Functional determinants for the second variation

Stefano Baranzini

Abstract. We study the determinant of the second variation of an optimal control problem for general boundary conditions. Generically, these operators are not trace class and the determinant is defined as a principal value limit. We provide a formula to compute this determinant in terms of the linearisation of the extrenal flow. We illustrate the procedure in some special cases, proving some Hill-type formulas.

Mathematics Subject Classification. 47A10, 47G10, 34L40, 49K99.

Keywords. Second variation, functional determinant, optimal control problem, Hill formula.

1. Introduction

The main focus of this paper is the study of the spectrum of a particular class of Fredholm operators that arise in the context of Optimal Control. Our main result is a formula that relates the determinant of these operators to the fundamental solutions of an ODE system in a finite-dimensional space, in the spirit of Gelfand–Yaglom Theorem.

For operators of the form $1 + K$, where K is a self-adjoint compact operator, various ways of defining a *determinant* function can be found in the literature. Going all the way back to Poincaré, Fredholm and Hilbert. If the operator K one considers is in the so called *trace class*, i.e. the sequence of its eigenvalues (with multiplicity) gives an absolutely convergent series, a definition of determinant which involves the (infinite) product of its eigenvalues is possible.

In our case, however, the classical approach does not immediately apply since, typically, the operators one encounters are not trace class. In fact, for a large class of optimal control problems, the second variation of the cost functional at a critical point, denoted by Q, can be written as:

$$Q(u) = \langle (1 + K)u, u \rangle = \int_0^1 \left(-\langle H_t u_t, u_t \rangle + \sigma\left(Z_t u_t, \int_0^t Z_\tau u_\tau \mathrm{d}\tau\right) \right) \mathrm{d}t,$$

where u_t is an infinitesimal variation of the control and lives in a finite codimension subspace of $L^2([0,1], \mathbb{R}^k)$, σ is the standard symplectic form on $\mathbb{R}^{2n}$, Z_t a $2n \times k$ matrix and $-H_t$ a strictly positive definite $k \times k$ symmetric matrix.

Whenever the compact part K has this particular integral form, there is a non negative real number $\xi(K)$, which we call *capacity*, for which the following asymptotic relation for the ordered sequence of eigenvalues holds:

$$\lambda_n(K) = \frac{\xi(K)}{n} + O(n^{5/3}), \quad n \in \mathbb{Z}. \tag{1}$$

This is shown in [3] and [10], under some hypothesis on the regularity of the coefficients H_t and Z_t. For more details see Sects. 1.1 and 2 and Appendix A.

This symmetry of the spectrum allows us to talk about trace and determinant of the operators K and $1+K$ in the sense of *principal value* limits. In particular, whenever the compact part of the second variation is trace class, this limit coincides exactly with the usual Fredholm determinant defined for trace class operators. A similar approach has been independently adopted in the works [19, 20] to study the spectrum of Hamiltonian differential operators.

There are, of course, many other ways to define a *determinant function* for classes of Fredholm operators. For instance, one could apply the theory of regularized determinants, see [27], or rely on the so-called $\zeta-$regularization, see [15] for details. The literature concerning these topics is vast. To mention a few works, one could refer to [24, 25] for Sturm-Liouville problems, to [17] for graphs and to [16] for similar results proved in the general framework of elliptic operators acting on sections of vector bundles. A relation between regularized determinants and ζ-regularization is given in [18].

It is worth pointing out another feature of the construction. Theorem 1 relates the determinant of $1 + K$ to the fundamental solution of a finite-dimensional system of (linear) ODEs. This provides a way to compute the determinant and allows to recover some classical results such as Hill's formula for periodic trajectories. This kind of formulas have important applications since they allow to relate variational properties of an extremal (i.e. the eigenvalues of the second variation), to dynamical properties such as *stability*. These latter properties are usually expressed through the eigenvalues of the linearisation of the Hamiltonian system of which the extremal we are considering is solution. Take the case of a periodic, non-degenerate trajectory. On one hand, knowing the sign of the determinant amounts to know the parity of the Morse index of the extremal. On the other hand, this sign is completely determined by the number of positive eigenvalues of the linearisation grater than one. Applications of this kind of ideas go back to Poincaré's result about the instability of closed geodesics and can be found, for example, in [13] or [21]. For several interesting examples of the interplay between parity of Morse index and stability see for instance [11, 12] or [8]. Other related works in this direction are [26, 28] and [22].

Being able to compute the determinant of an operator, without knowing explicitly its spectrum, is also the reason why Gelfand–Yaglom Theorem is an extremely useful tool. In theoretical physics, it finds application in the

semiclassical approximation of the wave function. Indeed, using the stationary phase expansion, the contributions to the path integral localize at classical solutions. The quadratic term of the expansion involves the determinant of the Hessian of the classical action. Theorem 1 could be of use in computing this expansion for a broader class of action functionals.

The results are formulated in a quite general framework and the techniques can be applied without virtually any modifications to treat constrained variational problems on compact graphs as already done in [9] to compute the Morse index.

In this paper we will deal exclusively with *strictly normal* extremals (see Definition 3 in Appendix A). This hypothesis is somewhat restrictive, nevertheless is a priori verified in a consistent number of cases. For instance in Riemannian, almost-Riemannian and contact manifolds. For generic normal extremals of step 2 sub-Riemannian manifolds and in the so called *linear quadratic regulator* problems to list a few.

The structure of the paper is the following: in Sect. 1.1 we recall the notation and the setting we will use through out the paper and give the full statements of our results. Section 2 contains some information about the second variation and the structure of the space of variation we will employ.

In Sect. 3 we deal with a couple of applications, such as Hill's formula and the eigenvalue problem for Schrödinger operators. The results deserve some interest in their own, however, the main focus of the section is to provide a worked out example of how to apply the formulas to concrete situations.

The last part of the paper is Sects. 4.1 and 4.2. They are devoted to the proof of Theorem 1. We first prove the result for boundary conditions of the type $N_0 \times N_1$ (Sect. 4.1) and than extend it to the general case (Sect. 4.2). Finally, we give formulas to compute the trace of the compact part of the second variation K (Lemmas 3 and 4) and some normalization constant appearing in Theorem 1.

1.1. Problem statement and main results

We begin this section briefly recalling the setting and the notations that will be used throughout the paper. The reader is referred to [4,6] for more information on optimal control and sub-Riemannian problems. By an optimal control problem we mean the following data: a *configuration space*, i.e. a smooth manifold M, a family of smooth (and complete if M is non compact) *vector fields* f_u defined on it and a *cost function* φ. The vector fields f_u depend on a parameter u living in some open set $U \subseteq \mathbb{R}^k$. We can think of it as our way of interacting with the system and moving a particle from one state to another. We will always assume that the f_u are smooth jointly in both u and the state variable.

To any function $u(\cdot) \in L^\infty([0,1], U)$ and to any $q_0 \in M$ we can associate a trajectory in the configuration space considering the solution of:

$$\dot{q} = f_{u(t)}(q), \quad q(0) = q_0. \tag{2}$$

We will usually call the function $u(\cdot)$ *control*.

We can impose further restrictions on solutions of (2) specifying proper boundary conditions. The most general situation that we are going to treat in the paper is the case in which the boundary conditions are given by a submanifold $N \subseteq M \times M$. We say that (a Lipschitz continuous) $\gamma : [0,1] \to M$ is admissible if it solves (2) for some control u and if $(\gamma(0), \gamma(1)) \in N$. We say that a control u is admissible if the corresponding trajectory, denoted by γ_u, is defined on the whole interval $[0,1]$ and it is admissible.

Given a smooth function $\varphi : U \times M \to \mathbb{R}$, we are interested in the following minimization problem on the space of *admissible* curves:

$$\min_{\gamma_u \text{ admissible}} \mathcal{J}(\gamma_u) = \min_{\gamma_u \text{ admissible}} \int_0^1 \varphi(u(t), \gamma_u(t)) \mathrm{d}t \qquad (3)$$

In classical mechanics problems φ corresponds to the *Lagrangian* of the system. In optimal control ones it is usually referred to as the *running cost*.

It is customary to parametrized the space of admissible curves using the control function $u(\cdot)$ and a finite-dimensional parameter space that takes into account the initial data. We are going to follow this approach. However, we postpone the discussion of the structure of our space of variations to Sect. 2 since it is independent of the main statements.

The wellknown Pontryagin Maximum Principle (see Appendix A) gives a geometric description of critical points u of the restriction of $\mathcal{J}$ to the space of admissible controls in terms of suitable lifts of γ_u to the cotangent bundle. This lifts are called *extremals* and satisfy, for some $\nu \leq 0$, a Hamiltonian ODE:

$$\dot{\lambda}_t = \vec{h}_{u(t)}(\lambda), \text{ where } h_u(\lambda) = \langle \lambda, f_u(q) \rangle + \nu \varphi(u, q).$$

There are two families of critical points. A point u can either be a critical point of the constraints determining the set of admissible controls (giving a so-called *abnormal extremal* having $\nu = 0$) or can be a *true* critical point of $\mathcal{J}$ (giving a *normal extremal* having $\nu < 0$). The two conditions are not mutually exclusive, a critical point can have multiple lifts. We will only consider critical points u which are regular points of the constraints. They are called *strictly normal* optimal controls and admit no abnormal lift. The associated normal extremal is termed *strictly normal extremal*.

In this case, the space of variations can be endowed, locally, with a smooth Banach manifold structure. Thus, it makes sense to consider its tangent space. It is a finite codimension subspace $\mathcal{V}$ of $L^\infty([0,1], \mathbb{R}^k) \oplus \mathbb{R}^{\dim(N)}$.

Suppose that u is critical point of the functional $\mathcal{J}$ restricted to the space of variations and consider the Hessian of $\mathcal{J}$ at u. It is a quadratic form on $\mathcal{V}$. We denote it by $Q(v) = d_u^2 \mathcal{J}|_{\mathcal{V}}(v, v)$ and we will refer to it as *the second variation* of $\mathcal{J}$ at u. Instead of working with the L^∞ topology, we will work with the weaker L^2 one since it turns out that the quadratic form Q extends by continuity.

We will further assume that $h_u(\lambda)$ defined above is *strictly convex* along $(u(t), \gamma_u(t))$ in the u variable, this is the so called *strong Legendre condition*. Under this assumption and for an appropriate choice of scalar product on $L^2([0,1], \mathbb{R}^k)$, it turns out that the quadratic form $(Q - I)(v) = Q(v) - \langle v, v \rangle$

is compact. However, as mentioned, it is in general not trace-class since the asymptotic relation in Eq. (1) holds.

Given an eigenvalue of Q, λ, denote by $m(\lambda)$ its multiplicity. We define the determinant of the second variation as the following limit:

$$\det(Q) = \lim_{\epsilon \to 0} \prod_{|\lambda - 1| > \epsilon} \lambda^{m(\lambda)}, \text{ where } \lambda \in \operatorname{Spec}(Q).$$

As stated in the introduction, the computation of this determinant for general boundary conditions is the main contribution of this work. We provide a formula for this determinant involving essentially two ingredients:

- the fundamental solution of a linear (non autonomous) system of ODE which we call *Jacobi* equation;
- the annihilator to the boundary condition manifold, a Lagrangian sub-manifold of T^*M.

To state the main Theorem and define precisely the objects above, we need to introduce a little bit of notation. We will just sketch here what is needed for this purpose, further details are collected in Appendix A or given along with the proofs. From now on, assume that a strictly normal extremal λ_t with optimal control $\tilde{u}$ is fixed (see Appendix A). As asserted by PMP, it satisfies the following (non autonomous) Hamiltonian ODE:

$$h_{\tilde{u}}^t(\lambda) = h_u(\lambda)|_{u=\tilde{u}(t)} = (\langle \lambda, f_u(q) \rangle - \varphi(u, q))|_{u=\tilde{u}(t)}, \quad \dot{\lambda} = \vec{h}_{\tilde{u}}^t(\lambda).$$

The first objects we introduce are the flow of $\vec{h}_{\tilde{u}}^t$ at time t, denoted by $\tilde{\Phi}_t$, and its differential, denoted by $(\tilde{\Phi}_t)_*$. Note that λ_t satisfy $\lambda_t = \tilde{\Phi}_t(\lambda_0)$. We will use the map $\tilde{\Phi}_t$ to connect the tangent spaces to each point of λ_t to the starting one, λ_0. This flow, in some sense, plays the role of the choice of a connection (or parallel transport as in Riemannian geometry, see [13]).

The second object we are going to introduce, is a kind of quadratic approximation of our starting system. It is given by a quadratic Hamiltonian on $T_{\lambda_0} T^*M$ (see for detail [1] or [6, Chapters 20 and 21]). To define it, we need to introduce two matrix-valued functions $Z_t \in \operatorname{Mat}_{k \times 2\dim(M)}(\mathbb{R})$ and $H_t \in \operatorname{Mat}_{k \times k}(\mathbb{R})$. They can be computed in terms of $\tilde{\Phi}_t$ and the Hamiltonians h_u as shown in (32). The matrix Z_t is the derivative of $\vec{h}_u$ with respect to the control variable along the extremal and, heuristically, represents a linear approximation of the Endpoint map of the original system. On the other hand, H_t is the hessian of h_u with respect to the control variable along the extremal and gives some sort of quadratic approximation of h_u. These matrices will be used to get an explicit integral expression of Q, given in Equation (5), and to define *Jacobi* equation below. Strong Legendre condition can be formulated in terms of H_t. We will assume that it is symmetric, positive definite and with uniformly bounded inverse.

Let $\pi : T^*M \to M$ be the natural projection and set $\Pi := \ker \pi_*$, the *fibers*. Define δ^s as the dilation by $s \in \mathbb{R}$ of Π. It is determined by the relations $\pi_*(\delta^s w) = \pi_*(w)$ for all $w \in T_{\lambda_0} T^*M$ and $\delta^s v = sv$ for all $v \in \ker(\pi_*)$. Let J be some coordinates representation of the standard symplectic form on

$T_{\lambda_0}T^*M$. Let us define the following quadratic form:

$$\beta_s(\lambda) = \frac{1}{2}\langle \lambda, J\delta^s Z_t H_t^{-1}(\delta^s Z_t)^* J\lambda\rangle, \quad \lambda \in T_{\lambda_0}T^*M.$$

We will call *Jacobi (or Jacobi type) equation* the following ODEs system on $T_{\lambda_0}T^*M$:

$$\dot{\lambda} = \vec{\beta}_s(\lambda), \quad \lambda \in T_{\lambda_0}T^*M.$$

Denote its fundamental solution at time t by Φ_t^s. Here, and for the rest of the paper, we will call *fundamental solution* any family Θ_t, $t \in \mathbb{R}$ of linear maps which satisfies a linear ODE and have initial condition $\Theta_0 = I$.

Remark 1. Whenever PMP's maximum condition determines a C^2 function $h = \max_u h_u$ on T^*M, normal extremals satisfy a Hamiltonian ODE on the cotangent bundle of the form $\dot{\lambda} = \vec{h}(\lambda)$. Jacobi equation for $s = 1$ is closely related to the linearisation of $\vec{h}$ along the extremal we are fixing. Suppose local coordinates are fixed and let $d_{\lambda_t}^2 h$ be the Hessian of h along the extremal. Let Ψ_t be the fundamental solution of:

$$\dot{\Psi}_t = d_{\lambda_t}^2 h\, \Psi_t.$$

It can be shown (see for example [6, Proposition 21.3]) that:

$$\Psi_t = (\tilde{\Phi}_t)_* \Phi_t^1.$$

The last maps we will need are a family of symplectomorphisms of $T_{\lambda_0}T^*M$ and $T_{\lambda_1}T^*M$. Their definition depends on the choice of a scalar product on each tangent space. Let g_0 and g_1 be two scalar products on these spaces. Assume that at each λ_i, $\Pi_i := \ker(\pi_*) \subseteq T_{\lambda_i}T^*M$ has a Lagrangian orthogonal complement with respect to g_i which we denote by $\Pi_i^\perp$. Denote by J_0 and J_1 the linear maps determined by $\sigma(v, w) = g_i(J_i v, w)$ for all $v, w \in T_{\lambda_i}T^*M$. For a subspace V, denote by pr_V the orthogonal projection onto V. We set:

$$A_0^s(\eta) = \eta + (1 - s)J_0^{-1}pr_{\Pi_0^\perp}\eta, \quad \eta \in T_{\lambda_0}(T^*M),$$
$$A_1^s(\eta) = \eta + (1 - s)\left(J_1^{-1} + \tilde{\Phi}_* \circ pr_{\Pi_0} \circ \tilde{\Phi}_*^{-1}\right)pr_{\Pi_1^\perp}\eta, \quad \eta \in T_{\lambda_1}(T^*M). \tag{4}$$

The datum of the boundary condition is encoded in a Lagrangian sub-manifold of $(T^*(M \times M), (-\sigma) \oplus \sigma)$, the annihilator of N. It can be thought of as the symplectic version of the normal bundle in Riemannian geometry and is defined as follows. Take a sub-manifold $N \subseteq M \times M$ and consider:

$$Ann(N) = \bigcup_{q \in N} \{(\lambda_0, \lambda_1) \in T_q^*(M \times M) :$$
$$\langle \lambda_0, X_0 \rangle = \langle \lambda_1, X_1 \rangle, \forall (X_0, X_1) \in T_q N\}.$$

In light of PMP (see Appendix A), critical points of $\mathcal{J}$ with boundary conditions given by N, lift to the cotangent bundle to curves λ_t such that $(\lambda_0, \lambda_1) \in Ann(N)$.

Fix now a complement to $T_{(\lambda_0,\lambda_1)}Ann(N)$, say V_N, and denote by π_N the projection on V_N having $T_{(\lambda_0,\lambda_1)}Ann(N)$ as kernel. We are ready now to

define a function that plays the role of the *characteristic polynomial* of the Hessian of $\mathcal{J}$. For a map f denote by $\Gamma(f)$ its graph, set:

$$\mathfrak{p}_Q(s) = \det(\pi_N|_{\Gamma(A_1^s \tilde{\Phi}_* \Phi_1^s A_0^s)}).$$

With this notation, our main result reads as follows:

Theorem 1. *Suppose that λ_t is a strictly normal extremal for problem (3) and $\tilde{u}$ is its optimal control. Moreover, suppose that λ_t satisfies Legendre strong condition, i.e. that $\exists \alpha > 0$ such that, $\forall v \in \mathbb{R}^k$*

$$\langle -H_t v, v \rangle \geq \alpha \langle v, v \rangle$$

and that at least one of the following holds:

- *the maps $t \mapsto Z_t$ and $t \mapsto H_t$ are piecewise analytic in t;*
- *the dimension of the space of controls k satisfies $k \leq 2$;*
- *the operator $I - Q$ is trace class;*

Let $\lambda \in \operatorname{Spec}(Q)$ and $m(\lambda)$ be its multiplicity. Then, the limits:

$$\det(Q) = \lim_{\epsilon \to 0} \prod_{|\lambda-1|>\epsilon} \lambda^{m(\lambda)}, \quad \operatorname{tr}(Q - I) = \lim_{\epsilon \to 0} \sum_{|\lambda-1|>\epsilon} m(\lambda)(\lambda - 1).$$

are well defined and finite. Moreover, for almost any choice of metrics g_0 and g_1 we have that $\mathfrak{p}_Q(0) \neq 0$ and that:

$$\det(I + s(Q - I)) = \mathfrak{p}_Q(0)^{-1} e^{s(\operatorname{tr}(Q-I)-\mathfrak{p}_Q'(0))} \mathfrak{p}_Q(s).$$

Remark 2. The hypothesis about the regularity of Z_t and H_t are needed to obtain the asymptotic for the spectrum of $Q-I$ that guarantees the existence of the trace and of the determinant as limits. They can be weakened somehow by requiring that the skew-symmetric $k \times k$ matrix $Z_t^* J Z_t$ is continuously diagonalizable (see [3]).

Remark 3. It is worth pointing out that, under Legendre strong condition, the matrices Z_t and H_t are analytic (or C^k) as soon as M, the dynamics $f_u(q)$, the cost $\varphi(u, q)$ and the Hamiltonian function $h = \max_u h_u$ appearing in the statement of PMP are analytic (or C^{k+1}) too. Compare with [6, Section 21.4].

Moreover, if Z_t and H_t are at least C^1, $Q - I$ is trace class as soon as $\sigma(Z_t v, Z_t w) = 0$ for all $t \in [0, 1]$ and for all $v, w \in \mathbb{R}^k$ (see [10]).

Remark 4. The constants $\mathfrak{p}_Q(0)$, $\mathfrak{p}_Q'(0)$ and $\operatorname{tr}(Q - I)$ are completely explicit and are given in terms of iterated integrals in Lemmas 4 and 3.

In particular we have the following corollary:

Corollary 1. *Under the assumption above, the determinant of the second variation Q satisfies:*

$$\det(Q) = \mathfrak{p}_Q(0)^{-1} e^{\operatorname{tr}(Q-I)-\mathfrak{p}_Q'(0)} \det(\pi_N|_{\Gamma(\Psi_t)}),$$

where $\Psi_t = \tilde{\Phi}_* \Phi_1$ and coincides with the fundamental solution of the linearisation of the extremal flow, whenever the latter is defined.

2. The second variation

The aim of this section is threefold: to define precisely what we mean by $d^2 \mathcal{J}|_\mathcal{V}$, to define precisely its domain and to provide the integral representation of this quadratic form we will use throughout the proof section of the paper.

Before going on, a little remark about topology is in order. Up until now we have considered Lipschitz continuous curves and L^∞ controls. Hence, it would be natural to work on the Banach space $L^\infty([0,1], \mathbb{R}^k) \oplus \mathbb{R}^{\dim(N)}$. However, it turns out that, even if $d^2 \mathcal{J}|_\mathcal{V}$ is defined on the latter space, it extends to a continuous quadratic form on $L^2([0,1], \mathbb{R}^k) \oplus \mathbb{R}^{\dim(N)}$. Moreover, critical points of $d^2 \mathcal{J}|_\mathcal{V}$ in $L^2([0,1], \mathbb{R}^k)$ are continuous and thus belong to $L^\infty([0,1], \mathbb{R}^k)$. For this reason (and Fredholm alternative), we will work with L^2 controls for the rest of the paper.

Let $n_0, n_1 \in \mathbb{N}$ and consider the Hilbert space $\mathcal{H}$ (whose scalar product will be defined in the next section) given by:
$$\mathcal{H} = \mathbb{R}^{n_0} \oplus L^2([0,1], \mathbb{R}^k) \oplus \mathbb{R}^{n_1}.$$
Let (Σ, σ) be a symplectic space and consider a linear map $Z : \mathcal{H} \to \Sigma$ defined as:
$$Z(u) = Z_0 u_0 + \int_0^1 Z_t u_t \mathrm{d}t + Z_1 u_1, \quad u = (u_0, u_t, u_1) \in \mathcal{H}.$$
Suppose that $\Pi \subset \Sigma$ is a Lagrangian subspace transverse to the image of the map Z and define:
$$\mathcal{V} = Z^{-1}(\Pi).$$

For an appropriate choice of Z and Π which depends on (2) and (3), the *second variation* (at a strictly normal critical point) is the quadratic form given in the following definition.

Definition 1. (*Second variation*) The second variation at $\tilde{u}$ is the quadratic form defined on $\mathcal{V} \subseteq \mathcal{H}$:
$$Q(u) = \int_0^1 \left(-\langle H_t u_t, u_t \rangle + \sigma(Z_t u_t, \int_0^t Z_\tau u_\tau d\tau + Z_0 u_0) \right) \mathrm{d}t \tag{5}$$
$$+ \sigma(Z_0 u_0 + \int_0^1 Z_t u_t \mathrm{d}t, Z_1 u_1),$$
where H_t stands for a symmetric matrix of dimension k and $\langle , \rangle$ for the standard euclidean inner product of $\mathbb{R}^k$.

The definition of this quadratic form may be a bit strange at first glance. Despite the appearances, the way one gets to such an expression is quite natural. The construction is explained in detail in [9]. We will sketch here just the main features, essentially to introduce the notation needed.

The idea is to reduce the problem with boundary conditions N to a fixed points (or Dirichlet) problem for an appropriate auxiliary system. We will consider just the case of separated boundary conditions $N_0 \times N_1$. The general case reduces to this one using the procedure explained in Sect. 4.2.

The first step of the construction is to build the auxiliary system. We always work with a fixed strictly normal extremal λ_t. Choose local foliations in neighbourhoods of its initial and final points having a portion of N_0 and N_1 as leaves. This determines two integrable distributions in a neighbourhood of those points. Suppose that said distributions are generated by some fields $\{X_i^j\}_{i=1}^{\dim(N_j)}$ and $j = 0, 1$. Consider the extended system:

$$\hat{f}_u^t(q) = \begin{cases} \sum_{i=0}^{\dim(N_0)} X_i^0(q)u_0^i, & \text{for } t \in [-1, 0) \\ f_u(q), & \text{for } t \in [0, 1] \\ \sum_{i=0}^{\dim(N_1)} X_i^1(q)u_1^i, & \text{for } t \in (1, 2]. \end{cases}$$

Denote the initial and final points of our original extremal curve by (q_0, q_1). We will use controls that are locally constant on $[0, 1]^c$, this will be enough to reach any neighbouring point of (q_0, q_1) in $N_0 \times N_1$. Minimizing our original functional is equivalent to minimize, with Dirichlet boundary conditions, the following one:

$$\mathcal{J}(\gamma_{(u_0, u, u_1)}) = \int_0^1 \varphi(u(t), \gamma_{(u_0, u, u_1)}(t))\mathrm{d}t.$$

The second step is to differentiate the Endpoint map (see Appendix A) of the auxiliary system. We employ the machinery of Chronological Calculus (see also [6, Section 20.3]), which is standard for fixed endpoints. One of the main steps of this differentiation, is to use a suitable family of symplecto-morphism to trivialize the cotangent bundle along the curve we are fixing. This allows us to write all the equations in the tangent space to the initial point, $T_{\lambda_0}(T^*M)$. Let us consider the following functions depending on the parameter u:

$$\hat{h}_u^t(\lambda) = \langle \lambda, \hat{f}_u^t(q) \rangle - \varphi_t(u, q), \quad q = \pi(\lambda).$$

When an optimal control $\tilde{u}(t)$ is given, we consider $\hat{h}_{\tilde{u}(t)}^t(\lambda)$ and the Hamiltonian system:

$$\dot{\hat{\Phi}}_t = \vec{\hat{h}}_{\tilde{u}(t)}^t(\hat{\Phi}_t).$$

We then define the following functions:

$$\hat{Z}_t = \partial_u \vec{h}_u^t(\hat{\Phi}_t(\lambda))|_{\lambda=\lambda_0}, \quad \hat{Z}_t : \mathbb{R}^k \to T_{\lambda_0}(T^*M),$$
$$\hat{H}_t = \partial_u^2 h_u^t(\hat{\Phi}_t(\lambda))|_{\lambda=\lambda_0}, \quad \hat{H}_t : \mathbb{R}^k \to \mathbb{R}^k.$$

The asymptotic expansions of Chronological Calculus tell us that the second variation at $\tilde{u}$ is the following quadratic form:

$$Q(u) = \int_{-1}^2 \langle -\hat{H}_t u, u \rangle - \int_{-1}^2 \int_{-1}^t \sigma(\hat{Z}_\tau u_\tau, \hat{Z}_t u_t)\mathrm{d}\tau\mathrm{d}t.$$

It is defined on the tangent space $\mathcal{V}$ to the variations fixing the endpoints of our curve. This space can be described explicitly as $\mathcal{V} = \{v : \int_{-1}^2 \hat{Z}_t v_t \mathrm{d}t \in \Pi\}$, where $\Pi = \ker \pi_*$.

The third step is to specialize this representation to our auxiliary system. Notice that an extremal of the original problem lifts naturally to an

extremal of the auxiliary one. If $\tilde{u}$ is the original optimal control, extending it by zero on $[0,1]^c$ gives the optimal control for the auxiliary problem. Applying the construction just sketched to the extended system, we find that $\hat{Z}_t$ and $\hat{H}_t$ are locally constant on $[0,1]^c$. We denote Z_0 to be its value on $[-1,0)$ and Z_1 the value on $(1,2]$. $\hat{H}_t$ is zero outside $[0,1]$. Summing up we have:

$$\hat{Z}_t = \begin{cases} Z_0 & \text{for } t \in [-1,0) \\ Z_t & \text{for } t \in [0,1] \\ Z_1 & \text{for } t \in (1,2]. \end{cases} \qquad \hat{H}_t = \begin{cases} H_t & \text{for } t \in [0,1] \\ 0 & \text{for } t \in [0,1]^c. \end{cases}$$

Thus, after the substitution, we recover precisely the operator given in (5).

Remark 5. We always assume that our extremal is *strictly normal* and satisfies *Legendre strong condition*. In terms of the matrices Z_t and H_t this means that for $t \in [0,1]$ and some $\alpha > 0$:

$$X_t := \pi_* Z_t \text{ satisfies } \int_0^1 X_t^* X_t > 0, \quad \langle -H_t v, v \rangle \geq \alpha \langle v, v \rangle, \forall v \in \mathbb{R}^k.$$

As a last remark, notice that, by the first order optimality conditions, the map Z_0 takes values in the space $T_{\lambda_0} Ann(N_0)$ and the map Z_1 in the space $(\tilde{\Phi}_1^{-1})_* (T_{\lambda_1} Ann(N_1))$ (see PMP in Appendix A and [9]).

2.1. The scalar product on the space of variations

As already mentioned, we will assume through out this paper *Legendre strong condition*. The matrix $-H_t$ is positive definite on $[0,1]$, with uniformly bounded inverse. This allows to use $-H_t$ to define an Hilbert structure on $L^2([0,1], \mathbb{R}^k)$ equivalent to the standard one. We have still to define the scalar product on a subspace transversal to $\mathcal{V}_0 = \{u_0 = u_1 = 0\}$. A natural choice would be to introduce two metrics, one on $T_{\lambda_0} T^* M$ and one on $T_{\lambda_1} T^* M$, and pull them back to the space of controls using the maps Z_0 and $\tilde{\Phi}_* Z_1 : \mathbb{R}^n \to T_{\lambda_i} T^* M$. Let us call any such metrics g_0 and g_1.

Definition 2. For any $u, v \in \mathcal{H}$ define:

$$\langle u, v \rangle = -\int_0^1 H_t(u_t, v_t) \mathrm{d}t + g_0(Z_0 u_0, Z_0 v_0) + g_1(\tilde{\Phi}_* Z_1 u_1, \tilde{\Phi}_* Z_1 v_1).$$

Since the symplectic form σ is a skew-symmetric bilinear form, there exists a g_i−skew-symmetric linear operator J_i such that:

$$g_i(J_i X_1, X_2) = \sigma(X_1, X_2), \quad \forall X_1, X_2 \in T_{\lambda_i} T^* M, i = 0, 1.$$

In terms of the symplectic form, the scalar product can be written as:

$$\langle u, v \rangle = -\int_0^1 H_t(u_t, v_t) \mathrm{d}t + \sigma(J_0^{-1} Z_0 u_0, Z_0 u_0) + \sigma(J_1^{-1} \tilde{\Phi}_* Z_1 u_1, \tilde{\Phi}_* Z_1 u_1).$$

Now, we use the Hilbert structure just introduced to write the operator K associated with the quadratic form $Q - I$, which is compact. To simplify notation, we can perform the change of coordinates in L^2 sending $v_t \mapsto (-H_t)^{\frac{1}{2}} v_t$ and substitute Z_t with $Z_t(-H_t)^{-\frac{1}{2}}$. In this way, the Hilbert structure on the interval becomes the standard one.

We introduce a further piece of notation, call pr_0 (respectively pr_1) the orthogonal projection on $\mathrm{Im}(Z_0)$ (respectively $\mathrm{Im}(\tilde{\Phi}_*Z_1)$) with respect to the scalar product g_0 (respectively g_1). Let L be a partial inverse to $\tilde{\Phi}_*Z_1$, i.e. a map $L: T_{\lambda_0}T^*M \to \mathbb{R}^n$ defined by the relation $L\tilde{\Phi}_*Z_1 v_1 = v_1$. Set:

$$\Lambda(u) = L\, pr_1\, J_1 \tilde{\Phi}_* \left(Z_0 u_0 + \int_0^1 Z_t u_t \mathrm{d}t + Z_1 u_1 \right). \tag{6}$$

Lemma 1. *The second variation, as a bilinear form, can be expressed as:* $Q(u,v) = \langle u + Ku, v \rangle$ *where* $u, v \in \mathcal{V}$ *and* K *is the operator defined by:*

$$Ku = \begin{pmatrix} -u_0 \\ -Z_t^* J \left(\int_0^t Z_\tau u_\tau \mathrm{d}\tau + Z_0 u_0 \right) \\ -u_1 - \Lambda(u) \end{pmatrix} \tag{7}$$

where $\Lambda(u)$ *is given above, in Eq.* (6).

Proof. A quick manipulation of the expression involving the symplectic form in Definition 1, yields the following:

$$\int_{-1}^2 \int_{-1}^t \sigma(Z_\tau u_\tau, Z_t v_t) \mathrm{d}\tau \mathrm{d}t = \int_{-1}^0 \int_{-1}^t \sigma(Z_\tau u_\tau, Z_t v_t) \mathrm{d}\tau \mathrm{d}t$$

$$+ \int_0^1 \int_{-1}^t \sigma(Z_\tau u_\tau, Z_t v_t) \mathrm{d}\tau \mathrm{d}t + \int_1^2 \int_{-1}^t \sigma(Z_\tau u_\tau, Z_t v_t) \mathrm{d}\tau \mathrm{d}t$$

$$= \int_0^1 \sigma \left(\int_0^t Z_\tau u_\tau \mathrm{d}\tau + Z_0 u_0, Z_t v_t \right) \mathrm{d}t$$

$$+ \sigma \left(Z_0 u_0 + \int_0^1 Z_t u_t \mathrm{d}t + Z_1 u_1, Z_1 v_1 \right)$$

$$= \int_0^1 \sigma \left(\int_0^t Z_\tau u_\tau \mathrm{d}\tau + Z_0 u_0, Z_t v_t \right) \mathrm{d}t$$

$$+ g_1 \left(J_1 \tilde{\Phi}_* \left(Z_0 u_0 + \int_0^1 Z_t u_t \mathrm{d}t + Z_1 u_1 \right), \tilde{\Phi}_* Z_1 v_1 \right).$$

Recall that Z_t is constant on $[0,1]^c$. Moreover, the images of the maps Z_0 and Z_1 are isotropic subspaces. We used this fact to simplify the expression in the first line. Now, it is clear that in the last term:

$$g_1 \left(J_1 \tilde{\Phi}_* \left(Z_0 u_0 + \int_0^1 Z_t u_t \mathrm{d}t + Z_1 u_1 \right), \tilde{\Phi}_* Z_1 v_1 \right),$$

only the projection onto the image of $\tilde{\Phi}_*Z_1$ plays a role. It is straightforward to check that:

$$g_1 \left(J_1 \tilde{\Phi}_* \left(Z_0 u_0 + \int_0^1 Z_t u_t \mathrm{d}t + Z_1 u_1 \right), \tilde{\Phi}_* Z_1 v_1 \right) = g_1(\tilde{\Phi}_* Z_1 \Lambda(u), \tilde{\Phi}_* Z_1 v_1).$$

Recall that we have normalized H_t to -1, thus the first summand can be rewritten as follows:

$$\int_0^1 \sigma \left(\int_0^t Z_\tau u_\tau \mathrm{d}\tau + Z_0 u_0, Z_t v_t \right) \mathrm{d}t = \int_0^1 \langle Z_t^* J \left(\int_0^t Z_\tau u_\tau \mathrm{d}\tau + Z_0 u_0 \right), v_t \rangle \mathrm{d}t.$$

Adding and subtracting $g(Z_0 u_0, Z_0 v_0)$ and $g(Z_1 u_1, Z_1 v_1)$ to single out the identity, we obtain the formula in the statement. $\qquad\qquad\square$

3. Hill-type formulas

Before going to the proof of Theorem 1, we present here some applications of the main result. We deduce Hill's formula for periodic trajectories and specify it to the eigenvalue problem for Schrödinger operators. In the second sub-section we present a variation of the classical Hill formula for systems with drift. We will mainly deal with periodic and quasi-periodic boundary conditions. Namely, we consider the case $N = \Gamma(f)$ for a diffeomorphism of the state space $f : M \to M$.

The proofs of this section rely quite heavily on the machinery introduced in Sect. 4.2, in particular in Lemmas 4 and 3. The statements, on the other hand, do not and could shed some light on Theorem 1. Despite the appearance, proofs are rather simple. They reduce to a (long) computation of the normalizing factors appearing in the statement of Theorem 1 and can be skipped at first reading.

3.1. Driftless systems and classical Hill's formula

In this section, we consider driftless systems with periodic boundary conditions on $\mathbb{R}^n$ and specify the formulas of Theorem 4 for this class of problems.

First of all, let us explain what we mean by *driftless* systems. Let $t \mapsto R_t$ be a continuous family of symmetric matrices of size $n \times n$ and let us denote by u a function in $L^\infty([0, 1], \mathbb{R}^n)$.

Consider the following family of vector fields $f_u(q)$, their associated trajectories $q_u(t)$ and the action functional $\mathcal{A}(u)$:

$$f_u(q) = u(t), \quad \begin{cases} \dot{q}_u = f_u(q) = u(t), \\ q(0) = q_0 \in \mathbb{R}^n. \end{cases} \tag{8}$$

$$\mathcal{A}(u) = \frac{1}{2}\int_0^1 |u|^2 - \langle R_t q_u(t), q_u(t)\rangle \mathrm{d}t.$$

The Hamiltonian coming from the Maximum Principle takes the following form:

$$H(p, q) = \max_{u \in \mathbb{R}^k}\langle p, u\rangle - \frac{1}{2}(|u|^2 - \langle R_t q, q\rangle) = \frac{1}{2}(\langle p, p\rangle + \langle R_t q, q\rangle). \tag{9}$$

Let us denote the flow generated by H by Ψ_t. Fix a normal extremal λ_t for periodic boundary conditions and its optimal control $\tilde{u}(t) = p(t)$.

We impose periodic boundary conditions, i.e. we take $N = \Delta = \{(q, q) \in \mathbb{R}^{2n} : q \in \mathbb{R}^n\}$. Let g be a scalar product on $\mathbb{R}^n$ which we identify, with a slight abuse of notation, with a symmetric matrix G.

Theorem 2. (Hill's formula) *Let $I + K$ be the second variation at $\tilde{u}$ and Ψ the fundamental solution of $\dot{\Psi} = \vec{H}\Psi$ and G the matrix representing the scalar product g. The following equality holds:*

$$\det(I + K) = (-1)^n (2e)^{-n} \det(G^{-1}) \det(1 - \Psi).$$

Remark 6. If we are working on the interval $[0, T]$ instead of $[0, 1]$ everything remains essentially unchanged. The only difference is that extra factor T^{-n} appears on the right-hand side. In the notation of the proof below this corresponds to the term $\det(\Gamma)^{-1}$.

We can apply the previous result to study boundary value problems for Sturm-Liouville operators. Let us illustrate the case of Schrödinger equation with periodic boundary conditions. Fix, without loss of generality, the normal extremal $(p(t), q(t)) = (0, 0)$ and with relative optimal control $\tilde{u} = 0$. Consider the cost $\tilde{R}_t = R_t + \lambda$, for $\lambda \in \mathbb{R}$. Consider the second variation of the functional

$$\mathcal{A}_\lambda(u) = \frac{1}{2} \int_0^1 |u_t|^2 + \langle (R_t + \lambda) q_u(t), q_u(t) \rangle \mathrm{d}t,$$

at the point $\tilde{u} = 0$. It is given by the operator $1 + K_\lambda$ where:

$$\langle K_\lambda(u), u \rangle = \lambda \left(\int_0^1 \int_0^t \langle (\tau - t) u(\tau), u(t) \rangle \mathrm{d}\tau \mathrm{d}t + \langle u_0, u_0 \rangle \right) + \langle K_0(u), u \rangle.$$

We have the following corollary:

Corollary 2. *Let $\lambda \in \mathbb{R}$, Ψ_λ the fundamental matrix of the lift to $\mathbb{R}^{2n}$ of the following ODE on $\mathbb{R}^{2n}$:*

$$\ddot{q}(t) = (R_t + \lambda) q(t).$$

The determinant of the operators $1 + K_\lambda$ can be expressed as:

$$\det(1 + K_\lambda) = (-1)^n (2e)^{-n} \det(G^{-1}) \det(1 - \Psi_\lambda),$$

where G is as in the previous statement.

Proof of Theorem 2. We are going now to describe explicitly all the objects involved in Theorem 1. Let us start with the flow $\tilde{\Phi}$ we use to re-parametrize the space and its differential. It is given by the Hamiltonian:

$$h_{\tilde{u}(t)}(p, q) = \langle p, \tilde{u}(t) \rangle + \frac{1}{2} \langle R_t q, q \rangle \quad \Rightarrow \quad \begin{cases} \dot{p} = -R_t q \\ \dot{q} = \tilde{u}(t). \end{cases}$$

$$\tilde{\Phi}_t(p, q) = \begin{pmatrix} 1 & -\int_0^t R_\tau \mathrm{d}\tau \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} -\int_0^t \int_0^\tau R_\tau \tilde{u}(r) \mathrm{d}r \mathrm{d}\tau \\ \int_0^t \tilde{u}(\tau) \mathrm{d}\tau \end{pmatrix}$$

$$(\tilde{\Phi}_t)_* = \begin{pmatrix} 1 & -\int_0^t R_\tau \mathrm{d}\tau \\ 0 & 1 \end{pmatrix}.$$

The matrix Z_t is the following:

$$Z_t = (\tilde{\Phi}_t^{-1})_* \partial_u \vec{h}_u^t = \begin{pmatrix} \int_0^t R_\tau \mathrm{d}\tau \\ 1 \end{pmatrix}.$$

To simplify notation, let us call $\hat{R} = -\int_0^1 R_\tau \mathrm{d}\tau$. The annihilator of the diagonal is simply the graph of the identity. Hence the following map, defined on $(T_{\lambda_0} T^* M)^2$, has the latter as kernel:

$$(\eta_0, \eta_1) \mapsto \eta_1 - \eta_0.$$

We will now define Q^s as in Eq. (28) (actually up to a scalar, but this is irrelevant). For $\eta \in T_{\lambda_0} T^* M$ set:

$$Q^s(\eta) = \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} \eta \\ A_1^s \tilde{\Phi}_* \Phi_1^s A_0^s \eta \end{pmatrix} = (A_1^s \tilde{\Phi}_* \Phi_1^s A_0^s - 1)\eta \in T_{\lambda_0} T^* M.$$

It is clear that the kernel of Q^s is precisely the intersection of the graph with the diagonal subspace. Since we are working on $\mathbb{R}^{2n}$ we can define the determinant of this map as:

$$\det(Q^s) = \det(A_1^s \tilde{\Phi}_* \Phi_1^s A_0^s - 1).$$

As already mentioned in Sect. 1.1 this function is a multiple of the *characteristic polynomial* of K. It satisfies (see Sects. 4.1 and 4.2):

$$\det(Q^s) = ae^{bs}\det(1 + sK), \quad a \in \mathbb{C}^*, b \in \mathbb{C}.$$

Let us compute the normalization factors. To do so, we have to evaluate $\det(Q^s)$ and its derivative at $s = 0$. This will give us the relations:

$$\det(Q^0) = a, \quad a(b + \operatorname{tr}(K)) = \partial_s \det(Q^s)|_{s=0}.$$

We have to work a bit to write down precisely all the quantities appearing in the formulas above. It is straightforward to compute the matrix representations of the maps A_0^s and A_1^s. In this setting the projections onto Π_0 and $\Pi_0^\perp$ are given by:

$$pr_{\Pi_0} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad pr_{\Pi_0^\perp} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Recall that the definition of A_0^s and A_1^s given in (4) depends on the choice of two scalar products g_0 and g_1. Denote by G_0 and G_1 their restriction to $\Pi_0^\perp$ and $\Pi_1^\perp$ respectively. We have that:

$$A_0^s = \begin{pmatrix} 1 & (1-s)G_0 \\ 0 & 1 \end{pmatrix}, \quad A_1^s = \begin{pmatrix} 1 & (1-s)(G_1 - \hat{R}) \\ 0 & 1 \end{pmatrix}$$

$$A_1^s \tilde{\Phi}_* = \begin{pmatrix} 1 & s\hat{R} + (1-s)G_1 \\ 0 & 1 \end{pmatrix}.$$

The value of Φ_t^s and its derivative at $s = 0$ is given in Lemma 4. Here, for the submatrices of Φ_t^0 and $\partial_s \Phi_t^s|_{s=0}$, we use the notation defined in the Lemma 4. In this case, since $Y_t = \int_0^t R_\tau d\tau$ and $X_t = 1$, we obtain:

$$\Phi_t^0 = \begin{pmatrix} 1 & 0 \\ \Gamma & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix},$$

$$\partial_s \Phi_t^s|_{s=0} = \begin{pmatrix} \Theta & 0 \\ \Omega & -\Theta^* \end{pmatrix} = \begin{pmatrix} \int_0^t \int_0^\tau R_\omega d\omega & 0 \\ \int_0^t \int_0^\tau \int_r^\tau R_\omega d\omega dr d\tau & -\int_0^t \int_0^\tau R_\omega d\omega \end{pmatrix}.$$

Let us compute the value of $\det(Q^s)$ in zero. Putting all together we have:

$$\det(Q^s)|_{s=0} = \det\left(\begin{pmatrix} 1 & G_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \Gamma & 1 \end{pmatrix} \begin{pmatrix} 1 & G_0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right).$$

After a little bit of computation we find that $Q_s|_{s=0}$ satisfies:

$$Q^s|_{s=0} = \begin{pmatrix} G_1\Gamma & G_0 + G_1 + G_1\Gamma G_0 \\ \Gamma & \Gamma G_0 \end{pmatrix}, \quad \det(Q^s|_{s=0}) = (-1)^n \det(\Gamma)\det(G_1 + G_0),$$

$$(Q^s|_{s=0})^{-1} = \begin{pmatrix} -G_0(G_1 + G_0)^{-1} & \Gamma^{-1} + G_0(G_1 + G_0)^{-1}G_1 \\ (G_1 + G_0)^{-1} & -(G_1 + G_0)^{-1}G_1 \end{pmatrix}.$$

We can compute the derivative $\det(Q^s)$ at $s = 0$, we find that:

$$\partial_s Q^s|_{s=0} = (\partial_s A_1^s)\tilde{\Phi}_*\Phi_1^s A_0^s + A_1^s\tilde{\Phi}_*(\partial_s\Phi_1^s)A_0^s + A_1^s\tilde{\Phi}_*\Phi_1^s(\partial_s A_0^s)|_{s=0}$$

$$= \begin{pmatrix} (G_1 - \hat{R})\Gamma & G_1 - \hat{R} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & G_0 \\ 0 & 1 \end{pmatrix}$$

$$+ \begin{pmatrix} 1 & G_1 \\ 0 & 1 \end{pmatrix}\left(\begin{pmatrix} \Theta & 0 \\ \Omega & -\Theta^* \end{pmatrix}\begin{pmatrix} 1 & G_0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & G_0 \\ 0 & \Gamma G_0 \end{pmatrix}\right).$$

We use now Jacobi formula for the derivative of the determinant of a family of invertible matrices. It reads:

$$\partial_s\det(M_s) = \det(M_s)\mathrm{tr}(\partial_s M_s M_s^{-1}).$$

Without going into the detail of the actual computation, which at this point is just matrix multiplication, we have that:

$$\partial_s\det(Q^s)|_{s=0} = \det(Q^s)|_{s=0}\,\mathrm{tr}(\partial_s(Q^s)(Q^s)^{-1})|_{s=0}$$

$$= \det(Q^s)|_{s=0}\,\mathrm{tr}((G_1 + G_0)^{-1}(G_0 - G_1 + \hat{R}) + \Gamma^{-1}\Omega).$$

The last quantity we have to compute is $\mathrm{tr}(K)$. To do so we use Lemma 3. Mind that in the statement of the Lemma one works with twice the variables, taking as state space $\mathbb{R}^n \times \mathbb{R}^n$ and using the symplectic form $(-\sigma)\oplus\sigma$ on $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$. The quantities with $\tilde{\ }$ on top always refer to the system in $\mathbb{R}^{4n}$, where we have a trivial dynamic on the first factor and the boundary condition we impose are in this case $\Delta \times \Delta$ (see the beginning of Sect. 4.2 for more details). The formula given in the lemma reads:

$$\mathrm{tr}(K) = -\dim(N) + \mathrm{tr}[\pi_*^1\tilde{\Phi}_*^{-1}\,pr_1\tilde{J}_1\tilde{\Phi}_*(\tilde{Z}_0)]$$

$$+ \mathrm{tr}\left[\Gamma^{-1}\left(\Omega + (\pi_*^2 - \pi_*^1)\tilde{\Phi}_*^{-1}\,pr_1\tilde{J}_1\tilde{\Phi}_*\left(\int_0^1 \tilde{Z}_t\tilde{Z}_t^* J|_\Pi dt\right)\right)\right]. \tag{10}$$

Let us explain all the objects appearing in the formula. $\tilde{\pi}_*^i$ denotes the differential of the natural projection on the i−th factor. The matrix $\tilde{\Phi}_*$ is given here by:

$$\tilde{\Phi}_* = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 1 & R \\ 0 & 1 \end{pmatrix} \end{pmatrix}.$$

Moreover the matrices $\tilde{Z}_0$, $\tilde{Z}_t$ and $\tilde{Z}_1$ are:

$$\tilde{Z}_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \tilde{Z}_t = \begin{pmatrix} 0 \\ 0 \\ \int_0^t R_\tau d\tau \\ 1 \end{pmatrix}, \quad \tilde{Z}_1 = \begin{pmatrix} 0 \\ 1 \\ -R \\ 1 \end{pmatrix}, \quad \tilde{\Phi}_*\tilde{Z}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

The map pr_1 denotes the orthogonal projection onto the image of $\tilde{Z}_1$. We are using the scalar product $g_0 \oplus g_1$ on $T_{\lambda_0}T^*M \times T_{\lambda_1}T^*M$ to define it. One can check that the following map is the coordinate representation of pr_1:

$$
pr_1 = (G_0 + G_1)^{-1}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & G_0 & 0 & G_1 \\
0 & 0 & 0 & 0 \\
0 & G_0 & 0 & G_1
\end{pmatrix},
$$

$$
pr_1 \tilde{J}_1 = (G_0 + G_1)^{-1}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0
\end{pmatrix}.
$$

Now everything reduces to some tedious matrix multiplications. The second term in the right hand side of (10) simplifies as follows:

$$
\mathrm{tr}[\pi_*^1 \tilde{\Phi}_*^{-1} pr_1 \tilde{J}_1 \tilde{\Phi}_*(\tilde{Z}_0)] = \mathrm{tr}(\hat{R}(G_0 + G_1)^{-1}).
$$

For the third term notice that $(\pi_*^2 - \pi_*^1)(\tilde{\Phi}_*)^{-1} pr_1$ is identically zero since $\tilde{\Phi}_*$ does not change the projection on the horizontal part and we are working with periodic boundary conditions. It follows we are left with $\mathrm{tr}(\Gamma^{-1}\Omega)$. Summing up we have that:

$$
\mathrm{tr}(K) = \mathrm{tr}(\Omega\Gamma^{-1} + \hat{R}(G_0 + G_1)^{-1}) - \dim(M).
$$

It is natural to think of g_0 and g_1 as restrictions of globally defined Riemannian metrics. Doing so, since we are working with periodic boundary conditions, amounts to choose $G_0 = G_1$. Hence the result. $\qquad\square$

3.2. System with drift and Hill-type formulas

In this section we give a version of Hill's formula for linear systems with drift. They are again linear system with quadratic cost of the following form:

$$
f_u(q) = A_t q + B_t u(t), \quad
\begin{cases}
\dot{q}_u = f_u(q) = A_t q_u + B_t u(t), \\
q(0) = q_0 \in \mathbb{R}^n
\end{cases}
\tag{11}
$$

$$
\mathcal{A}(u) = \frac{1}{2}\int_0^1 |u|^2 + \langle R_t q_u(t), q_u(t)\rangle \mathrm{d}t.
$$

Here A_t is $n \times n$ matrix and B_t a $n \times k$ one, both with possibly non-constant (but continuous) coefficients. The maximized Hamiltonian takes the form:

$$
H(p, q) = \langle p, A_t q\rangle + \frac{1}{2}\left(\langle B_t B_t^* p, p\rangle + \langle R_t q, q\rangle\right).
$$

Denote by $\hat{\Phi}_t$ the fundamental solution of $\dot{q} = A_t q$ at time t. We can lift this map to a symplectomorphism of the cotangent bundle which we denote by $\underline{\Phi}_t$. As boundary conditions, we take the following affine subspace of $\mathbb{R}^{4n}$:

$$
N = \Gamma\left(\hat{\Phi} + \hat{\Phi}\int_0^1 \hat{\Phi}_r^{-1} B_r \tilde{u}(r)\mathrm{d}r\right).
$$

Notice that, since only the tangent space matters in our formulas, the translation is irrelevant and it would be the same as if we considered $\Gamma(\hat{\Phi})$. An obvious choice of extremal is the point $\lambda_t = (0, 0)$, with control $\tilde{u} = 0$.

Theorem 3. (Hill's formula with drift) *Suppose that a critical point of the functional given in* (11) *is fixed and let $\tilde{u}$ be its optimal control. Let G_0 and G_1 be our choices of scalar product, $\Gamma := \int_0^1 \hat{\Phi}_\tau B_\tau B_\tau^* \hat{\Phi}_\tau^* \mathrm{d}\tau$ and Ψ_t the fundamental solution of $\dot{\Psi} = \vec{H}\Psi$. Moreover, define the following matrix:*

$$G = G_0 + \hat{\Phi}_t^* G_1 \hat{\Phi}_t.$$

Let $I + K$ be the second variation at $\tilde{u}$. The following equality holds:

$$\det(I + K) = \frac{(-1)^n e^{-2 \operatorname{tr}(G^{-1} G_0)}}{\det(G) \det(\Gamma)} \det(\Psi_t - \underline{\Phi}_t).$$

Proof. The proof is completely analogous to the one of Theorem 2. First of all the Hamiltonian we use to re-parametrize is given as follows:

$$h_{\tilde{u}(t)}^t = \langle p, B_t \tilde{u}(t) + A_t q \rangle + \frac{1}{2} \langle R_t q, q \rangle.$$

Hence the flow and its differential are given by:

$$\tilde{\Phi}_t(p, q) = \begin{pmatrix} (\hat{\Phi}_t^*)^{-1} & -(\hat{\Phi}_t^*)^{-1} \int_0^t \hat{\Phi}_\tau^* R_\tau \hat{\Phi}_\tau \mathrm{d}\tau \\ 0 & \hat{\Phi}_t \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

$$+ \begin{pmatrix} -(\hat{\Phi}_t^*)^{-1} \int_0^t \hat{\Phi}_\tau^* R_\tau \hat{\Phi}_\tau \int_0^\tau \hat{\Phi}_r^{-1} B_r \tilde{u}(r) \mathrm{d}r \mathrm{d}\tau \\ \hat{\Phi}_t \int_0^t \hat{\Phi}_r^{-1} B_r \tilde{u}(r) \mathrm{d}r \end{pmatrix},$$

$$(\tilde{\Phi}_t)_* = \begin{pmatrix} (\hat{\Phi}_t^*)_*^{-1} & -(\hat{\Phi}_t^*)^{-1} \int_0^t \hat{\Phi}_\tau^* R_\tau \hat{\Phi}_\tau \mathrm{d}\tau \\ 0 & \hat{\Phi}_t \end{pmatrix}.$$

We have to define the map Q^s. Similar to the previous case, we can define a map having as kernel the annihilator to the boundary conditions as follows:

$$(\eta_0, \eta_1) \mapsto \begin{pmatrix} \hat{\Phi}_t^* & 0 \\ 0 & (\hat{\Phi}_t)^{-1} \end{pmatrix} \eta_1 - \eta_0 =: T_1 \eta_1 - \eta_0.$$

Hence:

$$\det(Q_s) = \det(T_1 A_1^s \tilde{\Phi}_* \Phi_1^s A_0^s - 1).$$

Set as before $\hat{R} = -(\hat{\Phi}^*)^{-1} \int_0^1 \hat{\Phi}_\tau^* R_\tau \hat{\Phi}_\tau \mathrm{d}\tau$, the upper right minor of $\tilde{\Phi}_*$. A quick computation shows that:

$$T_1 A_1^s \tilde{\Phi}_* = \begin{pmatrix} 1 & (1-s)\hat{\Phi}_t^* G_1 \hat{\Phi}_t + s\hat{\Phi}_t^* \hat{R} \\ 0 & 1 \end{pmatrix} := \hat{A}_1^s \tilde{\Phi}_*.$$

Hence, up to renaming G_1 and Γ in the proof of Theorem 2, we have:

$$\det(Q^s)|_{s=0} = (-1)^n \det(\Gamma) \det(\hat{\Phi}_t^* G_1 \hat{\Phi}_t + G_0) = (-1)^n \det(\Gamma) \det(G),$$

$$\frac{\partial_s \det(Q^s)}{\det(Q^s)}|_{s=0} = \operatorname{tr}(((\hat{\Phi}_t^* G_1 \hat{\Phi}_t + G_0)^{-1}(G_0 - \hat{\Phi}_t^* G_1 \hat{\Phi}_t + \hat{\Phi}_t^* \hat{R}) + \Gamma^{-1}\Omega).$$

Now we have to apply Lemma 3 to compute the trace of the compact part of the second variation. Here pr_1 and $\tilde{Z}_1$ are different since we have

changed boundary conditions. However we have the same kind of simplification as in the previous case. Let us write explicitly the new objects:

$$Z_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \hat{\Phi}_t \end{pmatrix}, \quad pr_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & LG_0 & 0 & L\hat{\Phi}_t^* G_1 \\ 0 & 0 & 0 & 0 \\ 0 & \hat{\Phi}_t L G_0 & 0 & \hat{\Phi}_t L \hat{\Phi}_t^* G_1 \end{pmatrix}, \quad L = (G_0 + \hat{\Phi}_t^* G_1 \hat{\Phi}_t)^{-1}.$$

In the end, the trace reads:

$$\operatorname{tr}(K) = \operatorname{tr}(\Gamma^{-1}\Omega + \hat{\Phi}_t^* \hat{R}(\hat{\Phi}_t^* G_1 \hat{\Phi}_t + G_0)^{-1}) - \dim(M).$$

Contrary to the previous section the two terms do not simplify. We are left with the following equation for b:

$$b = \dim(M) + \operatorname{tr}((\hat{\Phi}_t^* G_1 \hat{\Phi}_t + G_0)^{-1}(G_0 - \hat{\Phi}_t^* G_1 \hat{\Phi}_t))$$
$$= n + 2\operatorname{tr}(G^{-1}G_0) - n = 2\operatorname{tr}(G^{-1}G_0).$$

Hence, the statement follows evaluating $\det(Q^s)$ at $s = 1$:

$$\det(Q^s) = \det(T_1 \tilde{\Phi}_* \Phi_1^1 - 1) = \det(\underline{\Phi}_t^{-1} \tilde{\Phi}_* \Phi_1^1 - 1)$$
$$= \det(\Psi_t - \underline{\Phi}_t).$$

$\square$

4. Proof of the main theorem

In this section, we provide a proof of Theorem 1. First, we work with separated boundary conditions and then reduce the general case to the former. The proof is a bit long so we try to give here a concise outline. The idea is to construct an analytic function f which vanishes precisely on the set $\{-1/\lambda : \lambda \in \operatorname{Spec}(K)\} \subseteq \mathbb{R}$. Particular care is needed to show that the multiplicity of the zeros of this function equals the multiplicity of the eigenvalues of K. We do this in Propositions 1 and 2 respectively. We show that this function decays exponentially and use a classical factorization Theorem by Hadamard to represent it as

$$f(s) = as^k e^{bs} \prod_{\lambda \in \operatorname{Spec}(K)} (1 + \lambda s)^{m(\lambda)}, \quad a, b \in \mathbb{C}, a \neq 0, k \in \mathbb{N}.$$

To prove the general case, we double the variables and consider general boundary conditions as separated ones. In this framework, we compute the value of the parameters a, b and k appearing in the factorization.

4.1. Separated boundary conditions

We briefly recall the notation. We are working with an extremal λ_t with initial and final point $(\lambda_0, \lambda_1) \in Ann(N)$, where $N = N_0 \times N_1$ are product boundary conditions. We are assuming that λ_t is strictly normal and satisfies Legendre strong conditions. We work in a fixed tangent space, namely $T_{\lambda_0} T^* M$, to do so we backtrack our curve to its starting point λ_0 using the flow generated by the time-dependent Hamiltonian:

$$h_{\tilde{u}}^t(\lambda) = \langle \lambda, f_{\tilde{u}(t)}(q) \rangle - \varphi_t(q, \tilde{u}(t)).$$

We denote the differential of said flow by $\tilde{\Phi}_*$. We have a scalar product g_i on $T_{\lambda_i} T^* M$, for each $i = 0, 1$. We assume that the orthogonal complement to the fiber at λ_i, $\Pi_i = T_{\lambda_i} T^*_{\pi(\lambda_i)} M$ is a Lagrangian subspace and that the range of Z_0 (and $\tilde{\Phi}_* Z_1$ respectively) is contained in $\Pi_0^\perp$ (resp. $\Pi_1^\perp$).

Remark 7. If we fix Darboux (i.e. *canonical*) coordinates coming from the splitting $\Pi_i \oplus \Pi_i^\perp$ it is straightforward to check that g_i takes a block diagonal form with symmetric $n \times n$ matrix G_i^j on the main diagonal. Similarly, we can write down the coordinate representation of the matrix J_i and find:

$$g_i(X, Y) = \left\langle \begin{pmatrix} G_i^1 & 0 \\ 0 & G_i^2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \right\rangle, \quad J_i = \begin{pmatrix} 0 & -(G_i^1)^{-1} \\ (G_i^2)^{-1} & 0 \end{pmatrix}.$$

For $s \in \mathbb{R}$ (or $\mathbb{C}$) we introduce the following symplectic maps:

$$\begin{aligned}
A_0^s(\eta) &= \eta + (1-s) J_0^{-1} pr_{\Pi_0^\perp} \eta, \quad \eta \in T_{\lambda_0}(T^* M), \\
A_1^s(\eta) &= \eta + (1-s)(J_1^{-1} + \tilde{\Phi}_* \circ pr_{\Pi_0} \circ \tilde{\Phi}_*^{-1}) pr_{\Pi_1^\perp} \eta, \quad \eta \in T_{\lambda_1}(T^* M).
\end{aligned} \tag{12}$$

Notice that the transformation A_i^s are indeed symplectomorphisms. In (canonical) coordinates given by Π_i and $\Pi_i^\perp$ they have the following matrix representation:

$$A_i^s = \begin{pmatrix} 1 & (1-s)S_i \\ 0 & 1 \end{pmatrix}, \quad S_i^* = S_i.$$

The last map we are going introduce is two families of dilation in $T_{\lambda_0}(T^* M)$, one of the vertical subspace and one of its orthogonal complement. Let $s \in \mathbb{R}$ (or $\mathbb{C}$) and let us define the following maps:

$$\begin{aligned}
\delta^s &: T_{\lambda_0} T^* M \to T_{\lambda_0} T^* M, \quad \delta^s \nu = s\, pr_{\Pi_0} \nu + pr_{\Pi_0^\perp} \nu, \\
\delta_s &: T_{\lambda_0} T^* M \to T_{\lambda_0} T^* M, \quad \delta_s \nu = pr_{\Pi_0} \nu + s\, pr_{\Pi_0^\perp} \nu.
\end{aligned} \tag{13}$$

Proposition 1. *Let A_i^s be the maps given in Eq. (12) and let Φ_1^s be the fundamental solution of the system:*

$$\dot{\eta} = Z_t^s (Z_t^s)^* J\eta, \quad Z_t^s = \delta^s Z_t,$$

The operator $I + sK$ restricted to $\mathcal{V}$ has non trivial kernel if and only if there exists a non zero $(\eta_0, \eta_1) \in T_{(\lambda_0, \lambda_1)}(Ann(N))$ such that

$$A_1^s \circ \tilde{\Phi}_* \circ \Phi_1^s \circ A_0^s \eta_0 = \eta_1,$$

In particular, the geometric multiplicity of the kernel of $I + sK$ equals the number of linearly independent solutions of the above equation.

Proof. $I + sK$ has a non trivial kernel if and only if

$$\langle u, v \rangle + \langle sKu, v \rangle = 0, \quad \forall u, v \in \mathcal{V}.$$

Equivalently if and only if $\exists u$ such that $u + sKu \in \mathcal{V}^\perp$ (see Lemma 2 below for a description of $\mathcal{V}^\perp$). This is in turn equivalent to the following system

$$\begin{cases} (1-s)u_0 = v_0 \\ u_t = sZ_t^* J \left(\int_0^t Z_\tau u_\tau d\tau + Z_0 u_0 \right) + Z_t^* J\nu \\ (1-s)u_1 = s\Lambda(u) + v_1 \end{cases} \tag{14}$$

Let us substitute Z_t with $Z_t^s = \delta^s Z_t$. It is straightforward to check, using the definition in (13), that:

$$sZ_t^* J \int_0^t Z_\tau u_\tau \mathrm{d}\tau = (Z_t^s)^* J \int_0^t Z_\tau^s u_\tau \mathrm{d}\tau.$$

Moreover $Z_t^* J\nu = (Z_t^s)^* J\nu$ for any $\nu \in \Pi$ and $sZ_t^* JZ_0 = (Z_t^s)^* JZ_0^s$ since $\delta^s Z_0 = Z_0$.

All the computations we will do from here on are aimed at rewriting (14) as a boundary value problem in $T_{\lambda_0}T^*M \times T_{\lambda_0}T^*M$. Let us start with the second equality in (14). Define:

$$\eta(t) = \int_0^t Z_\tau^s u_\tau \mathrm{d}\tau + Z_0^s u_0 + \nu, \quad \eta(0) = Z_0^s u_0 + \nu.$$

The linear constraint defining $\mathcal{V}$ implies:

$$Z_0 u_0 + \int_0^1 Z_t u_t \mathrm{d}t + Z_1 u_1 \in \Pi \iff Z_0 u_0 + \int_0^1 Z_t^s u_t \mathrm{d}t + Z_1^s u_1 \in \Pi$$
$$\iff \eta(1) + Z_1^s u_1 \in \Pi_0$$
$$\iff \tilde{\Phi}_*(\eta(1) + Z_1^s u_1) \in \Pi_1.$$

This imposes a condition on the initial and final value of $\eta(t)$. The initial one must lie in $\pi_*^{-1}TN_0$ and final one in $\pi_*^{-1}TN_1$. If we multiply by Z_t^s the second equation in (14), we are brought to consider the following problem:

$$\begin{cases} \dot{\eta}(t) = Z_t^s (Z_t^s)^* J\eta(t), \\ (\pi_*\eta(0), \pi_*\eta(1)) \in T(N_0 \times N_1). \end{cases} \quad (15)$$

Now we use the remaining equations in (14) to reduce the space $\pi_*^{-1}(T(N_0 \times N_1))$ to a Lagrangian one. Let us reformulate the first and third line in (14) as equations in $T_{\lambda_0}T^*M$ and $T_{\lambda_1}T^*M$. Using the maps Z_0 and $\tilde{\Phi}_* Z_1$ we obtain:

$$\begin{cases} (1-s)Z_0 u_0 = Z_0 v_0 = pr_0 J_0 \nu, \\ (1-s)\tilde{\Phi}_* Z_1 u_1 = \tilde{\Phi}_* Z_1 v_1 + s\tilde{\Phi}_* Z_1 \Lambda(u) = pr_1 J_1 \tilde{\Phi}_* \nu + s\tilde{\Phi}_* Z_1 \Lambda(u). \end{cases}$$

Where we used the fact that (v_0, v_t, v_1) is a vector in $\mathcal{V}^\perp$. Notice that, for $u \in \mathcal{V}$ we have that:

$$s\left(Z_0 u_0 + \int_0^1 Z_t u_t \mathrm{d}t + Z_1 u_1\right) = Z_0 u_0 + \int_0^1 Z_t^s u_t \mathrm{d}t + Z_1^s u_1.$$

This implies that the term $s\tilde{\Phi}_* Z_1 \Lambda(u)$ can be rewritten as:

$$s\tilde{\Phi}_* Z_1 \Lambda(u) = s\, pr_1 J_1 \tilde{\Phi}_* \left(Z_0 u_0 + \int_0^1 Z_t u_t \mathrm{d}t + Z_1 u_1\right)$$
$$= pr_1 J_1 \tilde{\Phi}_*(\eta(1) + Z_1^s u_1 - \nu).$$

If substitute $Z_0 u_0$ with $pr_{\Pi_0^\perp}\eta(0)$ we end up with the equations:

$$\begin{cases} (1-s)pr_{\Pi_0^\perp}\eta(0) = pr_0 J_0 \nu = pr_0 J_0 \eta(0), \\ (1-s)\tilde{\Phi}_* Z_1 u_1 = pr_1 J_1 \tilde{\Phi}_*(\eta(1) + Z_1^s u_1). \end{cases}$$

Now we do the same kind of substitution for the term $Z_1^s u_1$. Using the projections on Π_0 and $\Pi_0^\perp$ and recalling that $\tilde{\Phi}_*$ sends Π_0 to Π_1, we have:

$$Z_1^s u_1 = s \, pr_{\Pi_0} Z_1 u_1 + pr_{\Pi_0^\perp} Z_1 u_1 = (s-1) \, pr_{\Pi_0} Z_1 u_1 + Z_1 u_1.$$

$$pr_1 J_1 \tilde{\Phi}_* Z_1^s u_1 = (s-1) \, pr_1 J_1 \tilde{\Phi}_* pr_{\Pi_0} Z_1 u_1 + pr_1 J_1 \tilde{\Phi}_* Z_1 u_1$$

$$= (s-1) \, pr_1 J_1 \tilde{\Phi}_* pr_{\Pi_0} Z_1 u_1.$$

Last equality being due to the fact that the image of $\tilde{\Phi}_* Z_1$ is isotropic and thus $J_1 \mathrm{Im}(\tilde{\Phi}_* Z_1) \subset \mathrm{Im}(\tilde{\Phi}_* Z_1)^\perp$. Moreover $\tilde{\Phi}_* Z_1 u_1$ coincides with the projection of $-\tilde{\Phi}_* \eta(1)$ on $\Pi_1^\perp$. Thus we are left with:

$$\begin{cases} (1-s) pr_{\Pi_0^\perp} \eta(0) = pr_0 J_0 \nu = pr_0 J_0 \eta(0), \\ (1-s)(-pr_{\Pi_1^\perp} \tilde{\Phi}_* \eta(1) + pr_1 J_1 \tilde{\Phi}_* pr_{\Pi_0} Z_1 u_1) = pr_1 J_1 \tilde{\Phi}_* \eta(1). \end{cases} \tag{16}$$

It is straightforward to check that $pr_1 J_1 \tilde{\Phi}_* pr_{\Pi_0} Z_1 u_1$ depends only on the projection of $Z_1 u_1$ on $\Pi_0^\perp$. Moreover, expanding $1 = \tilde{\Phi}_* \circ \tilde{\Phi}_*^{-1}$ and using the relation $\tilde{\Phi}_* Z_1 u_1 = -pr_{\Pi_1^\perp} \tilde{\Phi}_* \eta(1)$, the second equality in (16) can be rewritten as:

$$(s-1) pr_{\Pi_1^\perp} \tilde{\Phi}_* \eta(1) = pr_1 J_1 \tilde{\Phi}_* \eta(1) + (1-s) pr_1 J_1 \tilde{\Phi}_* pr_{\Pi_0} \tilde{\Phi}_*^{-1} pr_{\Pi_1^\perp} \tilde{\Phi}_* \eta(1). \tag{17}$$

If $s = 1$, the equations reduce to $pr_0(J_0 \eta(0)) = 0$ and $pr_1 J_1 \tilde{\Phi}_* \eta(1) = 0$. Consider the first case, the relation is equivalent to:

$$\sigma(\eta(0), Z_0 w_0) = g_0(J_0 \eta(0), Z_0 w_0) = g_0(pr_0 J_0 \eta(0), Z_0 w_0) = 0, \ \forall w_0 \in \mathbb{R}^{\dim(N_0)}.$$

Thus we are looking for a solution starting from $T_{\lambda_0} Ann(N_0)$. Similarly setting $s = 1$ in the second equality we find that $\eta(1)$ must lie inside $T_{\lambda_1} Ann(N_1)$.

Now, for $s \neq 1$, we want to interpret the boundary conditions as an analytic family of Lagrangian subspaces depending on s. To do so we employ the following linear map defined in (12):

$$A_0^s(\eta) = \eta + J_0^{-1}(1-s) pr_{\Pi_0^\perp} \eta.$$

If $\eta \in T_{\lambda_0} Ann(N_0)$ we have that $pr_{\Pi_0^\perp} \eta = pr_0 \eta$ and $pr_0 J_0 \eta = 0$ and thus:

$$pr_{\Pi_0^\perp}(A_0^s(\eta)) = pr_{\Pi_0^\perp}(\eta) \in \mathrm{Im}(Z_0).$$

$$pr_0 J_0(A_0^s(\eta)) = pr_0(J_0 \eta + (1-s) pr_{\Pi_0^\perp} \eta)$$

$$= pr_0 J_0 \eta + (1-s) pr_{\Pi_0^\perp} \eta$$

$$= (1-s) pr_{\Pi_0^\perp}(A_0^s(\eta)).$$

So we have shown that $A_0^s(T_{\lambda_0} Ann(N_0))$ is precisely the space satisfying the first set of equations. A similar argument works for the final point. Let us recall the definition of A_1^s given in (12):

$$A_1^s(\eta) = \eta + (1-s)(J_1^{-1} + \tilde{\Phi}_* pr_{\Pi_0} \tilde{\Phi}_*^{-1}) pr_{\Pi_1^\perp} \eta.$$

Now, we check that the boundary condition for the final point are satisfied if and only if $A_1^s \circ \tilde{\Phi}_* \eta(1) \in T_{\lambda_1} Ann(N_1)$. In fact, take any η in

$T_{\lambda_1} Ann(N_1)$, it holds:

$$pr_{\Pi_1^\perp}(A_1^s)^{-1}\eta = pr_{\Pi_1^\perp}\eta,$$

$$pr_{\Pi_1}(A_1^s)^{-1}\eta = pr_{\Pi_1}\eta + (s-1)(J_1^{-1} + \tilde{\Phi}_* pr_{\Pi_0}\tilde{\Phi}_*^{-1})pr_{\Pi_1^\perp}\eta,$$

$$pr_1 J_1(A_1^s)^{-1}\eta = (s-1)pr_1(1 + J_1\tilde{\Phi}_* pr_{\Pi_0}\tilde{\Phi}_*^{-1})pr_{\Pi_1^\perp}\eta.$$

It is straightforward to substitute the last equality in (17) and check that indeed $(A_1^s)^{-1}(T_{\lambda_1} Ann(N_1))$ is the right space.

Let us call Φ_1^s the fundamental solution of (15) at time 1 and denote by $\Gamma(\Phi_1^s)$ its graph. It follows that $s \in \mathbb{R} \setminus \{0\}$ is in the kernel of $1 + sK$ if and only if:

$$\Gamma(\tilde{\Phi}_* \circ \Phi_1^s) \cap A_0^s(T_{\lambda_0} Ann(N_0)) \times (A_1^s)^{-1}(T_{\lambda_1} Ann(N_1)) \neq (0).$$

which is equivalent to the condition:

$$\Gamma(A_1^s \circ \tilde{\Phi}_* \circ \Phi_1^s \circ A_0^s) \cap T_{\lambda_0} Ann(N_0) \times T_{\lambda_1} Ann(N_1) \neq (0). \qquad (18)$$

Now we prove the part about the multiplicity. Suppose that two different controls u and v give the same trajectory η_t solving (15). Since the maps Z_0 and Z_1 are injective it must hold that $v_0 = u_0$ and $v_1 = u_1$. Moreover, $\int_0^t Z_\tau u_\tau d\tau = \int_0^t Z_\tau v_\tau d\tau \; \forall\, t \in [0,1]$ and thus:

$$Ku = Z_t^* J \int_0^t Z_\tau u_\tau d\tau = Z_t^* J \int_0^t Z_\tau v_\tau d\tau = Kv.$$

However, Volterra operators are always injective and thus $u = v$.

Vice-versa, consider $u = 0$ and see whether you get solutions of the system above that do not correspond to any variation. Since u_0 and u_1 are both zero we are considering solution starting from the fiber and reaching the fiber. Plugging in $u_t = 0$ we obtain:

$$0 = \dot{\eta} = Z_t^s (Z_t^s)^* J\eta = Z_t^s (Z_t^s)^* J\nu.$$

However $pr_{\Pi^\perp} Z_t^s (Z_t^s)^* J\nu = X_t X_t^* \nu$ and by assumption the matrix $\int_0^1 X_t X_t^* dt$ is invertible. Thus we get a contradiction. $\qquad \square$

The following Lemma was used in the proof of Proposition 1. Gives the orthogonal complement to $\mathcal{V}$ inside $\mathcal{H}$, using the Hilbert structure introduced in Definition 2. We will denote by the symbol $\perp_i$ the orthogonal complement, in $T_{\lambda_i} T^* M$, with respect to the scalar product g_i.

Lemma 2. *With this choice of scalar product the orthogonal complement to $\mathcal{V}$ is given by:*

$$\mathcal{V}^\perp = \{(v_0, Z_t^* J\nu, v_1) : \nu \in \Pi\},$$

where v_0 and v_1 are determined by the following conditions:

$$Z_0 v_0 - J_0 \nu \in \mathrm{Im} Z_0^{\perp_0}, \quad \tilde{\Phi}_* Z_1 v_1 - J_1 \tilde{\Phi}_* \nu \in \mathrm{Im} \tilde{\Phi}_* Z_1^{\perp_1}.$$

Proof. Suppose that $u \in \mathcal{V}^\perp$. Let us test it against infinitesimal variations that fix the endpoints, i.e. such that $u_i = 0$ and $\int_0^1 Z_t u_t dt \in \Pi$. Recall that Π is Lagrangian, thus the condition $\int_0^1 Z_t u_t dt \in \Pi$ can be equivalently formulated as $\sigma(\int_0^1 Z_t u_t dt, \nu) = 0$ for all $\nu \in \Pi$. Hence, the subspace $\{u : u_i =$

$0, \int_0^1 Z_t u_t \mathrm{d}t \in \Pi\}$ is the intersection of the kernels of $u_t \mapsto \int_0^1 \sigma(Z_t u_t, \nu)\mathrm{d}t$. It follows that:

$$\langle v, u \rangle = -\int_0^1 \langle v_t, u_t \rangle \mathrm{d}t = 0, \ \forall u \in V \iff v_t = \sigma(\nu, Z_t \cdot), \quad \nu \in \Pi.$$

Hence $v_t = Z_t^* J \nu$. Now, take $u \in \mathcal{V}$ and compute:

$$\langle v, u \rangle = \sigma(\nu, \int_0^1 Z_t u_t \mathrm{d}t) + \sigma(J_0^{-1} Z_0 v_0, Z_0 u_0) + \sigma(J_1^{-1} \tilde{\Phi}_* Z_1 v_1, \tilde{\Phi}_* Z_1 u_1).$$

Since $u \in \mathcal{V}$, we have:

$$\sigma\left(\nu, \int_0^1 Z_t u_t \mathrm{d}t\right) = -\sigma(\nu, Z_0 u_0 + Z_1 u_1).$$

It follows that:

$$\langle v, u \rangle = -\sigma(\nu, Z_0 u_0 + Z_1 u_1) + \sigma(J_0^{-1} Z_0 v_0, Z_0 u_0) + \sigma(J_1^{-1} \tilde{\Phi}_* Z_1 v_1, \tilde{\Phi}_* Z_1 u_1)$$
$$= \sigma(J_0^{-1} Z_0 v_0 - \nu, Z_0 u_0) + \sigma(J_1^{-1} \tilde{\Phi}_* Z_1 v_1 - \tilde{\Phi}_* \nu, \tilde{\Phi}_* Z_1 u_1).$$

Since we are assuming that $\int_0^1 X_t X_t^* \mathrm{d}t > 0$, the image of the map $u_t \mapsto \pi_* \int_0^1 Z_t u_t \mathrm{d}t$ is the whole $T_{q_0} M$. In particular, for any u_0, is possible to find variations of the form $(u_0, u_t, 0) \in \mathcal{V}$. An analogous statement holds for variations of the form $(0, u_t, u_1)$.

Hence, if $\langle u, v \rangle = 0 \ \forall u \in \mathcal{V}$, then both $\sigma(J_0^{-1} Z_0 v_0 - \nu, Z_0 u_0)$ and $\sigma(J_1^{-1} \tilde{\Phi}_* Z_1 v_1 - \tilde{\Phi}_* \nu, \tilde{\Phi}_* Z_1 u_1)$ must be zero at the same time.

Moreover, this implies that v_0 and v_1 are completely determined by the value of ν. Finally notice that:

$$\sigma(J_0^{-1} Z_0 v_0 - \nu, Z_0 u_0) = 0 \forall u_0 \iff Z_0 v_0 - J_0 \nu \in \mathrm{Im}(Z_0)^{\perp_0},$$
$$\sigma(J_1^{-1} \tilde{\Phi}_* Z_1 v_1 - \tilde{\Phi}_* \nu, Z_1 u_1) = 0 \forall u_1 \iff \tilde{\Phi}_* Z_1 v_1 - J_1 \tilde{\Phi}_* \nu \in \mathrm{Im}(\tilde{\Phi}_* Z_1)^{\perp_1}.$$

$\square$

Remark 8. If we complexify all the subspaces involved in the proof of Proposition 1, i.e. tensor with $\mathbb{C}$ we can take also $s \in \mathbb{C}$.

We can reformulate the intersection problem in the statement of Proposition 1 as follows. Let π_{N_1} the orthogonal projection, with respect to g_1, onto the subspace $T_{\lambda_1} Ann(N_1)^\perp$ and define a map Q^s as:

$$Q^s : T_{\lambda_0} Ann(N_0) \to T_{\lambda_1} Ann(N_1)^\perp, \quad Q^s(\eta) = \pi_{N_1}(A_1^s \Phi_1^s A_0^s)(\eta). \quad (19)$$

Let us fix now two bases, one of $T_{\lambda_0} Ann(N_0)$ and one of $T_{\lambda_1} Ann(N_1)$. Construct two $2n \times n$ matrices using the elements of the chosen basis. Let us call the resulting objects T_0 and T_1 respectively. It follows that $J_1 T_1$ is a basis of $T_{\lambda_1} Ann(N_1)^\perp$. Define the function $\det(Q^s)$ as the determinant of the $n \times n$ matrix $T_1^* J_1 A_1^s \Phi_1^s A_0^s T_0$. Clearly, different choices of basis give simply a scalar multiple of $\det(Q^s)$ and thus is well defined:

$$\det(Q^s) = \frac{\det(T_1^* J_1 A_1^s \Phi_1^s A_0^s T_0)}{\det(T_0^* T_0)^{1/2} \det(T_1^* J_1 J_1^* T_1)^{1/2}}.$$

Moreover $\det(Q^s)|_{s=s_0} = 0$ if and only if there exists at least a solution to our boundary problem. Notice that map $s \mapsto \det(Q^s)$ is analytic in s since

the fundamental matrix is an entire map in s (see [3, Proposition 4]). The following Proposition shows that the multiplicity of any root $s_0 \neq 0$ is equal to the number of independent solutions to the boundary value problem.

Proposition 2. *The multiplicity of any root $s_0 \neq 0$ of $\det Q^s$ equals the dimension of the kernel of Q^s.*

Proof. The proof is done in two steps. First, we show that the equation $\det(Q^s) = 0$ is equivalent to $\det(R^s) = 0$ where R^s is a symmetric matrix, analytic in s. Once one knows this, it suffices to compute $\partial_s R^s$ and show that it is non-degenerate to prove that the multiplicity of the equation is the same as the dimension of the kernel.

Step 1: Replace Q^s with a symmetric matrix Let us restrict to the case $s \in \mathbb{R}$, since all the roots are real. As shown in (18) and remarked above, the determinant of the matrix Q^s is zero whenever the graph of $A_1^s \tilde{\Phi}_* \Phi_1^s A_0^s$ intersect the subspace $L_0 = T_{(\lambda_0, \lambda_1)}(Ann(N_0 \times N_1))$. Suppose that s_0 is a time of intersection and choose as coordinates in the Lagrange Grassmannian L_0 and another subspace L_1 transversal to both L_0 and $\Lambda_s := \Gamma(A_1^s \tilde{\Phi}_* \Phi_1^s A_0^s)$. This means that, if $(T_{\lambda_0} T^* M)^2 \approx \{(p, q)|p, q \in \mathbb{R}^{2n}\}$, we identify $L_0 \approx \{q = 0\}$ and $L_1 \approx \{p = 0\}$.

In this coordinates Λ_s is given by the graph of a symmetric matrix, i.e. is the following subspace $\Lambda_s = \{(p, R^s p)\}$ where again R^s is analytic in s.

The quadratic form associated to the derivative $\partial_s R(s)$ can be interpreted as the velocity of the curve $s \mapsto \Lambda_s$ inside the Grassmannian, it is possible to compute it choosing an arbitrary base of Λ_s and an arbitrary set of coordinates. Invariants such as signature and nullity do not change (see for example [5,7] or [2]). Take a curve $\lambda_s = (p_s, R^s p_s)$ inside Λ_s then one has:

$$S(\lambda_s) = \sigma(\lambda_s, \dot{\lambda}_s) = \langle p_s, \partial_s R^s p_s \rangle.$$

Recall that we will be using the symplectic form given by $(-\sigma_{\lambda_0}) \oplus \sigma_{\lambda_1}$, in order to have that graph of a symplectic map is a Lagrangian subspace.

Step 2: Replace Λ_s with a positive curve We slightly modify our curve to exploit a hidden positivity of the Jacobi equation. We substitute the fundamental solution Φ_1^s with the following map:

$$\Psi^s = \Psi_1^s = \delta_s \Phi_1^s \delta_{\frac{1}{s}}.$$

It is straightforward to check that Ψ^s is again a symplectomorphism and that it is the fundamental solution of the following ODE system at time $t = 1$:

$$\dot{\Psi}_t^s = s Z_t Z_t^* J \Psi_t^s, \quad \Psi_0^s = Id. \tag{20}$$

On one hand, we are introducing a singularity at $s = 0$ but on the other hand we are going to show that the graph of Ψ^s becomes a monotone curve and its velocity is fairly easy to compute.

First of all, hoping that the slight abuse of notation will not create any confusion, let us introduce a family of dilations similar to the δ_s, δ^s also in $T_{\lambda_1} T^* M$. The definition is analogous to the one in (13) but with Π_1 and $\Pi_1^\perp$ instead of Π_0 and $\Pi_1^\perp$. We will denote them with the same symbol.

Let us consider the following symplectomorphisms:
$$\delta_s A_1^s \tilde{\Phi}_* \Phi_1^s A_0^s \delta_{\frac{1}{s}} = \delta_s A_1^s \tilde{\Phi}_* \delta_{\frac{1}{s}} \Psi_s \delta_s A_0^s \delta_{\frac{1}{s}}.$$

Notice that the dilations δ_s preserve the subspaces $T_{\lambda_i} Ann(N_i)$ and thus the intersection points between the graph of the above map and the subspace $T_{(\lambda_0, \lambda_1)} Ann(N_0 \times N_1)$ are unchanged. Let us rewrite the maps $\delta_s A_0^s \delta_{\frac{1}{s}}$ and $\delta_s A_1^s \tilde{\Phi}_* \delta_{\frac{1}{s}}$. For the former:

$$\delta_s A_0^s \delta_{\frac{1}{s}} = \delta_s (1 + (1-s) J_0^{-1} pr_{\Pi_0^\perp}) \delta_{\frac{1}{s}} = 1 + \frac{1-s}{s} J_0^{-1} pr_{\Pi_0^\perp} = B_0^s.$$

For the latter, a computation in local coordinates and the fact that the dilations δ_s and $\tilde{\Phi}_*$ do not commute yield:

$$\delta_s A_1^s \tilde{\Phi}_* \delta_{\frac{1}{s}} = \delta_s (1 + (1-s)(J_1^{-1} + \tilde{\Phi}_* pr_{\Pi_0} \tilde{\Phi}_*^{-1}) pr_{\Pi_1^\perp}) \tilde{\Phi}_* \delta_{\frac{1}{s}}$$
$$= \left(1 + \frac{1-s}{s} J_1^{-1} pr_{\Pi_1^\perp} \right) \tilde{\Phi}_* = B_1^s \tilde{\Phi}_*.$$

Thus we take, for $s \neq 0$, as curve $\Lambda_s := \Gamma(B_1^s \tilde{\Phi}_* \Psi^s B_0^s)$, the graph of the symplectomorphism just introduced. Notice that Ψ^s is actually analytic, the singularity at $s = 0$ comes only form the maps B_i^s.

Step 3: Computation of the velocity Now we compute the velocity of the graph of $B_1^s \tilde{\Phi}_* \Psi^s B_0^s$. Take a curve $\lambda_s = (\eta, B_1^s \tilde{\Phi}_* \Psi^s B_0^s \eta)$ inside the Λ_s and let us compute the quadratic form associated to the velocity:

$$S(\lambda_s) = -\sigma(\eta, \partial_s \eta) + \sigma(B_1^s \tilde{\Phi}_* \Psi^s B_0^s \eta, \partial_s(B_1^s \tilde{\Phi}_* \Psi^s B_0^s \eta))$$
$$= \sigma(B_1^s \tilde{\Phi}_* \Psi^s B_0^s \eta, (\partial_s B_1^s) \tilde{\Phi}_* \Psi^s B_0^s \eta) + \sigma(\Psi^s B_0^s \eta, (\partial_s \Psi^s) B_0^s \eta) +$$
$$+ \sigma(B_0^s \eta, \partial_s(B_0^s) \eta).$$

Let us consider the terms of the type $\sigma(B_i^s x, \partial_s B_i^s x)$. It immediate to compute the derivative in this case, recall that $B_i^s x = x + \frac{(1-s)}{s} J_i^{-1} pr_{\Pi_i} x$. It follows that $\partial_s B_i^s = -\frac{1}{s^2} J_i^{-1} pr_{\Pi_i^\perp}$ thus the first and last term read as:

$$\sigma(B_1^s \xi, (\partial_s B_1^s)\xi) = -\frac{1}{s^2} \sigma(\xi, J_1^{-1} pr_{\Pi_1^\perp} \xi)$$
$$= \frac{1}{s^2} g_1(pr_{\Pi_1^\perp} \xi, pr_{\Pi_1^\perp} \xi), \quad \text{where } \xi = \tilde{\Phi}_* \Psi^s B_0^s \eta,$$
$$\sigma(\partial_s(B_0^s)\eta, B_0^s \eta) = -\frac{1}{s^2} \sigma(B_0^s \eta, J_0^{-1} pr_{\Pi_0^\perp} \eta) = -\frac{1}{s^2} g_0(J_0 \eta, J_0^{-1} pr_{\Pi_0^\perp} \eta)$$
$$= \frac{1}{s^2} g_0(pr_{\Pi_0^\perp} \eta, pr_{\Pi_0^\perp} \eta).$$

Notice we used the fact that J_i (and thus J_i^{-1}) is g_i−skew symmetric. Now we rewrite the middle term. We present it as the integral of its derivative using the equation for Ψ_t^s. Let us use the shorthand notation $x = B_0^s \eta$. We obtain:

$$\frac{d}{dt}(\sigma(\Psi_t^s x, (\partial_s \Psi_t^s)x)) = \sigma(\partial_t \Psi_t^s x, (\partial_s \Psi_t^s)x) + \sigma(\Psi_t^s x, (\partial_s \partial_t \Psi_t^s)x)$$
$$= s\sigma(Z_t Z_t^* J \Psi_t^s x, \partial_s \Psi_t^s x) + s\sigma(\Psi_t^s x, Z_t Z_t^* J \partial_s \Psi_t^s x)$$
$$+ \sigma(\Psi_t^s x, \partial_s(s Z_t Z_t^*) J \Psi_t^s x).$$

The first and second term have opposite sign and thus cancel out. What remains is:

$$\frac{d}{dt}\left(\sigma((\partial_s\Psi_t^s)x,\Psi_t^sx)\right) = \sigma(\Psi_t^sx, Z_tZ_t^*J\Psi_t^sx) = g(Z_t^*J\Psi_t^sx, Z_t^*J\Psi_t^sx).$$

Integrating over $[0,1]$ and using the fact that $\partial_s\Psi_t^s|_{(0,0)} = 0$ we get that:

$$\sigma(\partial_s\Psi^sx, \Psi^sx) = \int_0^1 g(Z_t^*J\Psi_t^sx, Z_t^*J\Psi_t^sx)dt.$$

Using the notation $||\cdot||$ to denote the norm with respect to the corresponding metric and summing everything up we find the following expression for the velocity of our curve:

$$S_s(\lambda_s) = \frac{1}{s^2}\left(\|pr_{\Pi_0^\perp}\eta\|^2 + \|pr_{\Pi_1^\perp}\Psi^sB_0^s\eta\|^2\right) + \int_0^1 \|Z_t^*J\Psi_t^sB_0^s\eta\|^2dt.$$

Since each term of the sum is non-negative $S_s(\lambda_s)$ is zero if and only if each term is zero. From the first one we obtain that η must be contained in the fiber. Notice that B_0^s acts as the identity on Π_0 and thus in this case $\|pr_{\Pi_1^\perp}\tilde{\Phi}_*\Psi^sB_0^s\eta\|^2 = \|pr_{\Pi_1^\perp}\Psi^s\eta\|^2$.

It follows that $\Psi_t^sB^s\eta = \Psi_t^s\eta$ is a solution of the Jacobi equation (15) starting and reaching the fiber (recall that $\tilde{\Phi}_*(\Pi_0) = \Pi_1$). Let us consider now the third piece, since the integrand is positive it must hold that for almost any t, $Z_t^*J\Psi_t^s\eta = 0$. If we multiply this equation by Z_t we find that:

$$Z_tZ_t^*J\Psi_t^s\eta = 0 = \dot{\Psi}_t^s\eta.$$

It follows that we are dealing with a constant solution starting and reaching the fiber. However, this contradicts the assumption that the matrix $\int_0^1 X_tX_t^*dt$ is non degenerate. In fact, if we substitute a non-zero constant solutions starting from the fiber in (15), we find that $pr_{\Pi_0^\perp}(\eta) = \int_0^1 X_tX_t^*dt\,\eta \neq 0$ $\square$

The following proposition is proved in [3].

Proposition 3. *There exists $c_1, c_2 > 0$ such that:*

$$\|\Phi_1^s\| \leq c_1e^{c_2|s|} \quad \forall\, s \in \mathbb{C}.$$

Moreover Φ_t^s is analytic and the function $s \mapsto \det(Q^s)$ is entire and satisfy the same type of estimate.

This fact tells us that $\det Q^s$ is an entire function of order $\rho \leq 1$. We know its zeros, which are determined by the eigenvalues of K, and thus we can apply Hadamard factorization theorem (see [14]) to present it as an infinite product. It follows that we have the following identity:

$$\det(Q^s) = as^ke^{bs}\prod_{\lambda\in Sp(K)}(1+s\lambda)^{m(\lambda)} \quad a, b \in \mathbb{C}, a \neq 0, k \in \mathbb{N}. \quad (21)$$

where $m(\lambda)$ is the geometric multiplicity of the eigenvalue λ. To determine the remaining parameters it is sufficient to know the value of $\det(Q^s)$ and

a certain number of its derivatives at $s = 0$ (depending on the value of k). Assume for now that $k = 0$, a straightforward computation shows that:

$$\det(Q^s)|_{s=0} = a, \quad \partial_s \det(Q^s)|_{s=0} = a(b + \mathrm{tr}(K)). \tag{22}$$

We will compute these quantities in Lemmas 4 and 3 for general boundary conditions. The proofs can be adapted to the case of separated conditions easily.

4.2. General boundary condition

In this section we prove a determinant formula for general boundary conditions $N \subseteq M \times M$. First, we reduce this case to the case of separate boundary conditions. We have to slightly modify the proof of Proposition 1 since, after this reduction, the Endpoint map will not be a submersion anymore. Then, we compute the normalization factors given in (22).

Let us consider $M \times M$ as state space, with the following dynamical system:

$$f_u(q', q) = \begin{pmatrix} 0 \\ f_u(q) \end{pmatrix}, \quad (q', q) \in M \times M. \tag{23}$$

and boundary conditions $\Delta \times N$. With this definition, any extremal between two points q_0 and q_1 lifts naturally to an extremal between (q_0, q_0) and (q_0, q_1). However, the Endpoint map of the new system is no longer surjective. In fact, any trajectory is confined to a submanifold of the form $\{\hat{q}\} \times M$. Thus, even if we started with a strictly normal extremal, we do not get a strictly normal extremal of the new system. However, there is no real singularity of the Endpoint map here: we have just introduced a certain number of conserved quantities. All the proofs presented above work also in this case. We are going to discuss briefly how to adapt them.

Let us start with Pontryagin maximum principle. It implies that the lift of the extremal curve $\tilde{q}(t) = (q_0, q(t))$ is the curve $\tilde{\lambda}(t) = (-\lambda_0, \lambda(t))$. This is because the initial and final covector of the lift must annihilate the tangent space of the boundary conditions manifold. In this case $N_0 = \Delta$ and the annihilator of the diagonal subspace is $\{(\lambda, -\lambda) : \lambda \in T_{\lambda_0} T^*_{q_0} M\}$. Moreover, by the orthogonality condition in PMP (see Appendix A), we know that $(-\lambda(0), \lambda(1))$ annihilates the tangent space of N.

Thus, if we want to work in one fixed tangent space, we have to multiply the first covector by -1. This changes the sign of the symplectic form and we are thus brought to work on $T_{\lambda_0} T^* M \times T_{\lambda_0} T^* M$ with symplectic form $(-\sigma) \oplus \sigma$.

With this change of sign, the tangent space to the annihilator of the diagonal gets mapped to the diagonal subspace of $T_{\lambda_0} T^* M \times T_{\lambda_0} T^* M$ and the tangent space to the annihilator of the boundary conditions N is mapped to the tangent space of:

$$A(N) = \{(\mu_0, \mu_1) : \langle \mu_0, X_0 \rangle = \langle \mu_1, X_1 \rangle, \forall (X_0, X_1) \in TN\}.$$

Let us make now some notational remarks. We will still denote by $\tilde{\Phi}_*$ the map $1 \times \tilde{\Phi}_*$, which correspond to the new flow we are using to backtrack our trajectory to the starting point.

As explained in Sect. 2, to define the scalar product on our space of variations, it is necessary to introduce two metrics on the tangent spaces to the endpoints of our curve. We will choose them of the form $\tilde{g}_0 = g_0 \oplus g_0$ and $\tilde{g}_1 = g_0 \oplus g_1$ where g_0 and g_1 are two metrics on $T_{\lambda_0} T^* M$ and $T_{\lambda_1} T^* M$ respectively.

Now we compute the second variation of the new system (23). As a general rule, we will denote all the quantities relative to (23) on $M \times M$, putting a $\tilde{\ }$ on top. We have:

$$\tilde{Z}_0 u_0 = \begin{pmatrix} Z_0 u_0 \\ Z_0 u_0 \end{pmatrix}, \quad \tilde{Z}_1 = \begin{pmatrix} Z_1^0 u_1 \\ Z_1^1 u_1 \end{pmatrix}, \quad \tilde{Z}_t = \begin{pmatrix} 0 \\ Z_t \end{pmatrix}.$$

Notice that $\tilde{\Phi}_* \tilde{Z}_1$ maps $\mathbb{R}^{\dim(N)}$ to the tangent space to $A(N)$ and we can assume that its image is contained in $\Pi_0^\perp \times \Pi_1^\perp$. We will denote by $\tilde{pr}_1$ the orthogonal projection onto the image of $\tilde{Z}_1$.

The domain of the second variation is the subspace $\mathcal{V} = \{(u_0, u_t, u_1) : \tilde{Z}_0 u_0 + \int_0^1 \tilde{Z}_t u_t + \tilde{Z}_1 u_1 \in \Pi_0 \times \Pi_0\}$. Clearly, this equation is equivalent to:

$$Z_0 u_0 + Z_1^0 u_1 \in \Pi_0 \quad \text{and} \quad Z_0 u_0 + Z_1^1 u_1 + \int_0^1 Z_t u_t \mathrm{d}t \in \Pi_0.$$

It follows that the control u_0 is completely determined by u_1. Moreover, we can assume that $Z_0 u_0 = -Z_1^0 u_1$ since we are free to choose any system of coordinates and any trivialization of the tangent bundle of the manifolds Δ and N.

Let A_i^s be the maps given in (12). The following proposition is the counterpart of Proposition 1 for general boundary conditions.

Proposition 4. *Let Φ_1^s be the fundamental solution of the Jacobi system:*

$$\dot{\eta} = Z_t^s (Z_t^s)^* J \eta.$$

The operator $1 + sK$ restricted to $\mathcal{V}$ has non trivial kernel if and only if there exists a non zero $(\eta_0, \eta_1) \in T_{(\lambda_0, \lambda_1)} A(N)$ such that

$$A_1^s \circ \tilde{\Phi}_* \circ \Phi_1^s \circ A_0^s \eta_0 = \eta_1. \tag{24}$$

The geometric multiplicity of the kernel equals the number of linearly independent solutions of the above equation.

Proof. The proof is completely analogous to the one of Proposition 1. However, some slight modifications are in order since the Endpoint map is not surjective in this case.

Step 1: Characterize $\mathcal{V}^\perp$. The orthogonal complement to $\mathcal{V}$ is given by:

$$\mathcal{V}^\perp = \left\{ (v_0, \tilde{Z}_t^* J \tilde{\nu}, v_1) : \tilde{pr}_1 \left(\begin{pmatrix} 2 Z_0 v_0 \\ 0 \end{pmatrix} + \tilde{J}_1 \tilde{\nu} \right) = \tilde{\Phi}_* \tilde{Z}_1 v_1 \right\}.$$

The proof is the same as the one of Lemma 2 and yields $v_t = \tilde{Z}_t^* J\tilde{\nu}$. Here, however, we can not separate v_0 from v_1. Take $u \in \mathcal{V}$ and $v \in V^\perp$:

$$\langle u, v \rangle = \sigma\left(\int_0^1 Z_t u_t \mathrm{d}t, \nu\right) + \tilde{g}_0(\tilde{Z}_0 u_0, \tilde{Z}_0 v_0) + \tilde{\Phi}^* \tilde{g}_1(\tilde{Z}_1 u_1, \tilde{Z}_1 u_1)$$

$$= \tilde{g}_0(\tilde{Z}_0 u_0, \tilde{Z}_0 v_0 - \tilde{J}_0 \nu) + \tilde{\Phi}^* \tilde{g}_1(\tilde{Z}_1 u_1, \tilde{Z}_1 v_1 - \tilde{J}_1 \nu)$$

$$= g_0(Z_1^0 u_1, J_0\nu - 2Z_0 v_0 + Z_1^0 v_1) + \tilde{\Phi}^* g_1(Z_1^1 u_1, Z_1^1 v_1 - J_1 \nu) = 0.$$

Hence:

$$\tilde{pr}_1 \begin{pmatrix} 2Z_0 v_0 - J_0 \nu \\ J_1 \nu \end{pmatrix} = \tilde{pr}_1\left(\begin{pmatrix} 2Z_0 v_0 \\ 0 \end{pmatrix} + \tilde{J}_1 \tilde{\nu}\right) = \tilde{\Phi}_* \tilde{Z}_1 v_1. \tag{25}$$

Step 2: Derivation of Jacobi equation. Now we can write down Jacobi equation in a fashion similar to the one of Proposition 1. The system reads:

$$\begin{cases} (1-s)\tilde{Z}_0 u_0 = \tilde{Z}_0 v_0, \\ \dot{\tilde{\eta}}(t) = \tilde{Z}_t^s(\tilde{Z}_t^s) J\tilde{\eta}(t), \\ (1-s)\tilde{\Phi}_* \tilde{Z}_1 u_1 = \tilde{\Phi}_* \tilde{Z}_1 v_1 + s\, \tilde{pr}_1 \tilde{J}_1 \tilde{\Phi}_*(\tilde{Z}_0 u_0 + \int_0^1 \tilde{Z}_t u_t \mathrm{d}t + \tilde{Z}_1 u_1). \end{cases} \tag{26}$$

Where $\tilde{\eta}(t) = \int_0^t \tilde{Z}_t^s u_t \mathrm{d}t + \tilde{Z}_0 u_0 + \tilde{\nu}$. Arguing as in Proposition 1, we can rewrite the last term of the third equation. We have:

$$pr_1 \tilde{J}_1 \tilde{\Phi}_*\left(\tilde{Z}_0^s u_0 + \int_0^1 \tilde{Z}_t^s u_t \mathrm{d}t + \tilde{Z}_1^s u_1\right) = pr_1 \tilde{J}_1 \tilde{\Phi}_*(\eta(1) - \tilde{\nu} + \tilde{Z}_1^s u_1).$$

$$\tilde{\Phi}_* \tilde{Z}_1^s u_1 = \tilde{\Phi}_* Z_1^1 u_1 + s\tilde{\Phi}_* \tilde{pr}_\Pi \tilde{Z}_1 u_1 = \tilde{\Phi}_* Z_1^1 u_1 + s(\tilde{\Phi}_* \tilde{pr}_\Pi \tilde{\Phi}_*^{-1})\tilde{\Phi}_* \tilde{Z}_1 u_1.$$

By the equation defining $\mathcal{V}$ we have that:

$$\tilde{pr}_{\Pi^\perp} \tilde{\Phi}_* \eta(1) = -\begin{pmatrix} Z_1^0 u_1 \\ \tilde{\Phi}_* Z_1^1 u_1 \end{pmatrix} = -\tilde{\Phi}_* \tilde{Z}_1 u_1.$$

Lastly note the the first equation gives $(1-s)\tilde{Z}_0 u_0 = (1-s)\tilde{pr}_{\Pi^\perp}\eta(0) = \tilde{Z}_0 v_0$. Let us now plug all this equations into (26) recalling that $\tilde{pr}_\Pi + \tilde{pr}_{\Pi^\perp} = 1$. The third line now reads:

$$pr_1\left(\begin{pmatrix} 2(1-s)\tilde{pr}_{\Pi_0^\perp}\eta(0) \\ 0 \end{pmatrix} + \tilde{J}_1 \tilde{\Phi}_*(\tilde{pr}_\Pi \eta(1) - s\tilde{\Phi}_* \tilde{pr}_\Pi \tilde{\Phi}_*^{-1}\tilde{pr}_{\Pi^\perp}\tilde{\Phi}_*\eta(1))\right.$$

$$\left. + (1-s)\tilde{pr}_{\Pi^\perp}\tilde{\Phi}_*\eta(1)\right) = 0.$$

Writing the equation component-wise and using the fact that, since $\tilde{\Phi}_*$ preserve the fibers, it holds:

$$\tilde{\Phi}_* pr_{\Pi_0} = \tilde{\Phi}_* pr_{\Pi_0} \tilde{\Phi}_*^{-1} \tilde{pr}_{\Pi_1^\perp} \Phi_* + pr_{\Pi_1} \tilde{\Phi}_*,$$

we obtain a relation very similar to the one in (16). Namely:

$$\tilde{pr}_1 \begin{pmatrix} -J_0\left(3(s-1)J_0^{-1} pr_{\Pi_0^\perp}\eta(0) + \tilde{\Phi}_* pr_{\Pi_0}\eta(0)\right) \\ J_1\left(pr_{\Pi_1}\tilde{\Phi}_*\eta(1) + (1-s)(\tilde{\Phi}_* pr_{\Pi_1}\tilde{\Phi}_*^{-1} + J_1^{-1})pr_{\Pi_1^\perp}\tilde{\Phi}_*\eta(1)\right) \end{pmatrix} = 0. \tag{27}$$

At this point the argument is again the following. If X belongs to $A(N) + \Pi$ and $\tilde{pr}_1(\tilde{J}_1 X) = 0$, it follows that:

$$\tilde{g}_1(\tilde{pr}_1 \tilde{J}_1 X, \tilde{Z}_1 u_1) = \sigma(X, \tilde{Z}_1 u_1) = 0, \forall u_1 \in \mathbb{R}^{\dim(N)}.$$

Hence X must lie in $\mathrm{Im}(\tilde{Z}_1)^{\angle} \cap (A(N) + \Pi)$, which is $A(N)$. Consider the maps A_i^s defined in (12), they preserve $A(N) + \Pi$. Thus we can rewrite (27) as:

$$\begin{pmatrix} (A_0^s)^{-1}\eta(0) \\ A_1^s \tilde{\Phi}_* \eta(1) \end{pmatrix} \in A(N) \iff \Gamma(A_1^s \tilde{\Phi}_* \Phi_1^s A_0^s) \cap A(N).$$

Notice the presence of the inverse of A_0^s due to the sign of the symplectic form.

Step 3: Uniqueness Arguing again as in Proposition 1, to the trivial variation $(0,0,0)$ correspond constant solutions of Jacobi equation starting from the fiber. However, since the Endpoint map of the original system is regular, there are no such solutions. Hence the correspondence between $\ker(1 + sK)$ and $\Gamma(A_1^s \tilde{\Phi}_* A_0^s) \cap A(N)$ is one-to-one. $\qquad \square$

Now we define an analogous map to the one in Eq. (19). Let π_N be the orthogonal projection on the space $T_{(\lambda_0,\lambda_1)}A(N)^{\perp}$ and consider the map:

$$Q^s : \Gamma(A_1^s \tilde{\Phi}_* \Phi_1^s A_0^s) \to T_{(\lambda_0,\lambda_1)}A(N)^{\perp}, \quad Q^s(\eta) = \pi_N(\eta). \qquad (28)$$

Let $T = (T_0, T_1)$ be any linear invertible map from $\mathbb{R}^{2n}$ to the tangent space $T_{(\lambda_0,\lambda_1)}A(N)$. We denote by J the map $(-J_0) \oplus J_1$ representing the symplectic form $(-\sigma_{\lambda_0}) \oplus \sigma_{\lambda_1}$. As in the previous section we define the following function:

$$\det(Q^s) = \frac{\det(T_1^* J_1 A_1^s \tilde{\Phi}_* \Phi_1^s A_0^s - T_0^* J_0)}{\det(T_0^* J_0 J_0^* T_0 + T_1^* J_1 J_1^* T_1)^{1/2}}.$$

Remark 9. One could also define (28) as a bilinear form, using just the symplectic pairing. In fact, for $(\eta, (\xi_0, \xi_1))$ in $T_{\lambda_0}T^*M \times T_{(\lambda_0,\lambda_1)}A(N)$, define:

$$\tilde{Q}^s(\eta, (\xi_0, \xi_1)) = \sigma(A_1^s \tilde{\Phi}_* \Phi_1^s A_0^s \eta, \xi_1) - \sigma(\eta, \xi_0).$$

This form is degenerate exactly when $\Gamma(A_1^s \tilde{\Phi}_* A_0^s) \cap A(N) \neq (0)$.

Proposition 5. *The multiplicity of any roots $s_0 \neq 0$ of the equation $\det(Q^s)$ is equal to the geometric multiplicity of the boundary value problem.*

Proof. The same proof of Proposition 2 works verbatim. Indeed, we are working with the same curve, $\Gamma(A_1^s \tilde{\Phi}_* A_0^s)$. $\qquad \square$

In the remaining part of this section we carry out the computation of the normalizing factors of the function $\det(Q^s)$. As already mentioned at the end of the previous section a classical factorization theorem by Hadamard (see [14]) tells us that:

$$\det(Q^s) = as^k e^{bs} \prod_{\lambda \in Sp(K)} (1 + s\lambda)^{m(\lambda)} \quad a, b \in \mathbb{C}, a \neq 0, k \in \mathbb{N}$$

where $m(\lambda)$ is the geometric multiplicity of the eigenvalue. We are now going to compute the values of $a, b \in \mathbb{C}$ and k.

Theorem 4. *For almost any choice of metrics g_0, g_1 on $T_{\lambda_i} T^* M$, $\det(Q^s|_{s=0})$ $\neq 0$. Whenever this condition holds, the determinant of the second variation is given by:*

$$\det(1 + sK) = \det((Q^s|_{s=0})^{-1}) e^{s(\operatorname{tr}(K) - \operatorname{tr}(\partial_s Q^s (Q^s)^{-1}|_{s=0}))} \det(Q^s). \quad (29)$$

Proof. We prove the first assertion: for almost any choice of scalar product, $k = 0$ and thus $a = \det(Q_1^s|_{s=0}) \neq 0$. This is equivalent to a transversality condition between the graph of the symplectomorphism $A_1^s \tilde{\Phi}_* \Phi^s A_0^s$ and the annihilator of the boundary conditions N.

We can argue as follows: consider the following family of maps acting on the Lagrange Grassmannian of $T_{\lambda_0} T^* M \times T_{\lambda_1} T^* M$ depending on the choice of scalar products G_0 and G_1:

$$F_G = (A_0^s)^{-1} \times A_1^s|_{s=0}, \quad G = (G_0, G_1), G_i > 0.$$

It is straightforward to see that they define a family of algebraic maps of the Grassmannian to itself. For any chosen subspace L_0, $F_G^{-1}(L_0)$ is arbitrary close to $\Pi_0 \times \Pi_1$, for G_i large enough. Notice that $\Gamma(A_1^s \tilde{\Phi}_* \Phi^s A_0^s) \cap L_0 \neq (0)$ if and only if $\Gamma(\tilde{\Phi}_* \Phi^s) \cap F_G^{-1}(L_0) \neq (0)$. Using the formula in Lemma 4 one has that $\Gamma(\tilde{\Phi}_* \Phi^s)$ is transversal to $\Pi_0 \times \Pi_1$ and thus to $F_G^{-1}(L_0)$ for any fixed L_0 and G_i sufficiently large. Now, since everything is algebraic in G and there is a Zariski open set in which the transversality condition holds, the possible choices of G_i for which $k > 0$ are in codimension 1.

Let us assume that $k = 0$ and compute b. Differentiating the expression for $\det(Q^s)$ in Eq. (21) at $s = 0$ we find that:

$$\partial_s \det(Q^s)|_{s=0} = a(b + \operatorname{tr}(K)).$$

An integral formula for the trace of K is given in Lemma 3. The derivative of $\det(Q^s)$ can be computed using Jacobi formula:

$$\partial_s \det(Q^s)|_{s=0} = a \operatorname{tr}(\partial_s Q^s (Q^s)^{-1})|_{s=0}.$$

An explicit expression of the derivatives of the map Q^s can be computed using Lemma 4. It follows that $b = \operatorname{tr}(\partial_s Q^s (Q^s)^{-1}) - \operatorname{tr}(K)$ and we obtain precisely the formula in the statement. $\square$

Before giving the explicit formula for $\operatorname{tr}(K)$ and the derivatives of the fundamental solution to Jacobi equation at $s = 0$ we need to make some notational remark and write down a formula for the second variation in the same spirit of Sect. 2 and Eq. (5). We are working on the state space $M \times M$ with twice the number of variables of the original system and trivial dynamic on the first factor and separated boundary conditions. The left boundary condition manifold is the diagonal of $M \times M$ and the right one is our starting N. We apply the formula in Eq. (5) to this particular system, we denote by $\tilde{Z}_t$ and $\tilde{Z}_i$ the matrices for the auxiliary problem, in general everything pertaining to it will be marked by a tilde. Identifying $T_{(\lambda_0, \lambda_0)} T^*(M \times M)$ with $T_{\lambda_0} T^* M \times T_{\lambda_0} T^* M$ we have that:

$$\tilde{Z}_0 u_0 = \begin{pmatrix} Z_0 u_0 \\ Z_0 u_0 \end{pmatrix}, \quad \tilde{Z}_1 = \begin{pmatrix} Z_1^0 u_1 \\ Z_1^1 u_1 \end{pmatrix}, \quad \tilde{Z}_t = \begin{pmatrix} 0 \\ Z_t \end{pmatrix}.$$

We still work on the subspace $\mathcal{V} = \{(u_0, u_t, u_1) : \tilde{Z}_0 u_0 + \int_0^1 \tilde{Z}_t u_t + \tilde{Z}_1 u_1 \in \Pi\}$. However, it is clear that this equation implies that:

$$Z_0 u_0 + Z_1^0 u_1 \in \Pi_0 \quad \text{and} \quad Z_0 u_0 + Z_1^1 u_1 + \int_0^1 Z_t u_t \mathrm{d}t \in \Pi_0.$$

It follows that control u_0 is completely determined by u_1. Moreover, we can assume that $Z_0 u_0 = -Z_1^0 u_1$ since we are free to choose any system of coordinates and any trivialization of the tangent bundle of the manifolds Δ and N. Technically we are working with different scalar products on each of the copies of $T_{\lambda_0} T^* M$. However, it is easy to see that on the space $\mathcal{V}$ only the sum of these metrics plays a role. We will denote it g_0. Now we are ready to state the following:

Lemma 3. *The second variation of the extended system, as a quadratic form, can be written as $\langle (I + K) u, u \rangle$ where K is the symmetric (on $\mathcal{V}$) compact operator given by:*

$$-\langle Ku, u \rangle = \int_0^1 \int_0^t \sigma(Z_\tau u_\tau, Z_t u_t) \mathrm{d}\tau \mathrm{d}t - \sigma\left(Z_1^0 u_1, \int_0^1 Z_t u_t \mathrm{d}t\right)$$
$$+ \sigma\left(\int_0^1 Z_t u_t \mathrm{d}t - Z_1^0 u_1, Z_1^1 u_1\right) + g_0(Z_1^0 u_1, Z_1^0 u_1)$$
$$+ (\tilde{\Phi}^* g_1)(Z_1^1 u_1, Z_1^1 u_1).$$

Moreover, define the following matrices:

$$\Gamma = \int_0^1 X_t X_t^* \mathrm{d}t, \quad \Omega = \int_0^1 \int_0^t X_t Z_t^* J Z_\tau X_\tau^* \mathrm{d}\tau \mathrm{d}t.$$

Denote by pr_1 the projection onto $T_{(\lambda_0, \lambda_1)} A(N)$ and by π_*^i the differential of the natural projections $\pi^i : T^* M \to M$ relative to the $i-$th component, $i = 1, 2$. The trace of K has the following expression:

$$\mathrm{tr}(K) = -\dim(N) + \mathrm{tr}[\pi_*^1 \tilde{\Phi}_*^{-1} \mathrm{pr}_1 \tilde{J}_1 \tilde{\Phi}_*(\tilde{Z}_0)]$$
$$+ \mathrm{tr}\left[\Gamma^{-1}\left(\Omega + (\pi_*^2 - \pi_*^1) \tilde{\Phi}_*^{-1} \mathrm{pr}_1 \tilde{J}_1 \tilde{\Phi}_* \left(\int_0^1 \tilde{Z}_t Z_t^* J|_\Pi \mathrm{d}t\right)\right)\right].$$

Proof. The first part is a straightforward computation combining the expression obtained in Lemma 1 for the second variation with the observation concerning the structure of the maps Z_0 and Z_1 made before the statement and the choice of the Riemannian metrics.

Now, notice that the codimension of the space giving fixed endpoints variations $\mathcal{V}$ for the extended system in $\mathcal{H}$ is $2 \dim(M)$. Moreover, it is defined as the kernel of the linear functional:

$$\rho : (u_0, u_t, u_1) \mapsto \tilde{\pi}_* \left(\tilde{Z}_0 u_0 + \int_0^1 \tilde{Z}_t u_t \mathrm{d}t + \tilde{Z}_1 u_1\right) \in T_{\pi(\lambda_0)} M \times T_{\pi(\lambda_0)} M.$$

It is straightforward to check that the following subspace is $2 \dim(M)-$dimensional and transversal to $\ker \rho$:

$$\mathcal{V}' = \{(u_0, Z_t^* J \nu, 0) : \nu \in \Pi, u_0 \in \mathbb{R}^{\dim M}\}.$$

The trace of K on the whole space splits as a sum of two pieces, the trace of $K|_{\mathcal{V}}$ and the trace of $K|_{\mathcal{V}'}$. We can then further simplify and compute separately the trace on $\mathcal{V}' \cap \{u_0 = 0\}$ and its complement $\mathcal{V}' \cap \{\nu = 0\}$. We are going to compute the trace of K on the whole space and then the trace of $K|_{\mathcal{V}'}$, determining in this way the value of $K|_{\mathcal{V}}$.

Consider $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Where

$$\mathcal{H}_1 = \{u : u = (0, u_t, 0)\}, \quad \mathcal{H}_2 = \{u : u = (u_0, 0, u_1)\}.$$

It is straightforward to check that $\mathcal{H}_1^{\perp} = \mathcal{H}_2$ for our class of metrics and that $\mathcal{H}_1 \equiv L^2([0,1], \mathbb{R}^k)$. Using Eq. (5), the restrictions of the quadratic form $\hat{K}(u) = \langle u, Ku \rangle$ to each one of the former subspaces read:

$$\hat{K}|_{\mathcal{H}_1}(u) = \int_0^1 \int_0^t \sigma(Z_t u_t, Z_\tau u_\tau) \mathrm{d}t \mathrm{d}\tau, \quad \hat{K}|_{\mathcal{H}_2}(u) = \sigma(\tilde{Z}_0 u_0, \tilde{Z}_1 u_1) - \|u\|_{\mathcal{H}}^2.$$

The trace of the first quadratic form is zero (see [3, Theorem 2]) whereas the trace of the second part is just $-\dim(N)$. Thus we have that:

$$\operatorname{tr}(K|_{\mathcal{V}}) = -\dim(N) - \operatorname{tr}(K|_{\mathcal{V}'}).$$

To compute the last piece we apply K to a control $u \in \mathcal{V}' \cap \{u_0 = 0\}$ using the explicit expression of the operator given in Lemma 1. Recall that pr_1 is projection onto the image of $\tilde{\Phi}_* \tilde{Z}_1$. It follows that:

$$K(u) = \left(0, -Z_t^* J \int_0^t Z_\tau u_\tau \mathrm{d}\tau, -L\, pr_1 \tilde{J}_1 \tilde{\Phi}_* \left(\int_0^1 \tilde{Z}_t u_t \mathrm{d}t\right)\right)$$
$$= \left(0, -Z_t^* J \int_0^t Z_\tau u_\tau \mathrm{d}\tau, -\Lambda(u)\right).$$

Now we write $K(u)$ in coordinates given by the splitting $\mathcal{V} \oplus \mathcal{V}'$. To do so we have first to consider $\rho \circ K(u)$. It is given by:

$$-\rho \circ K(u) = \tilde{\pi}_* \left(\int_0^1 \tilde{Z}_t Z_t^* J \int_0^t Z_\tau Z_\tau^* J\nu + \tilde{\Phi}_*^{-1} pr_1 \tilde{J}_1 \tilde{\Phi}_* \left(\int_0^1 \tilde{Z}_t Z_t^* J\nu \mathrm{d}t\right)\right)$$
$$= \begin{pmatrix} 0 \\ \Omega\nu \end{pmatrix} + \tilde{\pi}_* \tilde{\Phi}_*^{-1} pr_1 \tilde{J}_1 \tilde{\Phi}_* \left(\int_0^1 \tilde{Z}_t Z_t^* J\nu \mathrm{d}t\right). \tag{30}$$

Set $\Gamma = \int_0^1 X_t^* X_t \mathrm{d}t$, it easy to check that, if $u \in \mathcal{V}'$, then

$$\rho(u) = \begin{pmatrix} X_0 u_0 \\ \Gamma\nu + X_0 u_0 \end{pmatrix}.$$

for an invertible matrix X_0 which, without loss of generality, can be taken to be identity. It follows that the projection on the first component of $\rho \circ K(u)$ is completely determined by the second term in Eq. (30). Let us call π_*^i for $i = 1, 2$ the projection on the i-th component. It follows that an element $(Z_0 \hat{u}_0, Z_t^* J\hat{\nu}, 0) = \hat{u} \in \mathcal{V}'$ has the same projection as $K(u)$ if and only if:

$$\begin{cases} \hat{u}_0 = X_0^{-1} \pi_*^1 \tilde{\Phi}_*^{-1} pr_1 \tilde{J}_1 \tilde{\Phi}_* (\int_0^1 \tilde{Z}_t Z_t^* J\nu \mathrm{d}t), \\ \hat{\nu} = \Gamma^{-1} \left(\Omega\nu + (\pi_*^2 - \pi_*^1) \tilde{\Phi}_*^{-1} pr_1 \tilde{J}_1 \tilde{\Phi}_* (\int_0^1 \tilde{Z}_t Z_t^* J\nu \mathrm{d}t)\right). \end{cases}$$

In particular the restriction to $\mathcal{V}' \cap \{u_0 = 0\}$ is given by:

$$\nu \mapsto \Gamma^{-1}\left(\Omega + (\pi_*^2 - \pi_*^1)\tilde{\Phi}_*^{-1}\, pr_1 \tilde{J}_1 \tilde{\Phi}_* \left(\int_0^1 \tilde{Z}_t Z_t^* J \mathrm{d}t\right)\right)\nu.$$

A similar strategy applied to $\mathcal{V}' \cap \{\nu = 0\}$ tells us that the last contribution for the trace is given by the following map:

$$u_0 \mapsto X_0^{-1}\pi_*^1 \tilde{\Phi}_*^{-1}\, pr_1 \tilde{J}_1 \tilde{\Phi}_* \tilde{Z}_0 u_0.$$

It is worth pointing out that indeed the trace does not depend on X_0 and that the vector $\int_0^1 \tilde{Z}_t Z_t J \nu \mathrm{d}t$ is the following:

$$\int_0^1 \tilde{Z}_t Z_t J \nu \mathrm{d}t = \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} \Theta\nu \\ \Gamma\nu \end{pmatrix} \end{pmatrix}.$$

In particular, if the boundary conditions are separated (i.e. $N = N_0 \times N_1$) the part of the trace coming from $\mathcal{V}' \cap \{u_0 = 0\}$ depends only on the projection onto $T_{\lambda_1} N_1$. $\qquad\square$

Lemma 4. *The flow* $\Phi_t^s|_{s=0}$ *and its derivative* $\partial_s \Phi_t^s|_{s=0}$ *are given by:*

$$\Phi_t^s|_{s=0} = \begin{pmatrix} 1 & 0 \\ \int_0^t X_\tau X_\tau^* \mathrm{d}\tau & 1 \end{pmatrix}, \quad \partial_s \Phi_t^s|_{s=0} = \begin{pmatrix} \int_0^t Y_\tau X_\tau^* \mathrm{d}\tau & 0 \\ \int_0^t \int_0^\tau X_\tau Z_\tau^* J Z_r X_r^* \mathrm{d}r \mathrm{d}\tau - \int_0^t X_\tau Y_\tau^* \mathrm{d}\tau & \end{pmatrix}.$$

Proof. It is straightforward to check that $\Phi_1^s|_{s=0}$ solves the following Cauchy problem:

$$\begin{cases} \dot{\Phi}_t^0 = \begin{pmatrix} 0 & 0 \\ X_t X_t^* & 0 \end{pmatrix} \Phi_t^0, \\ \Phi_0^0 = Id. \end{cases}$$

Similarly $\partial_s \Phi_1^s|_{s=0}$ solves:

$$\begin{cases} \partial_s \dot{\Phi}_t^s|_{s=0} = \begin{pmatrix} 0 & 0 \\ X_t X_t^* & 0 \end{pmatrix} \partial_s \Phi_t^s|_{s=0} + \begin{pmatrix} Y_t X_t^* & 0 \\ 0 & -X_t Y_t^* \end{pmatrix} \Phi_t^0, \\ \partial_s \Phi_0^s|_{s=0} = 0. \end{cases}$$

Solving the ODE one obtains the formula in the statement. $\qquad\square$

Declarations
Conflict of interest The authors declare no competing interests.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Appendix A

In this appendix we collect some information concerning Pontryagin Maximum Principle (PMP) and the differentiation of the endpoint map used and mentioned throughout the text. Everything is fairly standard material in geometric control theory, the reader is referred to [4, 6, 23] for further details.

A.1. Pontryagin maximum principle

Let us introduce a useful family of Hamiltonian functions on T^*M. They generate a family of Hamiltonian flows which we use to backtrack admissible trajectories γ to their initial point. Moreover, they appear in the formulation of PMP and extend the flow of the fields $f_{u(t)}$ to the cotangent bundle. Set:

$$h_u^{t,\nu} : T^*M \to \mathbb{R}, \quad h_u^{t,\nu}(\lambda) = \langle \lambda, f_u \rangle + \nu\varphi(u, \pi(\lambda)), \quad \nu \leq 0.$$

In particular, if $\tilde{\gamma}$ is an admissible curve, we can build a lift, i.e. a curve $\tilde{\lambda}$ in T^*M such that $\pi(\tilde{\lambda}) = \tilde{\gamma}$, solving $\dot{\lambda} = \vec{h}_u^{t,\nu}(\lambda)$. The following wellknown theorem, Pontryagin Maximum Principle, gives a characterization of critical points of $\mathcal{J}$ (as defined in (3)), for any set of boundary conditions N.

Theorem 1. (PMP) *If a control $\tilde{u} \in L^\infty([0,1], U)$ is a local minimizer for the functional in Eq. (3) there exists a curve $\lambda : [0,1] \to T^*M$, $\nu \in \mathbb{R}$ and an admissible curve $q : [0,1] \to M$ such that for almost all $t \in [0,1]$*

1. $\lambda(t)$ is a lift of $q(t)$:

$$q(t) = \pi(\lambda(t));$$

2. $\lambda(t)$ *satisfies the following Hamiltonian system:*

$$\frac{d\lambda}{dt} = \vec{h}^{t,\nu}_{\tilde{u}(t)}(\lambda);$$

3. *the control $\tilde{u}$ is determined by the maximum condition:*

$$h^{t,\nu}_{\tilde{u}(t)}(\lambda(t)) = \max_{u \in U} h^{t,\nu}_{u}(\lambda(t)), \quad \nu \leq 0;$$

4. *the non-triviality condition holds:* $(\lambda(t), \nu) \neq (0,0)$;
5. *transversality condition holds:*

$$(-\lambda(0), \lambda(1)) \in Ann(N).$$

We call $q(t)$ an extremal curve (or trajectory) and $\lambda(t)$ an extremal.

There are essentially two possibilities for the parameter ν, it can be either 0 or, after appropriate normalization of λ_t, -1. The extremals belonging to the first family are called *abnormal* whereas the ones belonging to the second *normal*.

A.2. The endpoint map and its differentiation

In this subsection, we write down the integral expression for the first and second derivative of the endpoint map. Further details can be found in [6, Section 20.3]. Denote by $\mathcal{U}_{q_0} \subset L^\infty([0,1], U)$ be the space of admissible controls at point q_0 and define the following map:

$$E^t : \mathcal{U}_{q_0} \to M, \quad u \mapsto \gamma_u(t).$$

It takes the control u and gives the position at time t of the solution starting from q_0 of:

$$\dot{q} = f_{u(\tau)}(q).$$

We call this map *Endpoint map*. It turns out that E^t is smooth, provided that the fields $f_u(q)$ are smooth too.

For a fixed control $\tilde{u}$ consider the function $h^{t,\nu}_{\tilde{u}}(\lambda) := h^{t,\nu}_{\tilde{u}(t)}(\lambda)$ and define the following non autonomous flow which plays the role of parallel transport in this context:

$$\frac{d}{dt}\tilde{\Phi}_t = \vec{h}^{t,\nu}_{\tilde{u}}(\tilde{\Phi}_t), \qquad \tilde{\Phi}_0 = Id. \tag{31}$$

It has the following properties:

- It extends to the cotangent bundle the flow which solves $\dot{q} = f_{\tilde{u}}(q)$ on the base. In particular if λ_t is an extremal with initial condition λ_0, $\pi(\tilde{\Phi}_t(\lambda_0)) = q_{\tilde{u}}(t)$ where $q_{\tilde{u}}$ is an extremal trajectory.
- $\tilde{\Phi}_t$ preserves the fiber over each $q \in M$. The restriction $\tilde{\Phi}_t : T^*_q M \to T^*_{\tilde{\Phi}_t(q)} M$ is an affine transformation.

We suppose now that $\lambda(t)$ is an extremal and $\tilde{u}$ a critical point of the functional $\mathcal{J}$. We use the symplectomorphism $\tilde{\Phi}_t$ to pull back the whole curve $\lambda(t)$ to the starting point λ_0. We can express all the first- and second-order information about the extremal using the following map and its derivatives:

$$b^t_u(\lambda) = (h^{t,\nu}_u - h^{t,\nu}_{\tilde{u}}) \circ \tilde{\Phi}_t(\lambda).$$

Notice that:

- $b_u^t(\lambda_0)|_{u=\tilde{u}(t)} = 0 = d_{\lambda_0} b_u^t|_{u=\tilde{u}(t)}$ by definition.
- $\partial_u b_u^t|_{u=\tilde{u}(t)} = \partial_u(h_u^t \circ \tilde{\Phi}_t)|_{u=\tilde{u}(t)} = 0$ since $\lambda(t)$ is an extremal and $\tilde{u}$ the relative control.

Thus the first non zero derivatives are the order two ones. We define the following maps:

$$
\begin{aligned}
Z_t &= \partial_u \vec{b}_u^t(\lambda_0)|_{u=\tilde{u}(t)} : \mathbb{R}^k = T_{\tilde{u}(t)}U \to T_{\lambda_0}(T^*M), \\
H_t &= \partial_u^2 b_t(\lambda_0)|_{u=\tilde{u}(t)} : \mathbb{R}^k = T_{\tilde{u}(t)}U \to T^*_{\tilde{u}(t)}U = \mathbb{R}^k.
\end{aligned}
\tag{32}
$$

We denote by $\Pi = \ker \pi_*$ the kernel of the differential of the natural projection $\pi : T^*M \to M$.

Proposition 6. (Differential of the endpoint map) *Consider the endpoint map* $E^t : \mathcal{U}_{q_0} \to M$. *Fix a point $\tilde{u}$ and consider the symplectomorphism $\tilde{\Phi}_t$ and the map Z_t defined above. The differential is the following map:*

$$
d_{\tilde{u}}E(v_t) = d_{\lambda(t)}\pi \circ d_{\lambda_0}\tilde{\Phi}_t\left(\int_0^t Z_\tau v_\tau d\tau\right) \in T_{q_t}M.
$$

In particular, if we identify $T_{\lambda_0}(T^*M)$ with $\mathbb{R}^{2m}$ and write $Z_t = \begin{pmatrix} Y_t \\ X_t \end{pmatrix}$, $\tilde{u}$ is a regular point if and only if $v_t \mapsto \int_0^t X_\tau v_\tau d\tau$ is surjective. Equivalently if the following matrix is invertible:

$$
\Gamma_t = \int_0^t X_\tau X_\tau^* d\tau \in Mat_{n\times n}(\mathbb{R}), \quad \det(\Gamma_t) \neq 0
$$

If $d_{\tilde{u}}E^t$ is surjective then $(E^t)^{-1}(q_t)$ is smooth in a neighbourhood of $\tilde{u}$ and is tangent space is given by:

$$
\begin{aligned}
T_{\tilde{u}}(E^t)^{-1}(q_t) &= \left\{ v \in L^\infty([0,1], \mathbb{R}^k) : \int_0^t X_\tau v_\tau d\tau = 0 \right\} \\
&= \left\{ v \in L^\infty([0,1], \mathbb{R}^k) : \int_0^t Z_\tau v_\tau d\tau \in \Pi \right\}.
\end{aligned}
$$

If the differential of the endpoint map is surjective, the set of admissible control becomes smooth (at least locally) and our minimization problem can be interpreted as a constrained optimization problem. We are looking for critical points of $\mathcal{J}$ on the submanifold $\{u \in \mathcal{U} : E^t(u) = q_1\}$.

Definition 3. We say that a normal extremal $\lambda(t)$ with associated control $\tilde{u}(t)$ are strictly normal if the differential of the endpoint map at $\tilde{u}$ is surjective.

It makes sense to go on and consider higher-order optimality conditions. At critical points is well defined (i.e. independent of coordinates) the Hessian of $\mathcal{J}$ (or the *second variation*). Using chronological calculus (see again [6] or [1]) it is possible to write the second variation of $\mathcal{J}$ on $\ker dE^t \subseteq L^\infty([0,1], \mathbb{R}^k)$.

Proposition 7. *(Second variation) Suppose that $(\lambda(t), \tilde{u})$ is a strictly normal extremal, i.e. a critical point of $\mathcal{J}$ for fixed initial and final point. For any $u \in L^{\infty}([0,1], \mathbb{R}^k)$ such that $\int_0^1 X_t u_t \mathrm{d}t = 0$ the second variation of $\mathcal{J}$ has the following expression:*

$$d_{\tilde{u}}^2 \mathcal{J}(u) = -\int_0^1 \langle H_t u_t, u_t \rangle \mathrm{d}t - \int_0^1 \int_0^t \sigma(Z_\tau u_\tau, Z_t u_t) \mathrm{d}\tau \mathrm{d}t.$$

The associated bilinear form is symmetric provided that u, v lie in a subspace that projects to a Lagrangian one via the map $u \mapsto \int_0^1 Z_t u_t \mathrm{d}t$.

$$d_{\tilde{u}}^2 \mathcal{J}(u, v) = -\int_0^1 \langle H_t u_t, v_t \rangle \mathrm{d}t - \int_0^1 \int_0^t \sigma(Z_\tau u_\tau, Z_t v_t) \mathrm{d}\tau \mathrm{d}t.$$

Throughout the paper we make the assumption, which is customarily called *strong Legendre condition*, that the matrix H_t is strictly negative definite and has uniformly bounded inverse. This guarantees that the term:

$$\int_0^1 -\langle H_t u_t, v_t \rangle \mathrm{d}t,$$

is equivalent to the L^2 scalar product.

References

[1] Agrachev, A., Stefani, G., Zezza, P.: An invariant second variation in optimal control. Int. J. Control **71**(5), 689–715 (1998)

[2] Agrachëv, A.A.: Quadratic mappings in geometric control theory. In: Problems in Geometry, vol. 20 (Russian), Itogi Nauki i Tekhniki, pp. 111–205. Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1988. Translated in J. Soviet Math. **5**(1), no. 6, 2667–2734 (1990)

[3] Agrachev, A.A.: Spectrum of the second variation. Tr. Mat. Inst. Steklova 304(Optimal noe Upravlenie i Differentsial nye Uravneniya):32–48 (2019)

[4] Agrachev, A., Barilari, D., Boscain, U.: A comprehensive introduction to sub-Riemannian geometry, Cambridge Studies in Advanced Mathematics, vol. 181. Cambridge University Press, Cambridge (2020). From the Hamiltonian viewpoint, With an appendix by Igor Zelenko

[5] Agrachev, A., Beschastnyi, I.: Jacobi fields in optimal control: Morse and Maslov indices. Nonlinear Anal. **214**, 112608 (2022)

[6] Agrachev, A.A., Sachkov, Y.L.: Control theory from the geometric viewpoint, Encyclopaedia of Mathematical Sciences, vol. 87. Springer, Berlin (2004). Control Theory and Optimization, II

[7] Agrachev, A.A., Beschastnyi, I.Y.: Symplectic geometry of constrained optimization. Regul. Chaotic Dyn. **22**(6), 750–770 (2017)

[8] Asselle, L., Portaluri, A., Wu, L.: Spectral stability, spectral flow and circular relative equilibria for the Newtonian n-body problem. J. Differ. Equ. **337**, 323–362 (2022)

[9] Agrachev, A., Baranzini, S., Beschastnyi, I.: Nonlinearity **36**, 2792–2838 (2023)

[10] Baranzini, S.: Operators arising as second variation of optimal control problems and their spectral asymptotics. J. Dyn. Control Syst. **29**, 659–389 (2022)

[11] Barutello, V., Jadanza, R., Portaluri, A.: Linear instability of relative equilibria for n-body problems in the plane. J. Differ. Equ. **257**, 1773–1813 (2014)

[12] Barutello, V., Jadanza, R., Portaluri, A.: Morse index and linear stability of the lagrangian circular orbit in a three-body-type problem via index theory. Arch. Ration. Mech. Anal. **219**, 06 (2014)

[13] Bolotin, S.V., Treshchëv, D.V.: Hill's formula. Uspekhi Mat. Nauk **65**(2(392)), 3–70 (2010)

[14] Conway, J.B.: Functions of One Complex Variable I. Functions of One Complex Variable. Springer, Berlin (1978)

[15] Elizalde, E., Odintsov, S.D., Romeo, A., Bytsenko, A.A., Zerbini, S.: Zeta Regularization Techniques with Applications. World Scientific Publishing, Singapore (1994)

[16] Forman, R.: Functional determinants and geometry. Invent. Math. **88**, 447–494 (1987)

[17] Friedlander, L.: Determinant of the Schrödinger operator on a metric graph. Contemp. Math. **415**, 151–160 (2006)

[18] Hartmann, L., Lesch, M.: Zeta and Fredholm determinants of self-adjoint operators. J. Funct. Anal. **283**(1), 109491 (2022)

[19] Hu, X., Ou, Y., Wang, P.: Hill-type formula for Hamiltonian system with Lagrangian boundary conditions. J. Differ. Equ. **267**(4), 2416–2447 (2019)

[20] Hu, X., Wang, P.: Conditional Fredholm determinant for the s-periodic orbits in Hamiltonian systems. J. Funct. Anal. **261**(11), 3247–3278 (2011)

[21] Hu, X., Wang, P.: Hill-type formula and Krein-type trace formula for s-periodic solutions in odes. Discrete Contin. Dyn. Syst. **36**(2), 763–784 (2016)

[22] Hu, X., Wu, L., Yang, R.: Morse index theorem of Lagrangian systems and stability of brake orbit. J. Dyn. Differ. Equ. **32**, 03 (2020)

[23] Jean, F.: Control of nonholonomic systems: from sub-Riemannian geometry to motion planning. SpringerBriefs in Mathematics. Springer, Cham (2014)

[24] Kirsten, K., McKane, A.: Functional determinants for general Sturm–Liouville problems. J. Phys. A Math. Gen. **37**, 4649 (2004)

[25] Kirsten, K., McKane, A.J.: Functional determinants by contour integration methods. Ann. Phys. **308**, 502–527 (2003)

[26] Offin, D.: Hyperbolic minimizing geodesics. Trans. Am. Math. Soc. **352**(7), 3323–3338 (2000)

[27] Simon, B., Hitchin, S.P.G.N.J.: Trace Ideals and Their Applications. Lecture Note Series/London Mathematical Society. Cambridge University Press, Cambridge (1979)

[28] Ureña, A.: The spectrum of reversible minimizers. Regul. Chaot. Dyn. **23**, 248–256 (2018)

Stefano Baranzini
Università degli Studi di Torino
Via Carlo Alberto 10
Turin
Italy
e-mail: stefano.baranzini@unito.it