_{Z, E}^{\sigma, s}$ and $(\mathcal{E}_{Z, E}^{\sigma, s})'$ can be completely characterized by estimates on their short-time Fourier transform. Thereafter we use these characterizations in Sect. 3.2 to show that the $L_{Z, E}^2$ form on $\mathcal{E}_{Z, E}^{\sigma, s}$ is uniquely extendable to a continuous sesqui-linear form on $(\mathcal{E}_{Z, E}^{\sigma, s})' \times \mathcal{E}_{Z, E}^{\sigma, s}$, and that the dual of $\mathcal{E}_{Z, E}^{\sigma, s}$ can be identified by $(\mathcal{E}_{Z, E}^{\sigma, s})'$ through this form. Finally, in Sect. 3.3 we discuss analogous duality issues for $W_{Z, E, E_0, (\omega)}^p$ spaces.

3.1 Characterizations of Quasi-Periodic Elements via Estimates on Their Short-Time Fourier Transform

The following results are analogous to Propositions 2.7 and 2.8 in [40] concerning characterizations of periodic elements in Gelfand–Shilov distribution spaces, and to Proposition 1.2.

Proposition 3.1 *Let $s, \sigma > 0$, E be an ordered basis of $\mathbf{R}^d$, F be a quasi-periodic Gelfand–Shilov distribution with respect to E and let $\phi \in \mathcal{S}_{s, \sigma}^{\sigma, s}(\mathbf{R}^{2d}) \setminus 0$ ($\phi \in \Sigma_{s, \sigma}^{\sigma, s}(\mathbf{R}^{2d}) \setminus 0$). Then the following conditions are equivalent:*

- (1) $F \in \mathcal{E}_{Z,E}^{\sigma,s}(\mathbf{R}^{2d})$ ($F \in \mathcal{E}_{Z,E}^{\sigma,s;0}(\mathbf{R}^{2d})$);
 (2) for some $r > 0$ (for every $r > 0$), it holds

$$|(V_\phi F)(x, \xi, \eta, y)| \lesssim e^{-r(|\eta|^{\frac{1}{\sigma}} + |y|^{\frac{1}{s}})}, \quad x \in Q_E, \xi, \eta, y \in \mathbf{R}^d; \quad (3.1)$$

- (3) for some $r > 0$ (for every $r > 0$), it holds

$$|(V_\phi F)(x, \xi, \eta, y)| \lesssim e^{-r(|\eta|^{\frac{1}{\sigma}} + |x-y|^{\frac{1}{s}})}, \quad x, \xi, \eta, y \in \mathbf{R}^d. \quad (3.2)$$

Proposition 3.2 *Let $s, \sigma > 0$, E be an ordered basis of $\mathbf{R}^d$, F be a quasi-periodic Gelfand–Shilov distribution with respect to E and let $\phi \in \mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d}) \setminus \{0\}$ ($\phi \in \Sigma_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d}) \setminus \{0\}$). Then the following conditions are equivalent:*

- (1) $F \in (\mathcal{E}_{Z,E}^{\sigma,s})'(\mathbf{R}^{2d})$ ($F \in (\mathcal{E}_{Z,E}^{\sigma,s;0})'(\mathbf{R}^{2d})$);
 (2) for every $r > 0$ (for some $r > 0$), it holds

$$|(V_\phi F)(x, \xi, \eta, y)| \lesssim e^{r(|\eta|^{\frac{1}{\sigma}} + |y|^{\frac{1}{s}})}, \quad x \in Q_E, \xi, \eta, y \in \mathbf{R}^d; \quad (3.3)$$

- (3) for every $r > 0$ (for some $r > 0$), it holds

$$|(V_\phi F)(x, \xi, \eta, y)| \lesssim e^{r(|\eta|^{\frac{1}{\sigma}} + |x-y|^{\frac{1}{s}})}, \quad x, \xi, \eta, y \in \mathbf{R}^d. \quad (3.4)$$

We only prove Proposition 3.2, and then only in the Roumieu case. The Beurling case of Proposition 3.2 as well as Proposition 3.1 follow by similar arguments and are left for the reader.

Proof Since it is obvious that (3) implies (2), the result follows if we prove that (2) implies (1) and (1) implies (3).

Suppose (3.3) holds for every $r > 0$, and let $k = k_x \in \Lambda_E$ and $\kappa = \kappa_\xi \in \Lambda'_E$ be chosen such that $x_0 = x - k \in Q_E$ and $\xi - \kappa \in Q_{E'}$, when $x, \xi \in \mathbf{R}^d$ are given. Then (2.14) and (3.3) give

$$\begin{aligned} |V_\phi F(x, \xi, \eta, y)| &= |V_\phi F(x_0 + k, \xi_0 + \kappa, \eta, y)| = |V_\phi F(x_0, \xi_0, \eta, y - k)| \\ &\lesssim e^{r(|\eta|^{\frac{1}{\sigma}} + |y-k|^{\frac{1}{s}})} \lesssim e^{Cr(|k|^{\frac{1}{s}} + |\eta|^{\frac{1}{\sigma}} + |y|^{\frac{1}{s}})} \\ &\asymp e^{Cr(|x|^{\frac{1}{s}} + |\eta|^{\frac{1}{\sigma}} + |y|^{\frac{1}{s}})} \leq e^{Cr(|x|^{\frac{1}{s}} + |\xi|^{\frac{1}{\sigma}} + |\eta|^{\frac{1}{\sigma}} + |y|^{\frac{1}{s}})}, \end{aligned}$$

for every $r > 0$. By Proposition 1.2, it now follows that $F \in (\mathcal{E}_{Z,E}^{\sigma,s})'(\mathbf{R}^{2d})$, and we have proved that (2) implies (1).

It remains to prove that (1) implies (3). Suppose that (1) holds, let $x, \xi, \eta, y \in \mathbf{R}^d$ be arbitrary, and choose $x_0 \in Q_E, \xi_0 \in Q_{E'}, k \in \Lambda_E$ and $\kappa \in \Lambda'_E$ as before. Then (1) and Proposition 1.2 give

$$\begin{aligned}
|(V_\phi F)(x, \xi, \eta, y)| &= |(V_\phi F)(x_0 + k, \xi_0 + \kappa, \eta, y)| = |(V_\phi F)(x_0, \xi_0, \eta, y - k)| \\
&\lesssim e^{r(|x_0|^{\frac{1}{s}} + |\xi_0|^{\frac{1}{\sigma}} + |\eta|^{\frac{1}{\sigma}} + |y - k|^{\frac{1}{s}})} \asymp e^{r(|\eta|^{\frac{1}{\sigma}} + |x - y|^{\frac{1}{s}})},
\end{aligned}$$

and we have proved that (1) implies (3). $\square$

3.2 Duality Properties of Gevrey Type Quasi-Period Elements

We shall next use the previous characterizations in Propositions 3.1 and 3.2 to show that the form (2.1) can be written as

$$(F, G)_{Z,E} \equiv (|Q_{E'}| \|\phi\|_{L^2}^2)^{-1} \cdot (V_\phi F, V_\phi G)_{L^2(\Omega_E)}, \quad (3.5)$$

when $F, G \in L^2_{Z,E}(\mathbf{R}^{2d})$. Here $\phi \in \mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d}) \setminus 0$ is fixed and

$$\Omega_E = Q_{E \times E'} \times \mathbf{R}^{2d} \subseteq \mathbf{R}^{4d} \quad (3.6)$$

when E is an ordered basis of $\mathbf{R}^d$. We use this identity to extend the definition of $(F, G)_{Z,E}$ to permit

$$F \in \mathcal{E}_{Z,E}^{\sigma,s}(\mathbf{R}^{2d}) \quad \text{and} \quad G \in (\mathcal{E}_{Z,E}^{\sigma,s})'(\mathbf{R}^{2d}).$$

We also show that the dual of $\mathcal{E}_{Z,E}^{\sigma,s}(\mathbf{R}^{2d})$ is equal to $(\mathcal{E}_{Z,E}^{\sigma,s})'(\mathbf{R}^{2d})$ through this form, and similarly when $\mathcal{E}_{Z,E}^{\sigma,s}$ are replaced by $\mathcal{E}_{Z,E}^{\sigma,s;0}$ at each occurrence.

Remark 3.3 By Propositions 3.1 and 3.2 it follows that for $V_\phi F$ and $V_\phi G$ in (3.5) we have

$$V_\phi F(x, \xi, \eta, y) \overline{V_\phi G(x, \xi, \eta, y)} \in L^1(\Omega_E).$$

Hence the right-hand side of

$$\begin{aligned}
(F, G)_{Z,E} &= (|Q_{E'}| \|\phi\|_{L^2}^2)^{-1} \iiint_{\Omega_E} V_\phi F(x, \xi, \eta, y) \overline{V_\phi G(x, \xi, \eta, y)} dx d\xi d\eta dy \quad (3.5)'
\end{aligned}$$

makes sense and we may evaluate the integrals with respect to x, ξ, η, y in any order. It also follows that the map $(F, G) \mapsto (F, G)_{Z,E}$ defines a continuous map from $\mathcal{E}_{Z,E}^{\sigma,s}(\mathbf{R}^{2d}) \times (\mathcal{E}_{Z,E}^{\sigma,s})'(\mathbf{R}^{2d})$ to $\mathbf{C}$.

Theorem 3.4 Let $(\cdot, \cdot)_{Z,E}$ be given by (3.5). Then the following is true:

- (1) $(\cdot, \cdot)_{Z,E}$ on $\mathcal{E}_{Z,E}^{\sigma,s}(\mathbf{R}^{2d}) \times \mathcal{E}_{Z,E}^{\sigma,s}(\mathbf{R}^{2d})$ is uniquely extendable to a continuous sesqui-linear map from $L^2_{Z,E}(\mathbf{R}^{2d}) \times L^2_{Z,E}(\mathbf{R}^{2d})$ and from $(\mathcal{E}_{Z,E}^{\sigma,s})'(\mathbf{R}^{2d}) \times \mathcal{E}_{Z,E}^{\sigma,s}(\mathbf{R}^{2d})$ to $\mathbf{C}$, and

$$(F, G)_{Z,E} = |Q_{E'}|^{-1} (F, G)_{L^2(Q_{E \times E'})}, \quad F, G \in L^2_{Z,E}(\mathbf{R}^{2d}); \quad (3.7)$$

- (2) if $f \in (\mathcal{S}_s^\sigma)'(\mathbf{R}^d)$, $g \in \mathcal{S}_s^\sigma(\mathbf{R}^d)$, $F = Z_E f$ and $G = Z_E g$, then $(F, G)_{Z,E} = (f, g)_{L^2(\mathbf{R}^d)}$;
- (3) the dual of $\mathcal{E}_{Z,E}^{\sigma,s}(\mathbf{R}^{2d})$ is equal to $(\mathcal{E}_{Z,E}^{\sigma,s})'(\mathbf{R}^{2d})$ through the form $(\cdot, \cdot)_{Z,E}$;

The same holds true with $\mathcal{E}_{Z,E}^{\sigma,s;0}$ and Σ_s^σ in place of $\mathcal{E}_{Z,E}^{\sigma,s}$ respective $\mathcal{S}_s^\sigma$ at each occurrence.

Proof By Proposition 1.19, Theorems 2.2, 2.5, and the facts that $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ is dense in $L^2(\mathbf{R}^d)$ and that the dual of $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ equals $(\mathcal{S}_s^\sigma)'(\mathbf{R}^d)$ through the form $(\cdot, \cdot)_{L^2(\mathbf{R}^d)}$, it suffices to prove (2).

Let

$$f \in (\mathcal{S}_s^\sigma)'(\mathbf{R}^d), \quad g \in \mathcal{S}_s^\sigma(\mathbf{R}^d), \quad \phi \in \mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d}) \setminus 0, \\ \psi(x, y) = \mathcal{F}^{-1}(\phi(x, \cdot))(y), \quad \text{and} \quad \psi_y(x) = \psi(x, y).$$

Then it follows by straight-forward computations that

$$V_\phi(Z_E f)(x, \xi, \eta, y) = e^{-i\langle y, \xi \rangle} F_X(\xi), \quad F_X(\xi) = \sum_{j \in \Lambda_E} c_X(f, j) e^{i\langle j, \xi \rangle}$$

and

$$V_\phi(Z_E g)(x, \xi, \eta, y) = e^{-i\langle y, \xi \rangle} G_X(\xi), \quad G_X(\xi) = \sum_{j \in \Lambda_E} c_X(g, j) e^{i\langle j, \xi \rangle}$$

where $X = (x, \eta, y) \in Q_E \times \mathbf{R}^{2d}$, and the Fourier coefficients $c_X(f, j)$ and $c_X(g, j)$ are given by

$$c_X(f, j) = (V_{\psi_{y-j}} f)(x - j, \eta) e^{-i\langle j, \eta \rangle} \quad (3.8)$$

and

$$c_X(g, j) = (V_{\psi_{y-j}} g)(x - j, \eta) e^{-i\langle j, \eta \rangle}. \quad (3.9)$$

Since the short-time Fourier transforms $V_\phi(Z_E f)$ and $V_\phi(Z_E g)$ are smooth, it follows that $\xi \mapsto F_X(\xi)$ and $\xi \mapsto G_X(\xi)$ are smooth periodic functions for every X . Hence the Fourier coefficients in (3.8) and (3.9) satisfy

$$|c_X(f, j)| \lesssim \langle j \rangle^{-N} \quad \text{and} \quad |c_X(g, j)| \lesssim \langle j \rangle^{-N} \quad (3.10)$$

for every $N \geq 0$, when $X \in Q_E \times \mathbf{R}^{2d}$ is fixed. By integrating

$$\begin{aligned} & V_\phi(Z_E f)(x, \xi, \eta, y) \overline{V_\phi(Z_E g)(x, \xi, \eta, y)} \\ &= \left(\sum_{j \in \Lambda_E} c_X(f, j) e^{i\langle j, \xi \rangle} \right) \overline{\left(\sum_{j \in \Lambda_E} c_X(g, j) e^{i\langle j, \xi \rangle} \right)}, \end{aligned}$$

with respect to the ξ variable and using (3.10), we obtain

$$\begin{aligned}
 & \int_{Q_{E'}} V_\phi(Z_E f)(x, \xi, \eta, y) \cdot \overline{V_\phi(Z_E g)(x, \xi, \eta, y)} d\xi \\
 &= \int_{Q_{E'}} \left(\sum_{j \in \Lambda_E} c_X(f, j) e^{i\langle j, \xi \rangle} \right) \overline{\left(\sum_{j \in \Lambda_E} c_X(g, j) e^{i\langle j, \xi \rangle} \right)} d\xi \\
 &= |Q_{E'}| \sum_{j \in \Lambda_E} c_X(f, j) \overline{c_X(g, j)} \\
 &= |Q_{E'}| \sum_{j \in \Lambda_E} (V_{\psi_{y-j}} f)(x - j, \eta) \overline{(V_{\psi_{y-j}} g)(x - j, \eta)}.
 \end{aligned}$$

Hence, integrating with respect to x, η, y , using Moyal's identity and Remark 3.3, we obtain

$$\begin{aligned}
 & \|\phi\|_{L^2(\mathbf{R}^d)}^2 \cdot (F, G)_{Z, E} \\
 &= \iint_{Q_E \times \mathbf{R}^d} \left(\sum_{j \in \Lambda_E} \int_{\mathbf{R}^d} (V_{\psi_{y-j}} f)(x - j, \eta) \overline{(V_{\psi_{y-j}} g)(x - j, \eta)} dy \right) dx d\eta \\
 &= \iint_{Q_E \times \mathbf{R}^d} \left(\sum_{j \in \Lambda_E} \int_{\mathbf{R}^d} (V_{\psi_y} f)(x - j, \eta) \overline{(V_{\psi_y} g)(x - j, \eta)} dy \right) dx d\eta \\
 &= \int_{\mathbf{R}^d} \left(\iint_{\mathbf{R}^{2d}} (V_{\psi_y} f)(x, \eta) \overline{(V_{\psi_y} g)(x, \eta)} dx d\eta \right) dy \\
 &= \int_{\mathbf{R}^d} \left((f, g)_{L^2(\mathbf{R}^d)} \|\psi_y\|_{L^2(\mathbf{R}^d)}^2 \right) dy = \|\phi\|_{L^2(\mathbf{R}^{2d})}^2 (f, g)_{L^2(\mathbf{R}^d)}, \quad (3.11)
 \end{aligned}$$

which gives (2). Here we observe that the estimates in Proposition 1.2 imply that the involved expressions in (3.11) possess suitable L^1 properties, which allow us to swap the orders of summations and integrations. This gives the result. $\square$

3.3 Duality Properties of Banach Spaces of Quasi-Periodic Elements

In the following we use the links Theorem 2.13 and (3.7) to carry over duality properties of Lebesgue and modulation spaces to quasi-periodic elements in Lebesgue and Wiener type spaces. Here $p' \in [1, \infty]$ denotes the conjugate exponent to $p \in [1, \infty]$, i.e. p and p' should satisfy $\frac{1}{p} + \frac{1}{p'} = 1$. Furthermore, we let $\mathbf{p}' = (p'_1, \dots, p'_d)$ when $\mathbf{p} = (p_1, \dots, p_d) \in [1, \infty]^d$.

Theorem 3.5 *Let $\mathbf{p} \in [1, \infty]^{2d}$, E and E_0 be ordered bases of $\mathbf{R}^d$, $\omega \in \mathcal{P}_E(\mathbf{R}^{4d})$ and $\omega_0 \in \mathcal{P}_E(\mathbf{R}^{2d})$ be such that (2.5) holds. Then the following is true:*

- (1) *the map $(F, G) \mapsto (F, G)_{Z, E}$ from $\mathcal{E}_{Z, E}^{1,1;0}(\mathbf{R}^{2d}) \times \mathcal{E}_{Z, E}^{1,1;0}(\mathbf{R}^{2d})$ to $\mathbf{C}$ is uniquely extendable to a continuous mapping from $W_{Z, E, E_0, (\omega)}^{\mathbf{p}}(\mathbf{R}^{2d}) \times W_{Z, E, E_0, (1/\omega)}^{\mathbf{p}'}(\mathbf{R}^{2d})$*

- to $\mathbf{C}$. If in addition $\max(\mathbf{p}) < \infty$, then the dual of $W_{Z,E,E_0,(\omega)}^{\mathbf{p}}(\mathbf{R}^{2d})$ can be identified with $W_{Z,E,E_0,(1/\omega)}^{\mathbf{p}'}(\mathbf{R}^{2d})$ through the form $(\cdot, \cdot)_{Z,E}$;
- (2) if $f \in M_{E \times E'_0,(\omega_0)}^{\mathbf{p}}(\mathbf{R}^d)$, $g \in M_{E \times E'_0,(1/\omega_0)}^{\mathbf{p}'}(\mathbf{R}^d)$, then $F = Z_E f \in W_{Z,E,E_0,(\omega)}^{\mathbf{p}}(\mathbf{R}^{2d})$, $G = Z_E g \in W_{Z,E,E_0,(1/\omega)}^{\mathbf{p}'}(\mathbf{R}^{2d})$ and $(F, G)_{Z,E} = (f, g)_{L^2(\mathbf{R}^d)}$.

Proof In order to prove (1) we first observe that if $F \in W_{Z,E,E_0,(\omega)}^{\mathbf{p}}(\mathbf{R}^{2d})$ and $G \in W_{Z,E,E_0,(1/\omega)}^{\mathbf{p}'}(\mathbf{R}^{2d})$, then the integrand on the right-hand side of (3.5)' belongs to $L^1(\Omega_E)$, in view of Theorem 2.13. This proves the extension assertions for $(\cdot, \cdot)_{Z,E}$ on $W_{Z,E,E_0,(\omega)}^{\mathbf{p}}(\mathbf{R}^{2d}) \times W_{Z,E,E_0,(1/\omega)}^{\mathbf{p}'}(\mathbf{R}^{2d})$. The duality assertion will follow after we have proved (2), using the fact that the dual of $M_{E \times E'_0,(\omega_0)}^{\mathbf{p}}(\mathbf{R}^d)$ is equal to $M_{E \times E'_0,(1/\omega_0)}^{\mathbf{p}'}(\mathbf{R}^d)$ (see e.g. [18, Theorem 11.3.6]).

It remains to prove (2). Let $f \in M_{E \times E'_0,(\omega_0)}^{\mathbf{p}}(\mathbf{R}^d)$, $g \in M_{E \times E'_0,(1/\omega_0)}^{\mathbf{p}'}(\mathbf{R}^d)$, $F = Z_E f$ and $G = Z_E g$. Then Theorem 2.13 shows that $F \in W_{Z,E,E_0,(\omega)}^{\mathbf{p}}(\mathbf{R}^{2d})$ and $G \in W_{Z,E,E_0,(1/\omega)}^{\mathbf{p}'}(\mathbf{R}^{2d})$. By straight-forward computations it follows that (3.11) holds for our choices of f, g, F and G . This gives (2). $\square$

Remark 3.6 Let $p \in [1, \infty]$ and E be an ordered basis of $\mathbf{R}^d$. Then recall that we may identify $L_{Z,E}^p(\mathbf{R}^{2d})$ with $L^p(Q_{E \times E'})$. Hence (2.1) and standard duality properties for Lebesgue spaces show that the map $(F, G) \mapsto (F, G)_{Z,E}$ from $C_{Z,E}^\infty(\mathbf{R}^{2d}) \times C_{Z,E}^\infty(\mathbf{R}^{2d})$ to $\mathbf{C}$ is uniquely extendable to a continuous mapping from $L_{Z,E}^p(\mathbf{R}^{2d}) \times L_{Z,E}^{p'}(\mathbf{R}^{2d})$ to $\mathbf{C}$. If in addition $p < \infty$, then the dual of $L_{Z,E}^p(\mathbf{R}^{2d})$ can be identified with $L_{Z,E}^{p'}(\mathbf{R}^{2d})$ through the form $(\cdot, \cdot)_{Z,E}$;

Remark 3.7 Let E_0 be an ordered basis of $\mathbf{R}^d$, $E = E_0 \times E'_0$, $\omega_0 \in \mathcal{P}_E(\mathbf{R}^d)$, $\omega(x, \xi) = \omega_0(\xi)$, $x, \xi \in \mathbf{R}^d$ and $\mathbf{q} \in [1, \infty)^d$. For periodic functions and distributions, Theorem 3.5 (2) together with Proposition 1.17 correspond to [40, Theorem 3.2], which among others asserts that the dual of $\mathcal{E}_E(\omega_0, \ell_{E'_0}^{\mathbf{q}}(\Lambda_{E'_0}))$ is equal to $\mathcal{E}_E(1/\omega_0, \ell_{E'_0}^{\mathbf{q}'}(\Lambda_{E'_0}))$ through a unique extension of the $L^2(Q_{E_0})$ form on $\mathcal{E}_E^0(\mathbf{R}^d) \times \mathcal{E}_E^0(\mathbf{R}^d)$.

Here we observe the misprint in (2) in [40, Theorem 3.2], where it stays $\mathcal{E}^E(1/\omega, \ell_{\kappa(E')}^{\mathbf{q}})$ instead of $\mathcal{E}^E(1/\omega, \ell_{\kappa(E')}^{\mathbf{q}'})$

4 Transitions of Operators Under the Zak Transform

In this section we show how linear operators are transformed by the Zak transform into corresponding operators acting on quasi-periodic functions or distributions. We also present a condition on linear operators which is both necessary and sufficient in order for these operators should map quasi-periodic elements into quasi-periodic elements.

Our results are described in the following two theorems, which explain how linear operators acting on functions and distributions on $\mathbf{R}^d$ are transferred by the Zak transform. Especially the operator representation

$$(U_{y,\eta}F)(x, \xi) = e^{-i\langle y, \xi + \eta \rangle} F(x + y, \xi + \eta) \\ \text{when } F \in (\mathcal{S}_{s,\sigma}^{\sigma,s})'(\mathbf{R}^{2d}), \quad y, \eta \in \mathbf{R}^d \quad (4.1)$$

is important for characterizing such operators.

Theorem 4.1 *Let $s, \sigma > 0$, $U_{y,\eta}$ be as in (4.1) and T be a linear operator from $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ to $(\mathcal{S}_s^\sigma)'(\mathbf{R}^d)$ with kernel $K \in (\mathcal{S}_s^\sigma)'(\mathbf{R}^{2d})$. Then there is a unique linear and continuous operator T_Z from $\mathcal{E}_{Z,E}^{\sigma,s}(\mathbf{R}^{2d})$ to $(\mathcal{E}_{Z,E}^{\sigma,s})'(\mathbf{R}^{2d})$ such that $Z_E \circ T = T_Z \circ Z_E$, for every ordered basis E of $\mathbf{R}^d$. The kernel of T_Z is given by*

$$K_Z(x, \xi, y, \eta) = (2\pi)^{-\frac{d}{2}} \mathcal{F}(K(x - \cdot, y - \cdot))(\eta - \xi) \in (\mathcal{S}_s^\sigma)'(\mathbf{R}^{4d}), \quad (4.2)$$

where $s = (s, \sigma, s, \sigma)$ and $\sigma = (\sigma, s, \sigma, s)$, and T_Z fulfills

$$T_Z \circ U_{y,\eta} = U_{y,\eta} \circ T_Z, \quad y, \eta \in \mathbf{R}^d. \quad (4.3)$$

The same holds true with $\mathcal{E}_{Z,E}^{\sigma,s;0}$, $(\mathcal{E}_{Z,E}^{\sigma,s;0})'$ and $(\Sigma_s^\sigma)'$, or with $C_{Z,E}^\infty$, $\mathcal{S}'_{Z,E}$ and $\mathcal{S}'$ in place of $\mathcal{E}_{Z,E}^{\sigma,s}$, $(\mathcal{E}_{Z,E}^{\sigma,s})'$ and $(\mathcal{S}_s^\sigma)'$, respectively at each occurrence.

The converse of Theorem 4.1 is the following

Theorem 4.2 *Let $s, \sigma > 0$, T_Z be a linear and continuous map from $\mathcal{S}_{s,\sigma}^{\sigma,s}(\mathbf{R}^{2d})$ to $(\mathcal{S}_{s,\sigma}^{\sigma,s})'(\mathbf{R}^{2d})$ with kernel K_Z and such that (4.3) holds. Then the following is true:*

- (1) *there is a unique $K \in (\mathcal{S}_s^\sigma)'(\mathbf{R}^{2d})$ such that (4.2) holds;*
- (2) *if E is an ordered basis of $\mathbf{R}^d$, then T_Z is uniquely extendable to a linear and continuous operator from $\mathcal{E}_{Z,E}^{\sigma,s}(\mathbf{R}^{2d})$ to $(\mathcal{E}_{Z,E}^{\sigma,s})'(\mathbf{R}^{2d})$;*
- (3) *If T is the linear operator with kernel K in (1) and E is an ordered basis of $\mathbf{R}^d$, then $Z_E \circ T = T_Z \circ Z_E$.*

The same holds true with $\Sigma_{s,\sigma}^{\sigma,s}$, $(\Sigma_s^\sigma)'$, $\mathcal{E}_{Z,E}^{\sigma,s;0}$ and $(\mathcal{E}_{Z,E}^{\sigma,s;0})'$, or with $\mathcal{S}$, $\mathcal{S}'$, $C_{Z,E}^\infty$ and $\mathcal{S}'_{Z,E}$ in place of $\mathcal{S}_{s,\sigma}^{\sigma,s}$, $(\mathcal{S}_s^\sigma)'$, $\mathcal{E}_{Z,E}^{\sigma,s}$ and $(\mathcal{E}_{Z,E}^{\sigma,s})'$, respectively at each occurrence.

Proof of Theorem 4.1 We only prove the result when the involved spaces are given by $\mathcal{E}_{Z,E}^{\sigma,s}$ or $(\mathcal{E}_{Z,E}^{\sigma,s})'$. The other cases follow by similar arguments and are left for the reader.

First suppose that K_Z is given by (4.2). Since pull-back results of the type [21, Theorem 6.1.2] for usual distribution, hold true for Gelfand–Shilov distributions concerning linear pull-backs, it follows that $K_1(x, y, z) \equiv K(x - z, y - z)$ belongs to $(\mathcal{S}_s^\sigma)'(\mathbf{R}^{3d})$ when $K \in (\mathcal{S}_s^\sigma)'(\mathbf{R}^{2d})$. By Fourier transformation, it follows that $K_2(x, y, \xi) \equiv \mathcal{F}(K_1(x, y, \cdot))(\xi) \in (\mathcal{S}_{s,s,\sigma}^{\sigma,\sigma,s})'(\mathbf{R}^{3d})$. By similar pull-back results it now follows that

$$K_3(x, y, \xi, \eta) \equiv K_2(x, y, \eta - \xi) \in (\mathcal{S}_{s,s,\sigma,\sigma}^{\sigma,\sigma,s,s})'(\mathbf{R}^{4d}).$$

Since $K_Z(x, \xi, y, \eta) = (2\pi)^{-\frac{d}{2}} K_3(x, y, \xi, \eta)$, we get

$$K_Z \in (\mathcal{S}_{s, \sigma, s, \sigma}^{\sigma, s, \sigma, s})'(\mathbf{R}^{4d}),$$

and the last property in (4.2) follows.

Let E be an ordered basis of $\mathbf{R}^d$. We only prove $Z_E \circ T = T_Z \circ Z_E$ when $K \in \mathcal{S}(\mathbf{R}^{2d})$. The general result follows by similar arguments and is left for the reader. We have

$$(2\pi)^{\frac{d}{2}} \mathcal{F}^{-1}(K_Z(x, \xi, y, \cdot))(z) = K(x - z, y - z) e^{i\langle z, \xi \rangle}.$$

This gives

$$\begin{aligned} T_Z(Z_E f)(x, \xi) &= \sum_{j \in \Lambda_E} \iint_{\mathbf{R}^{2d}} K_Z(x, \xi, y, \eta) f(y - j) e^{i\langle j, \eta \rangle} dy d\eta \\ &= \sum_{j \in \Lambda_E} (2\pi)^{\frac{d}{2}} \int_{\mathbf{R}^d} \mathcal{F}^{-1}(K_Z(x, \xi, y, \cdot))(j) f(y - j) dy \\ &= \sum_{j \in \Lambda_E} \int_{\mathbf{R}^d} K(x - j, y - j) e^{i\langle j, \xi \rangle} f(y - j) dy \\ &= \sum_{j \in \Lambda_E} \int_{\mathbf{R}^d} K(x - j, y) e^{i\langle j, \xi \rangle} f(y) dy \\ &= \sum_{j \in \Lambda_E} (Tf)(x - j) e^{i\langle j, \xi \rangle} = Z_E(Tf)(x, \xi). \end{aligned}$$

This shows that $Z_E \circ T = T_Z \circ Z_E$.

The continuity assertions of T_Z now follows from the latter identity and Theorem 2.5. $\square$

We need the following lemma for the proof of Theorem 4.2.

Lemma 4.3 *Let $s, \sigma > 0$ and $K \in (\mathcal{S}_s^\sigma)'(\mathbf{R}^{2d})$. Then the following conditions are equivalent:*

- (1) $K(\cdot + (z, z)) = K$ for every $z \in \mathbf{R}^d$;
- (2) there is a unique $K_0 \in (\mathcal{S}_s^\sigma)'(\mathbf{R}^d)$ such that $K(x, y) = K_0(x - y)$.

The same hold true with $(\Sigma_s^\sigma)'$, $\mathcal{S}'$ or $\mathcal{D}'$ in place of $(\mathcal{S}_s^\sigma)'$ at each occurrence.

Lemma 4.3 is at least implicitly available in the literature, e.g. in [21]. In order to be self-contained we give a proof in Appendix A.

Proof of Theorem 4.2 Again we only prove the result when the involved spaces are given by $\mathcal{E}_{Z, E}^{\sigma, s}$ or $(\mathcal{E}_{Z, E}^{\sigma, s})'$. The other cases follow by similar arguments and are left for the reader.

The condition (4.3) implies that

$$K_Z(x, \xi, y, \eta) = e^{-i\langle z, \xi - \eta \rangle} K_Z(x + z, \xi + \zeta, y + z, \eta + \zeta),$$

for every $x, y, z, \xi, \eta, \zeta \in \mathbf{R}^d$. By Lemma 4.3 it follows that

$$K_Z(x, \xi, y, \eta) = e^{i\langle y, \xi - \eta \rangle} K_0(x - y, \xi - \eta)$$

for some $K_0 \in (\mathcal{S}_{s,\sigma}^{\sigma,s})'(\mathbf{R}^{2d})$. It now follows that

$$K(x, y) = (2\pi)^{\frac{d}{2}} (\mathcal{F}_2^{-1} K_0)(x - y, y)$$

fulfills all required properties.

The assertion (2) now follows from the commutative diagram

$$\begin{array}{ccc} \mathcal{S}_s^\sigma(\mathbf{R}^d) & \xrightarrow{T} & (\mathcal{S}_s^\sigma)'(\mathbf{R}^d) \\ \mathcal{Z}_E \downarrow & & \downarrow \mathcal{Z}_E \\ \mathcal{E}_{Z,E}^{\sigma,s}(\mathbf{R}^{2d}) & \xrightarrow{T_Z} & (\mathcal{E}_{Z,E}^{\sigma,s})'(\mathbf{R}^{2d}) \end{array} \quad (4.4)$$

when T is the map with kernel K , and the fact that T is continuous from $\mathcal{S}_s^\sigma(\mathbf{R}^d)$ and $(\mathcal{S}_s^\sigma)'(\mathbf{R}^d)$. The assertions (1) and (3) follow from Theorem 4.1. $\square$

Remark 4.4 By combining Theorems 4.1 and 4.2, the commutative diagram (4.4) and kernel theorems for linear operators between Gelfand–Shilov spaces (cf. e.g. [23] and the references therein), it follows that the set of linear and continuous operators from $\mathcal{E}_{Z,E}^{\sigma,s}(\mathbf{R}^{2d})$ to $(\mathcal{E}_{Z,E}^{\sigma,s})'(\mathbf{R}^{2d})$ can be identified with operators with kernels satisfying (4.2) and (4.3). The same holds true with $\mathcal{E}_{Z,E}^{\sigma,s;0}$ and Σ_s^σ in place of $\mathcal{E}_{Z,E}^{\sigma,s}$ and $\mathcal{S}_s^\sigma$, respectively, at each occurrence.

Acknowledgements I am very grateful to Professor Hans Feichtinger for reading parts of the first version of the paper and giving valuable comments, leading to improvements of the content and the style.

Funding Open access funding provided by Linnaeus University. The project was supported by Vetenskapsrådet (Swedish Science Council) within the project 2019-04890.

Compliance with Ethical Standards

Conflict of interest No conflict or competing of interests.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Appendix A

In this appendix we present proofs of Proposition 2.11 and Lemma 4.3.

Proof of Proposition 2.11 First suppose that $p \in (0, 1]$ and let $F = Z_E f$ as before. Then

$$\begin{aligned} \|F\|_{L^p_{Z,E}}^p &= |Q_{E'}|^{-1} \iint_{Q_{E \times E'}} \left| \sum_{j \in \Lambda_E} f(x-j) e^{i\langle j, \xi \rangle} \right|^p dx d\xi \\ &\leq |Q_{E'}|^{-1} \iint_{Q_{E \times E'}} \left(\sum_{j \in \Lambda_E} |f(x-j) e^{i\langle j, \xi \rangle}|^p \right) dx d\xi \\ &= \int_{Q_E} \left(\sum_{j \in \Lambda_E} |f(x-j)|^p \right) dx = \|f\|_{L^p}^p, \end{aligned}$$

and (1) follows for $p \in (0, 1]$.

Since the result is true for $p = 2$ in view of Proposition 1.19, the result now follows in the case $p \in [1, 2]$ by interpolating the case $p = 1$ above with the case $p = 2$. This gives (1), and (2) now follows from (1) and duality.

Finally, by the assumptions we have that $\sum_{j \in \Lambda_E} v(x+j)^{-1}$ is bounded. Hence, by the inversion formula (1.21) we obtain

$$\begin{aligned} \|f \cdot v^{-1}\|_{L^1} &\asymp \int_{\mathbf{R}^d} \left| \int_{Q_{E'}} F(x, \xi) d\xi \right| v(x)^{-1} dx \\ &\leq \sum_{j \in \Lambda_E} \int_{Q_E} \left(\int_{Q_{E'}} |F(x+j, \xi)| d\xi \right) v(x+j)^{-1} dx \\ &= \int_{Q_E} \left(\int_{Q_{E'}} |F(x, \xi)| d\xi \right) \left(\sum_{j \in \Lambda_E} v(x+j)^{-1} \right) dx \asymp \|F\|_{L^1_{Z,E}(\mathbf{R}^{2d})}, \end{aligned}$$

and (3) follows in the case $p = 1$ by combining the previous estimate with (1). Since (3) is true for $p = 2$ in view of Proposition 1.19, it follows that it is true also for $p \in [1, 2]$ by interpolating between the cases $p = 1$ and $p = 2$. For $p \in [2, \infty]$, (3) now follows from the case $p \in [1, 2]$ and duality. $\square$

Proof of Lemma 4.3 We only prove the result when the involved spaces are of the forms $(\mathcal{S}_s^\sigma)'(\mathbf{R}^d)$ and $(\mathcal{S}_s^\sigma)'(\mathbf{R}^{2d})$. The other cases follow by similar arguments and are left for the reader.

It is evident that (2) implies (1). Suppose that (1) is true. Then $(x, y) \mapsto K(2^{-1}(x+y), 2^{-1}(x-y))$ is an element in $(\mathcal{S}_s^\sigma)'(\mathbf{R}^{2d})$ which is constant with respect to the x variable. Hence,

$$K(2^{-1}(x+y), 2^{-1}(x-y)) = (1 \otimes K_0)(x, y)$$

for some $K_0 \in (\mathcal{S}'_s)^\sigma(\mathbf{R}^d)$. By taking $2^{-1}(x + y)$ and $2^{-1}(x - y)$ as new variables, we obtain (2). $\square$

References

1. Aoki, T.: Locally bounded linear topological spaces. *Proc. Imp. Acad. Tokyo* **18**, 588–594 (1942)
2. Becnel, J.J.: Equivalence of topologies and Borel fields for countably-Hilbert spaces. *Proc. Am. Math. Soc.* **134**, 581–590 (2006)
3. Brezin, J.: Harmonic analysis on nilmanifolds. *Trans. Am. Math. Soc.* **150**, 611–618 (1970)
4. Bölcskei, H.: A necessary and sufficient condition for dual Weyl–Heisenberg frames to be compactly supported. *J. Fourier Anal. Appl.* **5**, 409–419 (1999)
5. Bölcskei, H., Janssen, A.J.E.M.: Gabor frames, unimodularity, and window decay. *J. Fourier Anal. Appl.* **6**, 255–276 (2000)
6. Chung, J., Chung, S.-Y., Kim, D.: Characterizations of the Gelfand–Shilov spaces via Fourier transforms. *Proc. Am. Math. Soc.* **124**, 2101–2108 (1996)
7. Cordero, E., Pilipović, S., Rodino, L., Teofanov, N.: Quasianalytic Gelfand–Shilov spaces with applications to localization operators. *Rocky Mt. J. Math.* **40**, 1123–1147 (2010)
8. Eijndhoven, S.J.L.: Functional analytic characterizations of the Gelfand–Shilov spaces S^β_α . *Nederl. Akad. Wetensch. Indag. Math.* **49**, 133–144 (1987)
9. Feichtinger, H.G.: Banach spaces of distributions of Wiener’s type and interpolation. In: Butzer, P., Nagy, B.Sz., Görlich, E. (eds.) *Proceeding Conference Oberwolfach, Functional Analysis and Approximation, August 1980, International Series Numerical Mathematics*, vol. 69, pp. 153–165. Birkhäuser Verlag, Basel, Boston, Stuttgart (1981)
10. Feichtinger, H.G.: Modulation spaces on locally compact abelian groups. Technical report, University of Vienna, Vienna, 1983; also. In: Krishna, M., Radha, R., Thangavelu, S. (eds.) *Wavelets and their applications*, Allied Publishers Private Limited, NewDehli Mumbai Kolkata Chennai Nagpur Ahmedabad Bangalore Hyderabad Lucknow. pp. 99–140 (2003)
11. Feichtinger, H.G.: Modulation spaces: looking back and ahead. *Sampl. Theory Signal Image Process.* **5**, 109–140 (2006)
12. Feichtinger, H.G., Gröchenig, K.H.: Banach spaces related to integrable group representations and their atomic decompositions, I. *J. Funct. Anal.* **86**, 307–340 (1989)
13. Feichtinger, H.G., Gröchenig, K.H.: Banach spaces related to integrable group representations and their atomic decompositions, II. *Monatsh. Math.* **108**, 129–148 (1989)
14. Feichtinger, H.G., Luef, F.: Wiener amalgam spaces for the fundamental identity of Gabor analysis. *Collect. Math.* **57**, 233–253 (2006)
15. Galperin, Y.V., Samarah, S.: Time-frequency analysis on modulation spaces $M_m^{p,q}$, $0 < p, q \leq \infty$. *Appl. Comput. Harmon. Anal.* **16**, 1–18 (2004)
16. Gelfand, I.M.: Eigenfunction expansions for equations with periodic coefficients. *Dokl. Akad. Nauk. SSSR* **73**, 1117–1120 (1950)
17. Gelfand, I.M., Shilov, G.E.: *Generalized Functions*, II–III. Academic Press, New York (1968)
18. Gröchenig, K.H.: *Foundations of Time-Frequency Analysis*. Birkhäuser, Boston (2001)
19. Gröchenig, K.H.: Weight functions in time-frequency analysis. In: Rodino, L., Wong, M.W. (eds.) *Pseudodifferential Operators: Partial Differential Equations and Time-Frequency Analysis*, vol. 52, pp. 343–366. Fields Institute Communications, Berlin (2007)
20. Gröchenig, K.H., Zimmermann, G.: Spaces of test functions via the STFT. *J. Funct. Spaces Appl.* **2**, 25–53 (2004)
21. Hörmander, L.: *The Analysis of Linear Partial Differential Operators*, vol. I–III. Springer, Berlin (1983). (1985)
22. Janssen, A.J.E.M.: Bargmann transform, Zak transform, and coherent states. *J. Math. Phys.* **23**, 720–731 (1982)
23. Nicola, F., Rodino, L.: *Global Pseudo-Differential Calculus on Euclidean Spaces, Pseudo-Differential Operators. Theory and Applications*. Birkhäuser Verlag, Basel (2010)
24. Pilipović, S.: Generalization of Zemanian spaces of generalized functions which have orthonormal series expansions. *SIAM J. Math. Anal.* **17**, 477–484 (1986)

25. Pilipović, S.: Structural theorems for periodic ultradistributions. *Proc. Am. Math. Soc.* **98**, 261–266 (1986)
26. Pilipović, S.: Tempered ultradistributions. *Boll. U.M.I.* **7**, 235–251 (1988)
27. Rauhut, H.: Wiener amalgam spaces with respect to quasi-Banach spaces. *Colloq. Math.* **109**, 345–362 (2007)
28. Rauhut, H.: Coorbit space theory for quasi-Banach spaces. *Studia Math.* **180**, 237–253 (2007)
29. Rolewicz, S.: On a certain class of linear metric spaces. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astrono. Phys.* **5**, 471–473 (1957)
30. Teofanov, N.: Ultra-distributions and time-frequency analysis. In: Boggiatto, P., Rodino, L., Toft, J., Wong, M.W. (eds.) *Pseudo-Differential Operators and Related Topics, Operator Theory: Advances and Applications*, vol. 164, pp. 173–191. Birkhäuser, Basel (2006)
31. Teofanov, N.: Gelfand–Shilov spaces and localization operators. *Funct. Anal. Approx. Comput.* **7**, 135–158 (2015)
32. Tinaztepe, R.: Modulation spaces, BMO and the Zak transform, and minimising IPH functions over the unit simplex, Thesis (2010)
33. Tinaztepe, R., Heil, C.: Modulation spaces, BMO, and the Balian–Low theorem. *Sampl. Theory Signal Image Process.* **11**, 25–41 (2012)
34. Toft, J.: The Bargmann transform on modulation and Gelfand–Shilov spaces, with applications to Toeplitz and pseudo-differential operators. *J. Pseudo-Differ. Oper. Appl.* **3**, 145–227 (2012)
35. Toft, J.: Gabor analysis for a broad class of quasi-Banach modulation spaces. In: Pilipović, S., Toft, J. (eds.) *Pseudo-Differential Operators, Generalized Functions, Operator Theory: Advances and Applications*, vol. 245. Birkhäuser, Basel (2015)
36. Toft, J.: Images of function and distribution spaces under the Bargmann transform. *J. Pseudo-Differ. Oper. Appl.* **8**, 83–139 (2017)
37. Toft, J.: Semi-continuous convolutions on weakly periodic Lebesgue spaces. In: Boggiatto, P., Cordero, E., de Gosson, M., Feichtinger, H.G., Nicola, F., Oliaro, A., Tabacco, A. (eds.) *Landscapes of Time-Frequency Analysis, Applied and Numerical Harmonic Analysis*, vol. 95, pp. 309–321. Birkhäuser, Basel (2019)
38. Toft, J.: Tensor products for Gelfand–Shilov and Pilipović distribution spaces. *J. Anal.* **28**, 591–613 (2020)
39. Toft, J.: Wiener estimates for modulation spaces. In: Boggiatto, P., Capiello, M., Cordero, E., Coriasco, S., Garello, G., Oliaro, A., Seiler, J. (eds.) *Advances in Microlocal and Time-Frequency Analysis, Applied and Numerical Harmonic Analysis*, vol. 103, pp. 475–505. Birkhäuser, Basel (2020)
40. Toft, J., Nabizadeh, E.: Periodic distributions and periodic elements in modulation spaces. *Adv. Math.* **323**, 193–225 (2018)
41. Zak, J.: Finite translations in solid state physics. *Phys. Rev. Lett.* **19**, 1385–1397 (1967)