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Existence results for singular strongly non-linear integro-differential BVPs on the half line

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Abstract. This work is devoted to the study of singular strongly non-linear integro-differential equations of the type

$$(\Phi(k(t)v'(t)))' = f\left(t, \int_0^t v(s)\,\mathrm{d}s, v(t), v'(t)\right), \text{ a.e. on } \mathbb{R}_0^+ := [0, +\infty[,$$

where f is a Carathéodory function, Φ is a strictly increasing homeomorphism, and k is a non-negative integrable function, which is allowed to vanish on a set of zero Lebesgue measure, such that $1/k \in L^p_{\mathrm{loc}}(\mathbb{R}_0^+)$ for a certain $p > 1$. By considering a suitable set of assumptions, including a Nagumo–Wintner growth condition, we prove existence and non-existence results for boundary value problems associated with the non-linear integro-differential equation of our interest in the sub-critical regime on the real half line.

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1. Introduction

The focus of this work is to prove existence and non-existence results (in a sense that we specify later on) for boundary value problems (in short: BVPs) associated with a strongly non-linear, non-autonomous ordinary differential equation involving the Φ-Laplacian operator in $\mathbb{R}$ on the half line $\mathbb{R}_0^+$. In particular, we study the following BVP:

$$\begin{cases} (\Phi(k(t)v'(t)))' = f\left(t, \int_0^t v(s)\,\mathrm{d}s, v(t), v'(t)\right), \text{ a.e. on } \mathbb{R}_0^+, \\ v(0) = b, \ v(+\infty) = c, \end{cases} \tag{1}$$

where b, $c \in \mathbb{R}$ and $v(+\infty) = c$ is a short-hand notation for $v(t) \to c$ as $t \to +\infty$, under the following structural assumptions.

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(A_1) $\Phi : \mathbb{R} \to \mathbb{R}$ *is a strictly increasing homeomorphism, such that* $\Phi(0) = 0$ *and*

$$\liminf_{z\to 0^+} \frac{\Phi(z)}{z^\rho} > 0 \quad \text{for some } \rho > 0.$$

(A_2) $k : \mathbb{R}_0^+ \to \mathbb{R}$ *is a measurable function and there exists a certain* $p > 1$, *such that*

$$\frac{1}{k} \in L^p_{\mathrm{loc}}(\mathbb{R}_0^+), \quad \textit{and} \quad k > 0 \ \textit{a.e. in}\ \mathbb{R}_0^+.$$

(A_3) $f : \mathbb{R}_0^+ \times \mathbb{R}^3 \to \mathbb{R}$ is a *Carathéodory function.*

1.1. Motivation and background

From the applications point of view, integro-differential equations arise in various research fields to model non-local phenomena in time and they naturally describe various types of dynamical systems, from population dynamics to visco-elastic fluids; see [27] and references therein. Indeed, both Lotka–Volterra and Fredholm type models belong to this class; see for instance [22,26].

On the other hand, the Φ-Laplacian operator considered in assumption (A_1) is a generalization of the classical r-Laplacian operator defined as $\Phi(z) = z|z|^{r-2}$, with $r > 1$. Hence, singular strongly non-linear BVPs of type (1) find many applications in non-Newtonian fluid theory, diffusion flow in porous media, non-linear elasticity and theory of capillary surfaces; see for instance [9,16,20]. In view of this, literature presents many contributions studying second-order differential equations without integral dependence on the right-hand side of the type

$$(\Phi(k(t)v'(t)))' = f\left(t, v(t), v'(t)\right)$$

under various assumptions for Φ and f, alongside with different types of boundary conditions; see for instance [4–8,10,11,13,21,24,30].

Additionally, the a.e. strict positivity assumption introduced in (A_2) for the term k can equivalently be written as

$$|\left\{t \in \mathbb{R}_0^+ : \ k(t) = 0\right\}| = 0,$$

where $|A|$ denotes the Lebesgue measure of the set A. Then, it directly follows that the ODE associated with the boundary value problem (1):

$$(\Phi(k(t)v'(t)))' = f\left(t, \int_0^t v(s)\,\mathrm{d}s, v(t), v'(t)\right) \tag{2}$$

may be singular. This fact combined with the assumption $1/k \in L^p_{\mathrm{loc}}(\mathbb{R}_0^+)$, for some $p > 1$, implies that it is natural to look for solutions of (1) in $W^{1,p}_{\mathrm{loc}}(\mathbb{R}_0^+)$. Hence, the problem we study is a non-linear non-local possibly singular second-order integro-differential equation, which can remarkably be employed to describe non-local phenomena in time also presenting a non-linear and possibly singular behavior in the diffusion.

Furthermore, our analysis includes the study of the existence of *heteroclinic solutions* for integro-differential ODEs of the form (2), which are obtained by considering (1) with $b \neq c$. These solutions find many applications to the study of biological, physical, and chemical models, since they represent a phase transition process in which the system transits from an unstable equilibrium to a stable one. For this reason, heteroclinic solutions are also referred to as transitional solutions, and for further information on this subject, we refer to [23,25] and the references therein.

It is now clear that model (1) is a generalization of the existing literature, and we point out that by performing the change of variables $u(t) = \int_0^t v(s)\,\mathrm{d}s$, see (9), our results apply to a third-order ordinary differential equation, see (10), which finds many applications in fluid dynamics as a generalization of the Blasius problem, which models the flat plate problem in boundary layer theory for viscous fluids, see [14]. We remark that third-order ODEs of type (10) are studied under various assumptions and boundary conditions. We refer the interested reader to [1,2,15,18,19,28,29], and the references therein.

Finally, among future possible development of the present analysis, we recall the study of (1) in the *singular case*, i.e., when $I, J \subset \mathbb{R}$ are bounded open intervals and the map $\Phi : I \to J$ is a strictly increasing homeomorphism. In this case, the model operator we consider is the *relativistic operator*

$$\Phi(s) = \frac{s}{\sqrt{1-s^2}}, \quad s \in (-1,1).$$

Furthermore, one can also consider the *non-surjective case*, i.e., when $J \subset \mathbb{R}$ is a bounded open interval and the map $\Phi : \mathbb{R} \to J$ is a strictly increasing homeomorphism and for which the toy model is the *mean curvature operator*

$$\Phi(s) = \frac{s}{\sqrt{1+s^2}}, \quad s \in \mathbb{R}.$$

For further information on these subjects, we refer the reader to [3,12,17].

1.2. Plan of the paper

This work is organized as follows. In Sect. 2, we present an existence result for problem (1) (see Theorem 2.4) and we provide the reader with a scheme of the proof of our statement. This proof is split into two steps, which are later on separately analyzed in Sects. 3 and 4, respectively. Section 5 is devoted to the proof of a non-existence result (see Theorem 5.1) allowing us to discuss the optimality of assumptions considered in Sect. 2. Finally, in Sect. 6 we provide some explicit criteria to prove the existence and the non-existence of a solution when the right-hand side is either in the separate variables case $f(t,x,y,z) = a(t)b(x)c(y)d(z)$, or in the coupled case $f(t,x,y,z) = g(t,x,y)h(x,y,z)$, and the Φ-Laplace operator is either the p-Laplace operator, or a general Φ-Laplace operator governed by an odd function.

2. Existence result

This section is devoted to the proof of an existence result regarding problem (1). Given our assumptions, we look for solutions, lower solutions and upper solutions to (2) of the following type.

Definition 2.1. A continuous function $v : \mathbb{R}_0^+ \to \mathbb{R}$ is a solution to (2) if

1. $v \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}_0^+)$ and $\Phi \circ (kv') \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_0^+)$;
2. $(\Phi(k(t)v'(t)))' = f\left(t, \int_0^t v(s)\,\mathrm{d}s, v(t), v'(t)\right)$ for a.e. $t \in \mathbb{R}_0^+$.

Definition 2.2. A bounded continuous function $\alpha : \mathbb{R}_0^+ \to \mathbb{R}$ is a lower solution to (2) if

1. $\alpha \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}_0^+)$ and $\Phi \circ (k\alpha') \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_0^+)$;
2. $(\Phi(k(t)\alpha'(t)))' \geq f\left(t, \int_0^t \alpha(s)\,\mathrm{d}s, \alpha(t), \alpha'(t)\right)$ for a.e. $t \in \mathbb{R}_0^+$.

A bounded continuous function $\beta : \mathbb{R}_0^+ \to \mathbb{R}$ is an upper solution to (2) if

1. $\beta \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}_0^+)$ and $\Phi \circ (k\beta') \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_0^+)$;
2. $(\Phi(k(t)\beta'(t)))' \leq f\left(t, \int_0^t \beta(s)\,\mathrm{d}s, \beta(t), \beta'(t)\right)$ for a.e. $t \in \mathbb{R}_0^+$.

Remark 2.3. Considering point 1. of Definition 2.1 together with the fact that Φ is a strictly increasing homeomorphism by (A_1), we infer there exists a unique $\mathcal{K}_v \in C(\mathbb{R}_0^+, \mathbb{R})$, such that

$$\mathcal{K}_v(t) = k(t)v'(t) \qquad \text{for a.e. } t \in \mathbb{R}_0^+.$$

An analogous observation holds true for lower and upper solutions.

In addition to previously introduced structural assumptions (A_1), (A_2) and (A_3), from now on, we consider also the following ones describing the behavior of the right-hand side f:

(B_1) There exist a well-ordered pair α, β of lower and upper solutions to (1), respectively, in the sense that $\alpha(t) \leq \beta(t)$ for every $t \in \mathbb{R}_0^+$. Moreover, there exists $T_0 > 0$, such that

$$\beta \text{ is increasing on } [T_0, +\infty) \quad \text{and} \quad \lim_{t \to +\infty} \beta(t) := c \in \mathbb{R}.$$

(B_2) The function $f : \mathbb{R}_0^+ \times \mathbb{R}^3 \to \mathbb{R}$ is decreasing in the x variable, i.e., such that

$$f(t, x_1, y, z) \geq f(t, x_2, y, z)$$

for every $t \in \mathbb{R}_0^+$ and every $x_1, x_2, y, z \in \mathbb{R}$, with $x_1 \leq x_2$.

(B_3) There exist a constant $H > 0$, a non-negative function $\nu \in L^q([0, T_0])$, with $1 < q \leq \infty$, a non-negative function $\ell \in L^1([0, T_0])$, and a measurable function $\psi : (0, +\infty) \to (0, +\infty)$ satisfying

$$\tfrac{1}{\psi} \in L^1_{\mathrm{loc}}(0, +\infty) \quad \text{and} \quad \int^{\infty} \tfrac{1}{\psi(s)}\,\mathrm{d}s = +\infty, \tag{3}$$

such that

$$|f(t,x,y,z)| \leq \psi\left(|\Phi(k(t)z)|\right)\left(\ell(t)+\nu(t)|z|^{\frac{q-1}{q}}\right) \text{ a.e. on } [0,T_0]$$

$$\text{for every } x,y\in\mathbb{R} \text{ such that } x\in\left[\int_0^t \alpha(s)\,ds, \int_0^t \beta(s)\,ds\right], y\in[\alpha(t),\beta(t)]$$

$$\text{and every } z\in\mathbb{R} \text{ with } |z|\geq H, \text{ where } (q-1)/q = 1 \text{ if } q=+\infty. \tag{4}$$

(B_4) There exists $\theta > 1$, such that for every fixed $L > 0$, there exists a non-negative function $\eta_L \in L^1(\mathbb{R}_0^+)$ and a function $K_L \in W^{1,1}_{\text{loc}}(\mathbb{R}_0^+)$, null on $J := [0, T_0]$ and strictly increasing on $[T_0, +\infty)$, such that

$$(*) \int_{T_0+1}^{\infty} \frac{1}{k(t)} K_L(t)^{-\frac{1}{\rho(\theta-1)}}\,dt < \infty; \tag{5}$$

$$(*)\, f(t,x,y,z(t)) \leq -K_L'(t)|\Phi(k(t)z)|^{\theta} \text{ for a.e. } t\geq T_0,$$

$$\text{for every } x,y\in\mathbb{R} \text{ such that } x\in\left[\int_0^t \alpha(s)\,ds, \int_0^t \beta(s)\,ds\right] \text{ and } y\in[\alpha(t),\beta(t)]$$

$$\text{and every } z\in\mathbb{R} \text{ such that } |z|\leq \mathcal{N}_L(t)/k(t), \text{ where}$$

$$\mathcal{N}_L(t) := \Phi^{-1}\{(\Phi(L)^{1-\theta} + (\theta-1)K_L(t))^{\frac{1}{1-\theta}}\}; \tag{6}$$

$$(*)\, |f(t,x,y,z(t))| \leq \eta_L(t) \text{ for a.e. } t\in\mathbb{R}_0^+,$$

$$\text{for every } x,y\in\mathbb{R} \text{ such that } x\in\left[\int_0^t \alpha(s)\,ds, \int_0^t \beta(s)\,ds\right] \text{ and } y\in[\alpha(t),\beta(t)]$$

$$\text{and every } z\in\mathbb{R} \text{ such that } |z|\leq (\mathcal{N}_L(t)/k(t)) + |\alpha'(t)| + |\beta'(t)|. \tag{7}$$

Assumptions (B_1), (B_2), (B_3) are rather technical, but very general and widely established in the existing literature for strongly non-linear boundary value problems of the form (1); see for instance [4]. In particular, assumption (B_1) allows us to derive a priori bounds on the L^∞ norm of any solution to (2). Assumption (B_2) is required to deal with the integral dependence of f, and its technical importance will be later on clarified in Sect. 3. Finally, assumption (B_3) is the renowned Nagumo–Wintner type growth condition, that in combination with assumption (B_4) is responsible of the control of the solution in the sub-critical regime. Regarding the last assumption, we also recall that $\mathcal{N}_L/k \in L^1([0,+\infty[)$, because by (5) and (A_2), the following estimate holds:

$$\int_0^\infty \frac{\mathcal{N}_L(t)}{k(t)}\,\mathrm{d}t \leq \mathbf{c}\left(\int_0^T \frac{1}{k(t)}\,\mathrm{d}t + \int_T^\infty \frac{1}{k(t)} K_L(t)^{-\frac{1}{\rho(\theta-1)}}\,\mathrm{d}t\right), \tag{8}$$

for every $T > T_0$ arbitrarily fixed; see [6, Remark 3.7].

Theorem 2.4. *Let (A_1), (A_2), (A_3) and (B_1), (B_2), (B_3), (B_4) hold. Then, for every $b, c \in \mathbb{R}$, such that $\alpha(0) \leq b \leq \beta(0)$ and $\alpha(+\infty) \leq c \leq \beta(+\infty)$ problem (1) admits a continuous solution $v \in W^{1,p}_{\text{loc}}(\mathbb{R}_0^+)$, such that*

$$\alpha(t) \leq v(t) \leq \beta(t) \qquad \forall t\in\mathbb{R}_0^+.$$

Remark 2.5. When $b < c$, Theorem 2.4 implies the existence of a heteroclinic solution for (2) on the half line. Hence, our result is an extension of [6, 23] to the case of singular non-linear integro-differential ODEs of second order.

Furthermore, it is possible to extend it to heteroclinic solutions on $\mathbb{R}$ by following the method proposed in [8,23].

Remark 2.6. An analogous statement holds true for a BVP associated with a singular strongly non-linear third-order differential equation of the form (10). Indeed, as more precisely stated in Sect. 2.1, the proof of Theorem 2.4 directly follows from an existence result for the third-order boundary value problem (see Theorem 2.9) associated with (2) via a suitable change of variables (see (9)).

2.1. Scheme of the proof

The proof of our existence result is based on four main ingredients: a suitable change of variables, the lower and upper solutions method, a fixed point theorem, and a limiting procedure.

First of all, starting from problem (1), we introduce an auxiliary BVP, see (10), involving a singular strongly non-linear third-order differential equation. Indeed, if we consider a solution v to (1), then the function

$$u(t) = \int_0^t v(s)\,\mathrm{d}s \tag{9}$$

is $C^1(\mathbb{R}_0^+, \mathbb{R})$, belongs to $W^{2,p}_{\mathrm{loc}}(\mathbb{R}_0^+)$ and is a solution to the equivalent BVP

$$\begin{cases} (\Phi(k(t)u''(t)))' = f(t, u(t), u'(t), u''(t)) & \text{a.a. } t \in \mathbb{R}_0^+, \\ u'(0) = b, \;\; u'(+\infty) = c, \; u(0) = 0. \end{cases} \tag{10}$$

In particular, u is a solution to the third-order ODE

$$(\Phi(k(t)u''(t)))' = f(t, u(t), u'(t), u''(t)) \tag{11}$$

in the sense of the following definition.

Definition 2.7. A function $u \in C^1(\mathbb{R}_0^+, \mathbb{R})$ is a solution to (10) if

1. $u \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}_0^+)$ and $\Phi \circ (ku'') \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_0^+)$;
2. $(\Phi(k(t)u''(t)))' = f(t, u(t), u'(t), u''(t))$ for a.e. $t \in \mathbb{R}_0^+$.

Then, as observed in Remark 2.3, there exists a unique $\mathcal{K}_u \in C(\mathbb{R}_0^+, \mathbb{R})$, such that

$$\mathcal{K}_u(t) = k(t)u''(t) \qquad \text{for a.e. } t \in \mathbb{R}_0^+.$$

Furthermore, if we consider α and β a well-ordered pair of bounded lower and upper solutions of (2), whose existence is ensured by (B_1), we are allowed to define

$$\widetilde{\alpha}(t) = \int_0^t \alpha(\tau)\,\mathrm{d}\tau \quad \text{and} \quad \widetilde{\beta}(t) = \int_0^t \beta(\tau)\,\mathrm{d}\tau. \tag{12}$$

Notice that, by definition, $\widetilde{\alpha}(0) = \widetilde{\beta}(0) = 0$. Moreover, functions $\widetilde{\alpha}$ and $\widetilde{\beta}$ are lower and upper solutions to (11), respectively, in the following sense.

Definition 2.8. A Lipschitz function $\widetilde{\alpha} \in C^1(\mathbb{R}_0^+, \mathbb{R})$ is a lower solution to (11) if

1. $\widetilde{\alpha} \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}_0^+)$ and $\Phi \circ (k\widetilde{\alpha}'') \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_0^+)$;

2. $(\Phi(k(t)\widetilde{\alpha}''(t)))' \geq f(t, \widetilde{\alpha}, \widetilde{\alpha}'(t), \widetilde{\alpha}''(t))$ for a.e. $t \in \mathbb{R}_0^+$.

A Lipschitz function $\widetilde{\beta} \in C^1(\mathbb{R}_0^+, \mathbb{R})$ is an upper solution to (11) if

1. $\widetilde{\beta} \in W^{2,p}_{\text{loc}}(\mathbb{R}_0^+)$ and $\Phi \circ (k\widetilde{\beta}'') \in W^{1,1}_{\text{loc}}(\mathbb{R}_0^+)$;
2. $(\Phi(k(t)\widetilde{\beta}''(t)))' \leq f(t, \widetilde{\beta}(t), \widetilde{\beta}'(t), \widetilde{\beta}''(t))$ for a.e. $t \in \mathbb{R}_0^+$.

Hence, considering our previous observations together with the change of variables (9), proving the existence of a solution to (1) is equivalent to prove the existence of a solution to (10). For this reason, proof of Theorem 2.4 boils down to prove the following result, where every assumption (B_i), with $i = 1, 2, 3, 4$, needs to be understood according to the change of variables introduced in (9).

Theorem 2.9. *Let (A_1), (A_2), (A_3) and (B_1), (B_2), (B_3), (B_4) hold. Then, for every $b, c \in \mathbb{R}$, such that $\widetilde{\alpha}'(0) \leq b \leq \widetilde{\beta}'(0)$ and $\widetilde{\alpha}'(+\infty) \leq c \leq \widetilde{\beta}'(+\infty)$ problem (10) admits a continuous solution $u \in W^{2,p}_{\text{loc}}(\mathbb{R}_0^+)$, such that*

$$\widetilde{\alpha}(t) \leq u(t) \leq \widetilde{\beta}(t) \quad \text{and} \quad \widetilde{\alpha}'(t) \leq u'(t) \leq \widetilde{\beta}'(t) \quad \forall t \in \mathbb{R}_0^+.$$

It is worth noting this theorem has an importance of its own given the useful physical applications of problem (10) that we already mentioned in the introduction of this work; see [14]. As far as we are concerned with its proof, it mainly consists of two steps, that for the sake of readability, we will exploit in forthcoming Sect. 3 and Sect. 4, respectively.

Step 1. For every $n \in \mathbb{N}$, $n > T_0$, we prove there exists a solution u_n on the compact interval $I_n := [0, n]$ to the auxiliary problem

$$\begin{cases} (\Phi(k(t)u_n''(t)))' = f(t, u_n(t), u_n'(t), u_n''(t)), \text{ a.e. on } I_n \\ \\ u_n'(0) = b, \; u_n'(n) = \widetilde{\beta}(n), \; u_n(0) = 0, \end{cases} \tag{13}$$

where $\widetilde{\beta}(n)$ is the supersolution $\widetilde{\beta}$ introduced in (12) computed at $t = n$.

Step 2. Once the first step is established, we denote by u_n the solution to (13) for a fixed $n \in \mathbb{N}$. Then, we show the sequence $\{u_n\}_{n\in\mathbb{N}}$ converges up to a subsequence to a continuous function $u \in W^{2,p}_{\text{loc}}(\mathbb{R}_0^+)$, which is a solution to (10) in the sense of Definition 2.1.

3. Step 1: solvability on compact intervals

Throughout this section, we assume (A_1), (A_2), (A_3) and (B_1), (B_2), (B_3), (B_4) hold, where every (B_i), with $i = 1, 2, 3, 4$, needs to be understood according to the change of variables introduced in (9). Moreover, we recall that $J := [0, T_0]$, where T_0 is the parameter introduced in (B_1). Then, for every fixed $n \in \mathbb{N}$, with $n > T_0$, our aim is to prove the existence of a solution to (13).

To do this, we recall an useful result proved in [1, Theorem 3.1], that is a fundamental ingredient for the proof of the existence of a solution to (13). From now on, for the sake of brevity, we denote

$$\mathcal{W}_0(I_n) := \left\{ u \in W^{2,p}(I_n) : u(0) = 0 \right\},$$

and we observe every solution u defined through the change of variables (9) belongs to the space $\mathcal{W}_0(I_n)$. Hence, it is natural to look for solutions to (10) in $\mathcal{W}_0(I_n)$.

Theorem 3.1. (Theorem 2.2, [1]) *Let (A_1), (A_2), (A_3) hold. Moreover, let*

$$\begin{aligned} F:\ \mathcal{W}_0(I_n) &\to L^1(I_n) \\ u\ &\longmapsto F_u \end{aligned}$$

be a continuous operator and let $\eta \in L^1(I_n)$ be such that

$$|F_u(t)| \leq \eta(t), \qquad a.e.\ \ on\, I_n, \quad \forall u \in \mathcal{W}_0(I_n).$$

Then, for every $\nu_1, \nu_2 \in \mathbb{R}$, there exists a solution $u_n \in \mathcal{W}_0(I_n)$ to

$$\begin{cases} (\Phi(k(t)u''(t)))' = F_u(t) \qquad a.e.\ on\ I_n \\ \\ u'(0) = \nu_1,\ \ u'(n) = \nu_2. \end{cases}$$

Remark 3.2. The boundary condition $u(0) = 0$ is hidden in the definition of the set $\mathcal{W}_0(I_n)$.

Hence, our aim is to apply this result to a truncated version of (13) to then prove that a solution to the truncated problem is also a solution to (13).

Thus, we begin by considering the well-ordered pair $\widetilde{\alpha}$, $\widetilde{\beta}$ of lower and upper solutions introduced in assumption (12). Then, $\widetilde{\alpha}$, $\widetilde{\beta} \in \mathcal{W}_0(I_n)$, for every $n \in \mathbb{N}$, with $n > T_0$. Now, we let $M > 0$ be such that

$$\|\widetilde{\alpha}\|_{L^\infty(J)}, \|\widetilde{\beta}\|_{L^\infty(J)}, \|\widetilde{\alpha}'\|_{L^\infty(J)}, \|\widetilde{\beta}'\|_{L^\infty(J)} \leq M.$$

From (A_2), Φ is a strictly increasing homeomorphism. Moreover, if $H > 0$ is the positive constant introduce in (B_3), then we introduce a real constant $N > 0$, such that

$$\Phi(N) > 0, \quad \Phi(-N) < 0, \quad \text{and} \quad N > \max\left\{H, \frac{2M}{T_0}\right\} \cdot \|k\|_{L^\infty(J)}.$$

According to this choice of N, and taking (B_3) into account, we fix $L = L(N, M) > N > 0$, such that

$$\min\left\{\int_{\Phi(N)}^{\Phi(L)} \frac{1}{\psi}\,\mathrm{d}s, \int_{-\Phi(-N)}^{-\Phi(-L)} \frac{1}{\psi}\,\mathrm{d}s\right\} > \|l\|_{L^1(J)} + \|\nu\|_{L^q(J)} \cdot (2M)^{\frac{q-1}{q}}, \qquad (14)$$

and we introduce a function $\gamma_L \in L^p(I_n)$ (hence, $\gamma_L \in L^p_{\text{loc}}(\mathbb{R}^+_0)$), defined by

$$\gamma_L(t) := \frac{\mathcal{N}_L(t)}{k(t)}. \qquad (15)$$

Now, we are in a position to introduce the truncating operators we will employ to construct the truncated problem associated with (13). Given a pair of functions $\xi, \zeta \in L^1(I_n)$ satisfying the ordering relation $\xi(t) \leq \zeta(t)$ for a.e. $t \in I_n$, we introduce the truncating operator

$$\mathcal{T}^{\xi,\zeta} : L^1(I_n) \to L^1(I_n), \qquad \mathcal{T}^{\xi,\zeta}_x(t) = \max\left\{\xi(t), \min\left\{x(t), \zeta(t)\right\}\right\}.$$

By [5, Lemma A.1], the following properties hold true:

- $|\mathcal{T}_x^{\xi,\zeta}(t) - \mathcal{T}_y^{\xi,\zeta}(t)| \le |x(t) - y(t)|$ for every $x, y \in L^1(I_n)$;
- if $\xi, \zeta \in W^{1,1}(I_n)$, then $\mathcal{T}^{\xi,\zeta}\left(W^{1,1}(I)\right) \subseteq W^{1,1}(I_n)$. Moreover, $\mathcal{T}^{\xi,\zeta}$ is continuous from $W^{1,1}(I_n)$ into itself (with respect to the usual norm).

In addition, for every $u \in W^{2,p}(I_n)$, we denote

$$\mathcal{D}_{u'}(t) := \mathcal{T}^{-\gamma_L,\gamma_L}_{(\mathcal{T}_{u'}^{\widetilde{\alpha}',\widetilde{\beta}'})'}(t), \tag{16}$$

where γ_L is the function defined in (15). For further information on truncating operators, we refer the reader to [5, Appendix A].

Then, we are in a position to introduce the truncated problem associated to (13) for every $n \in N$, with $n > T_0$, that is the same BVP where the right-hand side is replaced by a truncated version of f obtained via a suitable composition with operators $\mathcal{T}$ and $\mathcal{D}$ introduced above.

Proposition 3.3. *Let (A_1), (A_2), (A_3) and (B_1), (B_2), (B_3), (B_4) hold. Let $n \in \mathbb{N}$, with $n > T_0$, be fixed. Then, the truncated problem*

$$\begin{cases} (\phi(k(t)u_n''(t))' = F_{u_n}(t), & \text{a.e. on } I_n, \\ u_n'(0) = b,\ u_n'(n) = \widetilde{\beta}(n) \end{cases} \tag{17}$$

admits at least a solution, where the truncated operator

$$F : \mathcal{W}_0(I_n) \to L^1(I_n),$$
$$u_n \longmapsto F_{u_n},$$

is defined by

$$F_{u_n}(t) := f\left(t, \mathcal{T}_{u_n}^{\widetilde{\alpha},\widetilde{\beta}}(t), \mathcal{T}_{u_n'}^{\widetilde{\alpha}',\widetilde{\beta}'}(t), \mathcal{D}_{u_n'}(t)\right) + \arctan\left(u_n'(t) - \mathcal{T}_{u_n'}^{\widetilde{\alpha}',\widetilde{\beta}'}(t)\right).$$

Proof. By the definition of $\mathcal{T}$, for every $t \in I_n$ and $u_n \in \mathcal{W}_0(I_n)$, we get

$$\widetilde{\alpha}(t) \le \mathcal{T}_{u_n}^{\widetilde{\alpha},\widetilde{\beta}}(t) \le \widetilde{\beta}(t), \qquad \widetilde{\alpha}'(t) \le \mathcal{T}_{u_n}^{\widetilde{\alpha},\widetilde{\beta}}(t) \le \widetilde{\beta}'(t)$$

Moreover, from the definition of $\mathcal{D}$ (see (16)), for every $u_n \in \mathcal{W}_0(I_n)$ and a.e. $t \in I_n$, we have

$$|\mathcal{D}_{u_n'}(t)| \le \gamma_L(t).$$

Hence, by assumption (B_4), there exists a non-negative function $\eta_L \in L^1(I_n)$, such that

$$|F_{u_n}(t)| \le \left| f\left(t, \mathcal{T}_{u_n}^{\widetilde{\alpha},\widetilde{\beta}}(t), \mathcal{T}_{u_n'}^{\widetilde{\alpha}',\widetilde{\beta}'}(t), \mathcal{D}_{u_n'}(t)\right)\right| + \frac{\pi}{2} \le \eta_L(t) + \frac{\pi}{2} := \eta(t) \tag{18}$$

for every $u_n \in \mathcal{W}_0(I_n)$ and for a.e. $t \in I_n$. Since $\eta_L \in L^1(I_n)$, also $\eta \in L^1(I_n)$, and hence, we conclude $F_{u_n} \in L^1(I_n)$ for every $u_n \in \mathcal{W}_0(I_n)$. Eventually, F_{u_n} satisfies the boundedness assumption of Theorem 3.1.

Furthermore, F is continuous from $\mathcal{W}_0(I_n)$ into $L^1(I_n)$. Given a sequence $(w_m)_m \in \mathcal{W}_0(I_n)$ converging to $u_n \in \mathcal{W}_0(I_n)$ in $W^{2,p}(I_n)$, our aim is to show $F_{w_m}(t) \to F_{u_n}(t)$ in $L^1(I_n)$, as $m \to +\infty$ and up to a subsequence. First, we notice

$$w_m \to u_n,\ w_m' \to u_n' \text{ in } W^{1,1}(I_n), \text{ and } w_m'' \to u_n'' \text{ in } L^1(I_n). \tag{19}$$

Then, by [5, Lemma A.1], we have

$$\mathcal{T}_{w_m}^{\widetilde{\alpha},\widetilde{\beta}} \to \mathcal{T}_{u_n}^{\widetilde{\alpha},\widetilde{\beta}} \quad \text{in } W^{1,1}(I_n), \quad \text{and} \quad \mathcal{T}_{w_m'}^{\widetilde{\alpha}',\widetilde{\beta}'} \to \mathcal{T}_{u_n'}^{\widetilde{\alpha}',\widetilde{\beta}'} \quad \text{in } W^{1,1}(I_n),$$

which, up to a subsequence, also implies that

$$\left(\mathcal{T}_{w_m'}^{\widetilde{\alpha}',\widetilde{\beta}'}\right)' \to \left(\mathcal{T}_{u_n'}^{\widetilde{\alpha}',\widetilde{\beta}'}\right)' \quad \text{in } L^1(I_n).$$

Then

$$\left(\mathcal{T}_{w_m'}^{\widetilde{\alpha}',\widetilde{\beta}'}\right)'(t) \to \left(\mathcal{T}_{u'}^{\widetilde{\alpha}',\widetilde{\beta}'}\right)'(t) \quad \text{for a.e. } t \in I_n.$$

Thus, combining this convergence with [5, Lemma A.1], we have

$$\mathcal{D}_{w_m{}'}(t) \to \mathcal{D}_{u_n'}(t) \quad \text{for a.e. } t \in I_n. \tag{20}$$

Considering the convergence relations from (19) to (20) and recalling f is a Carathéodory function by assumption (A3), we then obtain

$$\begin{aligned}\lim_{m\to+\infty} F_{w_m}(t) &= \lim_{m\to+\infty} \left[f\left(t, \mathcal{T}_{w_m}^{\widetilde{\alpha},\widetilde{\beta}}(t), \mathcal{T}_{w_m'}^{\widetilde{\alpha}',\widetilde{\beta}'}(t), \mathcal{D}_{w_m'}(t)\right) \right. \\ &\qquad \left. + \arctan\left(w_m'(t) - \mathcal{T}_{w_m'}^{\widetilde{\alpha}',\widetilde{\beta}'}(t)\right)\right] \\ &= F_{u_n}(t) \quad \text{for a.e. } t \in I_n.\end{aligned}$$

By combining this pointwise result with a standard dominated convergence theorem based on (18), we conclude $F_{w_m} \to F_{u_n}$ in $L^1(I_n)$ as $m \to +\infty$, which is the desired result.

Eventually, we are allowed to apply Theorem 3.1 to (17) proving the existence of a solution to the auxiliary problem (17). □

We remark that if $u_n \in \mathcal{W}_0(I_n)$ is a solution of the truncated problem (17) in the sense of Definition 2.7, then there exists a unique continuous function on I_n such that

$$\mathcal{K}_{u_n}(t) = k(t)u_n''(t) \qquad \text{for a.e. } t \in I_n. \tag{21}$$

Since the solvability of the truncated problem (17) is now established, our next aim is to show every solution to (17) is indeed a solution to (13). Our idea is to adapt the proof of [1, Theorem 3.3] to this case, and for reader's convenience, we recall that $J := [0, T_0]$, see assumption (B_4).

Proposition 3.4. *Let (A_1), (A_2), (A_3) and (B_1), (B_2), (B_3), (B_4) hold. Let $n \in \mathbb{N}$, with $n > T_0$, be fixed. Let $u_n \in \mathcal{W}_0(I_n)$ be any solution to the truncated problem (17), then u_n is a solution to (13).*

Proof. First of all, by arguing as in [1, Claim 1–5 of Theorem 3.1], every solution to (17) is such that

* $\widetilde{\alpha}'(t) \le u_n'(t) \le \widetilde{\beta}'(t)$ and $\widetilde{\alpha}(t) \le u_n(t) \le \widetilde{\beta}(t)$ for every $t \in I_n$; (22)
* $\mathcal{T}_{u_n}^{\widetilde{\alpha},\widetilde{\beta}}(t) = u_n(t)$, $\mathcal{T}_{u_n'}^{\widetilde{\alpha}',\widetilde{\beta}'}(t) = u_n'(t)$, $\mathcal{D}_{u_n'}(t) = \mathcal{T}_{u_n'}^{-\gamma_L,\gamma_L}(t)$ for every $t \in I_n$;
* $|\mathcal{K}_{u_n}| \le L$ for every $t \in J$, where L is chosen as in (14);
* $|u_n''(t)| < L/k(t)$ for almost every $t \in J$; (23)

where $\mathcal{K}_{u_n}$ is the function introduced in (21). Since assumption (B_3) only holds on J, we need to separately check the behavior of u_n on $[T_0, n]$. Indeed, by arguing as in [6, Proposition 3.5], we are able to prove

$*$ $\Phi(\mathcal{K}_{u_n})$ is decreasing on $[T_0, n]$;

$*$ $\mathcal{K}_{u_n} \geq 0$ on $[T_0, n]$;

$*$ if there exists $\bar{t} \in [T_0, n]$ such that $\mathcal{K}_{u_n}(\bar{t}) = 0$, then $\mathcal{K}_{u_n}(t) = 0$ for every $t \in [\bar{t}, n]$;

$*$ $|\mathcal{K}_{u_n}| \leq \mathcal{N}_L$ on I_n; (24)

$*$ $|u_n^{''}| \leq \gamma_L$ almost everywhere on I_n, where γ_L is defined in (15). (25)

Then, since u_n is a solution to (17), it is obvious that $u_n^{'}(0) = b$ and $u_n^{'}(n) = \beta(n)$. Moreover, by bearing in mind definitions of $\mathcal{T}$ and $\mathcal{D}$, alongside with the properties listed above, we infer

$$\left(\Phi\left(k(t)u_n^{''}(t)\right)\right)^{'} = f\left(t, u_n(t), u_n^{'}(t), u_n^{''}(t)\right).$$

Thus, u_n is a solution to (13) on the compact interval I_n. □

4. Step 2: limit argument

Throughout this section, we assume assumptions (A_1), (A_2), (A_3) and (B_1), (B_2), (B_3), (B_4) are in place, alongside with the notation introduced in Sect. 3. Our aim is to conclude the proof of Theorem 2.9 via a limit argument that will ensure us that any sequence $\{u_n\}_{n\in\mathbb{N}}$, where u_n is a solution to (13) on I_n for every fixed $n \in \mathbb{N}$, with $n > T_0$, converges to a solution u to (10).

Let $T_0 > 0$ be fixed by assumption (B_1), and for every $n > T_0$, we choose a solution $u_n \in \mathcal{W}_0(I_n)$ to (17). Then, by Proposition 3.4, u_n is also a solution to (13) on I_n. Additionally, (24) and (25) hold for every $n \in \mathbb{N}$, with $n > T_0$, that is

$*$ $|\mathcal{K}_{u_n}| \leq \mathcal{N}_L$ on I_n; (26)

$*$ $|u_n^{''}| \leq \mathcal{N}_L/k = \gamma_L$ almost everywhere on I_n. (27)

Then we define the sequence $\{x_n\} \subseteq W^{2,p}_{\text{loc}}(\mathbb{R}_0^+)$ as follows: $x_n : \mathbb{R}_0^+ \to \mathbb{R}$, such that

$$x_n(t) = \begin{cases} u_n(t), & \text{if } t \in I_n; \\ u_n(n) + \widetilde{\beta}'(n)(t-n), & \text{if } t > n; \end{cases} \tag{28}$$

and we observe that

$$x_n'(t) = \begin{cases} u_n'(t), & \text{if } t \in I_n; \\ \widetilde{\beta}'(n), & \text{if } t > n. \end{cases}$$

Our aim is to show $\{x_n\}_{n\in\mathbb{N}}$ uniformly converges (up to a subsequence) to a solution $u \in W^{2,p}_{\text{loc}}(\mathbb{R}_0^+)$ to (10).

To do this, for every $n > T_0$, we define

$$z_n(t) := x_n^{''}(t) = \begin{cases} u_n^{''}(t), & \text{if } t \in [0,n] \text{ and } \exists u_n^{''}(t) \\ 0, & \text{if } t > n; \end{cases}$$

and

$$\Psi_n(t) := \begin{cases} \left(\Phi(\mathcal{K}_{u_n}(t))\right)', & \text{if } t \in I_n \text{ and } \exists \left(\Phi(\mathcal{K}_{u_n}(t))\right)' \\ 0, & \text{otherwise.} \end{cases}$$

Since u_n is a solution to (13), from (7), (22) and (27), we get

$$\begin{aligned} |\Psi_n(t)| &:= \begin{cases} f\left(t, u_n(t), u_n'(t), u_n''(t)\right), & \text{a.e. on } I_n \\ 0, & \text{if } t > n \end{cases} \\ &\leq \eta_L(t) \ \text{ for a.e. } t \in \mathbb{R}_0^+. \end{aligned} \tag{29}$$

As a consequence, $\{\Psi_n\}_n$ is a sequence of uniformly integrable functions in $L^1(\mathbb{R}_0^+)$, since by assumption, $\eta_L \in L^1(\mathbb{R}_0^+)$. In addition, by (27), we have

$$|z_n(t)| = \begin{cases} |u_n^{''}(t)|, & \text{if } t \in I_n \text{ and } \exists u_n^{''}(t) \\ 0, & \text{if } t > n \end{cases} \leq \frac{\mathcal{N}_L(t)}{k(t)} \ \text{ for a.e. } t \in \mathbb{R}_0^+. \tag{30}$$

Since $\mathcal{N}_L/k \in L^1(\mathbb{R}_0^+)$ (see [6, Remark 3.7]), we infer $\{z_n\}_n$ is uniformly integrable in $L^1(\mathbb{R}_0^+)$.

Then, by applying the Dunford-Pettis Theorem we obtain there exist two functions $g, h \in L^1(\mathbb{R}_0^+)$, such that

$$z_n \rightharpoonup g \ \text{ and } \ \Psi_n \rightharpoonup h \text{ in } L^1(\mathbb{R}_0^+) \text{ as } n \to \infty, \tag{31}$$

up to a subsequence. Now, we observe that, since u_n solves (13), we also have $x_n(0) = 0$, $x_n'(0) = b$ for every $n > T_0$. Hence, we obtain

$$\begin{aligned} x_n(t) &:= x_n(0) + x_n'(0)t + \int_0^t \int_0^s x_n''(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\ &\xrightarrow{n\to\infty} bt + \int_0^t \int_0^s g(\tau)\,\mathrm{d}\tau\,\mathrm{d}s =: x_0(t) \quad \forall t \in \mathbb{R}_0^+. \end{aligned} \tag{32}$$

Given its definition, x_0 has the following properties:

* x_0 is absolutely continuous on $\mathbb{R}_0^+$, $x_0(0) = 0$ and $x_0'(0) = b$;
* $x_0 \in C^1(\mathbb{R}_0^+, \mathbb{R})$;
* $x_0'' = g \in L^1(\mathbb{R}_0^+)$.

Moreover, we observe that $\widetilde{\alpha}(t) \leq x_n(t) = u_n(t) \leq \widetilde{\beta}(t)$ and $\widetilde{\alpha}'(t) \leq x_n'(t) = u_n'(t) \leq \widetilde{\beta}'(t)$ for every $n > T_0$ and every $t \in I_n$ by (28) and (22). Then, it straightforwardly follows that:

$$\widetilde{\alpha}(t) \leq x_0(t) \leq \widetilde{\beta}(t) \quad \text{and} \quad \widetilde{\alpha}'(t) \leq x_0'(t) \leq \widetilde{\beta}'(t) \quad \text{for every } t \in \mathbb{R}_0^+.$$

Since $|\mathcal{K}_{u_n}(0)| \leq L$ for every $n > T_0$, see (23), then up to a subsequence

$$\mathcal{K}_{u_n}(0) \to \nu \in \mathbb{R}.$$

In addition, for a.e. $t \in I_n$, there holds

$$u_n''(t) = x_n''(t) \quad \text{and} \quad k(t)u_n''(t) = \mathcal{K}_{u_n}(t).$$

Then, it is possible to find a set $E \subseteq \mathbb{R}_0^+$ of vanishing Lebsgue measure, independent of n, such that

$$\Phi(k(t)x_n''(t)) = \Phi(\mathcal{K}_{u_n}(t)) = \Phi(\mathcal{K}_{u_n}(0)) + \int_0^t \Psi_n(s)\,\mathrm{d}s$$

for every $n > T_0$ and every $t \in I_n \setminus E$. Now, since $z\Psi_n \rightharpoonup h$ in $L^1(\mathbb{R}_0^+)$ by (26) (31), $\mathcal{K}_{u_n}(0) \to \nu$ and Φ^{-1} is continuous, we get

$$k(t)x_n''(t) \xrightarrow{n\to\infty} \Phi^{-1}\left(\Phi(\nu) + \int_0^t h(s)\,\mathrm{d}s\right) =: \mathcal{U}(t) \tag{33}$$

for every $t \in \mathbb{R}_0^+ \setminus E$. By its definition, $\mathcal{U}$ enjoys the following properties:

* $\mathcal{U} \in C(\mathbb{R}_0^+, \mathbb{R})$ and $\Phi \circ \mathcal{U}$ is absolutely continuous on $\mathbb{R}_0^+$;
* $(\Phi \circ \mathcal{U})' = h \in L^1(\mathbb{R}_0^+)$.

On one hand, by (33) and (A_2), we get

$$z_n(t) = x_n''(t) \xrightarrow{n\to\infty} \frac{\mathcal{U}(t)}{k(t)} \quad \text{for a.e. } t \in \mathbb{R}_0^+; \tag{34}$$

on the other hand, taking in consideration (30), we are allowed to apply a standard dominated convergence argument and prove $x_n'' \to \mathcal{U}/k$ in $L^1(\mathbb{R}_0^+)$ in norm. Consequently, since $x_n'' = z_n \rightharpoonup g$ in $L^1(\mathbb{R}_0^+)$ as $n \to \infty$, we obtain

$$g(t) = \frac{\mathcal{U}(t)}{k(t)} \quad \text{for a.e. } t \in \mathbb{R}_0^+. \tag{35}$$

Now, recalling $g = x_0''$ by (32), we get

* $x_0'' = \mathcal{U}/k \in L^p(\mathbb{R}_0^+)$, with $x_0 \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}_0^+)$;
* $k(t)x_0''(t) = \mathcal{U}(t)$ for a.e. $t \in \mathbb{R}_0^+$;
* $\Phi \circ (kx_0'') = \Phi \circ \mathcal{U} \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_0^+)$ and $(\Phi \circ (kx_0''))' = h$.

By (35) and the L^1-norm convergence in (34), we proved

$$x_n'' \to g = x_0'' \quad \text{in } L^1(\mathbb{R}_0^+),$$

and also $x_n''(t) \to g(t) = x_0''(t)$ for a.e. $t \in \mathbb{R}_0^+$ as $n \to \infty$.

Hence, to complete the proof of our main result, we need to show x_0 is a solution to the differential equation associated with (10) and

$$\lim_{t\to\infty} x_0'(t) = \lim_{t\to\infty} \widetilde{\beta}'(t).$$

First, we show x_0 is a solution to (10). By taking in consideration that u_n solves (13) on I_n and $x_n'' = u_n''$ a.e. on I_n, then it is possible to find a set $F \subseteq \mathbb{R}_0^+$ of vanishing Lebesgue measure, independent of n, such that for every $n > T_0$ and every $t \in I_n \backslash F$, we have

$$\Psi_n(t) = (\Phi(k(t)u_n''(t)))' = f(t, u_n(t), u_n'(t), u_n''(t)) = f(t, x_n(t), x_n'(t), x_n''(t)).$$

Since $x_n \to x_0$ pointwise and f is a Caratheodory function, the above equality implies

$$\lim_{n\to\infty} \Psi_n(t) = f(t, x_0(t), x_0^{'}(t), x_0^{''}(t)) \quad \text{for every } t \in \mathbb{R}_0^+ \setminus F. \tag{36}$$

Then, on account of (30), we are allowed to apply Lebesgue's Dominated Convergence Theorem obtaining $\Psi_n(t) \to f(t, x_0, x_0^{'}, x_0^{''})$ in $L^1(\mathbb{R}_0^+)$ (in norm). As a consequence, by combining this with (31) and (36), we obtain

$$(\Phi(k(t)x_0''(t))' = h(t) = f(t, x_0(t), x_0^{'}(t), x_0^{''}(t)) \qquad \text{for a.e. } t \in \mathbb{R}_0^+.$$

From this, we conclude x_0 is a solution to (10).

Finally, since $x_n^{''} \to x_0^{''}$ in $L^1(\mathbb{R}_0^+)$ and

$$x_n(0) = 0,\ \ x_n^{'}(0) = b,\ \ \sup_{\mathbb{R}_0^+} |x_n^{'} - x_0^{'}| \le \|x_n^{''} - x_0^{''}\|_{L^1(\mathbb{R}_0^+)} \quad \forall n \in \mathbb{N},$$

we get $x_n^{'} \to x_0^{'}$ uniformly on $\mathbb{R}_0^+$. An analogous reasoning leads us to the conclusion that also $x_n \to x_0$ uniformly on $\mathbb{R}_0^+$. In particular

$$\lim_{t\to\infty} x_0'(t) = \lim_{n\to\infty} \left(\lim_{t\to\infty} x_n'(t)\right) = \lim_{n\to\infty} \widetilde{\beta}'(n) = \lim_{t\to\infty} \widetilde{\beta}'(t).$$

Hence, x_0 is a solution to (10) and the proof is complete if we choose $u \equiv x_0$.

5. Non-existence result

This section is devoted to the proof of a non-existence result for solutions to the BVP (1). In particular, we are interested in showing the optimality of assumption (B_4) of Theorem 2.4 by proving that a non-trivial solution does not exist when we consider the complementary assumption of (B_4).

Throughout this section, we thoroughly employ the condition $\Phi(0) = 0$ for the Φ-Laplacian operator and we consider solutions v to (1) in the sense of Definition 2.1 that also are in $W^{1,p}(\mathbb{R}_0^+)$. To state our result, we first need to introduce some notation. For the sake of simplicity, see Sect. 6, given $\alpha, \beta \in L^1(\mathbb{R}_0^+)$ a pair of lower/upper solutions, we introduce four well-defined quantities (see Definition 2.2)

$$\nu_- := \inf_{t\in\mathbb{R}_0^+} \int_0^t \alpha(s)ds, \qquad \nu_+ := \sup_{t\in\mathbb{R}_0^+} \int_0^t \beta(s)ds. \tag{37}$$

Theorem 5.1. *Let (A_1), (A_2), (A_3) and (B_1), (B_2), (B_3), (B_4) hold, with k bounded and the following modifications for (6) of (B_4). There exist $\theta > 1$, $\rho > 0$ and a function $K \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_0^+)$, null on $[0, T_0]$ and strictly increasing on $[T_0, +\infty)$, such that*

$$(*) \int_{T_0+1}^{\infty} \frac{1}{k(t)} K(t)^{-\frac{1}{\rho(\theta-1)}} dt = \infty; \tag{38}$$

$$(*)\, f(t, x, y, z) \ge -K^{'}(t)|\Phi(k(t)z)|^{\theta} \ \text{ for a.e.}\, t \ge T_0,$$
$$\text{for every}\, x \in [\nu_-, \nu_+]\,, y \in [b, c] \text{ and every}\, |z| \le \rho. \tag{39}$$

Furthermore, $\Phi(0) = 0$ and we assume

$$f(t, x, y, z) \le 0 \text{ for a.e. } t \ge T_0, \forall x, y \in \mathbb{R} \text{ and } \forall z \in \mathbb{R} : |z| \le \rho. \tag{40}$$

If v is a solution to (1), then v is constant in $[T_0, +\infty)$.

Remark 5.2. It is possible to prove an analogous result for solutions to (10) considering the change of variables (9). Moreover, it can also be extended to BVPs on the real line taking in consideration slight modifications to our assumptions along the lines of [23, Theorem 4].

Proof. Let v be a continuous function solution to (1), then by Remark 2.3, there exists a unique $\mathcal{K}_v \in C(\mathbb{R}_0^+, \mathbb{R})$, such that $\mathcal{K}_v(t) = k(t)v'(t)$ for a.e. $t \in \mathbb{R}_0^+$. Moreover, from now on, we denote by $M \in \mathbb{R}_0^+$ a positive constants, such that $0 \le k(t) \le M$ for every $t \in \mathbb{R}_0^+$.
STEP 1. Our first aim is to prove

$$\lim_{t\to+\infty} \mathcal{K}_v(t) = 0.$$

First of all, we observe that, since $v(+\infty) = c$ and c is a finite number, it is clear that either the limit does not exist, or it is equal to 0. Furthermore, by definition of lim inf and lim sup, it is true that

$$\liminf_{t\to+\infty} \mathcal{K}_v(t)v'(t) \le 0 \qquad \text{and} \qquad \limsup_{t\to+\infty} \mathcal{K}_v(t)v'(t) \ge 0.$$

By contradiction, let us assume the limit does not exist; hence, either the liminf is strictly negative, or the limsup is strictly positive. Let us begin by assuming

$$l := \liminf_{t\to+\infty} \mathcal{K}_v(t) < 0.$$

Then, there exists an interval $[\underline{t}_1, \underline{t}_2] \subset [T_0, +\infty)$, such that

$$-\rho < \mathcal{K}_v(t) = k(t)v'(t) < 0 \quad \text{for a.e. } t \in [\underline{t}_1, \underline{t}_2], \text{with } \mathcal{K}_v(\underline{t}_2) > \mathcal{K}_v(\underline{t}_1). \tag{41}$$

It is always possible to find such an interval, because considering our assumptions, if we denote by $d := \max\{l, -\rho\}$ and recall that by assumption $l < 0$ and $\limsup_{t\to+\infty} \mathcal{K}_v(t) \ge 0$, then there exists an interval $[t_1, t_2] \subset [T_0, +\infty)$, such that $\mathcal{K}_v(\underline{t}_1) < d + \epsilon$ and $\mathcal{K}_v(\underline{t}_2) > -\epsilon$, with $\epsilon > 0$. By choosing $\epsilon < |d|/2$, the relation is ensured.

But by virtue of (40), we deduce $\Phi(\mathcal{K}_v(t))$ is decreasing in $[t_1, t_2]$. Hence

$$\Phi(\mathcal{K}_v(t_2)) \le \Phi(\mathcal{K}_v(t_1)). \tag{42}$$

Additionally, since Φ is a strictly increasing homeomorphism by assumption (A_1), we have

$$\mathcal{K}_v(t_2) \le \mathcal{K}_v(t_1). \tag{43}$$

However, this is in contradiction with (41); then $\liminf_{t\to+\infty} \mathcal{K}_v(t) = 0$.

Analogously, if

$$\limsup_{t\to+\infty} \mathcal{K}_v(t) > 0,$$

then there exists an interval $[\bar{t}_1, \bar{t}_2] \subset [T_0, +\infty)$, such that

$$0 \le \mathcal{K}_v(t) < \rho \quad \forall t \in [\bar{t}_1, \bar{t}_2], \text{ with } \mathcal{K}_v(\bar{t}_1) < \mathcal{K}_v(\bar{t}_2).$$

By reasoning similarly to (42) and (43), we reach a contradiction. Hence, $\limsup_{t\to+\infty}\mathcal{K}_v(t)=0$.
STEP 2. Thanks to **STEP 1**, we are allowed to introduce the number

$$t^* := \inf\{t \geq T_0 : |\mathcal{K}_v(\tau)| < M\rho \quad \forall \tau \in [t,+\infty)\}.$$

Our aim is to show $\mathcal{K}_v(t) \geq 0$ for every $t \geq t^*$.

By contradiction, we assume there exists $\hat{t} \geq t^*$, such that $\mathcal{K}_v(\hat{t}\,) < 0$. Then, by reasoning as in (42)–(43), we get

$$\frac{1}{M}\mathcal{K}_v(t) \leq \frac{1}{M}\mathcal{K}_v(\hat{t}\,) < 0 \qquad \forall t \in \mathbb{R}_0^+,\ t \geq \hat{t}.$$

Diving by M and recalling the definition of $\mathcal{K}_v$, we infer

$$v'(t) < 0 \qquad \text{for a.e. } t \geq \hat{t}.$$

This is in contradiction with the boundedness of v.
STEP 3. Now, we define the number $\widetilde{t} \geq t^*$ as

$$\widetilde{t} := \inf\{t \geq t^* : v(\tau) \geq b \text{ for every } \tau \in [t,+\infty)\}.$$

Then, we want to prove $\mathcal{K}_v(t) = 0$ for every $t \geq \widetilde{t}$.

To do this, we assume by contradiction there exists $\mathring{t} \geq \widetilde{t}$, such that $\mathcal{K}_v(\mathring{t}) > 0$. Then, the supremum T defined as

$$T := \sup\{t \geq \mathring{t} : \mathcal{K}_v(\tau) > 0 \text{ for every } \tau \in [\,\mathring{t}, t]\} = +\infty.$$

Indeed, if $T < +\infty$, since by definition of t^*, we have

$$0 < \mathcal{K}_v(t) < M\rho \qquad \text{in } [\mathring{t}, T],$$

then by (39), it follows

$$(\Phi(\mathcal{K}_v(t)))' \geq -K'(t)|\Phi(\mathcal{K}_v(t))|^\theta \qquad \forall t \in [\mathring{t}, T].$$

Now, given our assumptions, it is not restrictive to assume $\Phi(M\rho) \leq 1$, and recalling $\theta > 1$, this implies

$$(\Phi(\mathcal{K}_v(t)))' \geq -K'(t)\Phi(\mathcal{K}_v(t)) \qquad \forall t \in [\mathring{t}, T].$$

Moreover, integrating both sides between t and T, recalling $\Phi(0) = 0$ and $\mathcal{K}_v(T) = 0$ by definition, we obtain

$$\Phi(\mathcal{K}_v(t)) \leq \int_t^T K'(s)\Phi(\mathcal{K}_v(s))\,\mathrm{d}s \qquad \forall t \in (\mathring{t}, T].$$

Then, by applying Gronwall's inequality, we get

$$\Phi(\mathcal{K}_v(t)) \leq 0 \qquad \forall t \in (\mathring{t}, T].$$

Then, since Φ^{-1} is a strictly increasing monotone function and $\Phi(0) = 0$, we get

$$\mathcal{K}_v(t) \leq 0 \quad \forall t \in (\mathring{t}, T].$$

This is in contradiction with the definition of T and it implies $T = +\infty$.

To conclude the proof of this step, we are left to show $\mathcal{K}_v(t) = 0$ for every $t \geq \widetilde{t}$ and lastly that $T_0 = \mathring{t} = t^*$. To do this, by (39), we have

$$(\Phi(\mathcal{K}_v(t)))' \geq -K'(t)|\Phi(\mathcal{K}_v(t))|^\theta \qquad \forall t \in [\mathring{t}, T].$$

We divide by $(\Phi(\mathcal{K}_v(t)))^\theta$ on both sides (it is well posed because $\Phi(0)=0$, Φ is a strictly increasing homeomorphism, and by the first part of the proof of this step, we know $\mathcal{K}_v(t)>0$ for every $t\geq \mathring{t}$)

$$\frac{(\Phi(\mathcal{K}_v(t)))'}{(\Phi(\mathcal{K}_v(t)))^\theta}\geq -K'(t) \qquad \forall t\geq \mathring{t}.$$

Integrating both sides between $\mathring{t}$ and t, with $\mathring{t}\leq t$, we obtain

$$-\frac{(\Phi(\mathcal{K}_v(t)))^{-(\theta-1)}}{(\theta-1)}+\frac{\left(\Phi(\mathcal{K}_v(\mathring{t}))\right)^{-(\theta-1)}}{(\theta-1)} \geq -\int_{\mathring{t}}^{t} K'(s)\,\mathrm{d}s=-K(t)+K(\mathring{t}).$$

Recalling $\theta>1$ by assumption, by multiplying both sides by $-(\theta-1)$, we get

$$(\Phi(\mathcal{K}_v(t)))^{-(\theta-1)}\leq \left(\Phi(\mathcal{K}_v(\mathring{t}))\right)^{-(\theta-1)}-(\theta-1)(-K(t)+K(\mathring{t})).$$

Hence, by taking the $-1/(\theta-1)$ power of both sides, it follows:

$$\Phi(\mathcal{K}_v(t))\geq \left[\left(\Phi(\mathcal{K}_v(\mathring{t}))\right)^{-(\theta-1)}-(\theta-1)(-K(t)+K(\mathring{t}))\right]^{-\frac{1}{\theta-1}}.$$

Now, by elementary properties of the $-1/(\theta-1)$ power and the fact that Φ is strictly increasing, it follows:

$$\mathcal{K}_v(t)\geq \Phi^{-1}\left(\Phi(\mathcal{K}_v(\mathring{t}))+((\theta-1)(K(t)))^{-\frac{1}{\theta-1}}\right).$$

Then, we divide both sides of the previous inequality by $k(t)$, which is strictly positive except for a set E of zero measure on which is null, and we obtain the following estimate:

$$\frac{\mathcal{K}_v(t)}{k(t)}\geq \frac{1}{k(t)}\Phi^{-1}\left(\Phi(\mathcal{K}_v(\mathring{t}))+((\theta-1)(K(t)))^{-\frac{1}{\theta-1}}\right) \quad \text{for a.e. } t\geq \mathring{t}.$$

Now, integrating both sides in $[\mathring{t},\mathcal{T})$, with $\mathcal{T}\in\mathbb{R}_0^+$ and $\mathcal{T}\geq\mathring{t}$ and recalling that $\mathcal{K}_v(t)=k(t)v'(t)$ for a.e. $t\in\mathbb{R}_0^+$, we then obtain

$$v(\mathcal{T})-v(\mathring{t})\geq \int_{\mathring{t}}^{\mathcal{T}}\frac{1}{k(t)}\Phi^{-1}\left(\Phi(\mathcal{K}_v(\mathring{t}))+((\theta-1)(K_L(t)))^{-\frac{1}{\theta-1}}\right).$$

Now, letting $\mathcal{T}\to+\infty$, recalling v is continuous and reasoning as in (8), we obtain

$$v(+\infty)-v(\mathring{t})\geq c\left(\int_{\mathring{t}}^{\widetilde{\mathcal{T}}}\frac{1}{k(t)}\,\mathrm{d}t+\int_{\widetilde{\mathcal{T}}}^{+\infty}\frac{1}{k(t)}K_L(t)^{-\frac{1}{\rho(\theta-1)}}\,\mathrm{d}t\right)=+\infty,$$

where $\mathcal{T}\geq T_0$ is arbitrarily fixed, c is a suitable positive constant, and the right-hand side is unbounded by (38). This is in contradiction with the boundedness of the left-hand side.

Therefore $\mathcal{K}_v(t)=0$ for every $t\geq\widetilde{t}$. Furthermore, by definition of $\widetilde{t}$, we conclude $\widetilde{t}=t^*$. Hence, $\mathcal{K}_v(t)=0$ in $[t^*,+\infty)$ and by definition of t^* this implies $t^*=T_0$. Now, recalling the definition of $\mathcal{K}_v$ and that $k(t)>0$ for a.e.

$t \in \mathbb{R}_0^+$, we infer $v'(t) = 0$ for a.e. $t \geq T_0$. The only distribution admitting null derivative is a constant distribution, and considering that v is continuous the statement of the theorem follows. □

6. Examples of application

In this section, we present some operative criteria to prove the existence and non-existence of a weak solution for BVPs of type (1) under analogous assumptions of Sect. 5 when the right-hand side f has a specific product structure. In particular, we will focus on two different cases

$$f(t,x,y,z) = a(t)b(x)c(y)d(z) \quad \text{and} \quad f(t,x,y,z) = g(t,x,y)h(x,y,z). \quad (44)$$

From now on, we will refer to the first one as the *separate variables case*, and to the second one as the *coupled case*. As it will be clear in a short, while, on one hand, in the separate variables case, there is a strong connection between the asymptotic behavior at $+\infty$ of a and the local behavior of d as $|z| \to 0^+$. On the other hand, when dealing with the coupled case, we find an analogous situation comparing the asymptotic behavior of $h(\cdot, x, y)$ and the local behavior of $g(x, y, \cdot)$.

6.1. The separate variables' case

The aim of this subsection is to explicitly exploit our existence and non-existence criteria for BVPs of type (1), when the Φ-Laplacian operator is the r-Laplacian and $f(t,x,y,z)$ is the first form presented in (44). In particular, we are interested in analyzing the case where the asymptotic behavior of the function a is not critical, which corresponds to an asymptotic condition of the type

$$\lim_{t\to+\infty} |a(t)|\, t^{-\delta} = l_1 \in (0,+\infty), \quad \text{with } \delta > -1. \quad (45)$$

Theorem 6.1. *Let us consider problem (1) under the assumptions (A_2) and (B_1), when $\Phi(z) = z|z|^{r-2}$ is the r-Laplacian operator, with $r > 1$, and k is bounded. Let $\mathring{q} \in \mathbb{R}^+$ be such that $1 < \mathring{q} < +\infty$ and let $f(t,x,y,z) = a(t)b(x)c(y)d(z)$ satisfy the following properties:*

- *(i) a is a measurable function, such that $a \in L^{\mathring{q}}_{\mathrm{loc}}(\mathbb{R}_0^+)$. Additionally, $a(t) \leq 0$ for every $0 \leq t < T_0$ and $a(t) < 0$ for $t \geq T_0$;*
- *(ii) b and c are positive continuous functions, with b increasing;*
- *(iii) d is a continuous function, such that $d(0) = 0$ and $0 < d(z) < c_d|z|^{1-\frac{1}{\mathring{q}}}$ for $z \neq 0$, where $c_d > 0$.*

Finally, if the following asymptotic relations hold:

$$\lim_{t\to+\infty} |a(t)|\, t^{-\delta} = l_1 \in (0,+\infty) \quad \text{and} \quad \lim_{|z|\to 0} d(z)|z|^{-\mathring{\theta}} = l_2 \in (0,+\infty), \quad (46)$$

$$\int_{T_0+1}^{\infty} \frac{1}{k(t)} t^{-\sigma} \mathrm{d}t < \infty \quad (47)$$

for some $\delta > -1$, $\mathring{\theta} > r-1$, $\sigma > 0$, *then problem* (1) *admits a (non-trivial) solution if and only if* $\delta \geq -1 + \sigma(\mathring{\theta} - (r-1))$.

Proof. To prove the first part of this result (existence result), we need to show assumptions of Theorem 2.4 are satisfied. First of all, we observe that (A_1) and (A_3) hold true provided our choice of f and Φ, with $\rho = r-1$. Moreover, we observe that (B_2) trivially holds thanks to (ii). Hence, we are left to check whether assumptions (B_3) and (B_4) hold.

From now on, let us denote by M a strictly positive constant, such that $|k(t)| \leq M$ for every $t \in R_0^+$. Note that, by assumption (A_2), this implies $0 < k(t) \leq M$ for a.e. $t \in \mathbb{R}_0^+$. As far as we are concerned with assumption (B_3), we introduce a constant $H > 0$ and a non-negative function Ψ, such that

$$H > T_0 \qquad \text{and} \qquad \Psi(r) = 1 \quad \forall r \in \mathbb{R}_0^+. \tag{48}$$

We note that Ψ satisfies (3) of assumption (B_3). Then, by assumption (ii) and (iii) of the present theorem, for a.e. fixed $t \in [0, T_0]$ for every $x \in [\int_0^t \alpha(s)ds, \int_0^t \beta(s)ds] := [x_\alpha(t), x_\beta(t)]$, $y \in [\alpha(t), \beta(t)] := [y_\alpha(t), y_\beta(t)]$ and $|z| > H$, we have

$$|f(t,x,y,z)| = |a(t)|\, b(x)\, c(y)\, d(z) \leq \overline{m}\ |a(t)|\ d(z) \leq \overline{m}|a(t)||z|^{1-\frac{1}{\mathring{q}}},$$

where

$$\overline{m} := \max\left\{ b(x)c(y) : \ \min_{t\in J} x_\alpha(t) \leq x \leq \max_{t\in J} x_\beta(t), \right.$$
$$\left. \min_{t\in J} y_\alpha(t) \leq y \leq \max_{t\in J} y_\beta(t) \right\}$$

is a well-defined positive constant thanks to assumption (B_1) and (ii). Now, if we define a positive $L^{\mathring{q}}_{\text{loc}}(\mathbb{R}_0^+)$ function

$$\nu(t) = \overline{m}|a(t)|,$$

and we choose $\ell \equiv 0$, we then obtain (4) of assumption (B_3) holds true with $q \equiv \mathring{q}$.

Next, we focus on (B_4). Since $b(x), c(y) > 0$ for every x and y, we introduce a positive number

$$\underline{m} := \min\left\{ b(x)c(y) : \ \inf_{[T_0,+\infty)} x_\alpha(t) \leq x \leq \sup_{[T_0,+\infty)} x_\beta(t), \right.$$
$$\left. \inf_{t\in[T_0,+\infty)} y_\alpha(t) \leq y \leq \sup_{t\in[T_0,+\infty)} y_\beta(t) \right\} > 0,$$

which is well defined thanks to assumption (B_1), the properties listed in Definition 2.2, (ii) and in this section. Therefore, by also considering that k is bounded by assumption, we are allowed to define

$$\mu := \underline{m}\, c_2\, \ell_2\, M^{-\mathring{\theta}} > 0,$$

where $c_2 > 0$ is a constant whose existence is ensured by the asymptotic relation (46) for d, alongside with a non-negative function K_L

$$K_L(t) = \begin{cases} 0, & \text{for } 0 \le t \le T_0, \\ \mu \int\limits_{T_0}^{t} (-a(\tau))\,\mathrm{d}\tau & \text{for } t \ge T_0, \end{cases} \tag{49}$$

that is well posed, because $a \in L^{\mathring{q}}_{\text{loc}}(\mathbb{R}^+_0)$ by (i). First of all, $K_L \in W^{1,1}_{\text{loc}}(\mathbb{R}^+_0)$. Furthermore, it is strictly increasing for every $t \ge T_0$, because by its definition and considering assumption (i), we have

$$K_L'(t) = -\mu a(t) > 0 \qquad \text{for } t \ge T_0. \tag{50}$$

Additionally, by combining (46) with (49) and (50), it additionally follows:

$$K_L'(t) \ge c_1 l_1 \mu t^\delta \ \text{ and } \ K_L(t) \ge \frac{c_1 l_1 \mu}{\delta + 1}\left(t^{\delta+1} - T_0^{\delta+1}\right) \ \text{ for a.e. } t \ge T_0, \tag{51}$$

where $c_1 > 0$ is a suitably chosen constant. Now, let us introduce a constant $\theta > 1$ that we will define later on. We are choosing it in such a way that condition (5) of (B_4) is satisfied. Indeed, if we apply the definition of K_L and (51), we get

$$\begin{aligned} &\int\limits_{T_0+1}^{\infty} \frac{1}{k(t)} K_L(t)^{-\frac{1}{(r-1)(\theta-1)}}\,\mathrm{d}t \\ &\quad \le \left(\frac{c_1 l_1 \mu}{\delta+1}\right)^{-\frac{1}{(r-1)(\theta-1)}} \int\limits_{T_0+1}^{\infty} \frac{1}{k(t)}\left(t^{\delta+1} - T_0^{\delta+1}\right)^{-\frac{1}{(r-1)(\theta-1)}}\,\mathrm{d}t, \end{aligned}$$

which is finite if and only if

$$\frac{\delta+1}{(r-1)(\theta-1)} \ge \sigma. \tag{52}$$

Next, we observe that since K_L is a strictly increasing function and (52) is in place, then

$$\frac{1}{k(t)} K_L(t)^{-\frac{1}{(r-1)(\theta-1)}} \longrightarrow 0 \qquad \text{as } t \to +\infty.$$

Thus, the following asymptotic relation for its reciprocal holds:

$$k(t) K_L(t)^{\frac{1}{(r-1)(\theta-1)}} \longrightarrow +\infty \qquad \text{as } t \to +\infty. \tag{53}$$

Hence, $\mathcal{N}_L(t) \to 0$ as $t \to +\infty$.

Then, by applying our assumptions on b, c, and d to the right-hand side f and considering (46), the following estimate holds:

$$\begin{aligned} f(t,x,y,z) = a(t)b(x)c(y)d(z) &\le -\frac{1}{\mu}\underline{m} K_L'(t)|z|^{\mathring{\theta}} = -\frac{1}{\mu}\frac{\underline{m} c_2 \ell_2}{(k(t))^{\mathring{\theta}}} K_L'(t)(k(t))^{\mathring{\theta}}|z|^{\mathring{\theta}} \\ &\le -K_L'(t)(k(t))^{\mathring{\theta}}|z|^{\mathring{\theta}} = -K_L'(t)|k(t)z|^{(r-1)\theta}. \end{aligned}$$

for a.e. $t \ge T_0$ and for every $z \in \mathbb{R}$, such that $M|z| < \rho$, with ρ a small positive number. Then, recalling that by (53) $\mathcal{N}_L(t) \to 0$ as $t \to +\infty$, we have that there exists $t^* > T_0$, such that $\mathcal{N}_L(t) \leq \rho$ for every $t \geq t^*$. Hence, condition (6) of (B_4) is satisfied by choosing and

$$\theta = \frac{\mathring{\theta}}{r-1}, \tag{54}$$

which is strictly greater than 1, because by assumption $\mathring{\theta} > r-1$. Moreover, if we recall that $r > 1$ and θ is chosen as in (54), then the previous condition (52) is equivalent to

$$\frac{\delta+1}{\mathring{\theta}-(r-1)} \geq \sigma \qquad \Longleftrightarrow \qquad \delta \geq -1 + \sigma(\mathring{\theta}-(r-1)).$$

Now, it remains to show there exists a non-negative function $\eta_L \in L^1(\mathbb{R}_0^+)$, such that $|f(t,x,y,z)| \leq \eta_L(t)$ for a.e. $t \in \mathbb{R}_0^+$ (see (7)). To do this, we define the constant

$$\hat{m} := \max\left\{ b(x)c(y) :\; x \in \left[\inf_{[T_0,+\infty)} x_\alpha(t), \sup_{[T_0,+\infty)} x_\beta(t)\right],\right.$$
$$\left. y \in \left[\inf_{t\in[T_0,+\infty)} y_\alpha(t), \sup_{t\in[T_0,+\infty)} y_\beta(t)\right]\right\},$$

that is well defined because of (ii), the properties listed in Definition 2.2 and the assumptions of this section. Then, we are able to introduce the non-negative function

$$\eta_L(t) = \begin{cases} \hat{m} \max\limits_{z\in[-\zeta,\zeta]} d(z)|a(t)| & \text{if } 0 \leq t \leq T_0 \\ \hat{m}\, c_d\, c_1\, \ell_1\, M\, t^\delta \left(\left|\frac{\mathcal{N}_L(t)}{k(t)}\right| + c\right)^{1-\frac{1}{\mathring{q}}} & \text{if } t > T_0, \end{cases}$$

where the constant c is a.e. bound from above of $|\alpha'|$ and $|\beta'|$, which are bounded thanks to our Definition of lower/upper solution, see Definition 2.2, assumption (B_1), Remark 2.3, and $\zeta \in \mathbb{R}^+$ is a small positive number.

Now, we want to prove $\eta_L \in L^1(\mathbb{R}_0^+)$. By its definition, η_L is bounded from above by a $L^{\mathring{q}}$ function in $[0,T_0]$, with $\mathring{q} > 1$; see assumption (i). Furthermore, if we denote by $\overline{M} := \hat{m}\, c_d\, c_1\, \ell_1\, M$, then by our assumptions and (51), when $t > T_0$, we get

$$\begin{aligned} 0 < \eta_L(t) &\leq \frac{\overline{M}t^\delta}{|k(t)|^{1-\frac{1}{\mathring{q}}}} \left(C_1^{1-\frac{1}{\mathring{q}}} + ((\theta-1)K_L(t))^{-\frac{\mathring{q}-1}{\mathring{q}(\theta-1)(r-1)}} \right) \\ &\leq \frac{\overline{M}t^\delta}{t^{\frac{\sigma(\mathring{q}-1)}{\mathring{q}}}} \left(C_1^{\frac{\mathring{q}-1}{\mathring{q}}} + \left(C_2(\theta-1)(t^{\delta+1} - T_0^{\delta+1}\right)^{-\frac{\mathring{q}-1}{\mathring{q}(\theta-1)(r-1)}} \right), \end{aligned}$$

where C_1 and C_2 are two suitably chosen positive constants. Therefore, there exists an additional constant $C_3 > 0$, such that

$$\int_{T_0}^{+\infty} \eta_L(t)\mathrm{d}t \leq C_3 \int_{T_0}^{+\infty} t^{\delta - \left(\frac{\delta+1}{(\theta-1)(r-1)}+\sigma\right)\left(\frac{\mathring{q}-1}{\mathring{q}}\right)} \mathrm{d}t < \infty,$$

which is finite if and only if

$$\delta < -1 + \left(\frac{\delta+1}{(\theta-1)(r-1)} + \sigma \right) \left(\frac{\mathring{q}-1}{\mathring{q}} \right)$$
$$\iff \quad \sigma(\theta-1)(r-1)(\mathring{q}-1) > (\delta+1)\left((\theta-1)(r-1)\mathring{q}-1\right),$$

which is always true given our assumptions. Finally, we conclude

$$|f(t,x,y,z)| \leq |a(t)|b(x)c(y)d(z) \leq \ell_1 t^\delta b(x)c(y)d(z) \leq \eta_L(t)$$

for a.e. $t \in \mathbb{R}_0^+$. This concludes the proof of (7), and thus, also assumption (B_4) holds.

To prove the second part of our statement (non-existence result), it is sufficient to show given the choice of $\delta < -2 + \sigma(\mathring{\theta} - (r-1))$, the assumptions of Theorem 5.1 are satisfied. In particular, by choosing $g(t) = \ell(t)$ for $t \leq T_0$, with $\ell \in L^1(\mathbb{R}_0^+)$, and $g(t) = c_* t^\delta$ for $t > T_0$, we have that $K(t) = c_* \int_0^t g(\tau)\,\mathrm{d}\tau$ for $t \geq 0$ is a strictly increasing function belonging to $W^{1,1}_{\mathrm{loc}}(\mathbb{R}_0^+)$ and c_* is a suitably chosen positive constant. Additionally

$$K_L(t) = \int_0^{T_0} \ell(\tau)\mathrm{d}\tau + \int_{T_0}^{t} c_* \tau^\delta \mathrm{d}\tau = \int_0^{T_0} \ell(\tau)\mathrm{d}\tau + \frac{c_*}{\delta+1}(t^{\delta+1} - T_0^{\delta+1}).$$

Then, we have

$$\int_0^\infty \frac{1}{k(t)} \left(\int_0^{T_0} \ell(\tau)\mathrm{d}\tau + \frac{c_*}{\delta+1}(t^{\delta+1} - T_0^{\delta+1}) \right)^{-\frac{1}{\rho(\theta-1)}} dt$$
$$\geq \mathbf{c} + \frac{c_*}{\delta+1} \int_0^\infty \frac{1}{k(t)} \left(t^{\delta+1} - T_0^{\delta+1}\right)^{-\frac{1}{\rho(\theta-1)}} \mathrm{d}t = \infty$$

when $\delta + 1 < \sigma(\mathring{\theta} - (r-1))$. □

6.2. The coupled case

In this subsection, we focus on existence and non-existence criteria for BVPs of type (1), when Φ is the r-Laplacian, $f(t,x,y,z) = g(t,x,y)h(x,y,z)$ and g has a non-critical growth, i.e., its asymptotic behavior as $t \to +\infty$ is of the type (45). It is clear that previous subsection's results on the separate variables case can be viewed as a corollary of the forthcoming theorems. Nevertheless, performing our computations directly in the separate variable case allows us to explicitly exploit the relation between the considered growth conditions and the exponent of the r-Laplacian.

Theorem 6.2. *Let us consider problem* (1) *under the assumptions* (A_2) *and* (B_1)*, when* $\Phi(z) = z|z|^{r-2}$ *is the* r*-Laplacian operator and* k *is bounded. Let* $\mathring{q} \in \mathbb{R}^+$ *be such that* $1 < \mathring{q} < +\infty$*.*

Furthermore, we assume $f(t,x,y,z) = g(t,x,y)h(x,y,z)$ *satisfies the following properties:*

(i) g *is a Carathéodory function decreasing in* x*. Additionally,* $g(t,x,y) \le 0$ *for every* $0 \le t < T_0$ *and* $g(t,x,y,z) < 0$ *for* $t \ge T_0$*. Moreover, there exists a function* $\lambda \in L^{\mathring{q}}_{\text{loc}}(\mathbb{R}^+_0)$*, such that*

$$|g(t,x,y)| \le \lambda(t) \qquad \text{for a.e. } t \in \mathbb{R}^+_0, \tag{55}$$

for every $x, y \in \mathbb{R}$*, such that* $x \in [\nu_-, \nu_+]$ *and* $y \in [b,c]$*, where* ν_-, ν_+ *were introduced in* (37)*;*

(ii) h *is a positive continuous function increasing in* x*.*

Finally, assume that there exist real constants $-1 < \delta_1 \le \delta_2$*,* $0 < \gamma_1 \le \gamma_2$ *and* $\sigma > 0$*, such that*

$$\frac{\delta_1 + 1}{\gamma_1 - (r-1)} \ge \sigma \quad \text{and} \quad (\delta_1 + 1)\frac{\gamma_2 - (r-1)}{\gamma_1 - (r-1)}(r-1) \ge \sigma + \delta_2,$$

and there exist positive constants $h_1, h_2, k_1 k_2 > 0$ *and* $\zeta, L > 0$*, such that*

$$* \int_{T_0+1}^{\infty} \frac{1}{k(t)} t^{-\sigma} \mathrm{d}t < \infty, \tag{56}$$

$$* \; h_1 t^{\delta_1} \le |g(t,x,y)| \le h_2 t^{\delta_2}, \qquad \text{for a.e. } |t| > L, \tag{57}$$

$$* \; k_1 |z|^{\gamma_1} \le h(x,y,z) \le k_2 |z|^{\gamma_2}, \qquad \text{whenever } |z| < \zeta, \tag{58}$$

$$* \; h(x,y,z) \le k_2 |z|^{1-\frac{1}{\mathring{q}}}, \qquad \text{whenever } |z| > L, \tag{59}$$

for every $x, y \in \mathbb{R}$*, such that* $x \in \left[\int_0^t \alpha(s)\,ds, \int_0^t \beta(s)\,ds\right]$ *and* $y \in [\alpha(t), \beta(t)]$*. Then, problem* (1) *admits a solution.*

Proof. As in the proof of Theorem 6.1, it is sufficient to show assumptions of Theorem 2.4 hold true. Then, we observe: (A_1) and (A_3) hold true provided our choice of f and Φ, with $\rho = r - 1$; (B_2) trivially holds thanks to (i) and (ii). Hence, we need to show that assumptions (B_3) and (B_4) are in place.

As in the proof of Theorem 6.1, from now on, let us denote by M a strictly positive constant, such that $|k(t)| \le M$ for every $t \in R^+_0$. Note that, by assumption (A_2), this implies $0 < k(t) \le M$ for a.e. $t \in \mathbb{R}^+_0$. As far as we are concerned with assumption (B_3), we consider a non-negative function Ψ defined as in (48), but with $H > \max\{T_0, L\}$. Hence, Ψ satisfies (3) of assumption (B_3) by definition. Then, by assumption (i) and (59), for a.e. $t \in [0, T_0]$ for every $x \in [\nu_-, \nu_+]$, $y \in [b,c]$ and $|z| > H$, we have

$$|f(t,x,y,z)| = |g(t,x,y)| h(x,y,z) \le \lambda(t) k_2 |z|^{1-\frac{1}{\mathring{q}}}.$$

Now, if we define $\nu(t) = k_2 \lambda(t)$, that thanks to our assumptions is in $L^{\mathring{q}}([0,T_0])$, then we obtain

$$|f(t,x,y,z)| \le C_\Psi \nu(t) |z|^{1-\frac{1}{\mathring{q}}} \le \Psi(|z|^{r-1}) \nu(t) |z|^{1-\frac{1}{\mathring{q}}}.$$

Then, if we proceed as in the proof of Theorem 6.1 and we consider $\ell \equiv 0$, taking into consideration the contribution of the term $|k(t)|^{r-1}$, we obtain that assumption (B_3) holds true with $q \equiv \mathring{q}$.

Next, we focus on (B_4). From now on, let us denote by F the following set:

$$F := \left\{ (x,y) \in \mathbb{R} \times \mathbb{R} : \; x \in \left[\inf_{t \in [T_0, +\infty)} x_\alpha(t), \sup_{t \in [T_0, +\infty)} x_\beta(t) \right], \right.$$
$$\left. y \in \left[\inf_{t \in [T_0, +\infty)} y_\alpha(t), \sup_{t \in [T_0, +\infty)} y_\beta(t) \right] \right\}, \tag{60}$$

which is a bounded set thanks to Definition 2.2 and assumption (B_1). Since $h(x, y, z) > 0$ for every x, y, and $z \neq 0$, we denote by

$$\underline{m_{\overline{C}}} := \min \left\{ h(x,y,z) : (x,y) \in F, \quad \zeta \leq |z| \leq \overline{C} \right\} > 0,$$

where $\overline{C} > 0$ and F is defined in (60). Therefore, by recalling k is bounded by assumption, we are allowed to define

$$\mu := \underline{m_{\overline{C}}}\, k_2\, h_2 M^{-\gamma_1} > 0$$
$$\text{and} \quad K_L(t) = \begin{cases} 0, & \text{for } 0 \leq t \leq T_0, \\ -\mu \int\limits_{T_0}^{t} \max\limits_{(x,y) \in F} g(t,x,y)\, \mathrm{d}\tau & \text{for } t \geq T_0. \end{cases}$$

K_L is well defined, because $g \in L^p_{\text{loc}}(\mathbb{R}_0^+)$ by (i) and assumption (B_1) is in place. Moreover, it is strictly increasing for every $t \geq T_0$. Indeed, by its definition and (i), we have

$$K_L'(t) = -\mu \max_{(x,y) \in F} g(t,x,y) > 0 \qquad \text{for } t \geq T_0. \tag{61}$$

Additionally, $K_L \in W^{1,1}_{\text{loc}}(\mathbb{R}_0^+)$ and by combining (57) with (61), it additionally follows:

$$K_L'(t) \geq h_1 \mu t^{\delta_1} \quad \text{and} \quad K_L(t) \geq \frac{h_1 \mu}{\delta_1 + 1} \left(t^{\delta_1 + 1} - T_0^{\delta_1 + 1} \right) \quad \text{for a.e. } t \geq T_0. \tag{62}$$

Then, the following estimate for the right-hand side holds true, when $t \geq T_0$ and $M|z| < \zeta$:

$$f(t,x,y,z) = g(t,x,y) h(x,y,z) \leq \max_{(x,y) \in F} g(t,x,y) h(x,y,z)$$
$$\leq -K_L'(t) (k(t))^{\gamma_1} |z|^{\gamma_1} = -K_L'(t) |k(t) z|^{(r-1)\theta}.$$

This implies that (6) of (B_4) is satisfied by choosing $\theta = \frac{\gamma_1}{r-1}$. Hence, combining this with the definition of K_L and (62) and by proceeding as in the proof of Theorem 6.1, one can show (5) holds if and only if

$$\frac{\delta_1 + 1}{\gamma_1 - (r-1)} \geq \sigma \qquad \Longleftrightarrow \qquad \delta_1 \geq -1 + \sigma(\gamma_1 - (r-1)).$$

Eventually, if we observe $\lim\limits_{t \to +\infty} \mathcal{N}_L(t) = 0$, then it is possible to find $t^* > T_0$, such that

$$0 < \mathcal{N}_L(t) \leq \zeta \qquad \text{for a.e. } t \geq t^*. \tag{63}$$

Hence, (6) is proved. Now, we are left with the proof of (7). Thus, we observe

$$\begin{aligned}|f(t,x,y,z)| &\le |g(t,x,y)h(x,y,z)|\\ &\le \eta_L(t) := \begin{cases}\hat{m}\lambda(t) & \text{if } t < t^* \text{ by (55)},\\ h_2 k_2 t^{\delta_2}|z|^{\gamma_2} & \text{if } t \ge t^* \text{ by (58) and (59)},\end{cases}\end{aligned}$$

where the constant $\hat{m}$ is defined as

$$\hat{m} := \max\left\{h(x,y,z) : (x,y) \in F,\ |z| < L\right\},$$

and $\eta_L \in L^1$. Indeed, by our assumptions and (51) and (63), we get

$$\int_0^\infty \eta_L(t)\mathrm{d}t \le \int_0^{t^*} \hat{m}\lambda(t)\mathrm{d}t + \int_{t^*}^\infty h_2 k_2 t^{\delta_2}|z|^{\gamma_2}\mathrm{d}t.$$

The first integral is finite by assumption (ii), and for the second one, it holds

$$\int_{t^*}^\infty h_2 k_2 t^{\delta_2}|z|^{\gamma_2}\mathrm{d}t \le \mathbf{c}\int_{t^*}^\infty \frac{h_2 k_2 t^{\delta_2}}{k(t)^{\gamma_2}}|\mathcal{N}_L(t)|^{\gamma_2},$$

which is finite if and only if

$$(\delta_1+1)\frac{\gamma_2-(r-1)}{\gamma_1-(r-1)}(r-1) \ge \sigma + \delta_2.$$

This is always true given our assumptions, and hence, assumption (B_4) holds. □

Theorem 6.3. *Let assumptions of Theorem 6.2 be in place. If*

$$\frac{\delta_1+1}{\gamma_1-(r-1)} < \sigma \quad \textit{and} \quad (\delta_1+1)\frac{\gamma_2-(r-1)}{\gamma_1-(r-1)}(r-1) \ge \sigma+\delta_2,$$

then problem (1) does not admit any non-trivial solution.

Proof. To prove our non-existence result, it is sufficient to show that assumptions of Theorem 5.1 are satisfied given our choice of parameters. In particular, by choosing $g(t) = \ell(t)$ for $t \le T_0$, with $\ell \in L^1(\mathbb{R}_0^+)$ and $g(t) = -c_* \min_F g(t,x,y)$ for $t > T_0$, we have that $K_L(t) = \int_0^t g(\tau)\,\mathrm{d}\tau$ for $t \ge 0$ is a strictly increasing function belonging to $W^{1,1}_{\mathrm{loc}}(\mathbb{R}_0^+)$. Additionally

$$K_L(t) = \int_0^{T_0}\ell(\tau)\mathrm{d}\tau - \int_{T_0}^t c_*\min_E g(t,x,y)\mathrm{d}\tau \ge \int_0^{T_0}\ell(\tau)\mathrm{d}\tau + \frac{c_*}{\delta_1+1}(t^{\delta_1+1} - T_0^{\delta+1}).$$

Then, we have

$$\begin{aligned}&\int_0^\infty \frac{1}{k(t)}\left(\int_0^{T_0}\ell(\tau)\mathrm{d}\tau + \frac{c_*}{\delta+1}(t^{\delta+1}-T_0^{\delta+1})\right)^{-\frac{1}{\rho(\theta-1)}}\mathrm{d}t\\ &\quad\ge \mathbf{c} + \frac{c_*}{\delta+1}\int_0^\infty \frac{1}{k(t)}\left(t^{\delta+1}-T_0^{\delta+1}\right)^{-\frac{1}{\rho(\theta-1)}}\mathrm{d}t = \infty\end{aligned}$$

when $\delta_1 + 1 < \sigma(\gamma_1 - (r-1))$. □

6.3. A class of examples: an odd Φ-Laplace operator

In this subsection, we discuss a wide class of examples to which Theorems 2.4 and 5.1 apply. More precisely, we consider (1) under the assumption (B_1) combined with the following structural ones:

(I) $\Phi : \mathbb{R} \to \mathbb{R}$ is an odd function fulfilling assumption (A_1);

(II) $k : \mathbb{R}_0^+ \to \mathbb{R}$ is a bounded function satisfying assumption (A_2). Moreover, there exists $\sigma > 0$, such that, for a suitably chosen $p > 1$, it holds

$$\int_1^{\infty} \frac{1}{t^{\sigma} k(t)^p} \mathrm{d}t < \infty;$$

(III) $f : \mathbb{R}_0^+ \times \mathbb{R}^3 \to \mathbb{R}$ of the form $f(t, x, y, z) = g(t, x, y)h(z)$;

(IV) g is a Carathéodory function. Furthermore:

$(\mathrm{IV})_a$ there exists a function $\lambda \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}_0^+)$, such that

$$|g(t, x, y)| \leq \lambda(t) \quad \text{for a.e. } t \in \mathbb{R}^+$$

for every $x, y \in \mathbb{R}$, such that $x \in \left[\int_0^t \alpha(s)\, ds, \int_0^t \beta(s)\, ds\right]$ and $y \in [\alpha(t), \beta(t)]$;

$(\mathrm{IV})_b$ there exist $T_0 > 0$, two positive constants $h_1, h_2 > 0$ and two numbers $\delta_1, \delta_2 \in \mathbb{R}$, such that $-1 < \delta_1 \leq \delta_2$ and

$$h_1 t^{\delta_1} \leq |g(t, x, y)| \leq h_2 t^{\delta_2} \quad \text{for a.e. } t \geq T_0$$

for every $x, y \in \mathbb{R}$, such that $x \in \left[\int_0^t \alpha(s)\, ds, \int_0^t \beta(s)\, ds\right]$ and $y \in [\alpha(t), \beta(t)]$;

$(\mathrm{III})_c$ $g(t, x, y) \leq$ for every $t \geq T_0$, for every $x, y \in \mathbb{R}$, such that $x \in \left[\int_0^t \alpha(s)\, ds, \int_0^t \beta(s)\, ds\right]$ and $y \in [\alpha(t), \beta(t)]$;

(V) h is a continuous function enjoying the following additional properties:

$(\mathrm{V})_a$ $h > 0$ on $\mathbb{R}\backslash\{0\}$ and $g(0) = 0$;

$(\mathrm{V})_b$ there exists $\zeta > 0$, two positive constants $k_1, k_2 > 0$ and two numbers $\gamma_1, \gamma_2 > 0$, such that $0 < \gamma_1 \leq \gamma_2$ and

$$k_1 |z|^{\gamma_1} \leq |h(z)| \leq k_2 |z|^{\gamma_2} \quad \text{for a.e. } |z| \leq \zeta;$$

$(\mathrm{V})_c$ there exist $H > 0$ and a constant $c > 0$, such that, if $z \in \mathbb{R}$ and $|z| \geq H$, then the following holds true:

$$h(y) \leq c|z|^{1 - \frac{1}{\mathring{q}}} \quad \text{for some } 1 < \mathring{q} \leq \infty;$$

$(\mathrm{V})_d$ h is homogeneous of degree $d > 0$ on $\mathbb{R}$, that is

$$b(sz) = s^d b(z) \quad \text{for every } s > 0 \text{ and every } z \in \mathbb{R}.$$

Then, it is possible to prove the following result.

Theorem 6.4. *Let us consider* (1) *under the assumptions of this subsection. If*

$$\frac{\delta_1 + 1}{\gamma_1 + 1} \geq \sigma\rho \quad \textit{and} \quad \frac{\gamma_2(\delta_1 + 1)}{\gamma_1 - 1} \geq \sigma + \delta_2,$$

then (1) *admits at least one solution. If*

$$\frac{\delta_1+1}{\gamma_1+1} < \sigma\rho \quad \textit{and} \quad \frac{\gamma_2(\delta_1+1)}{\gamma_1-1} \geq \sigma+\delta_2,$$

then (1) *does not admits any (non-trivial) solution.*

The proof of this statement boils down to show assumptions of Theorems 2.4 and 5.1 are satisfied. Hence, it directly follows as in the previous subsections provided suitable adaptations, such as the ones proposed in [6, Section 5], and for this reason, it is not explicitly reported here.

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