Bavrin's Type Factorization of the Temljakov Operator for Holomorphic Functions in Circular Domains of $\mathbb{C}^n$

Renata Długosz^{1,2} · Piotr Liczberski² ·
Edyta Trybucka³

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Abstract The paper concerns investigations of holomorphic functions of several complex variables with a factorization of their Temljakov transform. Firstly, there were considered some inclusions between the families $\mathcal{C}_{\mathcal{G}}$, $\mathcal{M}_{\mathcal{G}}$, $\mathcal{N}_{\mathcal{G}}$, $\mathcal{R}_{\mathcal{G}}$, $\mathcal{V}_{\mathcal{G}}$ of such holomorphic functions on complete n -circular domain $\mathcal{G}$ of $\mathbb{C}^n$ in some papers of Bavrin, Fukui, Higuchi, Michiwaki. A motivation of our investigations is a condensation of the mentioned inclusions by some new families of Bavrin's type. Hence we consider some families $\mathcal{K}_{\mathcal{G}}^k$, $k \geq 2$, of holomorphic functions $f : \mathcal{G} \rightarrow \mathbb{C}$, $f(0) = 1$, defined also by a factorization of $\mathcal{L}f$ onto factors from $\mathcal{C}_{\mathcal{G}}$ and $\mathcal{M}_{\mathcal{G}}$. We present some interesting properties and extremal problems on $\mathcal{K}_{\mathcal{G}}^k$.

Keywords Holomorphic functions on n -circular domains in $\mathbb{C}^n$ · Minkowski function · Estimates of homogeneous polynomials of Taylor series · Temljakov operator · Bavrin's families of functions

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✉ Piotr Liczberski
piotr.liczberski@p.lodz.pl

Renata Długosz
renata.dlugosz@p.lodz.pl

Edyta Trybucka
eles@ur.edu.pl

¹ Centre of Mathematics and Physics, Lodz University of Technology, Al. Politechniki 11, 90-924 Lodz, Poland

² Institute of Mathematics, Lodz University of Technology, Wólczajska 215, 93-005 Lodz, Poland

³ Faculty of Mathematics and Natural Sciences, University of Rzeszów, Prof. St. Pignonia 1, 35-310 Rzeszow, Poland

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1 Introduction

We say that a domain $\mathcal{G} \subset \mathbb{C}^n$, is complete n -circular if $z\lambda = (z_1\lambda_1, \dots, z_n\lambda_n) \in \mathcal{G}$ for each $z = (z_1, \dots, z_n) \in \mathcal{G}$ and every $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathcal{U}^n$, where $\mathcal{U}$ is the unit disc $\{\zeta \in \mathbb{C} : |\zeta| < 1\}$. From now by $\mathcal{G}$ will be denoted a bounded complete n -circular domain in $\mathbb{C}^n$, $n \geq 2$. By $\mathcal{H}_{\mathcal{G}}$ let us denote the space of all holomorphic functions $f : \mathcal{G} \rightarrow \mathbb{C}$ and by $\mathcal{H}_{\mathcal{G}}(1)$ the collection of all $f \in \mathcal{H}_{\mathcal{G}}$, normalized by $f(0) = 1$.

Many authors (cf., eg., [1, 2, 5–7, 11, 18, 19, 23]) considered some Bavrin's subfamilies $\mathcal{X}_{\mathcal{G}}$ of the family $\mathcal{H}_{\mathcal{G}}(1)$. In the definitions of these families $\mathcal{X}_{\mathcal{G}}$ the main role play the families $\mathcal{C}_{\mathcal{G}}(\alpha)$, $\alpha \in [0, 1)$,

$$\mathcal{C}_{\mathcal{G}}(\alpha) = \{f \in \mathcal{H}_{\mathcal{G}}(1) : \operatorname{Re} f(z) > \alpha, z \in \mathcal{G}\}$$

and the following invertible Temljakov [24] linear operator $\mathcal{L} : \mathcal{H}_{\mathcal{G}} \rightarrow \mathcal{H}_{\mathcal{G}}$

$$\mathcal{L}f(z) = f(z) + Df(z)(z), z \in \mathcal{G},$$

where $Df(z)$ is the Fréchet derivative of f at the point z . By a Bavrin's family $\mathcal{X}_{\mathcal{G}}$ we mean a collection of functions $f \in \mathcal{H}_{\mathcal{G}}(1)$ whose the Temljakov transform $\mathcal{L}f$ has a functional factorization $\mathcal{L}f = p \cdot g$, where $p \in \mathcal{C}_{\mathcal{G}} \equiv \mathcal{C}_{\mathcal{G}}(0)$ and g is from a fixed subfamily of $\mathcal{H}_{\mathcal{G}}(1)$. Below, we recall the factorizations which define a few well known Bavrin's families $\mathcal{X}_{\mathcal{G}}$, like as

$$\begin{aligned}\mathcal{V}_{\mathcal{G}} : \mathcal{L}f &= p \cdot 1, p \in \mathcal{C}_{\mathcal{G}}, \\ \mathcal{M}_{\mathcal{G}} : \mathcal{L}f &= p \cdot f, p \in \mathcal{C}_{\mathcal{G}}, \\ \mathcal{N}_{\mathcal{G}} : \mathcal{L}f &= p \cdot \mathcal{L}f, p \in \mathcal{C}_{\mathcal{G}}, \\ \mathcal{R}_{\mathcal{G}} : \mathcal{L}f &= p \cdot \mathcal{L}\varphi, \varphi \in \mathcal{N}_{\mathcal{G}}, p \in \mathcal{C}_{\mathcal{G}}.\end{aligned}$$

It is known that functions of these families were used to construct biholomorphic mappings in $\mathbb{C}^n$ (cf., eg., [10, 13, 20]). Let us note that the above families have geometric interpretation, in particular the functions $f \in \mathcal{M}_{\mathcal{G}}$ map biholomorphically some planar intersections $\mathcal{S}$ of $\mathcal{G}$ onto starlike domains in $\mathbb{C}$, (see [1]). It is very important, because the starlikeness plays a central role in many different subjects of geometry and topology and in particular, in geometric function theory.

Let us recall also that Bavrin showed the inclusions $\mathcal{N}_{\mathcal{G}} \subsetneq \mathcal{R}_{\mathcal{G}}$, $\mathcal{V}_{\mathcal{G}} \subsetneq \mathcal{R}_{\mathcal{G}}$ and pointed that the first of them can be complete to the following double inclusion $\mathcal{N}_{\mathcal{G}} \subsetneq \mathcal{M}_{\mathcal{G}} \subsetneq \mathcal{R}_{\mathcal{G}}$. Thus, it is natural to ask whether is possible to do the same in the case of the second above inclusion. In the paper [12] the authors defined a family $\mathcal{K}_{\mathcal{G}}^{-}$, which satisfies the inclusion $\mathcal{V}_{\mathcal{G}} \subsetneq \mathcal{K}_{\mathcal{G}}^{-} \subsetneq \mathcal{R}_{\mathcal{G}}$. An adequate definition of $\mathcal{K}_{\mathcal{G}}^{-}$ has the form: A function $f \in \mathcal{H}_{\mathcal{G}}(1)$ belongs to $\mathcal{K}_{\mathcal{G}}^{-}$ if its Temljakov transform $\mathcal{L}f$ has the factorization

$$\mathcal{L}f(z) = p(z) \cdot h(z) \cdot h(-z), z \in \mathcal{G}, h \in \mathcal{M}_{\mathcal{G}}\left(\frac{1}{2}\right), p \in \mathcal{C}_{\mathcal{G}},$$

where the family $\mathcal{M}_{\mathcal{G}}(\alpha)$, $\alpha \in [0, 1)$, is defined similarly as $\mathcal{M}_{\mathcal{G}}$, but in this case $p \in \mathcal{C}_{\mathcal{G}}(\alpha)$.

In the present paper we consider Bavrin's type families $\mathcal{K}_{\mathcal{G}}^k$, $k \geq 2$ ($\mathcal{K}_{\mathcal{G}}^2 = \mathcal{K}_{\mathcal{G}}^-$) separating also the families $\mathcal{V}_{\mathcal{G}}$, $\mathcal{R}_{\mathcal{G}}$, i.e., satisfying the inclusions $\mathcal{V}_{\mathcal{G}} \subsetneq \mathcal{K}_{\mathcal{G}}^k \subsetneq \mathcal{R}_{\mathcal{G}}$, $k \geq 2$.

The formal definition of such family has the following form.

A function $f \in \mathcal{H}_{\mathcal{G}}(1)$ belongs to $\mathcal{K}_{\mathcal{G}}^k$ if there exist a function $p \in \mathcal{C}_{\mathcal{G}}$ and a function $h \in \mathcal{M}_{\mathcal{G}}(\frac{k-1}{k})$ such that the Temljakov transform $\mathcal{L}f$ of f has the factorization

$$\mathcal{L}f(z) = p(z) \cdot \prod_{l=0}^{k-1} h(\varepsilon^l z), z \in \mathcal{G}, \quad (1.1)$$

where $\varepsilon = \varepsilon_k = \exp \frac{2\pi i}{k}$ is a generator of the cyclic group of k th roots of unity.

Let us observe that $\mathcal{K}_{\mathcal{G}}^k$, $k \geq 2$ are nonempty families. Indeed, the function $f = 1$ belongs to $\mathcal{K}_{\mathcal{G}}^k$, because it satisfies the factorization (1.1) with $p = 1 \in \mathcal{C}_{\mathcal{G}}$ and $h = 1 \in \mathcal{M}_{\mathcal{G}}(\frac{k-1}{k})$.

In the future, we will use a characterization of the family $\mathcal{K}_{\mathcal{G}}^k$ by a notion of (j, k) -symmetry, which is connected with a functional decomposition with respect to the above group.

Let us observe that bounded complete n -circular domains $\mathcal{G}$ are k -symmetric sets for $k = 2, 3, \dots$, that is $\varepsilon \mathcal{G} = \mathcal{G}$. For $j = 0, 1, \dots, k-1$ we define the collections $\mathcal{F}_{j,k}(\mathcal{G})$ of functions (j, k) -symmetrical, i.e., all functions $f: \mathcal{G} \rightarrow \mathbb{C}$ such that

$$f(\varepsilon z) = \varepsilon^j f(z), z \in \mathcal{G}.$$

If $n = 1$ and $\mathcal{G} = \mathcal{U}$, then we write $\mathcal{F}_{j,k}(\mathcal{U})$.

The mentioned functional decomposition appears in the following result from [14].

Theorem A *For every function $f: \mathcal{G} \rightarrow \mathbb{C}$ there exists exactly one sequence of functions $f_{j,k} \in \mathcal{F}_{j,k}(\mathcal{G})$, $j = 0, 1, \dots, k-1$, such that*

$$f = \sum_{j=0}^{k-1} f_{j,k}.$$

Moreover,

$$f_{j,k}(z) = \frac{1}{k} \sum_{l=0}^{k-1} \varepsilon^{-jl} f(\varepsilon^l z), z \in \mathcal{G}.$$

The functions $f_{j,k}$, which are uniquely determined by the above decomposition, will be called (j, k) -symmetrical components of the function f . Some interesting applications of the above partition may also be found in [15, 16] and [17].

2 Results

Now we can present a characterization of $f \in \mathcal{K}_{\mathcal{G}}^k$, simpler than (1.1).

Theorem 1 *A function $f \in \mathcal{H}_{\mathcal{G}}(1)$ belongs to the family $\mathcal{K}_{\mathcal{G}}^k$, $k \geq 2$ if and only if there exists a function $g \in \mathcal{M}_{\mathcal{G}} \cap \mathcal{F}_{0,k}(\mathcal{G})$ and a function $p \in \mathcal{C}_{\mathcal{G}}$ such that*

$$\mathcal{L}f = p \cdot g. \quad (2.1)$$

Proof Let $f \in \mathcal{K}_{\mathcal{G}}^k$. Then there exists $p \in \mathcal{C}_{\mathcal{G}}$ and $h \in \mathcal{M}_{\mathcal{G}}(\frac{k-1}{k})$ such that

$$\mathcal{L}f(z) = p(z) \cdot g(z), \quad z \in \mathcal{G},$$

where

$$g(z) = \prod_{l=0}^{k-1} h(\varepsilon^l z), \quad z \in \mathcal{G}.$$

It is obvious that $g \in \mathcal{F}_{0,k}(\mathcal{G})$. We show that $g \in \mathcal{M}_{\mathcal{G}}$. To do it, using the differentiation product rule and the form of the operator $\mathcal{L}$, we have at $z \in \mathcal{G}$

$$\frac{\mathcal{L}g(z)}{g(z)} = 1 + \frac{Dg(z)(z)}{g(z)} = 1 + \sum_{l=0}^{k-1} \frac{Dh(\varepsilon^l z)(\varepsilon^l z)}{h(\varepsilon^l z)} = 1 - k + \sum_{l=0}^{k-1} \frac{\mathcal{L}h(\varepsilon^l z)}{h(\varepsilon^l z)}.$$

Hence and by the fact that $h \in \mathcal{M}_{\mathcal{G}}(\frac{k-1}{k})$, we obtain that $\operatorname{Re} \frac{\mathcal{L}g(z)}{g(z)} > 1 - k + k \frac{k-1}{k} = 0$. Thus $g \in \mathcal{M}_{\mathcal{G}}$.

Now, let us suppose that f satisfies the equality (2.1), with a $p \in \mathcal{C}_{\mathcal{G}}$ and a $g \in \mathcal{M}_{\mathcal{G}} \cap \mathcal{F}_{0,k}(\mathcal{G})$. Let us put $h(z) = (g(z))^{\frac{1}{k}}$, $z \in \mathcal{G}$, with the power function taking the value 1 at the point 1. Since $g(z) \neq 0$ (see [1]), the function h is holomorphic. It remains to show that $h \in \mathcal{M}_{\mathcal{G}}(\frac{k-1}{k})$ and the equality (1.1) is fulfilled. To this end we compute step by step

$$\begin{aligned} \operatorname{Re} \frac{\mathcal{L}h(z)}{h(z)} &= \operatorname{Re} \frac{\mathcal{L}(g(z))^{\frac{1}{k}}}{(g(z))^{\frac{1}{k}}} = 1 + \frac{1}{k} \operatorname{Re} \frac{(g(z))^{\frac{1}{k}-1} Dg(z)(z)}{(g(z))^{\frac{1}{k}}} \\ &= 1 + \frac{1}{k} \operatorname{Re} \frac{Dg(z)(z)}{g(z)} = \frac{k-1}{k} + \frac{1}{k} \operatorname{Re} \frac{\mathcal{L}g(z)}{g(z)} > \frac{k-1}{k}. \end{aligned}$$

The formula (1.1) follows from the definition of the function h . Indeed,

$$g(z) = (h(z))^k = \prod_{l=0}^{k-1} h(\varepsilon^l z), \quad z \in \mathcal{G},$$

because $h \in \mathcal{F}_{0,k}(\mathcal{G})$.

The proof is complete. $\square$

Now we consider an extremal problem for $f \in \mathcal{K}_{\mathcal{G}}^k$. More precisely, we look for some estimates for $\mathcal{G}$ -balances of m -homogeneous polynomials $Q_{f,m}$ of its unique power series expansion

$$f(z) = 1 + \sum_{m=1}^{\infty} Q_{f,m}(z), z \in \mathcal{G}. \quad (2.2)$$

In our considerations the Minkowski function

$$\mu_{\mathcal{G}}(z) = \inf\{t > 0: \frac{1}{t}z \in \mathcal{G}\}, z \in \mathbb{C}^n,$$

will be very useful. This function gives a possibility to redefine the domain $\mathcal{G}$ and its boundary $\partial\mathcal{G}$ as follows:

$$\mathcal{G} = \{z \in \mathbb{C}^n: \mu_{\mathcal{G}}(z) < 1\}, \partial\mathcal{G} = \{z \in \mathbb{C}^n: \mu_{\mathcal{G}}(z) = 1\}.$$

The notion of $\mathcal{G}$ -balance of m -homogeneous polynomial $Q_m: \mathbb{C}^n \rightarrow \mathbb{C}$, $m \in \mathbb{N} \cup \{0\}$, was defined in [3] as the quantity

$$\mu_{\mathcal{G}}(Q_m) = \sup_{w \in \mathbb{C}^n \setminus \{0\}} \frac{|Q_m(w)|}{(\mu_{\mathcal{G}}(w))^m} = \sup_{v \in \partial\mathcal{G}} |Q_m(v)| = \sup_{u \in \mathcal{G}} |Q_m(u)|.$$

The $\mathcal{G}$ -balance $\mu_{\mathcal{G}}(Q_m)$ generalizes the norm $\|Q_m\|$ of the polynomial Q_m and if $\mathcal{G}$ is convex, then $\mu_{\mathcal{G}}(Q_m)$ reduces to $\|Q_m\|$, because

$$|Q_m(w)| \leq \mu_{\mathcal{G}}(Q_m)(\mu_{\mathcal{G}}(w))^m, w \in \mathbb{C}^n$$

and for bounded convex complete n -circular domains $\mathcal{G}$ also $\mu_{\mathcal{G}}(w) = \|w\|$ (see, e.g., [21]).

We present the announced estimates of $\mathcal{G}$ -balances $\mu_{\mathcal{G}}(Q_{f,m})$ of m -homogeneous polynomials $Q_{f,m}$ from the Taylor series of $f \in \mathcal{M}_{\mathcal{G}}^k$ in the following theorem.

Theorem 2 *If the expansion of the function $f \in \mathcal{K}_{\mathcal{G}}^k$, $k \geq 2$, into a series of m -homogeneous polynomials $Q_{f,m}$ has the form (2.2), then for the $\mathcal{G}$ -balances $\mu_{\mathcal{G}}(Q_{f,m})$ of polynomials $Q_{f,m}$ the following sharp estimate hold:*

$$\mu_{\mathcal{G}}(Q_{f,m}) \leq \begin{cases} \frac{2}{m} \prod_{p=1}^{\frac{m}{k}-1} \left(1 + \frac{2}{pk}\right) & \text{for } m = k, 2k, 3k, \dots \\ \frac{2}{m+1} \prod_{p=1}^{\lfloor \frac{m}{k} \rfloor} \left(1 + \frac{2}{pk}\right) & \text{for remaining } m \in \mathbb{N} \end{cases},$$

where $\lfloor q \rfloor$ means the integral part of the number q . We use a standard convention that the product $\prod_{l=1}^{l_2} a_l$ is equal to 1 for $l_2 < l_1$.

Proof Let $f \in \mathcal{K}_{\mathcal{G}}^k$ be arbitrarily fixed. Then, by Theorem 1, the factorization (2.1) holds with a function $p \in \mathcal{C}_{\mathcal{G}}$ of the form

$$p(z) = 1 + \sum_{v=1}^{\infty} Q_{p,v}(z), \quad z \in \mathcal{G}$$

and a function $g \in \mathcal{M}_{\mathcal{G}} \cap \mathcal{F}_{0,k}(\mathcal{G})$ of the form

$$g(z) = 1 + \sum_{v=1}^{\infty} Q_{g,kv}(z), \quad z \in \mathcal{G}. \quad (2.3)$$

From the above, by the series expansion of $\mathcal{L}f$

$$\mathcal{L}f(z) = 1 + \sum_{m=1}^{\infty} Q_{\mathcal{L}f,m}(z) = 1 + \sum_{m=1}^{\infty} (m+1)Q_{f,m}(z), \quad z \in \mathcal{G}$$

and by the equalities $Q_{f,0} = Q_{p,0} = Q_{g,0} = 1$, we obtain the recursive formula for $m \in \mathbb{N}$

$$(m+1)Q_{f,m}(z) = \sum_{l=0}^{\lfloor \frac{m}{k} \rfloor} Q_{g,kl}(z)Q_{p,m-kl}(z), \quad z \in \mathcal{G}.$$

Hence

$$(m+1)|Q_{f,m}(z)| \leq \sum_{l=0}^{\lfloor \frac{m}{k} \rfloor} |Q_{g,kl}(z)| |Q_{p,m-kl}(z)|, \quad z \in \mathcal{G}. \quad (2.4)$$

Since

$$|Q_{p,v}(z)| \leq 2, \quad v \in \mathbb{N}, \quad z \in \mathcal{G}, \quad (2.5)$$

(see [1]) we need some bounds for $|Q_{g,k\mu}(z)|$. We show that for $g \in \mathcal{M}_{\mathcal{G}} \cap \mathcal{F}_{0,k}(\mathcal{G})$ and $\mu \in \mathbb{N}$ there hold the inequalities

$$|Q_{g,k\mu}(z)| \leq \frac{2}{k\mu} \prod_{v=1}^{\mu-1} \left(1 + \frac{2}{kv}\right), \quad z \in \mathcal{G}. \quad (2.6)$$

For this purpose let us observe that for each $z \in \mathcal{G}$, the function

$$G(\zeta) = \zeta g(\zeta z), \quad \zeta \in \mathcal{U}$$

belongs to the family $\mathcal{S}^* \cap \mathcal{F}_{1,k}(\mathcal{U})$ of $(1, k)$ -symmetric univalent starlike mappings (in the unit disc $\mathcal{U}$) and its Taylor series has the form

$$G(\zeta) = \zeta + \sum_{\mu=1}^{\infty} b_{k\mu+1} \zeta^{k\mu+1} = 1 + \sum_{\mu=1}^{\infty} Q_{g,k\mu}(z) \zeta^{k\mu+1}, \quad \zeta \in \mathcal{U}.$$

Thus, in view of the estimates [25] of the coefficients of functions from $\mathcal{S}^* \cap \mathcal{F}_{1,k}(\mathcal{U})$ we get the announced bounds (2.6).

In two next parts of the proof we use also the fact [4] that for every $k, s \in \mathbb{N} \setminus \{1\}$ there holds the identity:

$$1 + \frac{2}{k} + \sum_{l=2}^s \frac{2}{lk} \prod_{v=1}^{l-1} \left(1 + \frac{2}{vk}\right) = \prod_{v=1}^s \left(1 + \frac{2}{vk}\right). \quad (2.7)$$

Now, we will estimate the quantities $|Q_{f,m}(z)|$, $z \in \mathcal{G}$, using all the conditions (2.4)-(2.7).

First let us assume that $m = ks$, where $s \in \mathbb{N}$. Since $Q_{p,m-kl}(z) = 1$ for $l = s$, we get from (2.4) that

$$(m+1) |Q_{f,m}(z)| \leq |Q_{g,ks}(z)| + 2 \sum_{l=0}^{s-1} |Q_{g,kl}(z)|, \quad z \in \mathcal{G}.$$

Thus for $z \in \mathcal{G}$, in view of (2.6) and (2.7),

$$\begin{aligned} (m+1) |Q_{f,m}(z)| &\leq \frac{2}{sk} \prod_{v=1}^{s-1} \left(1 + \frac{2}{vk}\right) + 2 \left[1 + \frac{2}{k} + \sum_{l=2}^{s-1} \frac{2}{lk} \prod_{v=1}^{l-1} \left(1 + \frac{2}{vk}\right) \right] \\ &= \frac{-2}{sk} \prod_{v=1}^{s-1} \left(1 + \frac{2}{vk}\right) + 2 \left[1 + \frac{2}{k} + \sum_{l=2}^s \frac{2}{lk} \prod_{v=1}^{l-1} \left(1 + \frac{2}{vk}\right) \right] \\ &\leq \frac{-2}{sk} \prod_{v=1}^{s-1} \left(1 + \frac{2}{vk}\right) + 2 \prod_{v=1}^s \left(1 + \frac{2}{vk}\right) \\ &= 2 \frac{(sk+1)}{sk} \prod_{v=1}^{s-1} \left(1 + \frac{2}{vk}\right). \end{aligned}$$

Hence, for $m = k, 2k, 3k, \dots$

$$|Q_{f,m}(z)| \leq \frac{2}{m} \prod_{v=1}^{\frac{m}{k}-1} \left(1 + \frac{2}{vk}\right), \quad z \in \mathcal{G}.$$

Now let us consider the case $m = ks + r$, where $s \in \mathbb{N} \cup \{0\}$ and $r \in \{1, 2, \dots, k-1\}$. In this case we apply in (2.4) the inequality $|Q_{p,m-kl}(z)| \leq 2$, $l = 0, \dots, s = \lfloor \frac{m}{k} \rfloor$, which follows from estimates (2.5), because $m - kl > 0$. Thus, in view of (2.6) and (2.7) we get step by step

$$\begin{aligned}
 (m+1) |Q_{f,m}(z)| &\leq 2 \sum_{l=0}^{\lfloor \frac{m}{k} \rfloor} |Q_{g,kl}(z)| \leq 2 \left[1 + \frac{2}{k} + \sum_{l=2}^{\lfloor \frac{m}{k} \rfloor} \frac{2}{lk} \prod_{v=1}^{l-1} \left(1 + \frac{2}{vk} \right) \right] \\
 &\leq 2 \prod_{v=1}^{\lfloor \frac{m}{k} \rfloor} \left(1 + \frac{2}{vk} \right).
 \end{aligned}$$

Summing up the results of both cases we get

$$|Q_{f,m}(z)| \leq \begin{cases} \frac{2}{m} \prod_{v=1}^{\frac{m}{k}-1} \left(1 + \frac{2}{vk} \right) & \text{for } m = k, 2k, 3k, \dots \\ \frac{2}{m+1} \prod_{v=1}^{\lfloor \frac{m}{k} \rfloor} \left(1 + \frac{2}{vk} \right) & \text{for remaining } m \in \mathbb{N} \end{cases}, z \in \mathcal{G}$$

and consequently

$$\sup_{z \in \mathcal{G}} |Q_{f,m}(z)| \leq \begin{cases} \frac{2}{m} \prod_{v=1}^{\frac{m}{k}-1} \left(1 + \frac{2}{vk} \right) & \text{for } m = k, 2k, 3k, \dots \\ \frac{2}{m+1} \prod_{v=1}^{\lfloor \frac{m}{k} \rfloor} \left(1 + \frac{2}{vk} \right) & \text{for remaining } m \in \mathbb{N} \end{cases}.$$

These inequalities and the definition of $\mathcal{G}$ -balances $\mu_{\mathcal{G}}(Q_{f,m})$ of m -homogeneous polynomials imply the estimates from the statement of the theorem.

Now, we will show the sharpness of the above estimates.

For the linear functional $I = (\mu_{\mathcal{G}}(J))^{-1} J$, with

$$J(z) = \sum_{l=1}^n z_l, z = (z_1, \dots, z_n) \in \mathbb{C}^n,$$

let us denote by $\mathcal{Z}$ an analytic set $\mathcal{G} \cap I^{-1}\{0\}$ and let $I^m(z) = (Iz)^m$, $z \in \mathcal{G}$, $m \in \mathbb{N} \cup \{0\}$. The equalities in our estimates are achieved for the following function $f \in \mathcal{K}_{\mathcal{G}}^k$, $k \geq 2$,

$$f(z) = \begin{cases} \frac{\sum_{l=0}^{k-1} I^{l-1}(z)}{(1-I^k(z))^{\frac{2}{k}}} - \frac{1}{I(z)} - \sum_{l=3}^{k-1} \frac{l-2}{l} I^{l-1}(z) H\left(\frac{2}{k}, \frac{l}{k}, \frac{l+k}{k}, I^k(z)\right) & \text{for } z \in \mathcal{G} \setminus \mathcal{Z}, \\ 1 & \text{for } z \in \mathcal{Z} \end{cases} \quad (2.8)$$

where $H(a, b, c, \zeta) : \mathcal{U} \rightarrow \mathbb{C}$ is a hypergeometric function

$$H(a, b, c, \zeta) = \sum_{v=0}^{\infty} \frac{(a)_v (b)_v \zeta^v}{(c)_v v!}, \zeta \in \mathcal{U},$$

defined by Pochhammer symbols $(a)_v, (b)_v, (c)_v$:

$$(a)_v = \begin{cases} a(a+1) \dots (a+v-1), & v \in \mathbb{N} \\ 1, & v = 0 \end{cases},$$

and the branch of the power function $x^{\frac{2}{k}}$ takes the value 1 at the point $x = 1$. In the case $k = 2, 3$ we use a standard convention that the sum

$$\sum_{l=3}^{k-1} \frac{l-2}{l} I^{l-1}(z) H\left(\frac{2}{k}, \frac{l}{k}, \frac{l+k}{k}, I^k(z)\right), \quad z \in \mathcal{G}$$

is equal to zero, if the superscript of the sum is smaller than the subscript.

In the paper [4], it was proven that the above function gives the equalities in the bounds from the statement of the theorem. It remains to show that $f \in \mathcal{K}_{\mathcal{G}}^k$ for $k \geq 2$. To do it, let us observe that as shown in [4]

$$\mathcal{L}f(z) = \frac{1+I(z)}{1-I(z)} \frac{1}{(1-I^k(z))^{\frac{2}{k}}}, \quad z \in \mathcal{G}.$$

This implies, in view of Theorem 1, the relation $f \in \mathcal{K}_{\mathcal{G}}^k$, because the functions

$$p(z) = \frac{1+I(z)}{1-I(z)}, \quad g(z) = \frac{1}{(1-I^k(z))^{\frac{2}{k}}}, \quad z \in \mathcal{G}$$

belong to $\mathcal{C}_{\mathcal{G}}$ and to $\mathcal{M}_{\mathcal{G}} \cap \mathcal{F}_{0,k}(\mathcal{G})$, respectively. $\square$

We use the estimates of $\mathcal{G}$ -balances $\mu_{\mathcal{G}}(Q_{f,m})$ of polynomials $Q_{f,m}$ to solve the mentioned separation problem for the families $\mathcal{V}_{\mathcal{G}}, \mathcal{K}_{\mathcal{G}}^k, \mathcal{R}_{\mathcal{G}}$. We prove the following theorem:

Theorem 3 *For every $k \geq 2$ there holds the double inclusion*

$$\mathcal{V}_{\mathcal{G}} \subsetneq \mathcal{K}_{\mathcal{G}}^k \subsetneq \mathcal{R}_{\mathcal{G}}.$$

Proof We start with the inclusion $\mathcal{V}_{\mathcal{G}} \subset \mathcal{K}_{\mathcal{G}}^k$. To do it, let us assume that $f \in \mathcal{V}_{\mathcal{G}}$, then $\mathcal{L}f \in \mathcal{C}_{\mathcal{G}}$. Putting $p = \mathcal{L}f$ and $h = 1$, we obtain the factorization (1.1) with $p \in \mathcal{C}_{\mathcal{G}}$ and $g = 1 \in \mathcal{M}_{\mathcal{G}} \cap \mathcal{F}_{0,k}(\mathcal{G})$. Hence $f \in \mathcal{K}_{\mathcal{G}}^k$. It remains to show the relation $\mathcal{V}_{\mathcal{G}} \neq \mathcal{K}_{\mathcal{G}}^k$. To do it, let us observe that for $f \in \mathcal{V}_{\mathcal{G}}$ there hold the sharp estimates $\mu_{\mathcal{G}}(Q_{f,m}) \leq \frac{2}{m+1}, m \in \mathbb{N}$ (cf., eg., [1]), while for $f \in \mathcal{K}_{\mathcal{G}}^k$ the sharp estimates $\mu_{\mathcal{G}}(Q_{f,m}) \leq B(m)$ (Theorem 2.), with the obvious bound $B(m) > \frac{2}{m+1}, m \in \mathbb{N} \setminus \{1\}$. Hence, the extremal function $f \in \mathcal{K}_{\mathcal{G}}^k$ does not belong to $\mathcal{V}_{\mathcal{G}}$.

Now we prove that $\mathcal{K}_{\mathcal{G}}^k \subset \mathcal{R}_{\mathcal{G}}$. To this end, let us suppose that $f \in \mathcal{K}_{\mathcal{G}}^k$. Then there exist functions $p \in \mathcal{C}_{\mathcal{G}}, g \in \mathcal{M}_{\mathcal{G}} \cap \mathcal{F}_{0,k}(\mathcal{G})$ such that $\mathcal{L}f = p \cdot g$. Denoting $\varphi = \mathcal{L}^{-1}g$, we have that $\varphi \in \mathcal{N}_{\mathcal{G}}$ (by the Aleksander type theorem [1]) and $\mathcal{L}f = p\mathcal{L}\varphi$. Thus

$f \in \mathcal{R}_{\mathcal{G}}$. It remains to show the relation $\mathcal{K}_{\mathcal{G}}^k \neq \mathcal{R}_{\mathcal{G}}$. For this purpose, let us observe that in the above estimates $\mu_{\mathcal{G}}(Q_{f,m}) \leq B(m)$, $m \in \mathbb{N}$, we have $B(m) \leq 1$, $m \in \mathbb{N}$ (see below), while for $f \in \mathcal{R}_{\mathcal{G}}$ there hold the sharp estimates $\mu_{\mathcal{G}}(Q_{f,m}) \leq m+1$ (see for instance [1]). Therefore, the extremal function $f \in \mathcal{R}_{\mathcal{G}}$ does not belong to $\mathcal{K}_{\mathcal{G}}^k$.

To complete the proof, we show that $B(m) \leq 1$, $m \in \mathbb{N}$. To do it, we consider two cases, according to the partition $m = ks + r$, $r \in \{0, 1, \dots, k-1\}$, from the proof of Theorem 2.

1. Let us suppose that $r = 0$. Then, if $s = \frac{m}{k} = 1$, we see that the superscript $s-1$ of the first product in Theorem 2 is smaller than its subscript 1. Hence, we replace the referred product by 1 and consequently, we get $\mu_{\mathcal{G}}(Q_{f,m}) \leq \frac{2}{m} \leq 1$, because $m = k \geq 2$. Next, if $s \geq 2$, then from Theorem 2, by the inequality $1 + \frac{2}{vk} \leq \frac{v+1}{v}$, $v \in \mathbb{N}$, $k \in \mathbb{N} \setminus \{1\}$, we obtain

$$\mu_{\mathcal{G}}(Q_{f,m}) \leq \frac{2}{m} \prod_{v=1}^{s-1} \frac{v+1}{v} \leq \frac{2}{m} s = \frac{2}{k} \leq 1.$$

2. Let us suppose that $r \in \{1, \dots, k-1\}$. Then, if $s = \lfloor \frac{m}{k} \rfloor = 0$, we see that the superscript of the second product in Theorem 2 is smaller than its subscript. Hence we replace the referred product by 1 and consequently, we get $\mu_{\mathcal{G}}(Q_{f,m}) \leq \frac{2}{m+1} \leq 1$, because $m \leq k-1$. Next, if $s = \lfloor \frac{m}{k} \rfloor \geq 1$, then similarly as in step 1, we obtain

$$\mu_{\mathcal{G}}(Q_{f,m}) \leq \frac{2}{m+1} \prod_{v=1}^s \frac{v+1}{v} \leq \frac{2}{m+1} (s+1) \leq \frac{2(s+1)}{ks+2} \leq \frac{2(s+1)}{2s+2} \leq 1.$$

□

Now, we give a growth theorem for $f \in \mathcal{K}_{\mathcal{G}}^k$ and its Temljakov transform $\mathcal{L}f$.

Theorem 4 For functions $f \in \mathcal{K}_{\mathcal{G}}^k$ there follow the following sharp estimates

$$\frac{1-r}{1+r} \frac{1}{(1+r^k)^{\frac{2}{k}}} \leq |\mathcal{L}f(z)| \leq \frac{1+r}{1-r} \frac{1}{(1-r^k)^{\frac{2}{k}}}, r = \mu_{\mathcal{G}}(z) \in [0, 1), \quad (2.9)$$

$$\frac{1}{r} \int_0^r \frac{1-\varrho}{1+\varrho} \frac{1}{(1+\varrho^k)^{\frac{2}{k}}} d\varrho \leq |f(z)| \leq \frac{1}{r} \int_0^r \frac{1+\varrho}{1-\varrho} \frac{1}{(1-\varrho^k)^{\frac{2}{k}}} d\varrho, \quad r = \mu_{\mathcal{G}}(z) \in [0, 1). \quad (2.10)$$

Proof First, let us observe that the above estimates are true for $z = 0$ (in (2.10) the values at $r = 0$, of the left and right hand sides, mean the limit if $r \rightarrow 0^+$). Thus, in the sequel we will assume that $z \in \mathcal{G} \setminus \{0\}$. We start with the estimates (2.9). Since $f \in \mathcal{K}_{\mathcal{G}}^k$, there exist a function $p \in \mathcal{C}_{\mathcal{G}}$ and a function $g \in \mathcal{M}_{\mathcal{G}} \cap \mathcal{F}_{0,k}(\mathcal{G})$ such that the factorization (2.1) holds. Therefore, we show for such functions g the following inequalities

$$\frac{1}{(1+r^k)^{\frac{2}{k}}} \leq |g(z)| \leq \frac{1}{(1-r^k)^{\frac{2}{k}}}, r = \mu_{\mathcal{G}}(z) \in (0, 1).$$

To this aim, let us fix arbitrarily a point $z \in \mathcal{G}$ such that $\mu_{\mathcal{G}}(z) = r \in (0, 1)$ and let us consider the function

$$G(\zeta) = \zeta g\left(\zeta \frac{z}{\mu_{\mathcal{G}}(z)}\right), \zeta \in \mathcal{U}.$$

Then G is $(1, k)$ -symmetric, holomorphic, normalized and satisfies the condition

$$\operatorname{Re} \frac{\zeta G'(\zeta)}{G(\zeta)} = \operatorname{Re} \frac{\mathcal{L}g\left(\zeta \frac{z}{\mu_{\mathcal{G}}(z)}\right)}{g\left(\zeta \frac{z}{\mu_{\mathcal{G}}(z)}\right)} > 0, \zeta \in \mathcal{U}.$$

Hence $G \in \mathcal{S}^* \cap \mathcal{F}_{1,k}(\mathcal{U})$ and by [9, Thm. 2.2.13]

$$\frac{|\zeta|}{(1+|\zeta|^k)^{\frac{2}{k}}} \leq |G(\zeta)| \leq \frac{|\zeta|}{(1-|\zeta|^k)^{\frac{2}{k}}}, \zeta \in \mathcal{U}.$$

Putting $\zeta = \mu_{\mathcal{G}}(z)$ in the above we obtain, by the definition of the function G , the announced inequality.

On the other hand, there hold for $p \in \mathcal{C}_{\mathcal{G}}$ the following estimates [1]

$$\frac{1-r}{1+r} \leq |p(z)| \leq \frac{1+r}{1-r}, r = \mu_{\mathcal{G}}(z) \in (0, 1),$$

Using the estimates of $|p(z)|$ and $|g(z)|$ we get the estimates (2.9). The sharpness of the upper bounds (2.9) confirms the function given by (2.8). Indeed, for $r \in (0, 1)$ and function $f \in \mathcal{K}_{\mathcal{G}}^k$ given by (2.8), we get

$$\mathcal{L}f(z) = \frac{1+r}{1-r} \frac{1}{(1-r^k)^{\frac{2}{k}}}$$

at points $z \in \mathcal{G}$, $\mu_{\mathcal{G}}(z) = r \in (0, 1)$ such that $I(z) = r$ (this condition is fulfilled by the points $z = rz^*$, where $z^* \in \partial\mathcal{G}$ and $I(z^*) = 1$).

The sharpness of the lower bounds (2.9) can be proven in a similar way.

Now, we prove the estimates (2.10). To obtain the upper bound (2.10), we use the proved above upper bound (2.9) and the fact that the Temljakov operator $\mathcal{L}$ is invertible and

$$\mathcal{L}^{-1}u(z) = \int_0^1 u(tz)dt, u \in \mathcal{H}_{\mathcal{G}}, z \in \mathcal{G}.$$

Indeed, we have for $f \in \mathcal{K}_{\mathcal{G}}^k$ and $z \in \mathcal{G}$, $\mu_{\mathcal{G}}(z) = r \in (0, 1)$,

$$\begin{aligned} |f(z)| &= \left| \mathcal{L}^{-1} \mathcal{L} f(z) \right| = \left| \int_0^1 \mathcal{L}(tz) dt \right| \leq \int_0^1 \frac{1+rt}{(1-rt)(1-(rt)^k)^{\frac{2}{k}}} dt \\ &= \frac{1}{r} \int_0^r \frac{1+\varrho}{(1-\varrho)(1-\varrho^k)^{\frac{2}{k}}} d\varrho. \end{aligned}$$

To prove the lower bound (2.10) let us consider the function

$$F(\zeta) = \zeta f \left(\zeta \frac{z}{\mu_{\mathcal{G}}(z)} \right), \zeta \in \mathcal{U},$$

with arbitrarily fixed $f \in \mathcal{K}_{\mathcal{G}}^k$ and $z \in \mathcal{G}$, $\mu_{\mathcal{G}}(z) = r \in (0, 1)$. Since

$$F'(\zeta) = \mathcal{L} f \left(\zeta \frac{z}{\mu_{\mathcal{G}}(z)} \right), \zeta \in \mathcal{U},$$

we get, by Theorem 1, that there exist functions $g \in \mathcal{M}_{\mathcal{G}} \cap \mathcal{F}_{0,k}(\mathcal{G})$ and $p \in \mathcal{C}_{\mathcal{G}}$ such that the factorization (2.1) is true. Thus

$$F'(\zeta) = P(\zeta) \cdot G(\zeta), \zeta \in \mathcal{U},$$

where for $\zeta \in \mathcal{U}$

$$G(\zeta) = \zeta g \left(\zeta \frac{z}{\mu_{\mathcal{G}}(z)} \right), P(\zeta) = p \left(\zeta \frac{z}{\mu_{\mathcal{G}}(z)} \right).$$

Moreover, $G \in \mathcal{S}^* \cap \mathcal{F}_{1,k}(\mathcal{U})$ (see the proof of the estimates (2.9)) and $P : \mathcal{U} \rightarrow \mathbb{C}$, $P(0) = 1$, is a holomorphic function with a positive real part. Therefore, F belongs to a subclass $\mathcal{K}^{(k)}$ (considered in [22] and for $k = 2$ in [8]) of the class of close-to-convex functions. Hence, F is univalent in the disc $\mathcal{U}$.

On the other hand, by the lower bound (2.9), we have that

$$|F'(\zeta)| \geq \frac{1-|\zeta|}{1+|\zeta|} \frac{1}{(1+|\zeta|^k)^{\frac{2}{k}}},$$

because $r = \mu_{\mathcal{G}} \left(\zeta \frac{z}{\mu_{\mathcal{G}}(z)} \right) = |\zeta|$. Now we show that

$$|F(\zeta)| \geq \int_0^r \frac{1-\varrho}{1+\varrho} \frac{1}{(1+\varrho^k)^{\frac{2}{k}}} d\varrho, |\zeta| = r \in (0, 1).$$

To this aim, it is sufficient to show that it holds for the nearest point $F(\zeta_0)$ from zero ($|\zeta_0| = r \in (0, 1)$), otherwise, we have $|F(\zeta)| \geq |F(\zeta_0)|$, $|\zeta| = r$. Since F is univalent in the disc $\mathcal{U}$, the original image of the line segment $0, F(\zeta_0)$ is a piece of arc $F^{-1}(0, F(\zeta_0))$ in the disc $r\bar{\mathcal{U}}$. Thus

$$\begin{aligned} |F(\zeta_0)| &= \int_{0, F(\zeta_0)} |dw| = \int_{F^{-1}(0, F(\zeta_0))} |F'(\zeta)| |d\zeta| \\ &\geq \int_0^r \frac{1-\varrho}{1+\varrho} \frac{1}{(1+\varrho^k)^{\frac{2}{k}}} d\varrho, r \in (0, 1). \end{aligned}$$

Thus, by the definition of F , we get

$$\left| \zeta f \left(\zeta \frac{z}{\mu_{\mathcal{G}}(z)} \right) \right| \geq \int_0^r \frac{1-\varrho}{1+\varrho} \frac{1}{(1+\varrho^k)^{\frac{2}{k}}} d\varrho, |\zeta| = r \in (0, 1).$$

Hence, putting $\zeta = \mu_{\mathcal{G}}(z) = r \in (0, 1)$, we have the lower bound (2.10).

Finally, let us note that we obtain the equalities in the inequalities (2.10) for the function (2.8) in adequate points $z \in \mathcal{G}$. $\square$

We close the paper with a sufficient condition guaranteeing that a function $f \in \mathcal{H}_{\mathcal{G}}(1)$ belongs to $\mathcal{K}_{\mathcal{G}}^k$. We formulate it in the term of $\mathcal{G}$ -balances of m -homogeneous polynomials in developments of functions from $\mathcal{H}_{\mathcal{G}}(1)$.

Theorem 5 *Let $f \in \mathcal{H}_{\mathcal{G}}(1)$ has the form (2.2). If there exists a function $g \in \mathcal{M}_{\mathcal{G}} \cap \mathcal{F}_{0,k}(\mathcal{G})$ of the form (2.3) such that*

$$\sum_{m=1}^{\infty} (m+1) \mu_{\mathcal{G}}(Q_{f,m}) + \sum_{m=1}^{\infty} \mu_{\mathcal{G}}(Q_{g,mk}) \leq 1,$$

then $f \in \mathcal{K}_{\mathcal{G}}^k$.

Proof Since g , as a function from $\mathcal{M}_{\mathcal{G}}$ omits zero [1], we will prove that

$$\operatorname{Re} \frac{\mathcal{L}f(z)}{g(z)} > 0, z \in \mathcal{G}.$$

To do it, we compute step by step

$$\begin{aligned}
& |\mathcal{L}f(z) - g(z)| - |\mathcal{L}f(z) + g(z)| \\
&= \left| \sum_{m=1}^{\infty} (m+1) Q_{f,m}(z) - \sum_{m=1}^{\infty} Q_{g,mk}(z) \right| - \left| 2 + \sum_{m=1}^{\infty} (m+1) Q_{f,m}(z) \right. \\
&\quad \left. + \sum_{m=1}^{\infty} Q_{g,mk}(z) \right| \\
&\leq 2 \left[\sum_{m=1}^{\infty} (m+1) |Q_{g,mk}(z)| + \sum_{m=1}^{\infty} |Q_{g,mk}(z)| - 1 \right] \\
&\leq 2 \left[\sum_{m=1}^{\infty} (m+1) \mu_{\mathcal{G}}(Q_{f,m}) + \sum_{m=1}^{\infty} \mu_{\mathcal{G}}(Q_{g,mk}) - 1 \right] \leq 0.
\end{aligned}$$

Thus

$$\left| \frac{\mathcal{L}f(z)}{g(z)} - 1 \right| \leq \left| \frac{\mathcal{L}f(z)}{g(z)} + 1 \right|, z \in \mathcal{G}$$

and hence

$$\operatorname{Re} \frac{\mathcal{L}f(z)}{g(z)} \geq 0, z \in \mathcal{G}.$$

This gives the mentioned inequality by a maximum principle for pluriharmonic functions of several complex variables. Putting $p(z) = \frac{\mathcal{L}f(z)}{g(z)}$, $z \in \mathcal{G}$, we obtain that the transform $\mathcal{L}f$ has the factorization (1) with $g \in \mathcal{M}_{\mathcal{G}} \cap \mathcal{F}_{0,k}(\mathcal{G})$ and $p \in \mathcal{C}_{\mathcal{G}}$. Consequently, $f \in \mathcal{K}_{\mathcal{G}}^k$. $\square$

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References

1. Bavin, I.I.: A class of regular bounded functions in the case of several complex variables and extreme problems in that class. Moskov Obl. Ped. Inst. Moskov (1976) (**in Russian**)
2. Długoś, R.: Embedding theorems for holomorphic functions of several complex variables. J. Appl. Anal. **19**, 153–165 (2013)
3. Długoś, R., Leś, E.: Embedding theorems and extreme problems for holomorphic functions on circular domains of $\mathbb{C}^n$. Complex Var. Elliptic Equ. **59**, 883–899 (2014)
4. Długoś, R., Liczberski, P.: An application of hypergeometric functions to a construction in several complex variables. J. Anal. Math. (2017) accepted for publication
5. Dobrowolska, K., Liczberski, P.: On some differential inequalities for holomorphic functions of many variables. Demonstr. Math. **14**, 383–398 (1981)
6. Dziubiński, I., Sitariski, R.: On classes of holomorphic functions of many variables starlike and convex on some hypersurfaces. Demonstr. Math. **13**, 619–632 (1980)
7. Fukui, S.: On the estimates of coefficients of analytic functions. Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **10**, 216–218 (1969)
8. Gao, C., Zhou, S.: On a class of analytic functions related to the starlike functions. Kyungpook Math. J. **45**, 123–130 (2005)

9. Graham, I., Kohr, G.: Geometric Function Theory in One and Higher Dimensions. Marcel Dekker Inc, New York (2003)
10. Hamada, H., Honda, T., Kohr, G.: Parabolic starlike mappings in several complex variables. *Manuscr. Math.* **123**, 301–324 (2007)
11. Higuchi, T.: On coefficients of holomorphic functions of several complex variables. *Sci. Rep. Tokyo Kyoiku Daigaku* **8**, 251–258 (1965)
12. Leś-Bomba, E., Liczberski, P.: On some family of holomorphic functions of several complex variables. *Sci. Bull. Chelm Sect. Math. Comput. Sci.* **2**, 7–16 (2007)
13. Liczberski, P.: On the subordination of holomorphic mappings in $\mathbb{C}^n$. *Demonstr. Math.* **2**, 293–301 (1986)
14. Liczberski, P., Połubiński, J.: On (j, k) -symmetrical functions. *Math. Bohem.* **120**, 13–28 (1995)
15. Liczberski, P., Połubiński, J.: Functions (j, k) -symmetrical and functional equations with iterates of the unknown function. *Publ. Math. Debr.* **60**, 291–305 (2002)
16. Liczberski, P., Połubiński, J.: Symmetrical series expansion of complex valued functions. *N. Z. J. Math.* **27**, 245–253 (1998)
17. Liczberski, P., Połubiński, J.: A uniqueness theorem of Cartan–Gutzmer type for holomorphic mappings in $\mathbb{C}^n$. *Ann. Pol. Math.* **79**, 121–127 (2002)
18. Marchlewska, A.: On a generalization of close-to-convexity for complex holomorphic functions in $\mathbb{C}^n$. *Demonstr. Math.* **4**, 847–856 (2005)
19. Michiwaki, Y.: Note on some coefficients in a starlike functions of two complex variables. *Res. Rep. Nagaoka Tech. Coll.* **1**, 151–153 (1963)
20. Pfaltzgraff, J.A., Suffridge, T.J.: An extension theorem and linear invariant families generated by starlike maps. *Ann. Univ. Mariae Curie Skłodowska Sect. A* **53**, 193–207 (1999)
21. Rudin, W.: Functional Analysis. McGraw-Hill Inc, New York (1991)
22. Seker, B.: On certain new subclass of close-to-convex functions. *Appl. Math. Comput.* **218**, 1041–1045 (2011)
23. Stankiewicz, J.: Functions of two complex variables regular in halfspace. *Folia Sci. Univ. Techn. Rzeszów. Math.* **19**, 107–116 (1996)
24. Temljakov, A.: Integral representation of functions of two complex variables. *Izv. Ross. Akad. Nauk. Ser. Mat.* **21**, 89–92 (1957)
25. Waadeland, H.: Über k -fach symmetrische sternförmige schlichte Abbildungen des Einheitskreises. *Math. Scand.* **3**, 150–154 (1955)