On Analytic Functions with Divergent Series of Taylor Coefficients

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Received: 30 August 2017 / Accepted: 24 October 2017 / Published online: 31 October 2017
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Abstract We consider a bounded, circular, strictly convex domain Ω with C^2 boundary. We show that there exists a holomorphic function f_1 , continuous to the boundary such that every slice function has a series of Taylor coefficients divergent with every power $p \in [0, 2)$. We also construct inner function f_2 which every slice is also inner and has series of Taylor coefficients with the same property. Next we generalize it to obtain $f_3 \in \mathcal{O}(\Omega)$ with given modulus a.e. on all slices and Taylor series as above.

Keywords Inner function · Taylor series · Slice function · Boundary values of holomorphic function

Mathematics Subject Classification Primary 32A40; Secondary 32A05

1 Introduction

1.1 Historical Background

In [8, 7.2] Rudin gives some examples of boundary behavior of holomorphic functions in complex unit balls with dimensions 2 and 3. Ryll and Wojtaszczyk observed [9, Theorem 1.2 + Remark 1.10] that similar examples could be constructed in arbitrary dimension. The crucial tool used in the reminded constructions is [9, Theorem 1.2]: there exist polynomials $\{p_n\}$ homogeneous of degree n on the unit ball B^d such that

Communicated by Ahmed Sebbar.

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$$\|p_n\|_2 = 1 \text{ and } \|p_n\|_\infty \leq \frac{2^d}{\sqrt{\pi}}. \quad (1.1)$$

This tool can be used to convert some one-dimensional examples into a multidimensional case. An interesting example of such an application is presented in the paper [10], where Wojtaszczyk generalizes the following one variable result:

- there exists a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ analytic for $|z| < 1$ and continuous to the boundary such that $\sum_{n=0}^{\infty} |a_n|^p = \infty$ for all $p < 2$

to a multi dimensional case.

1.2 Motivations

Our inspiration is the paper [10], where it was proven that there exists a function $f \in A(\mathbb{B}^d)$ such that almost every slice function of f has a series of Taylor coefficients divergent with every power $p < 2$.

We are going to strengthen the Wojtaszczyk's result [10] by showing that:

- (1) the unit ball $\mathbb{B}^d$ can be replaced by Ω bounded, circular, strictly convex domain with C^2 boundary,
- (2) almost every slice can be replaced by every slice,
- (3) it is possible to construct $f \in A(\Omega)$ in the elementary way,
- (4) it is possible to construct an inner function for whom every slice function is both inner and has a series of Taylor coefficients divergent.
- (5) it is possible to construct a holomorphic function with a given modulus a.e. on slices with a Taylor series with the properties as given above.

Note that the first construction of an inner function for whom all slices functions are also inner was presented in [1] for the unit ball and in [6, Theorem 3.2] for bounded, circular, strictly convex domain Ω with C^2 boundary.

The tool like the one (1.1) is too weak to achieve (1)–(5). Therefore, we use a generalization of Ryll-Wojtaszczyk polynomials (see [4, 5, 11]). However, the results obtained in these papers have a wider applications when we combine them with information about holomorphic functions with bounded growth from the paper [3]. Such a combination was used before in [6, Theorem 3.2].

Wojtaszczyk used (1.1) and he also needed surjectivity of the operator $T : A(\mathbb{B}^d) \rightarrow l_2$ defined by $T(f) = (\langle f, p_{2^n} \rangle)_{n=0}^{\infty}$, which was showed in [10, Proposition] (see also [7, Proposition 3.1]) by duality theorem. In one variable case, the constructive proof of such a surjectivity can be found in [2]. As it was mentioned by Wojtaszczyk, it would be interesting to have such a constructive proof in case of the unit ball $\mathbb{B}^d$. We do not know if a similar result can be obtained for different domains than $\mathbb{B}^d$.

Another crucial tool used in [10] was “scrambling lemma”. This lemma needs unitary maps presented on the unit ball. A lack of mentioned surjectivity or “scrambling lemma” makes the propagation of Wojtaszczyk proof impossible.

1.3 Notations

Let Ω be a bounded, circular, strictly convex domain with C^2 boundary.

For a given $f \in \mathbb{O}(\Omega)$ we study the slice function

$$f_z : \mathbb{B}^1 = \mathbb{D} \ni \lambda \rightarrow f(\lambda z).$$

We also define the middle value $\|f\|_z := \sup_{r < 1} \sqrt{\int_0^1 |f(re^{2\pi i t} z)|^2 dt}$ of holomorphic function f on a circle given by the point $z \in \partial\Omega$. By f^* we denote radial limit of f . It is known that $\|f\|_z < \infty$ implies that f has such limit a.e on boundary of the slice $\mathbb{D}z$. A similar definition can be given for a function g on $\partial\Omega$. Then, we define a slice function $g_z : \partial\mathbb{D} \ni \lambda \rightarrow g(\lambda z)$ and a norm of g on slice: $\|g\|_z = \sqrt{\int_0^1 |g(e^{2\pi i t} z)|^2 dt}$.

We need the following facts:¹

Theorem 1 [3, Theorem 3.2] *There exists a natural number $N_1 = N_1(\partial\Omega)$ such that, if $\varepsilon \in (0, 1)$, T is a compact subset of Ω , h is a continuous, strictly positive function on $\partial\Omega$, then there exist holomorphic functions $f_1, \dots, f_{N_1} \in A(\Omega)$ such that:*

- (1) $|f_j| < \varepsilon$ on T ,
- (2) $|f_j| < h$ on $\partial\Omega$,
- (3) $\frac{1}{2}h < \max_{j=1, \dots, N_1} |f_j|$ on $\partial\Omega$.

We can observe that it is possible to replace holomorphic functions in the above fact with the polynomials (see [6, Lemma 2.1]).

Theorem 2 [5, Theorem 2.5] *There exists a natural number N_2 such that there exists $N_3 \in \mathbb{N}$ such that for all integers N and $m_1, \dots, m_{N_2}$ with $N_3 \leq N \leq m_1 \leq \dots \leq m_{N_2} \leq 2N$ there exist homogeneous polynomials $p_{m_1}, \dots, p_{m_{N_2}}$ of a degree $m_1, \dots, m_{N_2}$ respectively such that $\frac{1}{2} < \max_{i=1, \dots, N_2} |p_{m_i}(z)| < 1$ for all $z \in \partial\Omega$.*

1.4 Main Result

We obtain the following fact (see Theorems 4, 5, 6):

Theorem *There exists a holomorphic function $f \in A(\Omega)$ such that every slice function has a series of Taylor coefficients divergent with every power $s \in [0, 2)$.*

In fact our methods work also in case of an inner function:

Theorem *There exists an inner function $f \in \mathbb{O}(\Omega)$ such that for all $z \in \partial\Omega$ we have the following properties:*

- $|f_z^*| = 1$ a.e. on $\partial\mathbb{D}$,
- f_z has a series of Taylor coefficients divergent with every power $s \in [0, 2)$.

The above result has the following generalization:

¹ Theorem [3, Theorem 3.2] and [5, Theorem 2.5] are proven in a more general situation than we need, so we simplify it here to satisfy our expectations.

Theorem Let ψ be a positive LSC function on $\partial\Omega$ for whom every slice function $\psi_z : \partial\mathbb{D} \ni \lambda \rightarrow \psi(\lambda z)$ is from L^2 . There exists a holomorphic function $f \in \mathbb{O}(\Omega)$ such that for all $z \in \partial\Omega$ we have the following properties:

- $|f_z^*| = \psi_z$ a.e. on $\partial\mathbb{D}$,
- f_z has a series of Taylor coefficients divergent with every power $s \in [0, 2)$.

In all the above facts a similar method was used which is presented in Lemma 3.

To obtain Taylor series of a given function $f \in \mathbb{O}(\Omega)$ it is necessary to find homogeneous expansion for f :

$$f(z) = \sum_{n=0}^{\infty} p_n(z)$$

where p_n is a homogeneous polynomial of degree n . Now we have Taylor coefficients expansion for a slice function:

$$f_z : \lambda \rightarrow f(\lambda z) = \sum_{n=0}^{\infty} p_n(z) \lambda^n,$$

so we can construct a holomorphic function $f \in \mathbb{O}(\Omega)$ whose every slice function f_z is in H^2 so that:

$$\sum_{n=0}^{\infty} |p_n(z)|^s = \infty$$

for $s \in [0, 2)$. Note that the H^2 property of f_z implies:

$$\sum_{n=0}^{\infty} |p_n(z)|^2 < \infty$$

for $z \in \partial\Omega$.

2 Divergent Taylor Series

Lemma 3 There exists a constant $\gamma > 0$ such that for $k \in \mathbb{N}$, $a, \theta \in (0, 1)$ and a continuous positive function g on $\partial\Omega$ we can choose $N \in \mathbb{N}$ and polynomials $q_1, \dots, q_N$ such that:

- (1) each non zero monomial in q_i has a degree greater than k and smaller than any monomial's degree in q_{i+1} ,
- (2) $|q_j| < ag$ on $\partial\Omega$,
- (3) $|\sum_{i=1}^m q_i| < g$ on $\partial\Omega$ for $m \geq 1$,

- (4) $\|q_{m+1}\|_z^2 > \gamma \left\| \min \left\{ ag, g - \left| \sum_{i=1}^m q_i \right| \right\} \right\|_z^2$ for $z \in \partial\Omega$ and $m \geq 1$,
 (5) $\sum_{i=1}^N \|q_i\|_z^2 \geq \theta \|g\|_z^2$ for $z \in \partial\Omega$.

Proof Let $\gamma = \frac{1}{16N_1^2N_2^2}$ where $N_1, N_2 \in \mathbb{N}$ are constants from Theorems 1 and 2 respectively. We define polynomials $q_1, \dots, q_N$ by induction so that properties 1–4 are fulfilled. Let $q_1 = 0$. Suppose that we have just defined $q_1, \dots, q_k$ so that (1)–(4) are fulfilled. Let

$$h = \min \left\{ ag, g - \left| \sum_{i=1}^k q_i \right| \right\}.$$

In particular $h > 0$ on $\partial\Omega$, so we can use Theorem 1 to obtain polynomials $f_1, \dots, f_{N_1}$ such that:

- $|f_i| < \frac{h}{N_1}$ on $\partial\Omega$,
- $\frac{h}{2N_1} < \max_{i=1, \dots, N_1} |f_i|$.

Now for a given j due to Theorem 2 we can choose homogeneous polynomials $p_{j,1}, \dots, p_{j,N_2}$ with degrees $m_{j,1}, \dots, m_{j,N_2}$ respectively so that

$$\frac{1}{2} < \max_{i=1, \dots, N_2} |p_{j,i}(z)| < 1.$$

Let

$$q_{k+1} := \frac{1}{N_2} \sum_{j=1}^{N_1} \sum_{i=1}^{N_2} f_j p_{j,i}.$$

We can enlarge degrees if necessary so that polynomials $f_j p_{j,i}$ are orthogonal and q_{k+1} fulfills property (1).

Since $h \leq ag$ we observe the property (2):

$$|q_{k+1}| \leq \frac{1}{N_2} \sum_{j=1}^{N_1} \sum_{i=1}^{N_2} |f_j| |p_{j,i}| < \frac{1}{N_2} \sum_{j=1}^{N_1} \sum_{i=1}^{N_2} \frac{h}{N_1} \leq \frac{1}{N_2} \sum_{j=1}^{N_1} \sum_{i=1}^{N_2} \frac{ag}{N_1} = ag.$$

The property (3) is obvious:

$$\begin{aligned} \left| \sum_{i=1}^{k+1} q_i \right| &\leq \left| \sum_{i=1}^k q_i \right| + |q_{k+1}| < \left| \sum_{i=1}^k q_i \right| + \frac{1}{N_2} \sum_{j=1}^{N_1} \sum_{i=1}^{N_2} \frac{h}{N_1} \leq \left| \sum_{i=1}^k q_i \right| + h \\ &\leq \left| \sum_{i=1}^k q_i \right| + g - \left| \sum_{i=1}^k q_i \right| = g. \end{aligned}$$

Since $f_j p_{j,i}$ are orthogonal we can estimate for $z \in \partial\Omega$:

$$\begin{aligned} \|q_{m+1}\|_z^2 &= \frac{1}{N_2^2} \sum_{j=1}^{N_1} \sum_{i=1}^{N_2} \|f_j p_{j,i}\|_z^2 = \frac{1}{N_2^2} \sum_{j=1}^{N_1} \|f_j\|_z^2 \sum_{i=1}^{N_2} |p_{j,i}(z)| \geq \frac{1}{N_2^2} \sum_{j=1}^{N_1} \|f_j\|_z^2 \frac{1}{4} \\ &\geq \frac{1}{4N_2^2} \left\| \max_{i=1, \dots, N_1} |f_i| \right\|_z^2 > \frac{1}{16N_1^2 N_2^2} \|h\|_z^2 = \gamma \|h\|_z^2 \end{aligned}$$

which is the property (4) and finishes a construction of polynomials with required properties (1)–(4).

Let us choose polynomials $q_1, \dots, q_N$ which fulfill properties (1)–(4). Let

$$s_N : \partial\Omega \ni z \rightarrow \left\| \sum_{k=1}^N q_k \right\|_z^2 = \sum_{k=1}^N \|q_k\|_z^2.$$

We have defined an increased sequence of continuous bounded functions. For a given $z \in \partial\Omega$ there exists $\lim_{N \rightarrow \infty} s_N(z) \leq \|g\|_z^2$. In particular $\lim_{k \rightarrow \infty} \|q_k\|_z^2 = 0$, which implies that $\lim_{m \rightarrow \infty} \|g - \sum_{i=1}^m q_i\|_z = 0$ and $\lim_{N \rightarrow \infty} s_N(z) = \|g\|_z^2$. Now, due to Dini's theorem s_N is convergent uniformly to $z \rightarrow \|g\|_z^2$, so there exists $N \in \mathbb{N}$ such that $s_N \geq \theta \|g\|_z^2$ on $\partial\Omega$ [property (5)]. $\square$

Theorem 4 *There exists a holomorphic function $f \in A(\Omega)$ such that every slice function has a series of Taylor coefficients divergent with every power $s \in [0, 2)$.*

Proof For a given $j \in \mathbb{N}$, $g \equiv 1$, $a = 2^{-j}$ and $\theta = 0.5$ let us use Lemma 3 to obtain a number N_j and polynomials $q_{j,1}, \dots, q_{j,N_j}$. We can assume that polynomials $q_{j,i}$ are chosen so that different polynomials contain monomials with different degrees.

Let us summarize all the necessary properties of polynomials $q_{j,i}$:

- (1) If $I(j, i)$ is a list of all degrees of monomials in homogeneous expansion of $q_{j,i}$ then $I(j, i) < I(j, i+1)$ and $I(j, *) < I(j+1, *)$.
- (2) $|q_{j,i}| < 2^{-j}$ on $\partial\Omega$,
- (3) $\left| \sum_{i=1}^{N_j} q_{j,i} \right| < 1$ on $\partial\Omega$,
- (4) $\sum_{i=1}^{N_j} \|q_{j,i}\|_z^2 \geq 0.5$ for $z \in \partial\Omega$.

Let us define

$$f = \sum_{j=1}^{\infty} \frac{1}{j^2} \sum_{i=1}^{N_j} q_{j,i}.$$

Since $\left| \sum_{i=1}^{N_j} q_{j,i} \right| < 1$, we just defined a holomorphic function which is continuous to the boundary ($f \in A(\Omega)$). Now for a given polynomial $q_{j,i}$ we have a homogeneous expansion:

² If A, B are sets then $A < B \Leftrightarrow \forall x \in A, y \in B, x < y$.

$$q_{j,i} = \sum_{k \in I(j,i)} p_{j,i,k}$$

where $I(j, i)$ is a list of degrees of all homogeneous polynomials $p_{j,i,k}$ in $q_{j,i}$ and $p_{j,i,k}$ has a degree equal to k .

In particular, using Taylor coefficients, we obtain the expansion:

$$f(\lambda z) = \sum_{j=1}^{\infty} \sum_{i=1}^{N_j} \sum_{k \in I(j,i)} \frac{p_{j,i,k}(z)}{j^2} \lambda^k = \sum_{j,i,k} \frac{p_{j,i,k}(z)}{j^2} \lambda^k$$

for $z \in \partial\Omega$.

Let $s \in [0, 2)$. The property (2) gives us $\|q_{j,i}\|_z 2^j < 1$ for $z \in \partial\Omega$ and:

$$\|q_{j,i}\|_z^s \geq \|q_{j,i}\|_z^s \left(\|q_{j,i}\|_z 2^j \right)^{2-s} = \|q_{j,i}\|_z^2 2^{j(2-s)}. \quad (2.1)$$

Since $\|q_{j,i}\|_z^2 = \sum_k \|p_{j,i,k}\|_z^2$ we have

$$\|q_{j,i}\|_z^s \leq \left(\sum_k \|p_{j,i,k}\|_z^2 \right)^{s/2} \leq \sum_k \|p_{j,i,k}\|_z^s,$$

which together with the property (4) and inequality (2.1) gives us:

$$\begin{aligned} \sum_{j,i,k} \left| \frac{p_{j,i,k}(z)}{j^2} \right|^s &= \sum_{j,i,k} j^{-2s} \|p_{j,i,k}\|_z^s \geq \sum_{j,i} j^{-2s} \|q_{j,i}\|_z^s \\ &\geq \sum_{j,i} j^{-2s} \|q_{j,i}\|_z^2 2^{j(2-s)} \geq \sum_j 2^{-1} j^{-2s} 2^{j(2-s)} = \infty. \end{aligned}$$

In other words any slice function $\lambda \rightarrow f(\lambda z)$ has a series of Taylor coefficients divergent with every power $s \in [0, 2)$. $\square$

As a second application of Lemma 3 we give a construction of inner function:

Theorem 5 *There exists an inner function $f \in \mathbb{O}(\Omega)$ such that for all $z \in \partial\Omega$ we have following properties:*

- $|f_z^*| = 1$ a.e. on $\partial\mathbb{D}$,
- f_z has a series of Taylor coefficients divergent with every power $s \in [0, 2)$.

Proof We construct a sequence of orthogonal polynomials $\{p_j\}$ with the following properties:

- (1) $\left| \sum_{j=1}^m p_j(z) \right| < 1$ for $z \in \partial\Omega$, $m \geq 1$,
- (2) $\|p_m\|_z^2 \geq \frac{1}{2} \left\| 1 - \left| \sum_{j=1}^{m-1} p_j \right| \right\|_z^2$ for $z \in \partial\Omega$, $m > 1$,
- (3) p_j is a sum of orthogonal polynomials: $p_j = \sum_{i=1}^{N_j} p_{j,i}$ with the following property:

$$|p_{m,i}| < \left(1 - \left| \sum_{j=1}^{m-1} p_j \right| \right)^m$$

on $\partial\Omega$, $m > 1$.

Let $N_1 = 1$, $p_1 = p_{1,1} \equiv 0$. Properties (1)–(3) are fulfilled so let us assume that we have defined $p_1, \dots, p_m$.

Due to Lemma 3 used for the data: $a = \inf_{z \in \partial\Omega} \left(1 - \left| \sum_{j=1}^m p_j(z) \right| \right)^m$, $\theta = \frac{1}{2}$, $g = 1 - \left| \sum_{j=1}^m p_j \right|$ we can choose number $N_{m+1} \in \mathbb{N}$ and orthogonal polynomials $p_{m+1,1}, \dots, p_{m+1,N_{m+1}}$ with the following properties:

- $p_1, \dots, p_{m+1}, \sum_{i=1}^{N_{m+1}} p_{m+1,i}$ are orthogonal,
- $|p_{m+1,i}| < ag \leq \left(1 - \left| \sum_{j=1}^m p_j \right| \right)^{m+1}$ on $\partial\Omega$,
- $\left| \sum_{i=1}^{N_{m+1}} p_{m+1,i} \right| < g$ on $\partial\Omega$,
- $\left\| \sum_{i=1}^{N_{m+1}} p_{m+1,i} \right\|_z^2 \geq \frac{1}{2} \|g\|_z^2$ for $z \in \partial\Omega$.

We can choose $p_{m+1} = \sum_{i=1}^{N_{m+1}} p_{m+1,i}$ and observe that properties (1)–(3) are fulfilled, which finishes the construction of a sequence $\{p_j\}$ by induction.

Now we can define:

$$f = \sum_{j=1}^{\infty} p_j.$$

Since $\{p_j\}$ are orthogonal and $\left| \sum_{j=1}^N p_j \right| < 1$ on $\partial\Omega$, we have: $\sum_{j=1}^N \|p_j\|_z^2 < 1$, which implies that: $\sum_{j=1}^{\infty} \|p_j\|_z^2 \leq 1$ for $z \in \partial\Omega$. In particular $\lim_{j \rightarrow \infty} \|p_j\|_z = 0$ for $z \in \partial\Omega$. The property (2) implies that $\lim_{N \rightarrow \infty} \left\| 1 - \left| \sum_{j=1}^N p_j \right| \right\|_z = 0$ and $\sum_{j=1}^{\infty} \|p_j\|_z^2 = 1$ for $z \in \partial\Omega$. We can conclude that $f \in \mathbb{O}(\Omega)$, $|f| \leq 1$ on $\partial\Omega$ and $\|f\|_z = 1$ for $z \in \partial\Omega$. Let us note that every slice of f is inner.

Now we estimate series of Taylor coefficients. For a given polynomial $p_{j,i}$ we have the homogeneous expansion:

$$p_{j,i} = \sum_{k \in I(j,i)} p_{j,i,k}$$

where $I(j, i)$ is a list of degrees of all homogeneous polynomials $p_{j,i,k}$ in $p_{j,i}$ and $p_{j,i,k}$ has a degree equal to k . Now, using Taylor coefficients, we have the following expansion:

$$f(\lambda z) = \sum_{j=1}^{\infty} \sum_{i=1}^{N_j} \sum_{k \in I(j,i)} p_{j,i,k}(z) \lambda^k = \sum_{j,i,k} p_{j,i,k}(z) \lambda^k$$

for $z \in \partial\Omega$.

Let $s \in [0, 2)$. The property (3) gives us $\|p_{m,i}\|_z \left\| 1 - \left| \sum_{j=1}^{m-1} p_j \right| \right\|_z^{-m} < 1$ and implies:

$$\begin{aligned} \|p_{m,i}\|_z^s &\geq \|p_{m,i}\|_z^s \left(\|p_{m,i}\|_z \left\| 1 - \left| \sum_{j=1}^{m-1} p_j \right| \right\|_z^{-m} \right)^{2-s} \\ &= \|p_{m,i}\|_z^2 \left\| 1 - \left| \sum_{j=1}^{m-1} p_j \right| \right\|_z^{m(s-2)} \end{aligned} \quad (2.2)$$

for $z \in \partial\Omega$, $m > 1$.

Since $\|p_{j,i}\|_z^2 = \sum_k \|p_{j,i,k}\|_z^2$ we have

$$\|p_{j,i}\|_z^s \leq \left(\sum_k \|p_{j,i,k}\|_z^2 \right)^{s/2} \leq \sum_k \|p_{j,i,k}\|_z^s,$$

which together with the property (2) and inequality (2.2) gives us:

$$\begin{aligned} \sum_{j,i,k} |p_{j,i,k}(z)|^s &= \sum_{j,i,k} \|p_{j,i,k}\|_z^s \geq \sum_{j,i} \|p_{j,i}\|_z^s \geq \sum_{m>1,i} \|p_{m,i}\|_z^2 \left\| 1 - \left| \sum_{j=1}^{m-1} p_j \right| \right\|_z^{m(s-2)} \\ &\geq \sum_{m>1} \|p_m\|_z^2 \left\| 1 - \left| \sum_{j=1}^{m-1} p_j \right| \right\|_z^{m(s-2)} \geq \sum_{m>1} \frac{1}{2} \left\| 1 - \left| \sum_{j=1}^{m-1} p_j \right| \right\|_z^{2+m(s-2)} \\ &\geq \sum_{m>\frac{2}{2-s}} \frac{1}{2} = \infty \end{aligned}$$

In other words, any slice function $\lambda \rightarrow f(\lambda z)$ has a series of Taylor coefficients divergent with every power $s \in [0, 2)$. $\square$

Now let us improve the last Theorem:

Theorem 6 Let ψ be a positive LSC function on $\partial\Omega$, whose every slice function $\psi_z : \partial\mathbb{D} \ni \lambda \rightarrow \psi(\lambda z)$ is from L^2 . There exists a holomorphic function $f \in \mathbb{O}(\Omega)$ such that for all $z \in \partial\Omega$ we have the following properties:

- $|f_z^*| = \psi_z$ a.e. on $\partial\mathbb{D}$,
- f_z has a series of Taylor coefficients divergent with every power $s \in [0, 2)$.

Proof There exists a sequence of positive, continuous functions $\{\psi_k\}$ such that (point-wise):

$$0 < \psi_1 < \dots < \psi_n < \psi_{n+1} < \dots \leq \lim_{n \rightarrow \infty} \psi_n = \psi.$$

Let $\phi_1 = \psi_1$. We construct a sequence of orthogonal polynomials $\{p_j\}$ and continuous, positive functions ($m > 1$):

$$\phi_m : \partial\Omega \ni z \rightarrow \min \left\{ \psi_m(z), \left| \sum_{j=1}^{m-1} p_j(z) \right| + 1 \right\}$$

with the following properties:

- (1) $\left| \sum_{j=1}^m p_j \right| < \phi_m$ on $\partial\Omega$, $m \geq 1$,
- (2) $\left| \sum_{j=1}^{m-1} p_j \right| < \phi_m$ on $\partial\Omega$, $m > 1$,
- (3) $\|p_m\|_z^2 \geq \frac{1}{2} \left\| \phi_m - \left| \sum_{j=1}^{m-1} p_j \right| \right\|_z^2$ for $z \in \partial\Omega$, $m > 1$,
- (4) p_j is a sum of orthogonal polynomials: $p_j = \sum_{i=1}^{N_j} p_{j,i}$ with: $|p_{m,i}| < \left(\phi_m - \left| \sum_{j=1}^{m-1} p_j \right| \right)^m$ on $\partial\Omega$, $m > 1$.

Let $N_1 = 1$, $p_1 = p_{1,1} \equiv 0$. The properties (1)–(4) are fulfilled for $m = 1$ so let us assume that we have defined $p_1, \dots, p_m$ with the properties (1)–(4).

Let us observe inequalities $\left| \sum_{j=1}^m p_j \right| < \phi_m \leq \psi_m < \psi_{m+1}$, which gives us the property (2): $\left| \sum_{j=1}^m p_j \right| < \phi_{m+1}$ on $\partial\Omega$ so we can use Lemma 3 for the data:

$$a = \inf_{\partial\Omega} \left(\phi_{m+1} - \left| \sum_{j=1}^m p_j \right| \right)^m, \theta = \frac{1}{2}, g = \phi_{m+1} - \left| \sum_{j=1}^m p_j \right|$$

and we can obtain a number $N_{m+1} \in \mathbb{N}$ and polynomial $p_{m+1} := \sum_{i=1}^{N_{m+1}} p_{m+1,i}$ where $p_{m+1,1}, \dots, p_{m+1,N_{m+1}}$ are orthogonal polynomials with the following properties:

- $p_1, \dots, p_{m+1}, p_{m+1,1}, \dots, p_{m+1,N_{m+1}}$ are orthogonal,
- $|p_{m+1,i}| < ag \leq \left(\phi_{m+1} - \left| \sum_{j=1}^m p_j \right| \right)^{m+1}$ on $\partial\Omega$, (the property 4),

- $|p_{m+1}| < g = \phi_{m+1} - \left| \sum_{j=1}^m p_j \right|$ on $\partial\Omega$,
- $\|p_{m+1}\|_z^2 \geq \frac{1}{2} \|g\|_z^2 = \frac{1}{2} \left\| \phi_{m+1} - \left| \sum_{j=1}^m p_j \right| \right\|_z^2$ for $z \in \partial\Omega$, (the property 3).

We can easily observe the property (1):

$$\left| \sum_{j=1}^{m+1} p_j \right| \leq |p_{m+1}| + \left| \sum_{j=1}^m p_j \right| < g + \left| \sum_{j=1}^m p_j \right| = \phi_{m+1}.$$

Now the properties (1)–(4) are fulfilled, which finishes the construction of the sequence $\{p_j\}$ by induction.

Let us define

$$f = \sum_{j=1}^{\infty} p_j.$$

Since $\{p_j\}$ are orthogonal and $\left| \sum_{j=1}^m p_j \right| < \phi_m \leq \psi_m \leq \psi$ on $\partial\Omega$ we have: $\sum_{j=1}^m \|p_j\|_z^2 < \|\psi\|_z^2 < \infty$, which implies that: $\sum_{j=1}^{\infty} \|p_j\|_z^2 \leq \|\psi\|_z^2$ for $z \in \partial\Omega$. In particular $\lim_{j \rightarrow \infty} \|p_j\|_z = 0$ and $\lim_{m \rightarrow \infty} \left\| \phi_m - \left| \sum_{j=1}^{m-1} p_j \right| \right\|_z = 0$ for $z \in \partial\Omega$.

Since $\phi_m \leq \psi$ and $\|\psi\|_z < \infty$ we can conclude that $f_z \in H^2(\mathbb{D})$ so there exists f_z^* a.e. on $\partial\mathbb{D}$ for $z \in \partial\Omega$. Suppose that there exists $\delta > 0$ and $z \in \partial\Omega$ such that $\|\psi - |f^*|\|_z > 3\delta > 0$. In particular, there exists $\delta_2 > 0$ and a set $A \subset \partial\mathbb{D}$ with a measure $\mu(A) > \delta_2$ such that there exist f_z^* on A and $|\psi(\lambda z) - |f^*(\lambda z)|| > 2\delta$ for $\lambda \in A$. Since $\lim_{m \rightarrow \infty} \left\| \phi_m - \left| \sum_{j=1}^{m-1} p_j \right| \right\|_z = 0$ we have $\lim_{m \rightarrow \infty} \|\phi_m - |f^*|\|_z = 0$, so decreasing a little A if necessary, we can assume that $\mu(A) > \delta_2$ and there exists $m_0 \in \mathbb{N}$ so large that $\psi_m - 2\delta > |f^*| \geq \phi_m - \delta$ on Az for $m \geq m_0$. In particular $\psi_m > \phi_m + \delta$ on Az for $m \geq m_0$, which implies that $\phi_m := \left| \sum_{j=1}^{m-1} p_j \right| + 1$ on Az for $m \geq m_0$. But

$$\|p_m\|_z^2 \geq \frac{1}{2} \left\| \phi_m - \left| \sum_{j=1}^{m-1} p_j \right| \right\|_z^2 \geq \frac{1}{2} \int_{Az} \left(\phi_m - \left| \sum_{j=1}^{m-1} p_j \right| \right)^2 d\mu = \frac{1}{2} \mu(A) > \frac{1}{2} \delta_2$$

for $m \geq m_0$ which is impossible because, as we mentioned, $\lim_{j \rightarrow \infty} \|p_j\|_z = 0$ for $z \in \partial\Omega$. In particular $\|\psi - |f^*|\|_z = 0$ for $z \in \partial\Omega$ i.e. $|f_z^*| = \psi_z$ a.e. on $\partial\mathbb{D}$.

Now we estimate a series of Taylor coefficients. For a given polynomial $p_{j,i}$ we have the homogeneous expansion:

$$p_{j,i} = \sum_{k \in I(j,i)} p_{j,i,k}$$

where $I(j, i)$ is a list of degrees of all homogeneous polynomials $p_{j,i,k}$ in $p_{j,i}$ and $p_{j,i,k}$ has a degree equal to k , which, using Taylor coefficients, gives us the expansion:

$$f(\lambda z) = \sum_{j=1}^{\infty} \sum_{i=1}^{N_j} \sum_{k \in I(j,i)} p_{j,i,k}(z) \lambda^k = \sum_{j,i,k} p_{j,i,k}(z) \lambda^k$$

for $z \in \partial\Omega$.

Let $s \in [0, 2)$. The property (4) gives us $\|p_{m,i}\|_z \left\| \phi_m - \left| \sum_{j=1}^{m-1} p_j \right|_z \right\|_z^{-m} < 1$ and implies:

$$\|p_{m,i}\|_z^s \geq \|p_{m,i}\|_z^s \left(\|p_{m,i}\|_z \left\| \phi_m - \left| \sum_{j=1}^{m-1} p_j \right|_z \right\|_z^{-m} \right)^{2-s} = \|p_{m,i}\|_z^2 \left\| \phi_m - \left| \sum_{j=1}^{m-1} p_j \right|_z \right\|_z^{m(s-2)} \quad (2.3)$$

for $z \in \partial\Omega$, $m > 1$.

Since $\|p_{j,i}\|_z^2 = \sum_k \|p_{j,i,k}\|_z^2$ we have

$$\|p_{j,i}\|_z^s \leq \left(\sum_k \|p_{j,i,k}\|_z^2 \right)^{s/2} \leq \sum_k \|p_{j,i,k}\|_z^s,$$

which together with $0 < \left| \phi_m - \left| \sum_{j=1}^{m-1} p_j \right| \right| \leq 1$ on $\partial\Omega$, the property (3) and inequality (2.3) gives us:

$$\begin{aligned} \sum_{j,i,k} |p_{j,i,k}(z)|^s &= \sum_{j,i,k} \|p_{j,i,k}\|_z^s \geq \sum_{j,i} \|p_{j,i}\|_z^s \geq \sum_{m>1,i} \|p_{m,i}\|_z^2 \left\| \phi_m - \left| \sum_{j=1}^{m-1} p_j \right|_z \right\|_z^{m(s-2)} \\ &\geq \sum_{m>1} \|p_m\|_z^2 \left\| \phi_m - \left| \sum_{j=1}^{m-1} p_j \right|_z \right\|_z^{m(s-2)} \geq \sum_{m>1} \frac{1}{2} \left\| \phi_m - \left| \sum_{j=1}^{m-1} p_j \right|_z \right\|_z^{2+m(s-2)} \\ &\geq \sum_{m>\frac{2}{2-s}} \frac{1}{2} = \infty \end{aligned}$$

In other words any slice function $\lambda \rightarrow f(\lambda z)$ has a series of Taylor coefficients divergent with every power $s \in [0, 2)$. $\square$

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