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Fredholm Equation in Smooth Banach Spaces and Its Applications

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Abstract The object of this paper is to consider the Fredholm equation (i.e., $x - ABx = y_o$) for smooth Banach spaces. In particular, we will prove that the classical assumption of compactness of AB is redundant in some circumstances. In this paper, we show that Coburn's theorem holds for another classes of generally of nonnormal operators. Moreover, as a corollary, we find the distance from some operator to compact operators.

Keywords Semi-inner product · Smooth Banach space · Fredholm equation · Spectrum · Compact operators

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1 Preliminaries

The terminology "Fredholm operator" recognizes the pioneering work of Erik Fredholm. In 1903 he published a paper that, in modern language, dealt with equations of the form

$$f(t) - \int_a^b k(t,u)f(u)du = h(t), \tag{1.1}$$

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where $k(t,u)$ is in $L^2([a,b]\times[a,b])$ and f,h are in $L^2[a,b]$. The Fredholm equation and the Fredholm operator have been intensively studied in connection with integral equations, as well as operator theory. Let X be a real or complex Banach space. Assume that $A\in\mathcal{K}(X)$, $y_o\in X$. From our modern perspective we can think of the Fredholm equation as

$$x - Ax = y_o,$$

with a single unknown vector $x\in X$. We recall the celebrated Fredholm Alternative. This great result has a number of applications in the theory of integral equations.

Theorem 1.1 (The Fredholm Alternative) *Let X be a Banach space. If $A\in\mathcal{K}(X)$, $\lambda\in\mathbb{K}$ and $\lambda\neq 0$, then for every y_o in X there is an x in X such that*

$$\lambda x - Ax = y_o \tag{1.2}$$

if and only if the only vector u such that $\lambda u - Au = 0$ is $u = 0$. If this condition is satisfied, then the solution to (1.2) is unique.

In this paper in Sect. 2 we consider a generalized Fredholm equation. Our main result (Theorems 2.4) is an extension of the Fredholm Alternative to the setting of possibly noncompact operators. The most important theorems of this paper will be contained in Sects. 2 and 4. In Sect. 2 we will give a characterization for the general idea of the Fredholm equation (i.e., $x - ABx = y_o$) under certain assumptions. In particular, we will prove that the assumption of compactness of AB is not needed (in some circumstances). The second part of this work (i.e., Sects. 3, 4, 5) is devoted to applications of Theorems 2.4 and 2.5. Our main goal in Sect. 3 is to get some information on the spectrum of a noncompact operator on smooth space. Moreover, in Sect. 4 we will compute the distance from some operator to the subspace $\mathcal{K}(X)$ (see Theorem 4.2).

1.1 Semi-inner Product

Let $(X,\|\cdot\|)$ be a normed space over $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$. Lumer [6] and Giles [2] proved that in a space X there always exists a mapping $[\cdot|\cdot]: X\times X\to\mathbb{K}$ satisfying the following properties:

(sip1) $\forall_{x,y,z\in X}\ \forall_{\alpha,\beta\in\mathbb{K}}:\quad [\alpha x+\beta y|z] = \alpha\,[x|z]+\beta\,[y|z]$;
(sip2) $\forall_{x,y\in X}\ \forall_{\alpha\in\mathbb{K}}:\quad [x|\alpha y] = \overline{\alpha}\,[x|y]$;
(sip3) $\forall_{x,y\in X}:\quad |\,[x|y]\,| \leq \|x\|\cdot\|y\|$;
(sip4) $\forall_{x\in X}:\quad [x|x] = \|x\|^2$.

Such a mapping is called a *semi-inner product* (s.i.p.) in X (generating the norm $\|\cdot\|$). There may exist infinitely many different semi-inner products in X. There is a unique one if and only if X is *smooth* (i.e., there is a unique supporting hyperplane at each point of the unit sphere). If X is an inner product space, the only s.i.p. on X is the inner-product itself.

We quote some additional result concerning s.i.p.. Let X be a smooth, reflexive Banach space. Then there exists a unique s.i.p. $[\cdot|\cdot]: X\times X\to\mathbb{K}$. If A is a bounded linear operator from X to itself, then $f_z(\cdot) := [A(\cdot)|z]$ is a continuous linear functional, and from the generalized Riesz-Fischer representation theorem it follows that there is a unique vector $A_*(z)$ such that $[Ax|z] = [x|A_*(z)]$ for all x in X. Of course, in a Hilbert space we have $A_* = A^*$. In general case the mapping $A_*\colon X \to X$ is not linear but it still has some good properties:

(sip5) $(AB)_* = B_* A_*$,
(sip6) $(\alpha A)_* = \overline{\alpha} A_*$,
(sip7) $\forall_{x \in X}: \quad \|A_*(x)\| \le \|A\| \cdot \|x\|$.For a vector x in a normed space X, we consider the set

$$J(x) := \{\varphi \in X^* : \varphi(x) = \|x\|, \ \|\varphi\| = 1\}. \tag{1.3}$$

By the Hahn-Banach Extension Theorem we get $J(x) \neq \emptyset$. In this paper, for a normed space X, we denote by $\mathbf{B}(X)$ the closed unit ball in X. By $\operatorname{ext} D$ we will denote the set of all extremal points of a set D.

Let $\mathcal{L}(X)$ denote the space of all bounded linear operators on a space X, and I the identity operator. We write $\mathcal{K}(X)$ for the space of all compact operators on X. For $A \in \mathcal{L}(X)$, we denote the set $\mathcal{M}(A) := \{x \in \mathbf{B}(X) : \|Ax\| = \|A\|, \|x\| = 1\}$.

1.2 Geometry of Space $\mathcal{L}(X)$

The main tool in our approach in the next section is a theorem due to Lima and Olsen [5] which characterizes the extremal points of the closed unit ball in $\mathcal{K}(X)$.

Theorem 1.2 [5] *Let X be a reflexive Banach spaces over the field $\mathbb{K}$. The following conditions are equivalent:*

(a) $f \in \operatorname{ext}\mathbf{B}\,(\mathcal{K}(X)^*)$,
(b) *there exist $y^* \in \operatorname{ext}\mathbf{B}(X^*)$ and $x \in \operatorname{ext}\mathbf{B}(X)$ such that $f(T) = y^*(Tx)$ for every $T \in \mathcal{K}(X)$.*

It is possible that there is space X such that $\mathcal{L}(X)^* = \mathcal{K}(X)^* \oplus_1 \mathcal{K}(X)^\perp$ where $\mathcal{K}(X)^\perp := \{f \in \mathcal{L}(X)^* : \mathcal{K}(X) \subset \ker f\}$. In this case, if $\varphi = \varphi_1 + \varphi_2$ is the unique decay of φ in $\mathcal{L}(X)^*$, then $\|\varphi\| = \|\varphi_1\| + \|\varphi_2\|$. This fact is referred to [4, p. 28] (see also [3, Theorem 4]), where it was proved that $\mathcal{L}(X)^* = \mathcal{K}(X)^* \oplus_1 \mathcal{K}(X)^\perp$ when $X = \left(\bigoplus_{n=1}^\infty X_n\right)_{l_p}$ or $X = \left(\bigoplus_{n=1}^\infty X_n\right)_{c_0}$, with $p \in (1,\infty)$, $\dim X_n < \infty$. In particular, $\mathcal{L}(X)^* = \mathcal{K}(X)^* \oplus_1 \mathcal{K}(X)^\perp$ if $X = l_p$ or $X = c_0$, with $p \in (1,\infty)$. Moreover, Lima has obtained the equality $\mathcal{L}(L_2(\mu))^* = \mathcal{K}(L_2(\mu))^* \oplus_1 \mathcal{K}(L_2(\mu))^\perp$ for some measure μ (see [3]). On the other hand, if $p \neq 2$ and μ is not purely atomic, then it is known that $\mathcal{L}(L_p(\mu))^* \neq \mathcal{K}(L_p(\mu))^* \oplus_1 \mathcal{K}(L_p(\mu))^\perp$ (see [3, Theorem 11]).

If $\dim X < \infty$, then the desired equality $\mathcal{L}(X)^* = \mathcal{K}(X)^* \oplus_1 \mathcal{K}(X)^\perp$ holds trivially, because in such a case, $\mathcal{L}(X) = \mathcal{K}(X)$ and $\mathcal{K}(X)^\perp = \{0\}$.

Recall that a *smooth point* x of the unit sphere of a Banach space X is defined by the requirement that $x^*(x) = 1$ for a uniquely determined $x^* \in B(X^*)$, i.e., $\operatorname{card} J(x) = 1$. In this case the norm of X is Gâteaux differentiable at x with derivative x^*. We will need a characterization of points of smoothness in the space of operators $\mathcal{L}(X)$ (cf. [7]).

Theorem 1.3 [7] *Suppose that* $X = \left(\bigoplus_{n=1}^{\infty} X_n\right)_{l_p}$ *with* $p \in (1, \infty)$, $\dim X_n < \infty$ *for some* $1 < p < \infty$. *Then* T *is a smooth point of the unit sphere of* $\mathcal{L}(X)$ *if and only if*

(s1) $\inf\{\|T + K\| : K \in \mathcal{K}(X)\} < 1$;
(s2) *there is exactly one* x_o *in the unit sphere of* X *(up to multiplication with scalars of modulus 1) for which* $\|T(x_o)\| = 1$;
(s3) *the point* Tx_o *is smooth.*

2 Fredholm Equation

The following theorem can be considered as an extension of the Fredholm Alternative (i.e., Theorem 1.1). We want to show that the assumption of compactness of operators may be redundant (in some circumstances). It is more convenient to consider two operator A, B instead of only one. We are ready to prove the main result of this section.

Theorem 2.1 *Assume that* X *is a reflexive smooth Banach space over* $\mathbb{K}$. *Suppose that* $\mathcal{L}(X)^* = \mathcal{K}(X)^* \oplus_1 \mathcal{K}(X)^{\perp}$. *Let* $A, B \in \mathcal{L}(X)$, $\|A\| \cdot \|B\| \leq 1$. *Suppose that there is an operator* $C \in \mathcal{K}(X)$ *such that* $\|AB - C\| < \|AB\|$. *The following five conditions are equivalent:*

(a) $I - AB$ *is invertible;*
(b) $I - AB$ *is surjective;*
(c) $I - AB$ *is injective;*
(d) $I - AB$ *has dense range;*
(e) $\|I + AB\| < 2$.

Proof The implications (a)⇒(b)⇒(d) and (a)⇒(c) are immediate. First we prove (e)⇒(a). We get $\left\|I - \frac{1}{2}(I - AB)\right\| = \frac{1}{2}\|I + AB\| < 1$, which means that $\frac{1}{2}(I - AB)$ is invertible. Therefore $I - AB$ is invertible.

Now, we prove (c)⇒(e). Suppose that $I - AB$ is injective. Assume, for a contradiction, that $\|I + AB\| \geq 2$. From the inequalities $2 \leq \|I + AB\| \leq 1 + \|AB\| \leq 1 + \|A\| \cdot \|B\| \leq 2$ we get

$$\|AB\| = \|A\| \cdot \|B\| \quad \text{and} \quad \|I + AB\| = 1 + \|AB\|. \tag{2.1}$$

Now, we prove that $J(I + AB) \subset J(I) \cap J(AB)$. Let $\varphi \in J(I + AB)$. This gives $\varphi \in \mathcal{L}(X)^*$, $\varphi(I + AB) = \|I + AB\|$ and $\|\varphi\| = 1$. We will prove that $\varphi(I) = 1$ and $\varphi(AB) = \|AB\|$. It follows that

$$1 + \|AB\| \overset{(2.1)}{=} \|I + AB\| = \varphi(I + AB) = \varphi(I) + \varphi(AB),$$

and $|\varphi(I)| \leq 1$ and $|\varphi(AB)| \leq \|AB\|$. This clearly forces $\varphi(I) = 1$ and $\varphi(AB) = \|AB\|$ and, in consequences, the inclusion

$$J(I + AB) \subset J(I) \cap J(AB) \tag{2.2}$$

is true. The set $J(I + AB)$ is convex. Furthermore, the set $J(I + AB)$ is a nonempty weak*-closed subset of weak*-compact unit ball $\mathbf{B}(\mathcal{L}(X)^*)$. From this it follows that $J(I+AB)$ is weak*-compact. Applying the Krein-Milman Theorem we see that there exists $\mu \in \mathrm{ext}J(I + AB)$. An easy computation shows that the set $J(I + AB)$ is an extremal subset of $\mathbf{B}(\mathcal{L}(X)^*)$. Therfore

$$\mathrm{ext}J(I + AB) \subset \mathrm{ext}\mathbf{B}(\mathcal{L}(X)^*),$$

and, in consequences, we get $\mu \in \mathrm{ext}\mathbf{B}(\mathcal{L}(X)^*)$.

Now we prove that $\mu \in \mathrm{ext}\mathbf{B}(\mathcal{K}(X)^*)$. Recall that $\mathcal{L}(X)^* = \mathcal{K}(X)^* \oplus_1 \mathcal{K}(X)^{\perp}$. From this it may be concluded that

$$\mathrm{ext}\mathbf{B}(\mathcal{L}(X)^*) = \mathrm{ext}\mathbf{B}(\mathcal{K}(X)^*)\oplus_1\{0\} \;\cup\; \{0\}\oplus_1\mathrm{ext}\mathbf{B}(\mathcal{K}(X)^{\perp}).$$

Suppose that $\mu = \mu_1 + \mu_2$ is the suitable decay of μ. Namely, we have $\mu_1 \in \mathcal{K}(X)^*$, $\mu_2 \in \mathcal{K}(X)^{\perp}$ and $1 = \|\mu\| = \|\mu_1\| + \|\mu_2\|$. This implies that

$$\mu = \mu_1 \in \mathrm{ext}\mathbf{B}(\mathcal{K}(X)^*) \;\mathit{or}\; \mu = \mu_2 \in \mathrm{ext}\mathbf{B}(\mathcal{K}(X)^{\perp}),$$

So it suffices to show that $\mu_2 = 0$. Assume, contrary to our claim, that $\mu = \mu_2 \in \mathrm{ext}\mathbf{B}(\mathcal{K}(X)^{\perp})$. Hence $\|\mu_2\| = 1$. From the assumption, we have that $\|AB - C\| < \|AB\|$ and $C \in \mathcal{K}(X)$. It follows that

$$\begin{aligned}\|AB\| &= \mu(AB) = \mu_2(AB) = \mu_2(AB) - 0 = \mu_2(AB) - \mu(C) = \\ &= \mu_2(AB - C) \le \|AB - C\| < \|AB\|\end{aligned}$$

and we obtain a contradiction. Thus we must have $\mu = \mu_1 \in \mathrm{ext}\mathbf{B}(\mathcal{K}(X)^*)$.

Therefore it follows from (2.2) and Theorem 1.2 that

$$\mu(I+AB)=y^*(x+ABx) \;\mathit{and}\; \mu(I)=y^*(Ix)=y^*(x) \;\mathit{and}\; \mu(AB)=y^*(ABx)$$

for some $y^* \in \mathrm{ext}\mathbf{B}(X^*)$ and $x \in \mathrm{ext}\mathbf{B}(X)$.

This is summarized as follows:

$$y^*(x+ABx)=\|I+AB\| \;\text{ and }\; y^*(Ix)=1 \;\text{ and }\; y^*(ABx)=\|AB\|.$$

Then we have $\|I+AB\| = y^*(x+ABx) \le \|x+ABx\| \le \|I+AB\|$ and $\|AB\| = y^*(ABx) \le \|ABx\| \le \|AB\|$. Thus we obtain the equalities:

$$\|x+ABx\|=\|I+AB\| \;\text{ and }\; \|ABx\|=\|AB\|, \tag{2.3}$$

whence $\mathcal{M}(AB) \ne \emptyset$.

From the above, we have $\|A\|\cdot\|B\| = \|AB\| = \|ABx\| \le \|A\|\cdot\|Bx\| \le \|A\|\cdot\|B\|$, whence $\|Bx\| = \|B\|$, and consequently $x \in \mathcal{M}(B)$, $\mathcal{M}(B) \ne \emptyset$. Furthermore, we obtain $\left\|A\left(\frac{Bx}{\|Bx\|}\right)\right\| = \frac{1}{\|Bx\|}\|ABx\| = \frac{1}{\|B\|}\|AB\| \overset{(2.1)}{=} \frac{1}{\|B\|}\|A\|\cdot\|B\| = \|A\|$, whence $\frac{Bx}{\|Bx\|} \in \mathcal{M}(A)$, $\mathcal{M}(A) \neq \emptyset$.

Combining (2.1) and (2.3), we immediately get

$$\|x + ABx\| = \|x\| + \|ABx\|. \tag{2.4}$$

Using the properties of s.i.p., we obtain

$$\begin{aligned}\|A\|^2\cdot\|B\|^2 = \|AB\|^2 = \|ABx\|^2 &\overset{(\text{sip4})}{=} [ABx|ABx] =\\ &= [Bx|A_*(ABx)] \overset{(\text{sip3})}{\leq} \|Bx\|\cdot\|A_*(ABx)\| \overset{(\text{sip7})}{\leq}\\ &\leq \|Bx\|\cdot\|A\|\cdot\|ABx\| \leq \|Bx\|\cdot\|A\|\cdot\|A\|\cdot\|Bx\| \leq \|A\|^2\cdot\|B\|^2.\end{aligned}$$

It follows from the above inequalities that

$$\|A_*(ABx)\| = \|A\|\cdot\|A\|\cdot\|Bx\| \tag{2.5}$$

and $[Bx|A_*(ABx)] = \|Bx\|\cdot\|A_*(ABx)\|$, whence $\left[Bx|\frac{A_*(ABx)}{\|A_*(ABx)\|}\right] = \|Bx\|$. Therefore

$$\left[\,\cdot\,|\frac{A_*(ABx)}{\|A_*(ABx)\|}\right] \in J\,(Bx)\,.$$

On the other hand it is easy to verify that

$$\left[\,\cdot\,|\frac{Bx}{\|Bx\|}\right] \in J\,(Bx)\,.$$

The Banach space X is smooth. Thus we get $\operatorname{card} J(Bx) = 1$, whence

$$\left[\,\cdot\,|\frac{A_*(ABx)}{\|A_*(ABx)\|}\right] = \left[\,\cdot\,|\frac{Bx}{\|Bx\|}\right]. \tag{2.6}$$

Using again the properties of s.i.p., we obtain

$$\begin{aligned}\|x+ABx\|^2 &= [x+ABx|x+ABx] = [x|x+ABx] + [ABx|x+ABx] \overset{(\text{sip3})}{\leq}\\ &\leq \|x\|\cdot\|x+ABx\| + \|ABx\|\cdot\|x+ABx\| =\\ &= (\|x\|+\|ABx\|)\cdot\|x+ABx\| \overset{(2.4)}{=}\\ &= \|x+ABx\|\cdot\|x+ABx\| = \|x+ABx\|^2\end{aligned}$$

From the above, we have

$$[x|x+ABx] = \|x\|\cdot\|x+ABx\| and\ [ABx|x+ABx] = \|ABx\|\cdot\|x+ABx\|,$$

whence

$$\left[x|\frac{x+ABx}{\|x+ABx\|}\right]=\|x\| and \left[ABx|\frac{x+ABx}{\|x+ABx\|}\right]=\|ABx\|.$$

Therefore,

$$\left[\cdot|\frac{x+ABx}{\|x+ABx\|}\right]\in J(x) \quad \text{and} \quad \left[\cdot|\frac{x+ABx}{\|x+ABx\|}\right]\in J(ABx). \tag{2.7}$$

On the other hand,

$$[\cdot|x]\in J(x) \quad \text{and} \quad \left[\cdot|\frac{ABx}{\|ABx\|}\right]\in J(ABx). \tag{2.8}$$

Smoothness of X yields that $\text{card}J(x)=1$ and $\text{card}J(ABx)=1$. So, combining (2.7) and (2.8), we immediately get

$$\left[\cdot|\frac{x+ABx}{\|x+ABx\|}\right]=[\cdot|x] \quad \text{and} \quad \left[\cdot|\frac{x+ABx}{\|x+ABx\|}\right]=\left[\cdot|\frac{ABx}{\|ABx\|}\right].$$

It follows from the above equalities that

$$[\cdot|x]=\left[\cdot|\frac{ABx}{\|ABx\|}\right]. \tag{2.9}$$

Fix $y\in X$. Finally, we deduce

$$\begin{aligned}\|B\|^2\cdot[Ay|x] &\overset{(2.9)}{=} \|B\|^2\cdot\left[Ay|\frac{ABx}{\|ABx\|}\right]\overset{(2.3)}{=}\|B\|^2\cdot\left[Ay|\frac{1}{\|AB\|}ABx\right]=\\ &\overset{(2.1)}{=}\|B\|^2\cdot\left[Ay|\frac{1}{\|A\|\cdot\|B\|}ABx\right]\overset{(\text{sip2})}{=}\frac{\|B\|}{\|A\|}\cdot[Ay|ABx]=\\ &=\frac{\|B\|}{\|A\|}\cdot[y|A_*(ABx)]\overset{(2.6),(2.5)}{=}\frac{\|B\|}{\|A\|}\cdot\left[y|\ \|A\|^2\cdot Bx\right]\end{aligned}$$

and by (sip2) we have $\|B\|\cdot[Ay|x]=\|A\|\cdot[y|Bx]$. So we conclude that

$$\exists_{x\in\mathcal{M}(B)}\ \forall_{y\in X}:\ \|B\|\cdot[Ay|x]=\|A\|\cdot[y|Bx]. \tag{2.10}$$

Putting Bx in place of y in the above equality we get

$$[ABx|x]\overset{(2.10)}{=}\frac{\|A\|}{\|B\|}\cdot[Bx|Bx]=\frac{\|A\|}{\|B\|}\cdot\|Bx\|^2=\frac{\|A\|}{\|B\|}\cdot\|B\|^2\cdot\|x\|^2=[x|x]=1$$

From the above, we have $[ABx|x]=[x|x]$. Since $\|AB\|=\|A\|\cdot\|B\|=1$, we have $[AB(\cdot)|x],[\cdot|x]\in J(x)$. Smoothness of X yields that $\text{card}J(x)=1$. So, we immediately get

$$[AB(\cdot)|x]=[\cdot|x]. \tag{2.11}$$

Define $Y := \{y \in X : [y|x] = 1\} \cap \mathbf{B}(X)$. By (2.11) we obtain $AB(Y) \subset Y$. It is known that $AB\colon (X, weak) \to (X, weak)$ is continuous. It is easy to check that Y is a closed convex subset of X. By a theorem of James, Y is a weak-compact subset. Let us consider a function $AB|_Y\colon Y \to Y$. By the Schauder Fixed-Point Theorem, there is a vector x_1 in Y such that $ABx_1 = x_1$. That means $0 = (I - AB)x_1$ which implies that also $\ker(I - AB) \neq \{0\}$. Since $I - AB$ was assumed injective, $\ker(I - AB) = \{0\}$, a contradiction.

We prove (d)⇒(e). Suppose that $I - AB$ has dense range. Assume, for a contradiction, that $\|I + AB\| \geq 2$. Using similar elementary techniques, one may prove $[AB(\cdot)|x] = [\cdot|x]$ for some $x \in \mathcal{M}(B)$, i.e., (2.11). Thus we have $[(I - AB)y|x] = 0$ for all $y \in X$. Since $I - AB$ has dense range, $x = 0$ and we obtain a contradiction. □

Careful reading of the proof of Theorem 2.1 (more precisely, the proof of (c)⇒(e)) shows that we can get the following.

Proposition 2.2 *Let X, A, B, C be as in Theorem 2.1. Suppose that $\|A\| \cdot \|B\| = 1$. Then the following three statements are equivalent:*

(i) $\|I + AB\| = 2$;
(ii) $\exists_{x \in \mathcal{M}(B)}\ \forall_{y \in X} :\ \|B\| \cdot [Ay|x] = \|A\| \cdot [y|Bx]$;
(iii) $\exists_{x \in \mathcal{M}(B)}\ \forall_{y \in X} :\ [AB(y)|x] = [y|x]$.

Proof In a similar way as in the proof of Theorem 2.1 (see the proof of (c)⇒(e)) we obtain the implications $\|I + AB\| = 2$ ⇒(2.10)⇒(2.11) and we may consider (i)⇒(ii)⇒(iii) as shown. Finally, if (iii) holds, then

$$\begin{aligned} 2 = [x|x] + [x|x] &\overset{\text{(iii)}}{=} |[x|x] + [ABx|x]| = |[x + ABx|x]| \leq \\ &\leq \|x + ABx\| \cdot \|x\| \leq \|I + AB\| \leq 1 + \|AB\| \leq 1 + \|A\| \cdot \|B\| = 2 \end{aligned}$$

which yields $\|I + AB\| = 1 + \|A\| \cdot \|B\|$. Thus we get (iii)⇒(i). □

We can add another proposition to our list.

Proposition 2.3 *Let X be as in Theorem 2.1. Let $E \in \mathcal{L}(X)$, $\|E\| \leq 1$. Suppose that there is an operator $C \in \mathcal{K}(X)$ such that $\|E - C\| < \|E\|$. Then the following three statements are equivalent:*

(i) $\|I + E\| = 2$;
(ii) $\exists_{x \in \mathcal{M}(E)}\ \forall_{y \in X} :\ [y|x] = [y|Ex]$;
(iii) $\exists_{x \in \mathbf{B}(X), \|x\|=1}\ \forall_{y \in X} :\ [Ey|x] = [y|x]$.

Proof Putting IE in place of AB in the above proposition we get (i)⇔(ii). Putting EI in place of AB in the above proposition we get (i)⇔(ii). □

As an immediate consequence of Theorem 2.1, we have the following. Actually, it is not necessary to assume that AB is a compact operator. In fact, it suffices to assume reflexivity, smoothness and the two properties: $\mathcal{L}^* = \mathcal{K}^* \oplus_1 \mathcal{K}^{\perp}$, $\|A\| \cdot \|B\| \leq 1$.

Theorem 2.4 *Assume that X is a reflexive smooth Banach space. Suppose that $\mathcal{L}(X)^* = \mathcal{K}(X)^* \oplus_1 \mathcal{K}(X)^{\perp}$. Let $A, B \in \mathcal{L}(X)$ and $\|A\| \cdot \|B\| \leq 1$. Suppose that there is an operator $C \in \mathcal{K}(X)$ such that $\|AB - C\| < \|AB\|$. Then the following four statements are equivalent:*

(i) *for every y_o in X there is an unique u in X such that $u - ABu = y_o$;*
(ii) *for every y_o in X there is a u in X such that $u - ABu = y_o$;*
(iii) *the only vector x such that $x - ABx = 0$ is $x = 0$;*
(iv) $\|I + AB\| < 2$;

Proof It is clear that $I - AB$ is invertible, [surjective, (injective)] if and only if (i) holds , [(ii) holds, ((iii) holds)]. Now by applying Theorem 2.1 we arrive at the desired assertion. □

The Fredholm Alternative, together with Theorem 2.1, allows us to give an extension of Theorem 2.4.

Theorem 2.5 *Let X be as in Theorem 2.1. Let $A, B, D \in \mathcal{L}(X)$. Suppose that $1 \notin \sigma_p(AB)$ and $\|A\| \cdot \|B\| \leq 1$. Suppose that there is an operator $C \in \mathcal{K}(X)$ such that $\|AB - C\| < \|AB\|$. Suppose that $AB - D \in \mathcal{K}(X)$ Then the following three statements are equivalent:*

(i) *for every y_o in X there is an unique u in X such that $u - Du = y_o$;*
(ii) *for every y_o in X there is a w in X such that $w - Dw = y_o$;*
(iii) *the only vector x such that $x - Dx = 0$ is $x = 0$.*

Proof The implications (i)⇒(ii), (i)⇒(iii) are trivial, so we look first at (iii)⇒(i). Assume that $I - D$ is injective. Since $I - AB$ is injective, we conclude that $I - AB$ is invertible (see Theorem 2.1). Suppose, for a contradiction, that $I - D$ is not invertible. Writing

$$I - D = (I - AB) + (AB - D),$$

we can assert that

$$I - D = (I - AB)\left(I + (I - AB)^{-1}(AB - D)\right). \tag{2.12}$$

The operator $I - AB$ is invertible and $I - D$ is not invertible. Therefore, it follows from (2.12) that $\left(I + (I - AB)^{-1}(AB - D)\right)$ is also not invertible. The operator $(I - AB)^{-1}(AB - D)$ is compact. By Theorem 1.1, the operator $I + (I - AB)^{-1}(AB - D)$ is not injective, and thus there is a vector $c \in X \setminus \{0\}$ such that $c + (I - AB)^{-1}(AB - D)c = 0$. So, it is not difficult to check that $c - Dc = 0$. Hence $\ker(I - D) \neq \{0\}$, a contradiction.

The proof of (ii)⇒(i) runs similarly, but it is presented here for the convenience. Now suppose that $I - D$ is surjective. Since $I - AB$ is injective, whence (applying again Theorem 2.1) $I - AB$ is a bijection. Suppose, for a contradiction, that $I - D$ is not invertible. In a similar way as in the proof of (ii)⇒(i) we obtain (2.12). The operator $I - AB$ is invertible and $I - D$ is not invertible. Thus, it follows from (2.12) that $I + (I - AB)^{-1}(AB - D)$ is also not invertible. Since the operator $(I - AB)^{-1}(AB - D)$ is compact, this tells us that $\left(I + (I - AB)^{-1}(AB - D)\right)$ is not surjective (see Theorem

1.1). Hence there exists a vector $e \in X$ so that $x + (I - AB)^{-1}(AB - D)x \neq e$ for all $x \in X$. Now, it is easy to check that $x - Dx \neq e - ABe$ (for all $x \in X$). Now we have a contradiction: $I - D$ is not surjective. The proof is complete. □

3 An Application: Spectrum

As an application of the results in the previous section, we consider the notion of a spectrum of an operator. Let $\mathbb{T} := \{\lambda \in \mathbb{K} : |\lambda| = 1\}$. We next explore some consequences of Theorem 2.1.

Theorem 3.1 *Let X, A, B, C be as in Theorem 2.1. Then*

$$\mathbb{T} \cap \sigma(AB) = \mathbb{T} \cap \sigma_p(AB).$$

Proof Clearly, $\sigma_p(AB) \subset \sigma(AB)$. Let us fix λ in $\mathbb{T} \cap \sigma(AB)$. Then $I - \frac{1}{\lambda}AB$ is not invertible. According to Theorem 2.1 the operator $I - \frac{1}{\lambda}AB$ is not injective. Thus $\lambda \in \sigma_p(AB)$ and so $\lambda \in \mathbb{T} \cap \sigma_p(AB)$. □

The following theorem was discovered by Weyl [8].

Theorem 3.2 [8] *Suppose that H is a complex Hilbert space. If $A \in \mathcal{L}(H)$ and K is a compact operator, then $\sigma(A + K) \setminus \sigma_p(A + K) \subset \sigma(A)$.*

It makes sense to replace the operator AB by AI. Some part of Theorem 3.2 can be strengthen as follows.

Theorem 3.3 *Let X be as in Theorem 2.1. Let $A \in \mathcal{L}(X)$, $\|A\| = 1$. Suppose that there is an operator $C \in \mathcal{K}(X)$ such that $\|A - C\| < 1$. Then*

$$\mathbb{T} \cap (\sigma(A + K) \setminus \sigma_p(A + K)) \subset \mathbb{T} \cap \sigma_p(A),$$

for each compact operator K.

Proof Fix a number $\lambda \in \mathbb{T}$. Assume that $\lambda \in \sigma(A + K) \setminus \sigma_p(A + K)$. Suppose, for a contradiction, that $\lambda \notin \sigma_p(A)$. Then $I - \frac{1}{\lambda}A$ is injective. Define an operator D by $D := \frac{1}{\lambda}(A + K)$. Since $\lambda \notin \sigma_p(A + K)$, it follows that $I - \frac{1}{\lambda}(A + K)$ is injective; this means that $I - D$ is injective.

The inequality $\|A - C\| < 1$ implies $\left\|\frac{1}{\lambda}A - \frac{1}{\lambda}C\right\| < \left\|\frac{1}{\lambda}A\right\|$. It is easy to see that $\frac{1}{\lambda}A - D \in \mathcal{K}(X)$. Moreover, we have $1 \notin \sigma_p\left(\frac{1}{\lambda}A\right)$. Using Theorem 2.5 we see that $I - D$ is also invertible, so $I - \frac{1}{\lambda}(A + K)$ is invertible. Hence $\lambda \notin \sigma(A + K)$. We have our desired contradiction. □

As a consequence of this result, we get a corollary.

Corollary 3.4 *Let X, A be as in Theorem 3.3. Then*

$$\mathbb{T} \cap \bigcup_{K \in \mathcal{K}(X)} (\sigma(A + K) \setminus \sigma_p(A + K)) \subset \mathbb{T} \cap \sigma_p(A).$$

In the following we will show how our knowledge on the smooth operators can be used in order to characterize some points in the spectrum of bounded operator. Note that the following applies to l^p-spaces and, most particularly, to Hilbert spaces.

Proposition 3.5 *Suppose that* $X = \left(\bigoplus_{n=1}^{\infty} X_n\right)_{l_p}$ *with* $p \in (1, \infty)$, $\dim X_n < \infty$. *Suppose that the spaces* X_n *are smooth. Let* $A_1, A_2, B \in \mathcal{L}(X)$, $\|A_1\| = \|A_2\| = \|B\| = 1$, $\|A_1 B + A_2 B\| = 2$. *Assume that* B *is a smooth point of the unit sphere of* $\mathcal{L}(X)$. *Suppose that* A_1, A_2 *are linearly independent. Then*

$$\mathbb{T} \cap \sigma(A_1 B) \cap \sigma(A_2 B) \subset \mathbb{T} \cap \sigma_p(\alpha A_1 B + (1-\alpha) A_2 B)$$

for any $\alpha \in [0, 1]$.

Proof By assumption, X is smooth. It follows from Theorem 1.3 (condition (s1)) that, with some $K \in \mathcal{K}(X)$, $\|B - K\| < 1 = \|B\|$. Fix $\alpha \in [0, 1]$. Fix $\lambda \in \mathbb{T}$ such that $\lambda \in \sigma(A_1 B) \cap \sigma(A_2 B)$. We define A, C as follows: $A := \alpha A_1 + (1-\alpha) A_2$ and $C := \alpha A_1 K + (1-\alpha) A_2 K$. Thus we have $C \in \mathcal{K}(X)$. We have $A \neq 0$, because A_1, A_2 are linearly independent.

Since $\|A_1\| = \|A_2\| = \|B\| = 1$ and $\left\|\frac{1}{2} A_1 B + \frac{1}{2} A_2 B\right\| = 1$ we see that $\|A_1 B\| = \|A_2 B\| = 1$. From this we conclude that $\|\alpha A_1 B + (1-\alpha) A_2 B\| = 1$. Hence $\|AB\| = 1$. We have

$$\begin{aligned} \|AB - C\| &= \|\alpha A_1 B + (1-\alpha) A_2 B - \alpha A_1 K - (1-\alpha) A_2 K\| \leq \\ &\leq \|\alpha A_1 + (1-\alpha) A_2\| \cdot \|B - K\| < \|\alpha A_1 + (1-\alpha) A_2\| \cdot 1 \leq \\ &\leq \alpha \|A_1\| + (1-\alpha)\|A_2\| = 1 = \|AB\| \end{aligned}$$

so that $\|AB - C\| < \|AB\|$. Since $\lambda \in \mathbb{T} \cap \sigma(A_1 B) \cap \sigma(A_2 B)$, it means that $I - \frac{1}{\lambda} A_1 B$, $I - \frac{1}{\lambda} A_1 B$ are not invertible.

We define C_1, C_2 as follows: $C_1 := A_1 K$ and $C_2 := A_2 K$. Thus we have $C_1, C_2 \in \mathcal{K}(X)$. It follows easily that $\|A_1 B - C_1\| < 1 = \|A_1 B\|$ and $\|A_2 B - C_2\| < 1 = \|A_2 B\|$. Now we may apply Theorem 2.1 to conclude that

$$\|I + \frac{1}{\lambda} A_1 B\| = 2, \quad \|I + \frac{1}{\lambda} A_2 B\| = 2. \tag{3.1}$$

Applying Theorem 1.3, we can find the unique vector x_o in the unit sphere of X (up to multiplication with scalars of modulus 1) for which $\|Bx_o\| = 1$. In other words, $\mathcal{M}(B) = \{\beta x_o : \beta \in \mathbb{K}, |\beta| = 1\}$. Combining (3.1) with to Proposition 2.2 gives that

$$\forall_{y \in X} : \left[\frac{1}{\lambda} A_1 B(y) | x_o\right] = [y | x_o], \quad \left[\frac{1}{\lambda} A_2 B(y) | x_o\right] = [y | x_o]. \tag{3.2}$$

As an immediate consequence we see that

$$\forall_{y \in X} : \left[\frac{1}{\lambda}(\alpha A_1 B + (1-\alpha) A_2 B)(y) | x_o\right] = [y | x_o]. \tag{3.3}$$

Summarizing, we have proved $\frac{1}{\lambda}C \in \mathcal{K}(X)$ and

$$\left\| \frac{1}{\lambda}AB - \frac{1}{\lambda}C \right\| < \left\| \frac{1}{\lambda}AB \right\|, \quad \forall_{y \in X} : \left[\frac{1}{\lambda}AB(y)|x_o \right] = [y|x_o]. \tag{3.4}$$

Combining (3.4) and Proposition 2.2 we see that $\left\| I + \frac{1}{\lambda}AB \right\| = 2$. It follows from Theorem 2.1 ((c)⇔(e)) that $I - \frac{1}{\lambda}AB$ is not injective. This means $\lambda \in \sigma_p(AB) = \sigma_p(\alpha A_1 B + (1-\alpha)A_2 B)$, and the proof is complete. □

4 Distance from Operator to Compact Operators

The problem of best approximation of bounded linear operators $\mathcal{L}(X)$ on a Banach space X by compact operators $\mathcal{K}(X)$ has been of great interest in the twentieth century. This section is a small sample of it.

Let $\mathcal{L}(H)$ be the algebra of all bounded operators on infinite-dimensional complex Hilbert space H. An operator $A \in \mathcal{L}(H)$ is *hyponormal* if $A^*A \geq AA^*$. Clearly every hyponormal operator is normal. The interesting result on hyponormal operators may be found in [1, Corollary 3.2].

Theorem 4.1 [1] *If $A \in \mathcal{L}(H)$ is hyponormal and has no isolated eigenvalues of finite multiplicity, then $\|A\| \leq \|A + K\|$, for each compact operator K.*

As an immediate consequence of the above result, we deduce that if $U \in \mathcal{L}(l^2)$ is the unilateral shift, then $\operatorname{dist}(U, \mathcal{K}(l^2)) = 1$. But, this fact can be obtained (or even more generally) by using our next result.

We prove some modifications of Theorem 4.1, where the Hilbert space H is replaced by a smooth Banach space X, and where the condition is satisfied for unit operator only.

Theorem 4.2 *Let X be as in Theorem 2.1. Let $A \in \mathcal{L}(X)$, $\|A\| = 1$. Suppose that*

$$\mathbb{T} \cap \big(\sigma(A) \setminus \sigma_p(A)\big) \neq \emptyset.$$

Then $\operatorname{dist}(A, \mathcal{K}(X)) = 1$. *In particular, $\|A\| \leq \|A + K\|$, for each compact operator $K \in \mathcal{L}(X)$.*

Proof Fix a number $\lambda \in \mathbb{T} \cap \big(\sigma(A) \setminus \sigma_p(A)\big)$. Assume, for a contradiction, that $\operatorname{dist}(A, \mathcal{K}(X)) \neq 1$. Since $\lambda \notin \sigma_p(A)$, we see that $I - \frac{1}{\lambda}A$ is injective. Since $\operatorname{dist}(A, \mathcal{K}(X)) \leq \|A - 0\| = \|A\| = 1$, we obtain $\operatorname{dist}(A, \mathcal{K}(X)) < \|A\|$. It follows that $\operatorname{dist}\left(\frac{1}{\lambda}A, \mathcal{K}(X)\right) < \left\|\frac{1}{\lambda}A\right\|$. Hence, there is $C \in \mathcal{K}(X)$ such that $\left\|\frac{1}{\lambda}A - C\right\| < \left\|\frac{1}{\lambda}A\right\|$. Applying Theorem 2.1 (more precisely, (a)⇔(c)) we see that $I - \frac{1}{\lambda}A$ is invertible. This is a contradiction, since $\lambda \in \sigma(A)$. □

Here a small application is given.

Corollary 4.3 *If $p \in (1, \infty)$, define $U \in \mathcal{L}(l^p)$ by $U(x_1, x_2, \ldots) := (0, x_1, x_2, \ldots)$. $U(x_1, x_2, \ldots) := (0, x_1, x_2, \ldots)$. Then* $\operatorname{dist}(U, \mathcal{K}(l^p)) = 1$.

Proof We know that $\sigma(U) = \{\lambda \in \mathbb{K} : |\lambda| \leq 1\}$. It is easy to check that $\sigma_p(U) = \emptyset$. From this it follows that $\mathbb{T} \cap \big(\sigma(U) \setminus \sigma_p(U)\big) = \mathbb{T}$. By Theorem 4.2 we get $\operatorname{dist}(U, \mathcal{K}(l^p)) = 1$. □

Theorem 4.4 *Let H be a (real or complex) Hilbert space. Suppose that $T \in \mathcal{L}(H)$ is injective but not invertible. Let $\alpha\overline{\beta} \in (0, +\infty)$ and $-\frac{\alpha}{\beta} \notin \sigma(T)$. Let T satisfy $T + T^* \geq 0$. If $A_{\alpha,\beta} := (\alpha I + \beta T)^{-1}$, then $\operatorname{dist}(A_{\alpha,\beta}, \mathcal{K}(H)) = \frac{1}{|\alpha|}$. In particular, $A_{\alpha,\beta}$ is a well-defined operator.*

Proof First it must be shown that $A_{\alpha,\beta}$ is well defined. By assumption, $-\alpha \notin \beta\sigma(T)$. This clearly forces $0 \notin \alpha + \sigma(\beta T) = \sigma(\alpha I + \beta T)$. From this we deduce that $\alpha I + \beta T$ is invertible. So $A_{\alpha,\beta}$ is a well-defined operator.

We want to show that $\|A\| = \frac{1}{|\alpha|}$. By assumption, βT is not invertible, and hence $0 \in \sigma(\beta T)$. From this it follows that $\alpha \in \alpha + \sigma(\beta T)$, and thus $\alpha \in \sigma(\alpha I + \beta T) = \sigma(A^{-1}_{\alpha,\beta})$. So we conclude that $\frac{1}{\alpha} \in \sigma(A_{\alpha,\beta})$ and hence $\frac{1}{|\alpha|} \leq \|A_{\alpha,\beta}\|$.

Let x be an arbitrary unit vector in H. Thus we have

$$\begin{aligned}
1 &= \|x\|^2 = \|A^{-1}_{\alpha,\beta}(A_{\alpha,\beta}x)\|^2 = \|(\alpha I+\beta T)(A_{\alpha,\beta}x)\|^2 = \\
&= \|\alpha A_{\alpha,\beta}x+\beta T(A_{\alpha,\beta}x)\|^2 = \big\langle \alpha A_{\alpha,\beta}x+\beta T(A_{\alpha,\beta}x) | \alpha A_{\alpha,\beta}x+\beta T(A_{\alpha,\beta}x)\big\rangle = \\
&= |\alpha|^2 \cdot \|A_{\alpha,\beta}x\|^2 + \overline{\alpha}\beta \big\langle T(A_{\alpha,\beta}x) | A_{\alpha,\beta}x\big\rangle + \alpha\overline{\beta}\big\langle A_{\alpha,\beta}x | T(A_{\alpha,\beta}x)\big\rangle + \\
&\quad + \|\beta T(A_{\alpha,\beta}x)\|^2 \overset{(\alpha\overline{\beta}\in(0,\infty),\ so\ \alpha\overline{\beta}=\overline{\alpha}\beta)}{=} \\
&= |\alpha|^2 \cdot \|A_{\alpha,\beta}x\|^2 + \alpha\overline{\beta}\big\langle T(A_{\alpha,\beta}x) | A_{\alpha,\beta}x\big\rangle + \alpha\overline{\beta}\big\langle T^*(A_{\alpha,\beta}x) | A_{\alpha,\beta}x\big\rangle + \\
&\quad + \|\beta T(A_{\alpha,\beta}x)\|^2 \geq \\
&\geq |\alpha|^2 \cdot \|A_{\alpha,\beta}x\|^2 + \alpha\overline{\beta}\big\langle (T+T^*)(A_{\alpha,\beta}x) | A_{\alpha,\beta}x\big\rangle \overset{(T+T^*\geq 0)}{\geq} |\alpha|^2 \cdot \|A_{\alpha,\beta}x\|^2.
\end{aligned}$$

We obtain $\|A_{\alpha,\beta}x\| \leq \frac{1}{|\alpha|}$. Passing to the supremum over $\|x\| = 1$ we get $\|A_{\alpha,\beta}\| \leq \frac{1}{|\alpha|}$. Therefore $\|A_{\alpha,\beta}\| = \frac{1}{|\alpha|}$.

We next show that $\operatorname{dist}(A_{\alpha,\beta}, \mathcal{K}(H)) = \frac{1}{|\alpha|}$. It suffices to show that dist $(\alpha A_{\alpha,\beta}, \mathcal{K}(H)) = 1$. Recalling that $\frac{1}{\alpha} \in \sigma(A_{\alpha,\beta})$, we see that $1 \in \sigma(\alpha A_{\alpha,\beta})$.

We want to show that $1 \notin \sigma_p(\alpha A_{\alpha,\beta})$. Assume, for a contradiction, that $\alpha A_{\alpha,\beta} w = w$ for some $w \in H \setminus \{0\}$. Thus

$$\alpha w = A^{-1}_{\alpha,\beta} w = (\alpha I + \beta T)w = \alpha w + \beta T w.$$

It follows that $Tw = 0$. As T is injective we have $w = 0$. This is a contradiction. Thus $1 \notin \sigma_p(\alpha A_{\alpha,\beta})$. Finally, we deduce that $1 \in \sigma(\alpha A_{\alpha,\beta}) \setminus \sigma_p(\alpha A_{\alpha,\beta})$.

This gives $\mathbb{T} \cap (\sigma(\alpha A_{\alpha,\beta}) \setminus \sigma_p(\alpha A_{\alpha,\beta})) \neq \emptyset$. Applying Theorem 4.2, we see that $\operatorname{dist}(\alpha A_{\alpha,\beta}, \mathcal{K}(H)) = 1$. □

If H is a complex Hilbert space and $T \in \mathcal{L}(H)$ such that $\langle Tx|x\rangle = 0$ for all x in H, then $T = 0$. This is not true for real Hilbert spaces. For example, let $U \in \mathcal{L}(\mathbb{R}^2)$ be the linear operator given by $U(x, y) := (-y, x)$. Then we we have $\langle Ux|x\rangle = 0$ for all x in H However, $U \neq 0$. So it makes sense to consider the following result.

Corollary 4.5 *Let H be a real Hilbert space. Suppose that $T \in \mathcal{L}(H)$ is injective but not invertible. Let $\alpha\beta \in (0, +\infty)$ and $-\frac{\alpha}{\beta} \notin \sigma(T)$. Let T satisfy $\langle Tx|x\rangle = 0$ for all x in H. If $A_{\alpha,\beta} := (\alpha I + \beta T)^{-1}$, then* $\operatorname{dist}(A_{\alpha,\beta}, \mathcal{K}(H)) = \frac{1}{|\alpha|}$.

Proof Fix $x \in H$. It follows that $\langle (T+T^*)x|x\rangle = \langle Tx|x\rangle + \langle x|T^*x\rangle = \langle Tx|x\rangle + \langle Tx|x\rangle = 0 \geq 0$. Theorem 4.4 now leads to $\operatorname{dist}(A_{\alpha,\beta}, \mathcal{K}(H)) = \frac{1}{|\alpha|}$. □

5 An Application: System of Linear Equations

Systems of linear equations (such as (5.1)) arise in a number of physical problems. In this section we will discuss solutions of such system by relating the system to a certain noncompact operator.

Theorem 5.1 *Suppose that $X = \left(\bigoplus_{n=1}^{\infty} X_n\right)_{l_p}$ with $p \in (1, \infty)$,* $\dim X_n < \infty$. *Suppose that the spaces X_n are smooth. Let $A \in \mathcal{L}(X)$, $K \in \mathcal{K}(X)$ be nonvanishing operators with $\|A\|^p + \|K\|^p \leq 1$, $\|A\| < \|K\|$. Let us consider the system of linear equations:*

$$\begin{cases} x_1 - Ax_2 = y_1, \\ Kx_1 - x_2 = y_2. \end{cases} \tag{5.1}$$

The following three statements are equivalent:

(a) *for every y_1, y_2 in X there is an unique solution $x_1, x_2 \in X$ of* (5.1)*;*
(b) *for every y_1, y_2 in X there is a solution $x_1, x_2 \in X$ of* (5.1)*;*
(c) *the only vectors x_1, x_2 such that $\begin{cases} x_1 - Ax_2 = 0 \\ Kx_1 - x_2 = 0 \end{cases}$ are $x_1 = 0 = x_2$.*

Proof We define $A \oplus_p K \in \mathcal{L}(X \oplus_p X)$ by the formula $(A \oplus_p K)(u_1, u_2) := (A(u_1), K(u_2))$. It is not difficult to show that $\|A \oplus_p K\| = \|K\| \leq 1$ and $0 \oplus_p K \in \mathcal{K}(X \oplus_p X)$ and

$$\|A \oplus_p K - 0 \oplus_p K\| = \|A \oplus_p 0\| = \|A\| < \|K\| = \|A \oplus_p K\|.$$

Thus we have $\|A \oplus_p K - 0 \oplus_p K\| < \|A \oplus_p K\|$. Moreover, $X \oplus_p X = \left(\bigoplus_{n=1}^{\infty} Y_n\right)_{l_p}$, $Y_n \in \{X_k : k \in \mathbb{N}\}$ and $\mathcal{L}(X \oplus_p X)^* = \mathcal{K}(X \oplus_p X) \oplus_1 \mathcal{K}(X \oplus_p X)^{\perp}$. Applying Theorem 2.4 again, this time to $A \oplus_p K$, we obtain (a)⇔(b)⇔(c). □

6 Remarks: Invertibility

Let $E \in \mathcal{L}(X)$, where X is a Banach space. It is standard that if $\|I - E\| < 1$, then E is invertible. The converse does not hold in general. We prove that under certain conditions it is true. We end this paper with a simple result. However, a few words motivating the proof are appropriate. Let us consider the operator $T := \frac{1}{2}(I - AB)$. If $\|A\| \cdot \|B\| < 1$, then

$$\|I - T\| = \left\| I - \frac{1}{2}I + \frac{1}{2}AB \right\| = \frac{1}{2}\,\|I + AB\| \leq \frac{1}{2}(1 + \|A\| \cdot \|B\|) < 1,$$

which means that T is invertible. So the assumption $\|A\| \cdot \|B\| \leq 1$ makes the invertibility problem more interesting. In particular, the implication (b)⇒(a) (in Theorem 6.1) seem to be amazing.

Proposition 6.1 *Let X, A, B, C be as in Theorem 2.1. If $T := \frac{1}{2}(I - AB)$, then the following three statements are equivalent:*

(a) $\|I - T\| < 1$;
(b) T *is invertible.*

Proof It is easy to check that $\|I + AB\| < 2$ if and only if $\|I - T\| < 1$. It is clear that $I - AB$ is invertible if and only if T is invertible. Applying Theorem 2.1 we at the desired conclusion. □

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