Complex Anal. Oper. Theory (2011) 5:1093–1111
DOI 10.1007/s11785-010-0048-7

Complex Analysis and Operator Theory

Kernel Decompositions for Schur Functions on the Polydisk

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Received: 7 September 2009 / Accepted: 27 January 2010 / Published online: 12 February 2010

Abstract A certain kernel (sometimes called the Pick kernel) associated to Schur functions on the disk is always positive semi-definite. A generalization of this fact is well-known for Schur functions on the polydisk. In this article, we show that the "Pick kernel" on the polydisk has a great deal of structure beyond being positive semi-definite. It can always be split into two kernels possessing certain shift invariance properties.

Keywords Schur function · Polydisk · Polydisc · Reproducing kernel · Agler decomposition · Pick interpolation

Mathematics Subject Classification (2000) Primary 47A57; Secondary 42B05

1 Introduction

Let $\mathbb{D}^d$ be the unit polydisk in $\mathbb{C}^d$. A *Schur function* is simply a holomorphic function $f : \mathbb{D}^d \to \mathbb{C}$ bounded by one in modulus. One of the most fundamental facts about Schur functions in one variable is that the following kernel is positive semi-definite:

$$\frac{1 - f(z)\overline{f(\zeta)}}{1 - z\bar{\zeta}} \geq 0. \tag{1.1}$$

Communicated by Mihai Putinar.

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(we say a function $K : \mathbb{D}^d \times \mathbb{D}^d \to \mathbb{C}$ is a positive semi-definite kernel and write $K \geq 0$ if for every finite subset $F \subset \mathbb{D}^d$, the matrix

$$(K(z, \zeta))_{z,\zeta \in F}$$

is positive semi-definite—to actually form a matrix we would need an ordering of F, but this is unimportant).

The positive semi-definiteness of (1.1) is significant because (1) it relates function theory to operator theory and (2) it turns out to have a very strong converse: if f is a function on a finite subset of $\mathbb{D}$ such that (1.1) is positive semi-definite on that finite set, then f is the restriction of a Schur function. This is the content of the Pick interpolation theorem.

It is not clear what the "best" generalization of (1.1) is to several variables. For a Schur function in d variables, it is a fact that

$$\frac{1 - f(z)\overline{f(\zeta)}}{\prod_{j=1}^{d}(1 - z_j \bar{\zeta_j})} \tag{1.2}$$

is positive semi-definite, however this does not seem to be extremely useful. Here $z = (z_1, \ldots, z_d), \zeta = (\zeta_1, \ldots, \zeta_d) \in \mathbb{C}^d$.

It was not until ca. 1988 that a more useful result was given in two variables by Agler [1]: for any Schur function f on $\mathbb{D}^2$ there exist positive semi-definite kernels $\Gamma_1, \Gamma_2 : \mathbb{D}^2 \times \mathbb{D}^2 \to \mathbb{C}$ such that

$$1 - f(z)\overline{f(\zeta)} = (1 - z_1\bar{\zeta_1})\Gamma_1(z, \zeta) + (1 - z_2\bar{\zeta_2})\Gamma_2(z, \zeta). \tag{1.3}$$

This formula, called an *Agler decomposition*, does not generalize to more variables in the way that its form suggests. Schur functions which satisfy

$$1 - f(z)\overline{f(\zeta)} = \sum_{j=1}^{d}(1 - z_j\bar{\zeta_j})\Gamma_j(z, \zeta) \tag{1.4}$$

for some positive semi-definite kernels $\Gamma_1, \ldots, \Gamma_d$, form a proper subclass of the set of Schur functions called the Schur–Agler class.

Very recently, Grinshpan et al. [4] proved a decomposition that *does* hold in general and which is still analogous to (1.3). We state it here in the scalar valued case (in [4] it was proved in the operator-valued case).

Theorem 1.1 (Grinshpan et al. [4]) *Let $f : \mathbb{D}^d \to \mathbb{D}$ be holomorphic. Then, for each $j, k \in \{1, \ldots, d\}$: $j \neq k$ there exist positive semi-definite kernels K and K' such that*

$$1 - f(z)\overline{f(\zeta)} = \prod_{r \neq j}(1 - z_r\bar{\zeta_r})K(z, \zeta) + \prod_{r \neq k}(1 - z_r\bar{\zeta_r})K'(z, \zeta)$$

It is our goal to strengthen this theorem and to alter the point of view slightly. Rather than looking for more decompositions analogous to (1.3), we instead attempt to illuminate the structure of the kernel in (1.2).

Before presenting our theorem we need the following definition.

Definition 1.2 If K is a positive semi-definite kernel on $\mathbb{D}^d$, we shall say K is z_j*-contractive* if

$$(1 - z_j \bar{\zeta_j}) K(z, \zeta) \geq 0.$$

If $S \subset \{1, \dots, d\}$, then we say a kernel K is S-contractive if it is z_j-contractive for all $j \in S$.

Theorem 1.3 *Let $d \geq 2$ and let $f : \mathbb{D}^d \to \mathbb{D}$ be holomorphic. Then, for each non-empty $S \subsetneq \{1, 2, \dots, d\}$, there exist positive semi-definite S-contractive kernels K_S, L_S, such that if $S \sqcup T = \{1, \dots, d\}$ is a non-trivial partition, then*

$$\frac{1 - f(z)\overline{f(\zeta)}}{\prod_{j=1}^d (1 - z_j \bar{\zeta_j})} = K_S(z, \zeta) + L_T(z, \zeta),$$
$$K_T - L_T = K_S - L_S \geq 0,$$

and if $S \subset S' \subset \{1, 2, \dots, d\}$ then

$$K_S \geq K_{S'}.$$

Kernel inequalities like the last line should be interpreted as saying $K_S - K_{S'}$ is positive semi-definite.

The proof of Theorem 1.1 in [4] amounts to the case where S is a singleton, however many of the decompositions provided by Theorem 1.3 can be used to reprove Theorem 1.1.

Indeed, let $S \sqcup T = \{1, \dots, d\}$ be any partition with $j \in S$ and $k \in T$. Theorem 1.1 follows from writing as in Theorem 1.3

$$1 - f(z)\overline{f(\zeta)} = \prod_{r \neq j} (1 - z_r \bar{\zeta_r}) \underbrace{((1 - z_j \bar{\zeta_j}) K_S(z, \zeta))}_{K(z,\zeta)} + \prod_{r \neq k} (1 - z_r \bar{\zeta_r}) \underbrace{((1 - z_k \bar{\zeta_k}) L_T(z, \zeta))}_{K'(z,\zeta)}$$

and K and K' are positive since K_S is z_j-contractive and L_T is z_k-contractive.

Our proof of Theorem 1.3 relies on proving the result first for rational inner functions continuous on $\overline{\mathbb{D}^d}$; these can be characterized as follows.

Let $p \in \mathbb{C}[z] = \mathbb{C}[z_1, \dots, z_d]$ have no zeros on the closed polydisk $\overline{\mathbb{D}^d}$ and suppose $\deg p \leq n = (n_1, \dots, n_d)$. Define

$$\tilde{p}(z) := z^n \overline{p(1/\bar{z})} = z_1^{n_1} \cdots z_d^{n_d} \overline{p(1/\bar{z}_1, \dots, 1/\bar{z}_d)} \tag{1.5}$$

(and notice $|p| = |\tilde{p}|$ on the d-torus $\mathbb{T}^d$).

Every regular rational inner function can be represented as $f(z) = \tilde{p}(z)/p(z)$ for some choice of p and some choice of $n \geq \deg(p)$ as above (see Rudin [5] Theorem 5.2.5). We state a theorem below describing the structure of the following kernel

$$\mathcal{P}(z, \zeta) := \frac{p(z)\overline{p(\zeta)} - \tilde{p}(z)\overline{\tilde{p}(\zeta)}}{\prod_{j=1}^{d}(1 - z_j\bar{\zeta}_j)},$$

a trivial modification of (1.2) in the case of $f = \tilde{p}/p$.

First, we need another definition.

Definition 1.4 Let us call $K(z, \zeta) : \mathbb{D}^d \times \mathbb{D}^d \to \mathbb{C}$ a $\mathcal{P}$*-kernel* if

- $\mathcal{P} \geq K \geq 0$ in the sense of kernels and
- whenever $\mathcal{P}(z, \zeta) \geq f(z)\overline{f(\zeta)}$ and $K(z, \zeta) \geq \epsilon f(z)\overline{f(\zeta)}$ for some $\epsilon > 0$, then we necessarily have $K(z, \zeta) \geq f(z)\overline{f(\zeta)}$.

See Lemma 7.5 in the Appendix for a precise description of what this means. The (aesthetic) point here is that we have a theorem which does not refer to our methods of proof. The follow theorem is similar to Theorem 1.3 but more precise.

Theorem 1.5 *Let $p \in \mathbb{C}[z]$ be as above. For every non-empty $S \subsetneq \{1, 2, \dots, d\}$, there exist S-contractive $\mathcal{P}$-kernels K_S, L_S, such that if $S \sqcup T = \{1, \dots, d\}$ is a non-trivial partition, then*

$$\mathcal{P} = K_S + L_T.$$

Moreover, K_S is maximal among all S-contractive kernels bounded above by $\mathcal{P}$.

This last condition makes these decompositions unique.

2 The Kernel $\mathcal{P}$

The theorems from the introduction are proved by analyzing orthogonality relations for a “Bernstein–Szegő measure”:

$$d\mu = \frac{1}{|p(z)|^2} d\sigma(z) \tag{2.1}$$

where $d\sigma$ is normalized Lebesgue measure on the d-torus $\mathbb{T}^d$ and $p \in \mathbb{C}[z]$ has no zeros on the closed polydisk $\overline{\mathbb{D}^d}$. We also use $d\sigma$ to represent normalized Lebesgue measure on different dimensional tori, and the dimension will be made apparent by the variable; e.g. $d\sigma(z_1)$ corresponds to normalized Lebesgue measure on $\mathbb{T}$ using the variable z_1.

Notice that the complex Hilbert space $L^2(\mu)$ is a renorming of $L^2(\mathbb{T}^d)$ and therefore is topologically isomorphic. The inner product on $L^2(\mu)$ is denoted

$$\langle f, g\rangle_\mu = \int_{\mathbb{T}^d} f(z)\overline{g(z)}d\mu(z).$$

For a subset X of the lattice $\mathbb{Z}^d$ we define the closed subspace

$$L^2_\mu(X) := \{f \in L^2(\mu) : \hat{f}(\alpha) = 0 \text{ for } \alpha \notin X\}$$

where $\hat{f}(\alpha)$ denotes the αth Fourier coefficient of f (and note we typically use $\alpha = (\alpha_1, \dots, \alpha_d)$ to denote a d-tuple of integers). We use the following non-traditional notation. If $Y \subset X \subset \mathbb{Z}^d$ then we write

$$L^2_\mu(X \ominus Y) := L^2_\mu(X) \ominus L^2_\mu(Y). \tag{2.2}$$

We use the following partial order on d-tuples of integers $\alpha = (\alpha_1, \dots, \alpha_d)$, $\beta = (\beta_1, \dots, \beta_d)$:

$$\alpha \le \beta \ \text{ if and only if } \alpha_j \le \beta_j \ \text{ for all } j = 1, \dots, d;$$

$n = (n_1, \dots, n_d)$ is a fixed d-tuple which bounds the multi-degree of p (i.e. the degree of p with respect to z_j is at most n_j); writing $\alpha < \beta$ means $\alpha \le \beta$ and $\alpha \ne \beta$.

We typically write elements of $\mathbb{C}^d$ with $z = (z_1, \dots, z_d)$. We use multi-index notation:

$$z^\alpha := z_1^{\alpha_1} \cdots z_d^{\alpha_d}$$

for $\alpha \in \mathbb{Z}^d$ and $z \in \mathbb{C}^d$.

We need to define various subsets of $\mathbb{Z}^d$:

$$\begin{aligned} \mathbb{Z}^d_+ &:= \{\alpha \in \mathbb{Z}^d : \alpha \ge 0\} \\ \mathbb{Z}^d_{n+} &:= \{\alpha \in \mathbb{Z}^d : \alpha \ge n\} \\ B &:= \mathbb{Z}^d_+ \setminus \mathbb{Z}^d_{n+} = \{\alpha \in \mathbb{Z}^d_+ : \exists j : \alpha_j < n_j\} = \{\alpha \in \mathbb{Z}^d_+ : \alpha \not\ge n\} \end{aligned} \tag{2.3}$$

Then, for example $L^2_\mu(\mathbb{Z}^d_+)$ denotes the closure of the polynomials with respect to $L^2(\mu)$, a space equal to the Hardy space $H^2(\mathbb{T}^d)$ although it has a different inner product.

The first thing we prove provides the connection to the kernel $\mathcal{P}$. See [2] for background on reproducing kernel Hilbert spaces. The *Szegő kernel* will be denoted:

$$\mathcal{S}_d(z, \zeta) = \prod_{j=1}^d \frac{1}{(1 - z_j\bar{\zeta_j})}.$$

As $H^2(\mathbb{T}^d)$ is a reproducing kernel Hilbert space with kernel $\mathcal{S}_d$ and since $L^2_\mu(\mathbb{Z}^d_+)$ is a renorming of $H^2(\mathbb{T}^d)$, $L^2_\mu(\mathbb{Z}^d_+)$ and all of its closed subspaces are also reproducing kernel Hilbert spaces.

Proposition 2.1 *Let $p \in \mathbb{C}[z]$ have degree at most n, let $\tilde{p}(z) = z^n \overline{p(1/\bar{z})}$, and let*

$$d\mu = \frac{1}{|p(z)|^2} d\sigma(z).$$

Then, with B as in (2.3) the reproducing kernel for $L^2_\mu(B)$ is

$$\mathcal{P}(z, \zeta) = (p(z)\overline{p(\zeta)} - \tilde{p}(z)\overline{\tilde{p}(\zeta)})\mathcal{S}_d(z, \zeta).$$

Proof The kernel for $L^2_\mu(\mathbb{Z}^d_+)$ is $p(z)\overline{p(\zeta)}\mathcal{S}_d(z, \zeta)$. This is a simple computation; if $f \in H^2(\mathbb{T}^d)$ and $\zeta \in \mathbb{D}^d$ then

$$\begin{aligned}
\int_{\mathbb{T}^d} f(z)\overline{p(z)\overline{p(\zeta)}\mathcal{S}_d(z, \zeta)} d\mu(z) &= \int_{\mathbb{T}^d} f(z)\overline{p(z)} p(\zeta)\overline{\mathcal{S}_d(z, \zeta)} \frac{d\sigma(z)}{|p(z)|^2} \\
&= \int_{\mathbb{T}^d} \frac{f(z)}{p(z)} p(\zeta)\overline{\mathcal{S}_d(z, \zeta)} d\sigma(z) \\
&= \frac{f(\zeta)}{p(\zeta)} p(\zeta) = f(\zeta)
\end{aligned}$$

The third equality is the reproducing property of $\mathcal{S}_d$ (or just the Cauchy integral formula).

We prove in Lemma 2.2 below that $L^2_\mu(\mathbb{Z}^d_+ \ominus B) = \tilde{p} L^2_\mu(\mathbb{Z}^d_+)$ and a computation similar to that above proves that the reproducing kernel of $\tilde{p} L^2_\mu(\mathbb{Z}^d_+)$ is $\tilde{p}(z)\overline{\tilde{p}(\zeta)}$ $\mathcal{S}_d(z, \zeta)$. The result then follows from the fact that:

$$L^2_\mu(\mathbb{Z}^d_+) = L^2_\mu(B) \oplus L^2_\mu(\mathbb{Z}^d_+ \ominus B)$$

and that the reproducing kernel of a direct sum is the sum of the reproducing kernels of each direct summand. Namely,

$$\underbrace{p(z)\overline{p(\zeta)}\mathcal{S}_d(z, \zeta)}_{\text{kernel for } L^2_\mu(\mathbb{Z}^d_+)} - \underbrace{\tilde{p}(z)\overline{\tilde{p}(\zeta)}\mathcal{S}_d(z, \zeta)}_{\text{kernel for } L^2_\mu(\mathbb{Z}^d_+ \ominus B)} = \text{kernel for } L^2_\mu(B).$$

□

The following lemma was used above.

Lemma 2.2

$$\tilde{p} L^2_\mu(\mathbb{Z}^d_+) = L^2_\mu(\mathbb{Z}^d_+ \ominus B) = L^2_\mu(\mathbb{Z}^d \ominus (\mathbb{Z}^d \setminus \mathbb{Z}^d_{n+}))$$

Proof Observe that $\tilde{p}(z) = z^n \overline{p(z)}$ on $\mathbb{T}^d$ and so

$$\begin{aligned}\langle z^\alpha, \tilde{p}\rangle_\mu &= \int_{\mathbb{T}^d} z^\alpha \overline{\tilde{p}(z)} \frac{1}{|p(z)|^2} d\sigma(z)\\ &= \int_{\mathbb{T}^d} z^\alpha \frac{\bar{z}^n p(z)}{|p(z)|^2} d\sigma(z)\\ &= \int_{\mathbb{T}^d} \frac{z^{\alpha-n}}{\overline{p(z)}} d\sigma(z).\end{aligned}$$

This equals zero if any component of $\alpha - n$ is negative (i.e., if $\alpha \ngeq n$) since $1/\bar{p}$ is anti-analytic in $\mathbb{D}^d$. In particular, if $\alpha \ngeq n$, then for $\beta \geq 0$, $\alpha \ngeq n + \beta$ and therefore

$$\langle z^\alpha, z^\beta \tilde{p}\rangle_\mu = 0.$$

This shows

$$\tilde{p} L^2_\mu(\mathbb{Z}^d_+) \perp L^2_\mu(\mathbb{Z}^d \setminus \mathbb{Z}^d_{n+})$$

which means

$$\tilde{p} L^2_\mu(\mathbb{Z}^d_+) \subset L^2_\mu(\mathbb{Z}^d \ominus (\mathbb{Z}^d \setminus \mathbb{Z}^d_{n+})) \cap L^2_\mu(\mathbb{Z}^d_+ \ominus B).$$

Conversely, if $f \in L^2_\mu(\mathbb{Z}^d_+ \ominus B)$ and $f \perp \tilde{p} L^2_\mu(\mathbb{Z}^d_+)$, then we can show $f \perp L^2_\mu(\mathbb{Z}^d_+)$ as follows.

Since $p(0) \neq 0$, $\tilde{p}(z) = az^n + q(z)$ with $a = \overline{p(0)} \neq 0$ and q of degree at most n with no z^n term. By assumption on f, $f \perp \tilde{p}$ and $f \perp q$ ($q \in L^2_\mu(B)$). Therefore, $f \perp z^n$. From here we can give an inductive proof on the lattice $\mathbb{Z}^d_+$. If f is orthogonal to all non-negative frequencies less than some $\alpha \geq n$, then f is orthogonal to

$$z^{\alpha-n}\tilde{p}(z) = az^\alpha + z^{\alpha-n}q(z) \quad \text{and} \quad z^{\alpha-n}q(z)$$

as the latter contains only frequencies less than α. This implies $f \perp z^\alpha$, and by induction $f \perp L^2_\mu(\mathbb{Z}^d_+)$. (As this is a non-traditional way of doing induction we should explain using the contrapositive: if f is *not* perpendicular to some z^α, then f must also *not* be perpendicular to some z^β with $\beta < \alpha$. This can be continued until f is *not* perpendicular to a monomial supported in B—a contradiction.) This forces $f \equiv 0$.

Hence, $L^2_\mu(\mathbb{Z}^d_+ \ominus B) = \tilde{p} L^2_\mu(\mathbb{Z}^d_+) \subset L^2_\mu(\mathbb{Z}^d \ominus (\mathbb{Z}^d \setminus \mathbb{Z}^d_{n+}))$. By Lemma 2.3 given below, we automatically have

$$L^2_\mu(\mathbb{Z}^d_+ \ominus B) = \tilde{p} L^2_\mu(\mathbb{Z}^d_+) = L^2_\mu(\mathbb{Z}^d \ominus (\mathbb{Z}^d \setminus \mathbb{Z}^d_{n+})).$$

$\square$

Lemma 2.3 *Suppose $W, Y \subset \mathbb{Z}^d$ and set $X = W \cup Y$. Then,*

$$L^2_\mu(X) \ominus L^2_\mu(Y) \subset L^2_\mu(W) \tag{2.4}$$

if and only if

$$L^2_\mu(W) \ominus L^2_\mu(Y \cap W) \subset (L^2_\mu(Y))^\perp \tag{2.5}$$

and in either case

$$L^2_\mu(X) \ominus L^2_\mu(Y) = L^2_\mu(W) \ominus L^2_\mu(Y \cap W).$$

Proof This is essentially a result of the decomposition

$$L^2_\mu(X \ominus (Y \cap W)) = L^2_\mu(X \ominus Y) \oplus L^2_\mu(Y \ominus (Y \cap W)) \tag{2.6}$$

$$= L^2_\mu(X \ominus W) \oplus L^2_\mu(W \ominus (Y \cap W)). \tag{2.7}$$

Suppose $L^2_\mu(X \ominus Y) \subset L^2_\mu(W)$ which necessarily means $L^2_\mu(X \ominus Y) \subset L^2_\mu(W \ominus (Y \cap W))$. If $f \in L^2_\mu(W \ominus (Y \cap W)) \ominus L^2_\mu(X \ominus Y)$, then $f \in L^2_\mu(Y \ominus (Y \cap W))$ by (2.6). Hence, $f \in L^2_\mu(Y \cap W \ominus Y \cap W) = \{0\}$ showing that $L^2_\mu(X \ominus Y)$ fills out all of $L^2_\mu(W \ominus (Y \cap W))$.

Suppose $L^2_\mu(W \ominus (Y \cap W)) \subset L^2_\mu(Y)^\perp$ which necessarily means $L^2_\mu(W \ominus (Y \cap W)) \subset L^2_\mu(X \ominus Y)$. If $f \in L^2_\mu(X \ominus Y) \ominus L^2_\mu(W \ominus (Y \cap W))$, then $f \in L^2_\mu(X \ominus W)$ by (2.7). Hence, $f \perp L^2_\mu(Y) + L^2_\mu(W) = L^2_\mu(X)$, forcing $f \equiv 0$. This shows that $L^2_\mu(W \ominus (Y \cap W))$ fills out all of $L^2_\mu(X \ominus Y)$. □

So, we have shown that $\mathcal{P}$ represents the reproducing kernel of $L^2_\mu(B)$. Any orthogonal decomposition of $L^2_\mu(B)$ then gives a decomposition of $\mathcal{P}$. Our goal is to prove that $L^2_\mu(B)$ has a decomposition with very special properties.

3 Orthogonal Decompositions of $L^2_\mu(B)$

We recall the definition of B and define several subsets of B below:

Notation 3.1

$$X_j := \{\alpha \in \mathbb{Z}^d_+ : \alpha_j < n_j\}$$

$$X_S := \bigcup_{j \in S} X_j = \{\alpha \in \mathbb{Z}^d_+ : \exists j \in S : \alpha_j < n_j\}$$

$$B = \bigcup_{j=1}^{d} X_j = \{\alpha \in \mathbb{Z}^d_+ : \exists j : \alpha_j < n_j\}$$

where $S \subset \{1, 2, \ldots, d\}$.

Proposition 3.2 *With the same setup as Proposition 2.1 let $S \sqcup T = \{1, 2, \ldots, d\}$ be a partition. Then,*

$$L^2_\mu(B) = L^2_\mu(X_T) \oplus L^2_\mu(X_S \ominus (X_S \cap X_T)).$$

The content of the above proposition is that the subspaces listed in the orthogonal decomposition are *actually orthogonal*, something which would not hold for a general finite measure on $\mathbb{T}^d$. This proposition is still valid if S or T are empty if we interpret $X_\varnothing = \{0\}$. This makes the proposition sensible (although trivial) in the case $d = 1$ (something useful later).

We need the following notation for use in dividing up all of structures according to the partition $S \sqcup T = \{1, \ldots, d\}$. There is no harm in assuming $S = \{1, \ldots, s\}$, $T = \{s + 1, \ldots, d\}$, and $t := d - s$.

$$\begin{array}{lll} z_S = (z_1, \ldots, z_s) \in \mathbb{C}^s, & z_T = (z_{s+1}, \ldots, z_d) \in \mathbb{C}^t, & z = (z_S, z_T) \\ n_S = (n_1, \ldots, n_s) \in \mathbb{Z}^s, & n_T = (n_{s+1}, \ldots, n_d) \in \mathbb{Z}^t, & n = (n_S, n_T) \\ \alpha_S = (\alpha_1, \ldots, \alpha_s) \in \mathbb{Z}^s, & \alpha_T = (\alpha_{s+1}, \ldots, \alpha_d) \in \mathbb{Z}^t, & \alpha = (\alpha_S, \alpha_T) \\ B_S = \{\alpha_S \in \mathbb{Z}^s_+ : \alpha_S \not\geq n_S\}, & B_T = \{\alpha_T \in \mathbb{Z}^t_+ : \alpha_T \not\geq n_T\} & \end{array}$$

It is useful to note here that

$$\begin{array}{ll} X_S = B_S \times \mathbb{Z}^t_+, & X_T = \mathbb{Z}^s_+ \times B_T, \\ B_S \times B_T = X_S \cap X_T, & B = X_S \cup X_T \end{array}$$

Proof of Proposition 3.2 The proposition is really a type of inclusion-exclusion principle as it can be rewritten as saying

$$L^2_\mu((X_S \cup X_T) \ominus X_T) = L^2_\mu(X_S \ominus (X_S \cap X_T)).$$

To prove it, consider following the measures μ_{z_S} on $\mathbb{T}^t$ which are indexed by $z_S \in \mathbb{T}^s$:

$$d\mu_{z_S}(z_T) = \frac{1}{|p(z_S, z_T)|^2} d\sigma(z_T),$$

i.e. for each $z_S \in \mathbb{T}^s$ we get a measure on $\mathbb{T}^t$, and points in $\mathbb{T}^t$ are denoted by z_T.

By Proposition 2.1, the reproducing kernel for $L^2_{\mu_{z_S}}(B_T)$ is

$$\mathcal{P}^T_{z_S}(z_T, \zeta_T) := (p(z_S, z_T)\overline{p(z_S, \zeta_T)} - \tilde{p}(z_S, z_T)\overline{\tilde{p}(z_S, \zeta_T)})\mathcal{S}_t(z_T, \zeta_T)$$

where again $\mathcal{S}_t$ is the t-dimensional Szegő kernel. Notice that $\mathcal{P}^T_{z_S}(z_T, \zeta_T)$ is a trigonometric polynomial of degree at most n_S as a function of z_S, while as a function of z_T this function only has Fourier coefficients corresponding to points of B_T. For these reasons, the function of $z = (z_S, z_T) \in \mathbb{T}^d$ defined for each fixed $\zeta \in \mathbb{D}^d$ by

$$L_\zeta(z) = L(z, \zeta) = z_S^{n_S} \bar{\zeta}_S^{n_S} \mathcal{S}_s(z_S, \zeta_S) \mathcal{P}^T_{z_S}(z_T, \zeta_T)$$

is in $L^2_\mu(\mathbb{Z}^s_+ \times B_T) = L^2_\mu(X_T)$. (Specifically, as a function of (z_S, z_T)

$$\mathcal{S}_s(z_S, \zeta_S) \in L^2_\mu(\mathbb{Z}^s_+ \times \{0_T\})$$
$$\mathcal{P}^T_{z_S}(z_T, \zeta_T) \in L^2_\mu([-n_S, n_S] \times B_T)$$

Here 0_T is the zero t-tuple in $\mathbb{Z}^t$ and $[-n_S, n_S] = \{\alpha_S \in \mathbb{Z}^s : -n_S \leq \alpha_S \leq n_S\}$.) So, if $f \perp L^2_\mu(X_T)$, then

$$\langle f, L_\zeta \rangle_\mu = 0 \text{ for all } \zeta \in \mathbb{D}^d. \tag{3.1}$$

On the other hand, L can be thought of as a difference of two terms:

$$\begin{aligned} L_\zeta(z) = & \underbrace{p(z_S, z_T)\overline{p(z_S, \zeta_T)}(z_S^{n_S}\bar{\zeta_S}^{n_S})\mathcal{S}_d(z, \zeta)}_{A_\zeta} \\ & -\underbrace{\tilde{p}(z_S, z_T)\overline{\tilde{p}(z_S, \zeta_T)}(z_S^{n_S}\bar{\zeta_S}^{n_S})\mathcal{S}_d(z, \zeta)}_{B_\zeta}. \end{aligned}$$

(We used $\mathcal{S}_d(z, \zeta) = \mathcal{S}_s(z_S, \zeta_S)\mathcal{S}_t(z_T, \zeta_T)$ above.)

Since $z_S^{n_S}\overline{\tilde{p}(z_S, \zeta_T)}$ has only non-negative Fourier coefficients in z_S, the second term B_ζ is an element of $\tilde{p}L^2_\mu(\mathbb{Z}^d_+) = L^2_\mu(\mathbb{Z}^d_+ \ominus B)$. So, if $f \in L^2_\mu(B)$, then $B_\zeta \perp f$ and we have

$$\langle f, L_\zeta \rangle_\mu = \langle f, A_\zeta \rangle_\mu. \tag{3.2}$$

Finally, if $f \in L^2_\mu(\mathbb{Z}^d_+)$ then

$$\begin{aligned} \langle f, A_\zeta \rangle_\mu &= \int_{\mathbb{T}^s}\int_{\mathbb{T}^t} f(z)\overline{p(z)}p(z_S, \zeta_T)\overline{\mathcal{S}_t(z_T, \zeta_T)}\frac{d\sigma(z_T)}{|p(z)|^2}(\bar{z_S}^{n_S}\zeta_S^{n_S})\overline{\mathcal{S}_s(z_S, \zeta_S)}d\sigma(z_S) \\ &= \int_{\mathbb{T}^s}\int_{\mathbb{T}^t} \frac{f(z)}{p(z)}p(z_S, \zeta_T)\overline{\mathcal{S}_t(z_T, \zeta_T)}d\sigma(z_T)(\bar{z_S}^{n_S}\zeta_S^{n_S})\overline{\mathcal{S}_s(z_S, \zeta_S)}d\sigma(z_S) \\ &= \int_{\mathbb{T}^s} \frac{f(z_S, \zeta_T)}{p(z_S, \zeta_T)}p(z_S, \zeta_T)(\bar{z_S}^{n_S}\zeta_S^{n_S})\overline{\mathcal{S}_s(z_S, \zeta_S)}d\sigma(z_S) \\ &= \int_{\mathbb{T}^s} f(z_S, \zeta_T)(\bar{z_S}^{n_S}\zeta_S^{n_S})\overline{\mathcal{S}_s(z_S, \zeta_S)}d\sigma(z_S) \\ &= \sum_{\alpha_S \geq n_S}\sum_{\alpha_T \geq 0} \hat{f}(\alpha_S, \alpha_T)\zeta^\alpha \end{aligned} \tag{3.3}$$

which is the $L^2(\mathbb{T}^d)$ projection of f to $z_S^{n_S}H^2(\mathbb{T}^d)$. (The second and fourth equalities are algebra, the third is the reproducing property of $\mathcal{S}_t$, and the fifth is a Fourier series computation.)

If we combine the observations (3.1)–(3.3) above we see that if

$$f \perp L^2_\mu(X_T) \quad \text{and} \quad f \in L^2_\mu(B)$$

then

$$\hat{f}(\alpha) = 0$$

for $\alpha_S \geq n_S, \alpha_T \geq 0$ and therefore $f \in L^2_\mu(X_S)$. So, $L^2_\mu(B \ominus X_T) \subset L^2_\mu(X_S)$.
By Lemma 2.3, this proves

$$L^2_\mu(B \ominus X_T) = L^2_\mu(X_S \ominus (X_S \cap X_T))$$

since $B = X_S \cup X_T$. □

4 Closed Under Shifts

The goal of this section is to prove two facts.

Proposition 4.1 *With the setup of Proposition 3.2, $L^2_\mu(X_S)$ is closed under multiplication by z_j for all $j \notin S$, and contains all subspaces of $L^2_\mu(B)$ with this property.*

Proposition 4.2 *With the setup of Proposition 3.2, $L^2_\mu(X_S \ominus (X_S \cap X_T))$ is closed under multiplication by z_j for all $j \in T$.*

The first fact is not difficult.

Proof of Proposition 4.1 An element $f \in L^2_\mu(B)$ is in $L^2_\mu(X_S)$ if and only if $\hat{f}(\alpha) = 0$ whenever $\alpha_k \geq n_k$ for all $k \in S$. This property is obviously unaffected by multiplying f by variables z_j for $j \notin S$.
On the other hand, if $f \in L^2_\mu(B)$, has the property that

$$z^\alpha f \in L^2_\mu(B)$$

for all $\alpha \geq 0$ satisfying $\alpha_j = 0$ for $j \in S$, then f must be an element of $L^2_\mu(X_S)$. Otherwise, $\hat{f}(\alpha) \neq 0$ for some $\alpha \geq 0$, with $\alpha_k \geq n_k$ for all $k \in S$. But then if we set $m = (m_1, \dots, m_d)$ where

$$m_j = \begin{cases} 0 & \text{for } j \in S \\ n_j & \text{for } j \notin S \end{cases}$$

then $z^m f \notin L^2_\mu(B)$—a contradiction. This proves that $L^2_\mu(X_S)$ contains all subspaces closed under multiplication by all z_j for $j \notin S$. □

As for Proposition 4.2, it is convenient to prove the proposition by adjoining a variable and using results in d variables that have already been proven. Elements of $\mathbb{C}^{d+1} = \mathbb{C} \times \mathbb{C}^d$ will be written as (z_0, z). So, now $p \in \mathbb{C}[z_0, z]$ is a polynomial of $d+1$ variables of degree at most (n_0, n) with no zeros in $\overline{\mathbb{D}^{d+1}}$. The measure μ corresponds to $|p(z_0, z)|^{-2} d\sigma(z_0, z)$.

Notation already defined for d variables will retain its meaning, while we will use the following notation for certain $d+1$-variable objects:

$$Y_j = \{(\alpha_0, \alpha) \in \mathbb{Z}_+^{d+1} : \alpha_j < n_j\}$$
$$Y_S = \bigcup_{j \in S} Y_j \quad \text{for } S \subset \{0, 1, \dots, d\}$$

We also find it convenient to use interval notation for subsets of integers (as opposed to real numbers):

$$(a, b) = \{k \in \mathbb{Z} : a < k < b\}$$
$$[a, b) = \{k \in \mathbb{Z} : a \leq k < b\}, \text{ etc.}$$

We never make use of intervals of real numbers, so there should be no confusion.

Now, let $S \sqcup T$ be a partition of $\{1, \dots, d\}$, and let $T_0 = T \cup \{0\}$. We will prove that

$$L^2_\mu(Y_S \ominus (Y_{T_0} \cap Y_S))$$

is closed under multiplication by z_0. This is enough to prove the proposition.

Proof of Proposition 4.2 For each $z_0 \in \mathbb{T}$, let $d\mu_{z_0}(z)$ be the measure on $\mathbb{T}^d$

$$d\mu_{z_0}(z) = \frac{1}{|p(z_0, z)|^2} d\sigma(z).$$

Let

$$\Gamma_{z_0}(z, \zeta)$$

denote the reproducing kernel for $L^2_{\mu_{z_0}}(X_T \ominus (X_T \cap X_S))$, and let

$$\Delta_{z_0}(z, \zeta)$$

denote the reproducing kernel for $L^2_{\mu_{z_0}}(X_S)$.

By Proposition 3.2,

$$(p(z_0, z)\overline{p(z_0, \zeta)} - \tilde{p}(z_0, z)\overline{\tilde{p}(z_0, \zeta)})\mathcal{S}_d(z, \zeta)$$
$$= \Gamma_{z_0}(z, \zeta) + \Delta_{z_0}(z, \zeta).$$

The left hand side is a trigonometric polynomial in z_0 of degree at most n_0, while $\Delta_{z_0}(z,\zeta)$ as a function of z is the only function on the right hand side with any Fourier support in $X_S \setminus X_T$. This means the coefficients of z^α in Δ_{z_0} for $\alpha \in X_S \setminus X_T$ are trigonometric polynomials with respect to z_0; i.e. as a function of (z_0, z)

$$\begin{aligned}\Delta_{z_0}(z,\zeta) &\in L^2_\mu(\mathbb{Z}\times(X_S\cap X_T)\cup[-n_0,n_0]\times(X_S\setminus X_T)) \\ &= L^2_\mu(\mathbb{Z}\times(X_S\cap X_T)\cup[-n_0,n_0]\times X_S) \end{aligned} \tag{4.1}$$

(Perhaps it needs to be explicitly stated that $\Delta_{z_0}(z,\zeta)$ is actually in $L^2(\mathbb{T}^{d+1})$ as a function of $(z_0, z)\in\mathbb{T}^{d+1}$. See Lemma 4.3 below.)

Define for each $Z=(\zeta_0,\zeta)\in\mathbb{D}^{d+1}$

$$L_Z(z_0,z) = L((z_0,z),Z) = \frac{\bar{z_0}\zeta_0}{1-\bar{z_0}\zeta_0}\Delta_{z_0}(z,\zeta). \tag{4.2}$$

By (4.1) and (4.2),

$$L_Z \in L^2_\mu(\mathbb{Z}\times(X_S\cap X_T)\cup(-\infty,n_0)\times X_S).$$

Now, let $f\in L^2_\mu(\mathbb{Z}\times X_S)$, then for each $Z=(\zeta_0,\zeta)\in\mathbb{D}^{d+1}$

$$\begin{aligned}\langle f, L_Z\rangle_\mu &= \int_{\mathbb{T}}\int_{\mathbb{T}^d} f(z_0,z)\overline{\Delta_{z_0}(z,\zeta)}d\mu_{z_0}(z)\frac{z_0\bar{\zeta_0}}{1-z_0\bar{\zeta_0}}d\sigma(z_0) && (4.3)\\ &= \int_{\mathbb{T}} f(z_0,\zeta)\frac{z_0\bar{\zeta_0}}{1-z_0\bar{\zeta_0}}d\sigma(z_0) && (4.4)\\ &= \sum_{\alpha_0=-\infty}^{-1}\sum_{\alpha\geq 0}\hat{f}(\alpha_0,\alpha)\bar{\zeta_0}^{-\alpha_0}\zeta^\alpha; && (4.5)\end{aligned}$$

the equality (4.3) is by definition, (4.4) is because Δ_{z_0} is a reproducing kernel for X_S with respect to μ_{z_0}, and (4.5) is a Fourier series computation. If

$$\begin{aligned}&f\in L^2_\mu(\mathbb{Z}\times X_S) \quad \text{and}\\ &f\perp L^2_\mu(\mathbb{Z}\times(X_S\cap X_T)\cup(-\infty,n_0)\times X_S)\end{aligned}$$

then $f\perp L_Z$ and therefore the expression in (4.5) is zero which implies $f\in L^2_\mu(\mathbb{Z}_+\times X_S) = L^2_\mu(Y_S)$. Hence, by Lemma 2.3

$$L^2_\mu(\mathbb{Z}\times X_S)\ominus L^2_\mu(\mathbb{Z}\times(X_S\cap X_T)\cup(-\infty,n_0)\times X_S) \tag{4.6}$$

is unchanged if we intersect all sets with Y_S. This proves (4.6) equals

$$L^2_\mu(Y_S)\ominus L^2_\mu(Y_S\cap Y_{T_0}) \tag{4.7}$$

where we are using the facts that

$$(\mathbb{Z} \times X_S) \cap Y_S = Y_S$$

and

$$\begin{aligned}(\mathbb{Z} \times (X_S \cap X_T) \cup (-\infty, n_0) \times X_S) \cap Y_S \\ = (Y_S \cap Y_T) \cup (Y_S \cap Y_{\{0\}}) \\ = Y_S \cap Y_{T_0}.\end{aligned}$$

This proves

$$L^2_\mu(Y_S \ominus (Y_S \cap Y_{T_0})) \perp L^2_\mu((-\infty, 0) \times X_S)$$

since (4.6) = (4.7) and since

$$(-\infty, 0) \times X_S \subset \mathbb{Z} \times (X_S \cap X_T) \cup (-\infty, n_0) \times (X_S \setminus X_T).$$

This is enough to show $L^2_\mu(Y_S \ominus (Y_S \cap Y_{T_0}))$ is closed under multiplication by z_0, as follows.

Let $f \in L^2_\mu(Y_S \ominus (Y_S \cap Y_{T_0}))$. By Proposition 4.1, it is clear that $z_0 f \in L^2_\mu(Y_S)$. To show $z_0 f \perp L^2_\mu(Y_S \cap Y_{T_0})$, let $(\alpha_0, \alpha) \in Y_S \cap Y_{T_0}$. If $\alpha_0 > 0$ then $(\alpha_0 - 1, \alpha) \in Y_S \cap Y_{T_0}$ in which case

$$\langle z_0 f, z_0^{\alpha_0} z^\alpha \rangle_\mu = \langle f, z_0^{\alpha_0 - 1} z^\alpha \rangle_\mu = 0. \tag{4.8}$$

If $\alpha_0 = 0$, then $(\alpha_0 - 1, \alpha) \in (-\infty, 0) \times X_S$ in which case we again have (4.8) because $f \perp L^2_\mu((-\infty, 0) \times X_S)$. Hence, $z_0 f \in L^2_\mu(Y_S \ominus (Y_S \cap Y_{T_0}))$, proving that this subspace is closed under multiplication by z_0. □

We used the following lemma in the above proof.

Lemma 4.3 *Let $X \subset \mathbb{Z}^d_+$, $\zeta \in \mathbb{D}^d$. The reproducing kernel of $L^2_{\mu_{z_0}}(X)$, written $K_{z_0}(X)(z, \zeta)$ is in $L^2(\mathbb{T}^{d+1})$ as a function of (z_0, z).*

Proof For each $\alpha \in \mathbb{Z}^d$, let

$$C_\alpha(z_0) = \int_{\mathbb{T}^d} \frac{z^\alpha}{|p(z_0, z)|^2} d\sigma(z)$$

and define the following (generally infinite) self-adjoint matrix indexed by X

$$C_X(z_0) = (C_{\alpha - \beta}(z_0))_{\alpha, \beta \in X}.$$

The entries of $C_X(z_0)$ are clearly continuous on $\mathbb{T}$. Since $|p|$ is bounded above and below on the circle, it turns out $C_X(z_0)$ is bounded above and below as an operator on $\ell^2(X)$. Indeed, for $(v_\alpha) \in \ell^2(X)$

$$\sum_{\alpha,\beta \in X} C_{\alpha-\beta}(z_0) v_\alpha \bar{v_\beta} = \int_{\mathbb{T}^d} \frac{|\sum_{\alpha\in X} v_\alpha z^\alpha|^2}{|p(z_0, z)|^2} d\sigma(z)$$

is bounded above and below by

$$\int_{\mathbb{T}^d} \left| \sum_{\alpha \in X} v_\alpha z^\alpha \right|^2 d\sigma(z) = \sum_{\alpha\in X} |v_\alpha|^2$$

with constants $c_1 = (\inf_{\mathbb{T}^{d+1}} |p|)^{-2}$ and $c_2 = (\sup_{\mathbb{T}^{d+1}} |p|)^{-2}$, respectively.

Let

$$B_{\alpha,\beta}(z_0) = (C_X(z_0))^{-1}_{\alpha,\beta}$$

be the (α, β) entry of the inverse of $C_X(z_0)$. The reproducing kernel $K_{z_0}(X)(z, \zeta)$ can be given explicitly as

$$K_{z_0}(X)(z, \zeta) = \sum_{\alpha,\beta\in X} B_{\beta,\alpha}(z_0) z^\alpha (\bar{\zeta})^\beta.$$

The proof of this fact is a direct computation; if $\gamma \in X$, then

$$\left\langle z^\gamma \sum_{\alpha,\beta\in X} B_{\beta,\alpha}(z_0) z^\alpha (\bar{\zeta})^\beta \right\rangle_{\mu_{z_0}} = \sum_{\alpha,\beta\in X} C_{\gamma-\alpha}(z_0) B_{\alpha,\beta}(z_0) \zeta^\beta = \zeta^\gamma.$$

Since $C_X(z_0)$ is bounded above and below,

$$\sum_{\alpha\in X} \left(\sum_{\beta\in X} B_{\alpha,\beta}(z_0)(\bar{\zeta})^\beta \right) z^\alpha$$

is in $L^2(\mathbb{T}^{d+1})$ as a function of (z_0, z) for each $\zeta \in \mathbb{D}^d$. □

5 Proof of Theorem 1.5

So far we have shown (in Proposition 3.2)

$$L^2_\mu(B) = L^2_\mu(X_T) \oplus L^2_\mu(X_S \ominus (X_S \cap X_T))$$

for each partition $S \sqcup T = \{1, \dots, d\}$. In addition, $L^2_\mu(X_S)$ and $L^2_\mu(X_S \ominus (X_S \cap X_T))$ are closed under multiplication by all variables z_j for $j \in T$ and $L^2_\mu(X_S)$ is maximal among subspaces with this property (Propositions 4.1 and 4.2).

Theorem 1.5 now reduces to bookkeeping and facts about reproducing kernels. Namely, a kernel is a $\mathcal{P}$-kernel if it is the reproducing kernel for a closed subspace of $L^2_\mu(B)$ (Lemma 7.5). For a non-empty $S \subset \{1, \dots, d\}$, set $T = \{1, \dots, d\} \setminus S$ and let

- K_S be the reproducing kernel for $L^2_\mu(X_T)$ and
- L_S be the reproducing kernel for $L^2_\mu(X_T \ominus (X_S \cap X_T))$

(these definitions look like S and T have been mistakenly switched but they have not). Both K_S and L_S are S-contractive $\mathcal{P}$-kernels by Lemma 7.7 and Propositions 4.1 and 4.2.

By Proposition 3.2 we have

$$\mathcal{P} = K_S + L_T.$$

To prove the maximality property of K_S, suppose $\mathcal{P} \geq K \geq 0$ for some S-contractive kernel K. By Lemmas 7.2 and 7.6 below, $z^\alpha K_\zeta \in L^2_\mu(B)$ for all $\zeta \in \mathbb{D}^d$ and all $\alpha \geq 0$ satisfying $\alpha_j = 0$ for $j \notin S$. By Proposition 4.1, $K_\zeta \in L^2_\mu(X_T)$ and therefore by Lemma 7.4, K_S must dominate K:

$$K_S \geq K.$$

This completes the proof of Theorem 1.5.

6 Proof of Theorem 1.3

We have already proven the theorem for rational inner functions which are regular on $\overline{\mathbb{D}^d}$, since such functions can always be represented by $f = \tilde{p}/p$ where $p \in \mathbb{C}[z]$ with no zeros on $\overline{\mathbb{D}^d}$. Namely, we have by Theorem 1.5

$$\frac{1 - f(z)\overline{f(\zeta)}}{\prod_{j=1}^d (1 - z_j \bar{\zeta}_j)} = \frac{K_S(z,\zeta)}{p(z)\overline{p(\zeta)}} + \frac{L_T(z,z)}{p(z)\overline{p(\zeta)}}.$$

Let us agree to absorb the denominators into the definitions of K_S and L_T so that we really have the formula

$$\frac{1 - f(z)\overline{f(\zeta)}}{\prod_{j=1}^d (1 - z_j \bar{\zeta}_j)} = K_S(z,\zeta) + L_T(z,\zeta).$$

By Theorem 1.5, $K_S + L_T = K_T + L_S$ and by maximality of K_S, K_T among S and T-contractive $\mathcal{P}$-kernels, respectively, we have

$$K_S - L_S = K_T - L_T \geq 0$$

and

$$K_S \geq K_{S'} \text{ for } S \subset S'.$$

To prove the theorem for a general holomorphic function $f : \mathbb{D}^d \to \mathbb{D}$, we use a theorem of Rudin [5, Theorem 5.5.1] which says that such f can be approximated uniformly on compact subsets of $\mathbb{D}^d$ by rational inner functions, regular on $\overline{\mathbb{D}^d}$. So, say $f_k \to f$ uniformly on compacta, with each f_k rational, inner, and continuous up to $\overline{\mathbb{D}^d}$. We have corresponding decompositions:

$$\frac{1 - f_k(z)\overline{f_k(\zeta)}}{\prod_{j=1}^d (1 - z_j \bar{\zeta}_j)} = K_S^{(k)}(z, \zeta) + L_T^{(k)}(z, \zeta).$$

Since

$$|K_S^{(k)}(z, \zeta)|^2 \leq K_S^{(k)}(z, z) K_S^{(k)}(\zeta, \zeta) \leq \frac{1}{\prod_{j=1}^d (1 - |z_j|^2)(1 - |\zeta_j|^2)}$$

(with $L_T^{(k)}$ satisfying a similar estimate), we see that the $K_S^{(k)}$'s and $L_T^{(k)}$'s are holomorphic on $\mathbb{D}^d \times \mathbb{D}^d$ and locally uniformly bounded and hence they are in a normal family. Taking subsequences, we may assume $K_S^{(k)}$ converges to some K_S and $L_T^{(k)}$ converges to some L_T locally uniformly. Positive semi-definiteness, S and T contractivity, and the identities/inequalities

$$K_S - L_S = K_T - L_T \geq 0$$
$$K_S \geq K_S' \text{ for } S \subset S'$$

are all preserved under such limits.

Therefore we conclude that

$$\frac{1 - f(z)\overline{f(\zeta)}}{\prod_{j=1}^d (1 - z_j \bar{\zeta}_j)} = K_S(z, \zeta) + L_T(z, \zeta)$$

is a valid decomposition.

Appendix: Reproducing Kernels

We record a number of facts about reproducing kernels which we used above. We are sketchy since much of this is well-known. For general references see [2,3]. As before, $\mathcal{P}$ is the reproducing kernel for $L^2_\mu(B)$, where $d\mu = |p|^{-2} d\sigma$ and $B = \{\alpha \in \mathbb{Z}^d_+ : \alpha \ngeq n\}$. (The details of μ and B are by no means essential for what follows.)

Lemma 7.1 ([3, Theorem 2.2]) *A function $f : \mathbb{D}^d \to \mathbb{C}$ is in a reproducing kernel Hilbert function space $\mathcal{H}$ on $\mathbb{D}^d$ with kernel K if and only if*

$$K(z, \zeta) \geq \epsilon f(z)\overline{f(\zeta)}$$

for some $\epsilon > 0$. The largest possible ϵ is equal to $||f||^{-2}$.

Lemma 7.2 *Let K be a positive semi-definite kernel on $\mathbb{D}^d$, and let f be a finite linear combination of functions of the form $K_\eta(z) := K(z, \eta)$. Then, there is an $\epsilon > 0$ such that*

$$K(z, \zeta) \geq \epsilon f(z)\overline{f(\zeta)}.$$

In the case of a single kernel function we can say

$$K(z, \zeta) \geq \epsilon K_\eta(z)\overline{K_\eta(\zeta)}$$

if and only if $1 \geq \epsilon K(\eta, \eta)$.

Proof Follows from Lemma 7.1. □

Lemma 7.3 *A positive semi-definite kernel K satisfying $\mathcal{P} \geq K$ is a $\mathcal{P}$-kernel (as in Definition 1.4) if and only if for every function $f : \mathbb{D}^d \to \mathbb{C}$*

$$K(z, \zeta) \geq \epsilon f(z)\overline{f(\zeta)}$$

implies

$$K(z, \zeta) \geq \frac{f(z)\overline{f(\zeta)}}{||f||_\mu^2}$$

in which case we necessarily have $||f||_\mu^{-2} \geq \epsilon$. In particular, $K(\zeta, \zeta) = ||K_\zeta||_\mu^2$ holds for all $\zeta \in \mathbb{D}^d$ whenever K is a $\mathcal{P}$-kernel. (Here $K_\zeta(z) = K(z, \zeta)$.)

Proof Follows from the definition of a $\mathcal{P}$-kernel and Lemma 7.1. □

Lemma 7.4 *Suppose $\mathcal{P} \geq K \geq 0$. Let $\mathcal{H} = \vee\{K_\zeta : \zeta \in \mathbb{D}^d\}$ be the closed span in $L^2_\mu(B)$ of the functions $K_\zeta(z) = K(z, \zeta)$, and let L be the reproducing kernel for $\mathcal{H}$. Then, $L \geq K$.*

Proof This essentially follows from Corollary 2.6 of [3]. □

Lemma 7.5 *If K is a reproducing kernel for a closed subspace of $L^2_\mu(B)$, then K is a $\mathcal{P}$-kernel.*

Proof This follows from Lemmas 7.1 and 7.3 and the fact that the norm on a subspace is the same as the norm in the original space. □

Lemma 7.6 *If a kernel K with $\mathcal{P} \geq K$ is z_j-contractive, then*

$$K(z,\zeta) \geq \epsilon f(z)\overline{f(\zeta)}$$

implies $f, z_j f \in L^2_\mu(B)$.

Proof By assumption, $(1 - z_j\bar{\zeta_j})K(z,\zeta) \geq 0$ and therefore

$$\mathcal{P}(z,\zeta) \geq K(z,\zeta) \geq z_j\bar{\zeta_j}K(z,\zeta) \geq \epsilon z_j\bar{\zeta_j} f(z)\overline{f(\zeta)}$$

which shows $z_j f \in L^2_\mu(B)$ (see Lemma 7.1). □

Lemma 7.7 *If $\mathcal{H}$ is a closed subspace of $L^2_\mu(B)$ and $\mathcal{H}$ is closed under multiplication by z_j, then the reproducing kernel for $\mathcal{H}$ is z_j-contractive.*

See for example Corollary 2.37 of [2].

Acknowledgments I would like to thank the referee for a close reading.

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