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The Verlinde traces for $SU_X(2,\xi)$ and blow-ups

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Abstract. Given a compact Riemann surface X of genus at least 2 with automorphism group G, we provide formulae that enable us to compute traces of automorphisms of X on the space of global sections of G-linearized line bundles defined on certain blow-ups of projective spaces along the curve X. The method is an adaptation of one used by Thaddeus to compute the dimensions of those spaces. In particular, we can compute the traces of automorphisms of X on the Verlinde spaces corresponding to the moduli space $SU_X(2,\xi)$ when ξ is a line bundle G-linearized of suitable degree.

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1. Introduction

Let X be a complex, irreducible, smooth, projective curve of genus at least 2 and automorphism group $G = Aut(X)$. Let ξ be a $G-$linearized line bundle over X.

By the Verlinde traces, we refer to the traces of automorphisms of X on the space $H^0(SU_X(r,\xi),\mathscr{O}(n\Theta))$, where $SU_X(r,\xi)$ is the moduli space of semi-stable rank r vector bundles with determinant ξ and where $\mathscr{O}(\Theta)$ is the determinantal line bundle of $SU_X(r,\xi)$. In this work, we address the problem of computing the Verlinde traces for the case $r = 2$ and we give a way to compute them which is alternative to the approach of J. E. Andersen for case of the trace $\tau_{\mathscr{G}}^{(k)}(f)$ in [2] pg. 3 ($\mathscr{G} = SU(2)$ in our case).

Our method and result will be explained in the paragraphs below. Before that, we would like to mention a few things related to this problem. As the reader may be aware the case of the Verlinde traces for the identity of G is already solved. A formula for $dim\ H^0(SU_X(r,\xi),\mathscr{O}(n\Theta))$ was conjectured by E. Verlinde [34] and there are many proofs of it(see for instance the following works concerning this case [7,8,12,15,17,28,31–33,35] and [36]). The action of non-trivial automorphism groups G on the Verlinde spaces had already been considered in the work of Dolgachev ([13] in his proof of Cor. 6.3), in the work of the first author [23] (see Tables 1–4 there) where some Verlinde traces are computed for the cases $\xi = \mathscr{O}_X$, $n = 1$ and arbitrary rank r by computing them on $H^0(J_X^{g_X-1}, \mathscr{O}(r\Theta_{J_X^{g_X-1}}))^* \cong H^0(SU_X(r,\xi), \mathscr{O}(\Theta))$ and later in the work of Andersen [2] the Verlinde traces correspond to the traces $\tau_{\mathscr{G}}^{(k)}(f)$ which we will briefly refer to at the end of this introduction.

As there are automorphisms of $SU_X(r,\xi)$ that are not induced by G (for a description $Aut(SU_X(r,\xi))$ and related results see [9,20,21]) we should also mention that the action of torsion elements of the Jacobian of X acting on the Verlinde spaces of some moduli spaces of vector bundles had also been considered in the works [6,27,29]. Explicit formulae for the corresponding Verlinde traces are provided in [6] and [29].

Coming back to our problem, in the rank 2 case, we followed the method used by Thaddeus to derive the Verlinde formula in [32]. We shall see that his method can be extended to compute the Verlinde traces (see formula (4.9) and Sect. 5) by just replacing the use of the Riemann–Roch Theorem for the use of the Atiyah–Singer Holomorphic Lefschetz Theorem (see Sect. 3) and in this work, we derive some formulae (Theorem 8.1) required to apply the Holomorphic Lefschetz Theorem in Thaddeus' method.

Let K_X be the canonical line bundle of X. Suppose that $K_X\xi$ is very ample. Let $X \hookrightarrow \mathbb{P}^N$ be the embedding defined by the complete linear system $\mid K_X\xi \mid$. Let $\pi : \widetilde{\mathbb{P}_X^N} \mapsto \mathbb{P}^N$ be the blow-up of $\mathbb{P}^N$ with center X and let E be the corresponding exceptional divisor. The Picard group of $\widetilde{\mathbb{P}_X^N}$ is generated by $\mathscr{O}(E)$ and the hyperplane line bundle $\mathscr{O}(H)$. For integers m, n let $\mathscr{O}_1(m,n) = \mathscr{O}((m+n)H - nE)$ and let $V_{m,n} = H^0(\widetilde{\mathbb{P}_X^N}, \mathscr{O}_1(m,n))$. Let ξ have degree d. Thaddeus shows that for $d > 2g_X - 2$ there is a natural isomorphism

$$H^0(SU_X(2,\xi), \mathscr{O}(k\Theta)) \cong V_{k,k(d/2-1)}. \tag{1.1}$$

Under some mild conditions on m, n he finds a formula for the dimension of $V_{m,n}$ (see Theorem 4.1 below). The cases not covered by those conditions can be dealt with easily.

Now, when we assume that ξ is G-linearized it induces an action of G on $\mathbb{P}^N$ and on the blow-up $\widetilde{\mathbb{P}_X^N}$ such that the embedding $X \hookrightarrow \mathbb{P}^N$ and the blow-up map π are G-equivariant. The line bundles $\mathscr{O}_1(m,n)$ can be equipped with a linearization induced by that of ξ. That is because $\mathscr{O}(H) = \pi^*\mathscr{O}_{\mathbb{P}^N}(1)$ comes equipped with a linearization induced by that of ξ; also since E is a G-invariant divisor $\mathscr{O}(E)$ admits a linearization of G with trivial action on $H^0(\widetilde{\mathbb{P}_X^N}, \mathscr{O}(E))$, and since $h^0(\widetilde{\mathbb{P}_X^N}, \mathscr{O}(E)) = 1$ we can see that this linearization is unique(with trivial action on $H^0(\widetilde{\mathbb{P}_X^N}, \mathscr{O}(E)))$ because there is a bijection between the G-linearizations of $\mathscr{O}(E)$ and the G-invariant divisors linearly equivalent to E by G-invariant rational functions (see [13] Prop. 2.1 and [14] Ex. 7.4). So one can consider the problem of computing the traces of elements of G on the spaces $V_{m,n}$ (Thaddeus traces). As we will see in Sect. 4, the formula for $dimV_{m,n}$ is a linear combination of Euler Characteristics of sheaves $B_{i,m,n}$ defined over symmetric products S^iX of the curve (see Theorem 4.1 below) and these sheaves are naturally G-linearized. By tracking back the proof of Theorem 4.1 one notice that the homomorphisms between the cohomology groups involved are G-equivariant and that the trace of an element $h \in G$ on $V_{m,n}$ is in fact obtained by replacing the Euler characteristics of the sheaves $B_{i,m,n}$ by their corresponding Lefschetz numbers $N_i(h) = L(h, B_{i,m,n})$ (see formula (4.9) below and Sect. 5 for its proof). These Lefschetz numbers $L(h, B_{i,m,n})$ can be computed using the Holomorphic Lefschetz Theorem (see Sect. 3, Theorem 3.1 and formula (3.6)) because the sheaves $B_{i,m,n}$ are defined on smooth varieties. To do that we need to know for each component Z of the fixed point set the following data: the generalized Chern Character $\mathrm{ch}_h(i_Z^* B_{i,m,n})$ (see (3.1)), the stable characteristic classes $U(N_{Z/S^iX}(\nu^j))$ of the normal bundle N_{Z/S^iX} (see (3.2)), the Todd class $\mathrm{Td}(T_Z)$ of the tangent bundle T_Z and $\det(Id - h_{|}N^\vee_{Z/S^iX})$ (see Theorem 3.1). In this way by equation (3.6), the solution of the problem of computing $N_i(h) = L(h, B_{i,m,n})$ is reduced to the calculation of the following intersection numbers, namely, the contributions

$$C_{i,Z}(h) := \int_Z \frac{\mathrm{ch}_h(i_Z^* B_{i,m,n})[\prod_{j=1}^{o(h)-1} \mathrm{U}(N_{Z/S^iX}(\nu^j))]\mathrm{Td}(T_Z)}{\det(Id - h_{|}N^\vee_{Z/S^iX})} \tag{1.2}$$

of each component $Z \subseteq (S^iX)^h$ of the fixed point set of h.

As we pointed out earlier, the main goal of this paper (Theorem 8.1) is the calculation of the generalized Chern Character $\mathrm{ch}_h(i_Z^* B_{i,m,n})$. The other data required to apply (1.2) have been dealt with in the works [23](Prop. 3.2) and [24] (Theorem 2.3). However, in Sect. 7, we present a generalization of the formula for the stable characteristic classes $U(N_{Z/S^iX}(\nu^j))$ that was given in [24]. The Theorem 6.1 is required for the proof of Theorem 8.1 and part of its proof is modelled on the proof of Proposition 2.1 in [24]. At the end of the paper we illustrate the use of formula (4.9), when X is a hyperelliptic curve of genus 2, by computing the Verlinde traces corresponding to the hyperelliptic involution.

The problem of computing Verlinde traces may be considered interesting in its own right and following the work of Thaddeus on moduli of pairs there have been various constructions of series flips using moduli of pairs, triplets, that one should expect to be able to use to compute Verlinde traces on moduli spaces other than $SU_X(2,\xi)$. Some of our motivations are to determine the isotypic decomposition of the Verlinde spaces $H^0(SU_X(r,\xi), \mathcal{O}(n\Theta))$ to study the varieties defined as the zero locus of the invariant submodules in, for instance, $\mathbb{P}H^0(SU_X(r,\xi), \mathcal{O}(2\Theta))^\vee$; to study the Cox Ring of Blow-ups(in this case with the Thaddeus traces). Also, as it can be seen from the work of Andersen, Verlinde traces are used to compute invariants of 3-Manifolds, namely, the Witten-Reshetikhin-Turaev invariants of finite order mapping tori $Z_G^{(k)}(\sum f) = Det(f)^{-\frac{1}{2}\zeta}\tau_{\mathscr{G}}^{(k)}(f)$. Andersen considers moduli spaces of semi-stable $\mathscr{G}$-bundles on curves and by applying directly the

Lefschetz–Riemann–Roch Theorem for singular varieties (see [5]) to the determinantal line bundle of the moduli spaces he obtained an expression for the traces $\tau_{\mathscr{G}}^{(k)}(f)$ because the higher cohomologies of the determinantal line bundle vanish. The formula is given by the equation 8.1 or Theorem 1.3 in [2] up to the correction factor $Det(f)^{-\frac{1}{2}\zeta}$, and depends on data coming from the components of the fixed point set. The moduli spaces $SU_X(r, \mathscr{O}_X)$ correspond to the case $\mathscr{G} = SU(r)$ and a description of the components of fixed point set for this case is given in Theorem 6.10 of [2] (see also Theorem 3.4 of [1]). The contribution of the smooth components to the traces $\tau_{\mathscr{G}}^{(k)}(f)$ is also determined in his work. In the rank 2 case, the advantage of our method is that one does not have to deal with singular components of fixed points and one can compute the Verlinde traces if one knows the action of the automorphism on the tangent spaces of the fixed points in the curve.
Some results of this work are based on results of the Ph.D Thesis of the second author [30].

2. Notation

Let p be a positive integer and let $\nu = \exp(2i\pi/p)$. Given a finite cyclic group $H = \langle h \rangle$ of order p acting trivially on a variety Z and given an H-linearized vector bundle F on Z there is a decomposition into eigenbundles

$$F = \bigoplus_{j=0}^{p-1} F(\nu^j),$$

that is, $F(\nu^j)$ is the sub-bundle of F where the action of h on the fibres is multiplication by ν^j. Most of the times we will say that F is h-linearized.

For a divisor D on a variety W we write $\mathscr{O}_W(D)$ (or just $\mathscr{O}(D)$) to denote corresponding line-bundle of D. If F is a sheaf on W we usually write $F(D)$ rather than $F \otimes \mathscr{O}(D)$. Usually $F^n = F^{\otimes n}$, also if K is another sheaf some times we could write $KF = K \otimes F$. If ξ is a sheaf on W and ξ' is a sheaf on X we use the notation $\xi \boxtimes \xi' = \pi_W^* \xi \otimes \pi_X^* \xi'$ on the product $W \times X$, where π_W and π_X are the natural projections and if G is a sheaf on $W \times X$ we sometimes write $\xi G = \pi_W^* \xi \otimes G$ (or $\xi(D)$ instead of ξG if $G = \mathscr{O}_{W \times X}(D)$ with $D \subset W \times X$ is a divisor).

If $i : \Delta \hookrightarrow W$ is the inclusion of a subvariety of W then $\mathscr{O}_\Delta F = \mathscr{O}_\Delta \otimes F$ is the pull-back $i^* F$ for any sheaf F on W and if Δ is a divisor of W then we denote $\mathscr{O}_\Delta(\Delta) = i^* \mathscr{O}_W(\Delta)$. If F is locally free then $F^\vee$ is the dual sheaf and we also denote by F the vector bundle whose sheaves of sections is F. If $x \in W$ then F_x is the fibre of F at the point x. For a vector bundle F over X we denote by $\pi : \mathbb{P}(F) \to X$ the projective bundle of one dimensional subspaces of F thus $\mathbb{P}(F) = Proj_{\mathscr{O}_X}(Sym(F^\vee))$ where $Sym(E) = \oplus_m S^m E$ is the symmetric algebra of a locally free sheaf E over X. The exponential function $exp(x) = \sum_{n=0}^{\infty} x^n/n!$ is also be represented by e^x and in some formulae we mixed both notations.

3. The Holomorphic Lefschetz theorem

Consider a finite cyclic group $H = \langle h \rangle$ of order $p \geq 1$.
Suppose that H is acting trivially on a variety Z and also consider an h-linearized vector bundle F on Z. The generalized Chern character of F is given by

$$ch_h(F) = \sum_{j=0}^{p-1} \nu^j ch[F(\nu^j)], \tag{3.1}$$

where $ch[F(\nu^j)]$ is the Chern character of the eigenbundle $F(\nu^j)$. For each vector bundle $F(\nu^j)$ with $\nu^j \neq 1$ define the stable characteristic class $U(F(\nu^j))$ as

$$U(F(\nu^j)) = \prod_{i=1}^{r_j} \left(\frac{1 - \frac{e^{-x_i}}{\nu^j}}{1 - \frac{1}{\nu^j}} \right)^{-1}, \tag{3.2}$$

where r_j is the rank $F(\nu^j)$ and the x_i's are the Chern roots of $F(\nu^j)$.
We now make some observations about fixed point sets of h acting on non singular Varieties (see pg. 537 in [3] and Lemma 4.1 in [16]). Let X be an irreducible, non-singular complex algebraic variety on which h acts as an automorphism of X. Let X^h denote the set of fixed points of h in X. One has that X^h is non-singular and if $x \in X^h$ then as an analytic variety X^h is (in a neighbourhood of x) the image, under a suitable equivariant exponential map, of the eigenspace $T_{X,x}(\nu^0)$ of the tangent space $T_{X,x}$ to X at x. As $T_{X^h,x} = T_{X,x}(\nu^0)$ one has that the linear transformation $h_{|}N_{X^h/X,x}$ induced by h on the normal space $N_{X^h/X,x}$ to X^h at x has no eigenvalue ν^0 and therefore

$$\det(Id - h \mid N_{X^h/X,x}) \neq 0. \tag{3.3}$$

One also notice from what was said above that X^h is locally irreducible so a point $x \in X^h$ belongs to a unique irreducible component of X^h and so

$$X^h = \coprod_{Z \in CX^h} Z, \tag{3.4}$$

where CX^h is the set of irreducible components of X^h.

Let $i_{X^h} : X^h \hookrightarrow X$ the inclusion $X^h \subseteq X$. Let E be an h-linearized vector bundle on X. The Lefschetz number $L(h,E)$ (sometimes written $L(h,X,E)$) of h on E is

$$L(h,E) = \sum_i (-1)^i trz\ h|H^i(X,E)$$

and can be computed by the *Holomorphic Lefschetz Theorem*:

Theorem 3.1. (Atiyah–Singer, Theorem 4.6, pg. 566, [4])

$$L(h,E) = \int_{X^h} \frac{\mathrm{ch}_h(i^*_{X^h}E)[\prod_{j=1}^{o(h)-1} \mathrm{U}(N_{X^h/X}(\nu^j))]\mathrm{Td}(T_{X^h})}{\det(Id - h_{|}N^\vee_{X^h/X})} \tag{3.5}$$

where $\det(Id - h_{|}N^{\vee}_{X^h/X})$ *assigns to the component of* $x \in X^h$ *the value of* $\det(Id - h_{|}N^{\vee}_{X^h/X,x})$. *Also,* $U(N_{X^h/X}(\nu^j))$ *is the stable characteristic class of* $N_{X^h/X}(\nu^j)$, $\mathrm{ch}_h(i^*_{X^h}E)$ *is the generalized Chern character of the pull-back* $i^*_{X^h}E$ *of* E *to* X^h *and* $\mathrm{Td}(T_{X^h})$ *is the Todd class of the tangent sheaf* T_{X^h} *of* X^h.

On the right-hand side of (3.5), one is evaluating a cohomology class $u \in H^*(X^h, \mathbb{C})$ (here u is the integrand). By (3.4) on has that

$$H^*(X^h, \mathbb{C}) = \bigoplus_{Z \in CX^h} H^*(Z, \mathbb{C})$$

and then the evaluation of u is the sum of the evaluations of the components $u_Z \in H^*(Z, \mathbb{C})$ of u, that is,

$$L(h, E) = \sum_{Z \in \mathrm{CX}^h} \int_Z \frac{\mathrm{ch}_h(i^*_Z E)[\prod_{j=1}^{o(h)-1} \mathrm{U}(N_{Z/X}(\nu^j))]\mathrm{Td}(T_Z)}{\det(Id - h_{|}N^{\vee}_{Z/X})}, \tag{3.6}$$

A particular case of Theorem 3.1 is the *Atiyah-Bott formula* also known as the *Woods Hole Fixed Point Theorem* which is obtained when X^h is a finite set:

$$L(h, E) = \sum_{x \in X^h} \frac{trz\ h|E_x}{det(Id - h|T^{\vee}_{X,x})}, \tag{3.7}$$

where $trz\ h|E_x$ is the trace of $h|E_x$ (*see* [10] *pgs. 631 and 121*).

4. The trace formula

From now on what follows X will denote the curve considered in the introduction, that is, a complex, irreducible, smooth curve of genus $g \geq 2$ embedded in $\mathbb{P}^N = \mathbb{P}H^1(\xi^{-1})$ via the linear system $|K_X\xi|$, where ξ is a line bundle on X of degree d.

Let $\pi_{S^iX} : X \times S^iX \mapsto S^iX$ and $\pi_X : X \times S^iX \mapsto X$ be the natural projections. Let $\Delta_i \subset X \times S^iX$ be the universal divisor and let j' denote its inclusion into $X \times S^iX$. Consider the *Thaddeus bundles*

$$\begin{aligned} W_i^- &= (R^0_{\pi_{S^iX}})_* \mathcal{O}_{\Delta_i}\xi(-\Delta_i) \\ &= (R^0_{\pi_{S^iX}})_* \{\{j'_*\mathcal{O}_{\Delta_i}\} \otimes \pi^*_X(\xi) \otimes \mathcal{O}_{X \times S^iX}(-\Delta_i)\} \\ &= (R^0_{\pi_{S^iX}})_* \{\{j'_*\mathcal{O}_{\Delta_i}\} \otimes (\xi \boxtimes \mathcal{O}_{S^iX})(-\Delta_i)\} \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} W_i^+ &= (R^1_{\pi_{S^iX}})_* \xi^{-1}(2\Delta_i) \\ &= (R^1_{\pi_{S^iX}})_* \{\pi^*_X(\xi^{-1}) \otimes \mathcal{O}_{X \times S^iX}(2\Delta_i)\} \\ &= (R^1_{\pi_{S^iX}})_* \{(\xi^{-1} \boxtimes \mathcal{O}_{S^iX})(2\Delta_i)\}. \end{aligned} \tag{4.2}$$

One has that W_i^- is a vector bundle of rank i and that if $-d + 2i < 0$ then W_i^+ is a vector bundle of rank $d + g - 1 - 2i$.

Define

$$L_i = det^{-1}[(\pi_{S^iX})_!\{\xi\mathscr{O}_{X\times S^iX}(-\Delta_i)\}]\otimes det^{-1}[(\pi_{S^iX})_!\{\mathscr{O}_{X\times S^iX}(\Delta_i)\}], \tag{4.3}$$

where $(\pi_{S^iX})_! : K(X\times S^iX)\to K(S^iX)$ is the direct image homomorphism between Grothendieck groups given by

$$(\pi_{S^iX})_!(\mathscr{F}) = \sum_i (-1)^i R^i_{\pi_{S^iX}*}(\mathscr{F}) \tag{4.4}$$

for a coherent sheaf $\mathscr{F}$ over $X\times S^iX$ (See [19] pg. 436). Also, det : $K(S^iX)\to Pic(S^iX)$ is the group homomorphism between the additive group $K(S^iX)$ and the multiplicative group $Pic(S^iX)$ given by

$$det\left(\sum_{j=1}^m a_j\mathscr{F}_j\right) = \bigotimes_{j=1}^m det(\mathscr{F}_j)^{a_j}, \tag{4.5}$$

where $a_1,\cdots,a_m\in\mathbb{Z}$, the $\mathscr{F}_j's$ are coherent sheaves and $\det(\mathscr{F}_j)$ is the usual determinant of coherent sheaves defined by means of a locally free resolution of $\mathscr{F}_j$.

Let $U_i\to S^iX$ be the bundle

$$U_i = W_i^-\oplus(W_i^+)^\vee. \tag{4.6}$$

Let $q_i = n-(i-1)m$. For $i>0$ consider the Euler characteristic

$$N_i = \chi(S^iX, B_{i,m,n}), \tag{4.7}$$

where

$$B_{i,m,n} = L_i^m\otimes\wedge^iW_i^-\otimes S^{q_i-i}U_i. \tag{4.8}$$

For $i=0$ we define $B_{0,m,n} = S^{m+n}H^0(X,K_X\xi)$ and $N_0 = dimB_{0,m,n}$. It is a convention that $B_{i,m,n}=0$ when $q_i-i<0$.

Theorem 4.1. (See (6.9) in [32]) *Let $m,n\geq 0$ and suppose that $m(d-2)-2n > -d+2g-2$. Then*

$$dimV_{m,n} = \sum_{i=0}^\infty(-1)^iN_i = \sum_{i=0}^w(-1)^iN_i,$$

where $w=[(d-1)/2]$.

Under our hypothesis (namely, those of Theorem 4.1 together with the one that X has automorphism group G and that ξ is G-linearized so that line bundles $\mathscr{O}_1(m,n)$ are G-linearized) one has that for any $h\in G$ (see Sect. 5)

$$Trace(h_{|_{V_{m,n}}}) = \sum_{i=0}^\infty(-1)^iN_i(h) = \sum_{i=0}^w(-1)^iN_i(h), \tag{4.9}$$

where $N_i(h)$ stands for the Lefschetz number

$$L(h,B_{i,m,n}) = \sum_{j=0}^i(-1)^jTrace(h_|H^j(S^iX,B_{i,m,n})), \tag{4.10}$$

and for $i = 0$,

$$N_0(h) = Trace(h_| B_{0,m,n}) = \underset{t^{m+n}}{coef} \left[\frac{1}{\det(I - t \cdot h_| H^0(X, K_X \xi))} \right]. \quad (4.11)$$

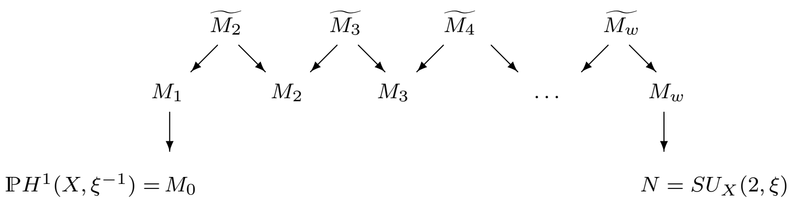
The far right-hand side of (4.11) follows from the proof of *Molien's Theorem 1.10* in [25]. So the Lefschetz numbers $N_i(h)$, $i > 0$, can be computed by means of the Holomorphic Lefschetz Theorem if one (see formula (3.6)) can determine the contribution

$$C_{i,Z}(h) = \int_Z \frac{\mathrm{ch}_h(i_Z^* B_{i,m,n})[\prod_{j=1}^{o(h)-1} \mathrm{U}(N_{Z/S^iX}(\nu^j))]\mathrm{Td}(T_Z)}{\det(Id - h_| N^\vee_{Z/S^iX})}$$

of each component $Z \subseteq (S^iX)^h$ of the fixed point set. It will be explained in Sect. 6 that the components Z of $(S^iX)^h$ are parametrized by certain set of h-invariant divisors D on the curve X so in the subsequent sections we will write Z_D rather than just Z because although components associated to distinct divisors may be isomorphic the data associated to them may not be the same. So far, the main obstruction to compute $C_{i,Z}(h)$ is the generalized Chern character $\mathrm{ch}_h(i_Z^* B_{i,m,n})$ and it will be determined in Sect. 8. In Sect. 7, Theorem 7.1 we present a new formula for the stable characteristic classes $\mathrm{U}(N_{Z/S^iX}(\nu^j))$ and in (7.5) one for $\det(Id - h_| N^\vee_{Z/S^iX})$. The Todd class of a component is given in formula (6.8).

5. The proof of Thaddeus

Here we shall derive formula (4.9) so we are assuming that a finite group G is acting on the curve X and that ξ is a G-linearized line bundle of degree d. Recall also that $K_X\xi$ is very ample and X is embedded into $\mathbb{P}H^1(X, \xi^{-1})$ by the complete linear system $\mid K_X\xi \mid$. We should remark that ξ plays the role of the line bundle Λ considered in [32]. Thaddeus' proof of Theorem 4.1 uses 4 results ((6.2),(6.6),(6.7) and (6.8) in [32]) which we will state here in terms of Lefschetz numbers (see Lemmas 5.1–5.4 below), their proofs follow from those of Thaddeus so we are essentially sketching his proof.
We start by introducing some notation related to the main tool which is a series of flips, depicted in the diagram (5.1) below, that connects the space $\mathbb{P}H^1(X, \xi^{-1})$ and the moduli space $SU_X(2, \xi)$.



(5.1)

The spaces M_i are constructed as moduli spaces of σ_i-stable pairs $(\mathscr{E},\phi)$ where $\mathscr{E}$ is a rank 2 vector bundle on the curve X such that $\wedge^2\mathscr{E} = \xi$, $\phi \in H^0(X,\mathscr{E})\backslash\{0\}$ and $\sigma_i \in (max(0, d/2-i-1), d/2-i)$ (for the definition of the stability condition see (1.1) in [32]). The space M_1 is $\widetilde{\mathbb{P}}^N_X$ the blow-up of $\mathbb{P}H^1(X,\xi^{-1})$ along X embedded via $\mid K_X\xi \mid$. When $i = w = [(d-1)/2]$ the map $M_w \mapsto N$ has fibre $\mathbb{P}H^0(X,\mathscr{E})$ over a stable bundle $\mathscr{E} \in N$ and it is surjective if $d > 2g-2$. All the spaces M_i turn out to be smooth, integral, rational projective varieties of dimension $d+g-1$ and for $i > 0$ there is a birational map $M_i \leftrightarrow M_1$ which is an isomorphism except on closed sets of codimension at least 2. In fact there are embeddings $\mathbb{P}W_i^+ \hookrightarrow M_i$ and $\mathbb{P}W_i^- \hookrightarrow M_{i-1}$, for $0 < i \leq (d-1)/2$, whose images correspond to the pairs represented in M_i but not in M_{i-1} and pairs represented in M_{i-1} but not in M_i respectively and there is an isomorphism $M_i\backslash\mathbb{P}W_i^+ \cong M_{i-1}\backslash\mathbb{P}W_i^-$. As for the spaces $\widetilde{M}_i$, for $i > 1$, the arrows between M_{i-1}, $\widetilde{M}_i$ and M_i make up each of the announced flips, specifically, $\widetilde{M}_i$ is the blow-up of M_{i-1} along $\mathbb{P}W_i^-$ and it is also the the blow-up of M_i along $\mathbb{P}W_i^+$, furthermore M_i can be obtained by blowing-up M_{i-1} along $\mathbb{P}W_i^-$ and then blowing down the same exceptional divisor to $\mathbb{P}W_i^+ \subset M_i$. Let $E_i \subset \widetilde{M}_i$ denote the exceptional divisor for $i = 2,\cdots,w$. If $i = 1$ one can also consider $\widetilde{M}_1 \cong M_1$ the blow-up of M_1 along $\mathbb{P}W_1^+$ with exceptional divisor $E_1 \cong \mathbb{P}W_1^+ \subset M_1$.

For $0 < i \leq w$ the embeddings $\mathbb{P}W_i^+ \hookrightarrow M_i$ and $\mathbb{P}W_i^- \hookrightarrow M_{i-1}$ induce respectively the following exact sequences (see (3.9) and (3.11) in [32])

$$\begin{aligned} &0 \to T_{\mathbb{P}W_i^+} \to T_{M_i} \mid_{\mathbb{P}W_i^+} \to W_i^-(-1) \to 0 \\ &\text{and} \\ &0 \to T_{\mathbb{P}W_i^-} \to T_{M_{i-1}} \mid_{\mathbb{P}W_i^-} \to W_i^+(-1) \to 0, \end{aligned} \tag{5.2}$$

we recall that according to our notation here $W_i^-(-1)$ is the pull-back of W_i^- to $\mathbb{P}W_i^+$ under the projection to S^iX tensored with the tautological bundle $\mathscr{O}_{\mathbb{P}W_i^+}(-1)$. From the exact sequences (5.2) one notice that

$$\mathbb{P}W_i^+(-1) \cong E_i \cong \mathbb{P}W_i^-(-1)$$

and in fact

$$E_i = \mathbb{P}W_i^- \times_{S^iX} \mathbb{P}W_i^+$$

is the fibred product of the projections from $\mathbb{P}W_i^-$ and $\mathbb{P}W_i^+$ to S^iX and we represent it in the diagram (5.3) for future reference in the proof Lemma 5.3

below.

$$\begin{array}{ccc} E_i & \xrightarrow{\mathsf{p}_2} & \mathbb{P}W_i^+ \\ {\scriptstyle \mathsf{p}_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ \mathbb{P}W_i^- & \xrightarrow{\pi_1} & S^iX, \end{array} \tag{5.3}$$

One denotes by $\mathcal{O}_{E_i}(j,k)$ the line bundle on E_i given by

$$\mathcal{O}_{E_i}(j,k) := \mathcal{O}_{\mathbb{P}W_i^-}(j) \boxtimes \mathcal{O}_{\mathbb{P}W_i^+}(k). \tag{5.4}$$

The twisting sheaf of E_i happens to be

$$\mathcal{O}_{E_i}(1) = \mathcal{O}_{E_i}(1,1)$$

therefore

$$\mathcal{O}_{E_i}(E_i) = \mathcal{O}_{E_i}(-1,-1).$$

Now we consider the line bundles $\mathcal{O}_1(m,n) = \mathcal{O}_{M_1}((m+n)H - nE)$ over M_1. Since M_1 and M_i are isomorphic outside closed sets of codimension at least 2 there is (as a consequence of Hartogs' theorem), for $i > 0$, an isomorphism

$$Pic\ M_1 \cong Pic\ M_i.$$

Let $\mathcal{O}_i(m,n)$ be the image of $\mathcal{O}_1(m,n)$ under this isomorphism. One also has (another consequence of Hartogs' theorem) that

$$H^0(M_1, \mathcal{O}_1(m,n)) = H^0(M_i, \mathcal{O}_i(m,n)). \tag{5.5}$$

Recall that we denoted in the introduction $V_{m,n} = H^0(M_1, \mathcal{O}_1(m,n)$. Abusing notation one also writes $\mathcal{O}_i(m,n)$ and $\mathcal{O}_{i-1}(m,n)$ for the pull-backs to the space $\widetilde{M_i}$ of the line bundles $\mathcal{O}_i(m,n)$ and $\mathcal{O}_{i-1}(m,n)$ defined above. In the case $i = 0$, we have $M_0 = \mathbb{P}H^1(X, \xi^{-1})$ and one defines $\mathcal{O}_0(m,n) = \mathcal{O}_{\mathbb{P}H^1(X,\xi^{-1})}(m+n)$. On $\widetilde{M_i}$ one has (see (5.6) in [32])

$$\mathcal{O}_i(m,n) = \mathcal{O}_{i-1}(m,n)(((i-1)m - n)E_i). \tag{5.6}$$

When we assume that a group G is acting on the curve X one has by functoriality an action on all the moduli spaces M_i and N. This action is compatible with the isomorphisms $M_i \backslash \mathbb{P}W_i^+ \cong M_{i-1} \backslash \mathbb{P}W_i^-$ because pairs representing points on the left-hand side also represent points on the right-hand side. In the introduction, we endowed the line bundles $\mathcal{O}_1(m,n)$ with a G-linearization and because of the isomorphism $M_1 \leftrightarrow M_i$ outside closed subsets of codimension at least 2 one has that this linearization also induces a linearization on the other line bundles $\mathcal{O}_i(m,n)$. In this way, the action of

G induced on $H^0(M_1,\mathcal{O}_1(m,n))$ and on $H^0(M_i,\mathcal{O}_i(m,n))$ through these linearizations is the same. The action of G on the spaces M_i also lifts to an action on the spaces $\widetilde{M}_i$ and one also endows the line bundles $\mathcal{O}_i(m,n),\mathcal{O}_{i-1}(m,n)$ in $Pic(\widetilde{M}_i)$ with a linearization induced by that of $\mathcal{O}_i(m,n)$ in $Pic(M_i)$ and $\mathcal{O}_{i-1}(m,n)$ in $Pic(M_{i-1})$. Now, we can state the following lemmas.

Lemma 5.1. *If $m,n\geq 0$ and $m(d-2)-2n>-d+2g-2$, there exist an integer $b\leq w$ such that for any $h\in G$*

$$Trace(h_{|}V_{m,n}) = L(h,M_b,\mathcal{O}_b(m,n)),$$

in fact, $b=[\frac{n+d+g-4}{m+3}]+1$.

Proof. Thaddeus showed that for $b=[\frac{n+d+g-4}{m+3}]+1$ the cohomology groups $H^i(M_b,\mathcal{O}_b(m,n))$ vanish for $i>0$ (see proof of (6.2) in [32]). Then, for any $h\in G$

$$\begin{aligned}L(h,M_b,\mathcal{O}_b(m,n)) &= \sum_{i\geq 0}(-1)^i Trace\ h_{|}H^i(M_b,\mathcal{O}_b(m,n))\\ &= Trace\ h_{|}H^0(M_b,\mathcal{O}_b(m,n)).\end{aligned}$$

Using (5.5), we have

$$Trace\ h_{|}H^0(M_b,\mathcal{O}_b(m,n)) = Trace\ h_{|}H^0(M_1,\mathcal{O}_1(m,n))$$

and since $V_{m,n}=H^0(M_1,\mathcal{O}_1(m,n))$ we are done. □

For the remainder of this section, we shall assume that $m,n\geq 0$, $m(d-2)-2n>-d+2g-2$ and $b=[\frac{n+d+g-4}{m+3}]+1$.

Lemma 5.2. $N_0(h)=L(h,M_0,\mathcal{O}_0(m,n))$.

Proof. This follows from the definitions of $N_0(h)$ and $\mathcal{O}_0(m,n)$. Namely, we have $M_0=\mathbb{P}H^1(X,\xi^{-1})$ and $\mathcal{O}_0(m,n)=\mathcal{O}_{\mathbb{P}H^1(X,\xi^{-1})}(m+n)$ (see definition before (5.6)). Notice that when $i=0$, we have $q_i-i=m+n\geq 0$ (see definition before (4.7)). Therefore, we have the following

$$\begin{aligned}H^0(M_0,\mathcal{O}_0(m,n)) &= S^{m+n}H^0(\mathbb{P}H^1(X,\xi^{-1}),\mathcal{O}_{\mathbb{P}H^1(X,\xi^{-1})}(1))\\ &= S^{m+n}H^0(X,K_X\xi)\end{aligned}$$

and $H^i(M_0,\mathcal{O}_0(m,n))=0$ for $i>0$. Then

$$L(h,M_0,\mathcal{O}_0(m,n)) = Trace(h_{|}S^{m+n}H^0(X,K_X\xi))$$

and by the definition of $B_{0,m,n}$ in Sect. 4 the last is

$$= Trace(h_{|}B_{0,m,n})$$

which in turn is the definition of $N_0(h)$ in (4.11). □

We can also restate (6.7) of [32] as the following

Lemma 5.3. *For $0<i\leq b$ we have for any $h\in G$*

$$L(h,M_i,\mathcal{O}_i(m,n)) - L(h,M_{i-1},\mathcal{O}_{i-1}(m,n)) = (-1)^i N_i(h)$$

Proof. When h is the identity Thaddeus first shows that $\chi(M_i, \mathcal{O}_i(m,n)) = \chi(\widetilde{M}_i, \mathcal{O}_i(m,n))$. But this follows because the higher direct images of $\mathcal{O}_i(m,n)$ under the map $\pi : \widetilde{M}_i \mapsto M_i$ vanish so the Leray spectral sequence implies $H^j(\widetilde{M}_i, \mathcal{O}_i(m,n)) = H^j(M_i, R^0_{\pi *}\mathcal{O}_i(m,n))$ and then since $R^0_{\pi *}\mathcal{O}_{\widetilde{M}_i} = \mathcal{O}_{M_i}$ the projection formula implies $H^j(\widetilde{M}_i, \mathcal{O}_i(m,n)) = H^j(M_i, \mathcal{O}_i(m,n))$. Similarly $H^j(\widetilde{M}_i, \mathcal{O}_{i-1}(m,n)) = H^j(M_{i-1}, \mathcal{O}_{i-1}(m,n))$. Then, we can write for any $h \in G$

$$L(h, M_i, \mathcal{O}_i(m,n)) = L(h, \widetilde{M}_i, \mathcal{O}_i(m,n))$$

and

$$L(h, M_{i-1}, \mathcal{O}_{i-1}(m,n)) = L(h, \widetilde{M}_i, \mathcal{O}_{i-1}(m,n))$$

and one can work on $\widetilde{M}_i$.
Recall that $q_i = n - (i-1)m$. Next, one considers two cases.
1) When $q_i \leq 0$, by definition of $B_{i,m,n}$ (see equation (4.8)), one has $N_i(h) = 0$.
If $q_i = 0$ then we are done because $\mathcal{O}_i(m,n) = \mathcal{O}_{i-1}(m,n)$ on $\widetilde{M}_i$ by (5.6). So we assume now that $q_i < 0$.
Since E_i is a G-invariant divisor on $\widetilde{M}_i$, the exact sequence induced by the embedding $E_i \hookrightarrow \widetilde{M}_i$

$$0 \to \mathcal{O}_{\widetilde{M}_i}(-E_i) \to \mathcal{O}_{\widetilde{M}_i} \to \mathcal{O}_{\widetilde{M}_i} \otimes \mathcal{O}_{E_i} \to 0 \tag{5.7}$$

is a G-equivariant exact sequence of linearized sheaves so for each j we have an equivariant exact sequence

$$\begin{aligned} 0 &\to \mathcal{O}_{i-1}(m,n)((j-1)E_i) \to \mathcal{O}_{i-1}(m,n)(jE_i) \\ &\to \mathcal{O}_{i-1}(m,n) \otimes \mathcal{O}_{E_i}(jE_i) \to 0. \end{aligned} \tag{5.8}$$

Thaddeus identify $\mathcal{O}_{i-1}(m,n) \otimes \mathcal{O}_{E_i}(jE_i)$ with the sheaf(or rather its push-forward to $\widetilde{M}_i$) $L_i^m(-q_i-j,-j)$. The last corresponds to a the tensor product $F_1 \otimes F_2$ of sheaves on E_i where F_1 is the pull-back of the sheaf L_i^m (defined in (4.3)) under the projection $\pi : E_i \twoheadrightarrow S^iX$ (here π is $\pi_1 \circ \mathsf{p}_1 = \pi_2 \circ \mathsf{p}_2$ in the diagram (5.3)) and $F_2 = \mathcal{O}_{E_i}(-q_i-j,-j)$ (see equation (5.4)).
The exact sequence (5.8) induces a G-equivariant long exact sequence of cohomology groups from which we can write (using the above identification):

$$\begin{aligned} &L(h, \widetilde{M}_i, \mathcal{O}_{i-1}(m,n)(jE_i)) - L(h, \widetilde{M}_i, \mathcal{O}_{i-1}(m,n)((j-1)E_1)) = \\ &= L(h, E_i, L_i^m(-q_i-j,-j)). \end{aligned} \tag{5.9}$$

Summing (5.9) over j, with $0 < j \leq -q_i$, and using (5.6) one arrives to

$$\begin{aligned} &L(h, \widetilde{M}_i, \mathcal{O}_i(m,n)) - L(h, \widetilde{M}_i, \mathcal{O}_{i-1}(m,n)) \\ &= \sum_{j=1}^{-q_i} L(h, E_i, L_i^m(-q_i-j,-j)). \end{aligned} \tag{5.10}$$

When h is the identity Thaddeus proves the vanishing of the right-hand side by showing that all the cohomology groups

$$H^s(E_i, L_i^m(-q_i-j,-j))$$

vanish (all the direct images of $L_i^m(-q_i-j,-j)$ under the projection $\mathsf{p}_1 : E_i \to \mathbb{P}W_i^-$ vanish because $R^t_{\pi*}L_i^m(-q_i-j,-j) = L_i^m(-q_i-j) \otimes R^t_{\pi*}\mathcal{O}_{E_i}(0,-j)$ and $R^t_{\pi*}\mathcal{O}_{E_i}(0,-j) = 0$ for all t because $0 < j < d+g-1-2i = \text{rank }(W_i^+)$ implies that a fibre $H^t(\mathbb{P}^{d+g-2-2i}, \mathcal{O}(-j)) = 0$). Therefore the right-hand side of (5.10) vanish for any $h \in G$.
2) Now, we assume $q_i > 0$. Summing (5.9) over j, with $-q_i < j \le 0$, and using (5.6) one obtains

$$\begin{aligned} L(h, \widetilde{M}_i, \mathcal{O}_{i-1}(m,n)) - L(h, \widetilde{M}_i, \mathcal{O}_i(m,n)) &= \sum_{j=-q_i+1}^{0} L(h, E_i, L_i^m(-q_i-j,-j)) \\ &= \sum_{j=0}^{q_i-1} L(h, E_i, L_i^m(-q_i+j,+j)). \end{aligned} \tag{5.11}$$

Now, the only non-zero direct image of $L_i^m(-q_i+j,+j)$ under the projection $\pi : E_i \twoheadrightarrow S^iX$ is the (i-1)-th (see (5.12) below):
given $D \in S^iX$ a fibre of π is of the form $E_{iD} = \mathbb{P}^{i-1} \times \mathbb{P}^{d+g-2-2i}$,
since $-q_i + j < 0$ one has $H^s(\mathbb{P}^{i-1}, \mathcal{O}(-q_i+j)) = 0$ for $s \neq i-1$ and
since $j \geq 0$, $H^s(\mathbb{P}^{d+g-1-2i}, \mathcal{O}(+j)) = 0$ for $s \neq 0$.
So $H^{i-1}(\mathbb{P}^{i-1}, \mathcal{O}(-q_i+j)) \otimes H^0(\mathbb{P}^{d+g-1-2i}, \mathcal{O}(+j))$ is the only non-zero Künneth component of $H^{i-1}(E_{iD}, \mathcal{O}_{E_i}(-q_i+j,+j) \mid_{E_{iD}})$.
To compute a direct image $R^{i-1}_{\pi*}F$, one uses the fact that the Leray Spectral sequence $E_2^{t,s} = R^t_{\pi_1*}R^s_{\mathsf{p}_1*}F \Rightarrow R^{t+s}_{\pi*}F$. It will be enough to work with $F = \mathcal{O}_{E_i}(-q_i+j,+j)$ because $R^{i-1}_{\pi*}(L_i^m(-q_i+j,+j)) = L_i^m \otimes R^{i-1}_{\pi*}\mathcal{O}_{E_i}(-q_i+j,+j)$. We shall see that the only non-zero term of the Leray spectral sequence is $E_2^{i-1,0}$.
Notice that for all sheaves $\mathcal{O}_{\mathbb{P}W_i^-}(l)$, $\mathcal{O}_{\mathbb{P}W_i^+}(l)$ the base change morphisms induced by the diagram (5.3) are isomorphisms, that is,
$R^t_{\mathsf{p}_1*}\mathsf{p}_2^*\mathcal{O}_{\mathbb{P}W_i^+}(l) \cong \pi_1^*R^t_{\pi_2*}\mathcal{O}_{\mathbb{P}W_i^+}(l)$ and $R^t_{\mathsf{p}_2*}\mathsf{p}_1^*\mathcal{O}_{\mathbb{P}W_i^-}(l) \cong \pi_2^*R^t_{\pi_1*}\mathcal{O}_{\mathbb{P}W_i^-}(l)$ for all $t \geq 0$.
Now, by the projection formula one has
$R^t_{\pi_1*}R^s_{\mathsf{p}_1*}\mathcal{O}_{E_i}(-q_i+j,+j) = R^t_{\pi_1*}\{\mathcal{O}_{\mathbb{P}W_i^-}(-q_i+j) \otimes R^s_{\mathsf{p}_1*}\mathsf{p}_2^*\mathcal{O}_{\mathbb{P}W_i^+}(+j)\} =$ (by the base change isomorphism) $R^t_{\pi_1*}\{\mathcal{O}_{\mathbb{P}W_i^-}(-q_i+j) \otimes \pi_1^*R^s_{\pi_2*}\mathcal{O}_{\mathbb{P}W_i^+}(+j)\}$ $=$
$R^t_{\pi_1*}\{\mathcal{O}_{\mathbb{P}W_i^-}(-q_i+j)\} \otimes R^s_{\pi_2*}\{\mathcal{O}_{\mathbb{P}W_i^+}(+j)\}$.
Since $j \geq 0$, for $s > 0$ one has $R^s_{\pi_2*}\{\mathcal{O}_{\mathbb{P}W_i^+}(+j)\} = 0$ and for $s = 0$ $R^s_{\pi_2*}\{\mathcal{O}_{\mathbb{P}W_i^+}(+j)\} = S^j(W_i^+)^\vee$.
Since $-q_i + j < 0$ we have that $R^t_{\pi_1*}\{\mathcal{O}_{\mathbb{P}W_i^-}(-q_i+j)\} = 0$ unless $t = \text{rank}(W_i^-) - 1 = i-1$ in which case
$R^{i-1}_{\pi_1*}\{\mathcal{O}_{\mathbb{P}W_i^-}(-q_i+j)\} = \wedge^i W_i^- \otimes S^{q_i-j-i}(W_i^-)$.
It follows that

$$R^{i-1}_{\pi*}(L_i^m(-q_i+j,+j)) = L_i^m \otimes \wedge^i W_i^- \otimes S^{q_i-j-i}(W_i^-) \otimes S^j(W_i^+)^\vee. \tag{5.12}$$

By the Leray spectral sequence(the usual one) one has

$$H^s(E_i, L_i^m(-q_i+j,+j)) = H^{s-(i-1)}(S^iX, R^{i-1}_{\pi *}(L_i^m(-q_i+j,+j))) \tag{5.13}$$

and the rest is a rather straightforward verification. One can write

$$\begin{aligned} &L(h, E_i, L_i^m(-q_i+j,+j)) \\ &\quad= \sum_s (-1)^s trace\left(h_| H^{s-(i-1)}(S^iX, R^{i-1}_{\pi *}(L_i^m(-q_i+j,+j)))\right) \\ &\quad= (-1)^{i-1} L\left(h, S^iX, R^{i-1}_{\pi *}(L_i^m(-q_i+j,+j))\right). \end{aligned} \tag{5.14}$$

Multiplying (5.11) by -1 and using (5.14) we have

$$\begin{aligned} &L(h, \widetilde{M}_i, \mathscr{O}_i(m,n)) - L(h, \widetilde{M}_i, \mathscr{O}_{i-1}(m,n)) \\ &\quad= -\left[L(h, \widetilde{M}_i, \mathscr{O}_{i-1}(m,n)) - L(h, \widetilde{M}_i, \mathscr{O}_i(m,n))\right] = \\ &\quad= (-1)^i \sum_{j=0}^{q_i-1} L\left(h, S^iX, R^{i-1}_{\pi *}(L_i^m(-q_i+j,+j))\right) \end{aligned}$$

from (5.12) this is

$$= (-1)^i \sum_{j=0}^{q_i-1} L\left(h, S^iX, L_i^m \otimes \wedge^i W_i^- \otimes S^{q_i-j-i}(W_i^-) \otimes S^j(W_i^+)^\vee\right)$$

and if $j > q_i - i$ then $S^{q_i-j-i}(W_i^-) = 0$ so in the last equation the sum becomes a sum that runs only from $j=0$ to $j = q_i - i$, that is

$$\begin{aligned} &= (-1)^i \sum_{j=0}^{q_i-i} L\left(h, S^iX, L_i^m \otimes \wedge^i W_i^- \otimes S^{q_i-j-i}(W_i^-) \otimes S^j(W_i^+)^\vee\right) \\ &= (-1)^i L\left(h, S^iX, L_i^m \otimes \wedge^i W_i^- \otimes \left\{\bigoplus_{j=0}^{q_i-i} S^{q_i-j-i}(W_i^-) \otimes S^j(W_i^+)^\vee\right\}\right) \\ &\qquad \left(\text{using } \bigoplus_{j=0}^{q_i-i} S^{q_i-j-i}(W_i^-) \otimes S^j(W_i^+)^\vee = S^{q_i-i}\left\{W_i^- \oplus (W_i^+)^\vee\right\}\right) \\ &= (-1)^i L\left(h, S^iX, L_i^m \otimes \wedge^i W_i^- \otimes S^{q_i-i}\left\{W_i^- \oplus (W_i^+)^\vee\right\}\right) \\ &= (-1)^i L\left(h, S^iX, L_i^m \otimes \wedge^i W_i^- \otimes S^{q_i-i} U_i\right) \\ &= (-1)^i L\left(h, S^iX, B_{i,m,n}\right) \\ &= (-1)^i N_i(h). \end{aligned}$$

□

Lemma 5.4. *For $i > b$, $N_i(h) = 0$.*

Proof. This is the same proof of (6.8) in [32] since one shows that $i > b$ implies $q_i - i < 0$ so $B_{i,m,n} = 0$. □

Finally, from Lemmas 5.1–5.4, one has

$$\begin{aligned}\sum_{i=1}^{w}(-1)^i N_i(h) &= L(h, M_b, \mathscr{O}_b(m,n)) - L(h, M_0, \mathscr{O}_0(m,n)) \\ &= Trace(h_{|_{V_{m,n}}}) - N_0(h).\end{aligned}$$

Therefore $Trace(h_{|_{V_{m,n}}}) = \sum_{i=0}^{w}(-1)^i N_i(h)$.

6. The Chern classes

Let h be an automorphism of the curve X and assume that h has order $p \neq 1$. We shall explain below that the k-dimensional components of fixed points of h in the symmetric product S^iX are parametrized by certain kind of h-invariant divisors so we represent such a component by Z_D, where D is the corresponding invariant divisor. Let ι_D denote the inclusion $Z_D \subset S^iX$. Consider the decompositions into eigenbundles

$$\iota_D^* W_i^- = \bigoplus_{j=1}^{p} \iota_D^* W_i^-(\nu^j) \tag{6.1}$$

and

$$\iota_D^* W_i^+ = \bigoplus_{j=1}^{p} \iota_D^* W_i^+(\nu^j). \tag{6.2}$$

For the proof of Theorem 8.1 in Sect. 8, we need to know the Chern classes of all these eigenbundles and before we compute them we recall from [23] Section 3, that a k-dimensional component of fixed points of h in S^iX is isomorphic to the symmetric product S^kY, here Y is the quotient curve $X/\langle h\rangle$. These components are parametrized by a set of certain kind of h-invariant divisors A_k of degree $d_k = i - pk$, hence the notation Z_D. More precisely, define A_k as the set of divisors $D \in \left(S^{d_k}X\right)^h$ satisfying the following property: if $x \in X$ is a point in the support of D then $D - \sum_{j=0}^{p-1} h^j x$ is not an effective divisor nor the zero divisor. For each $D \in A_k$ there is an embedding

$$\iota_D : S^kY \overset{\iota}{\hookrightarrow} S^{pk}X \overset{\mathscr{A}_D}{\hookrightarrow} S^{pk+d_k}X \tag{6.3}$$

where ι sends $P \in S^kY$ to the divisor $f^*P \in S^{pk}X$ ($f : X \to Y = X/\langle h\rangle$ is the quotient map) and $\mathscr{A}_D$ sends $P \in S^{pk}X$ to $P + D \in S^{pk+d_k}X$.
Then, the Chern classes of our eigenbundles can be expressed in terms of the cohomology classes θ, x and $\sigma_i \in H^2(S^kY, \mathbb{Z})$ (see [22] for details on cohomology of symmetric products), where x represents the class of a divisor $q + S^{k-1}Y \subset S^kY$ in $H^2(S^kY,\mathbb{Z})$ and θ represents the class of the pull back of the theta divisor class $\Theta \in H^2(J_Y,\mathbb{Z})$ of the Jacobian J_Y of the curve Y under the Abel Jacobi map. We recall some relations of these cohomology classes:

$$\theta = \sum_{i=1}^{g_Y} \sigma_i,\ \sigma_i\sigma_j = \sigma_j\sigma_i,\ \text{and } \sigma_i^2 = 0. \tag{6.4}$$

If $0 \le a \le g_Y$ and $0 \le d$, then

$$\sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_a}x^d = x^{a+d}, \text{ for distinct } i_1, i_2, \dots, i_a. \text{ Also} \tag{6.5}$$

$$\theta^a x^d = a!\binom{g_Y}{a}x^{a+d} \text{ and} \tag{6.6}$$

$$e^{z\theta} = \prod_{i=1}^{g_Y}(1+z\sigma_i). \tag{6.7}$$

The Todd class $Td(Z_D)$ of a k-dimensional component is given by

$$Td(Z_D) = Td(S^kY) = \left(\frac{x}{1-e^{-x}}\right)^{k-g_Y+1} exp\left(\frac{\theta}{e^x-1} - \frac{\theta}{x}\right), \tag{6.8}$$

see for instance (7.3) in [32], there $\sigma = \theta$ and $\eta = x$.

Let $D \in A_k$. We will consider as well the following decomposition into eigenbundles on Y

$$f_*(\xi^s(-nD)) = \bigoplus_{j=0}^{p-1}\lambda_{s,j,n}, \tag{6.9}$$

here $\lambda_{s,j,n} := f_*(\xi^s(-nD))(\nu^j)$. We have the following

Theorem 6.1. *Let Z_D be a k-dimensional component of fixed points of h in S^iX. Let $m_{j,1}, m_{j,2}, m'_{j,n}$ denote the degrees of the bundles $\lambda_{1,j,1}, \lambda_{1,j,2}$ and $\lambda_{-1,j,n}$, respectively, in formula (6.9). Let g_Y be the genus of the quotient curve Y. Then,*

(a) *For the eigenbundles in (6.2), their corresponding Chern characters and classes are given by:*

$$ch(\iota_D^* W_i^+(\nu^j)) = -e^{2x}(1+m'_{j,-2}-(-2k+g_Y+4\theta)) \tag{6.10}$$

and

$$c(\iota_D^* W_i^+(\nu^j)) = \frac{e^{\frac{4\theta}{1+2x}}}{(1+2x)^{(1+m'_{j,-2}+2k-g_Y)}}. \tag{6.11}$$

(b) *For the eigenbundles in equation (6.1) we have:*

$$\begin{aligned} ch(\iota_D^* W_i^-(\nu^j)) &= e^{-x}(1+m_{j,1}-(k+g_Y+\theta)) \\ &\quad -e^{-2x}(1+m_{j,2}-(2k+g_Y+4\theta)) \end{aligned} \tag{6.12}$$

and

$$c(\iota_D^* W_i^-(\nu^j)) = \frac{(1-x)^{1+m_{j,1}-k-g_Y}}{(1-2x)^{1+m_{j,2}-2k-g_Y}}e^{-\frac{\theta}{1-x}+\frac{4\theta}{1-2x}}. \tag{6.13}$$

In the diagrams (6.14), (6.16)and (6.18) below we introduce notation for some morphisms that appear in the proof of Theorem 6.1 and of Lemma 6.1. In the diagram (6.14) ρ_{S^kY} and π_{S^iX} are the natural projections, ι_D is the embedding (6.3) corresponding to the component Z_D, j' stands for the embedding of the universal divisor Δ_i of S^iX,

$$\overline{\Delta} := (Id_X \times \iota_D)^*\Delta_i$$

and β' is the corresponding embedding into $X \times S^kY$.

$$
\begin{array}{ccc}
\overline{\Delta} & & \Delta_i \\
\beta' \big\downarrow & & \big\downarrow j' \\
X \times S^kY & \xrightarrow{\;Id_X \times \iota_D\;} & X \times S^iX \\
\rho_{S^kY} \big\downarrow & & \big\downarrow \pi_{S^iX} \\
S^kY & \xrightarrow{\;\iota_D\;} & S^iX,
\end{array}
\tag{6.14}
$$

According to (6.3), we have

$$(Id_X \times \iota_D) = (Id_X \times \mathscr{A}_D) \circ (Id_X \times \iota) \tag{6.15}$$

and the diagram (6.14) can be subdivided as in (6.16) below

$$
\begin{array}{ccccc}
 & & \Delta_{pk} & & \\
 & & \big\downarrow & & \\
X \times S^kY & \xrightarrow{\;Id_X \times \iota\;} & X \times S^{pk}X & \xrightarrow{\;Id_X \times \mathscr{A}_D\;} & X \times S^iX \\
\rho_{S^kY} \big\downarrow & & \big\downarrow \mathsf{p}_{S^{pk}X} & & \big\downarrow \pi_{S^iX} \\
S^kY & \xrightarrow[\;\iota\;]{} & S^{pk}X & \xrightarrow[\;\mathscr{A}_D\;]{} & S^iX
\end{array}
\tag{6.16}
$$

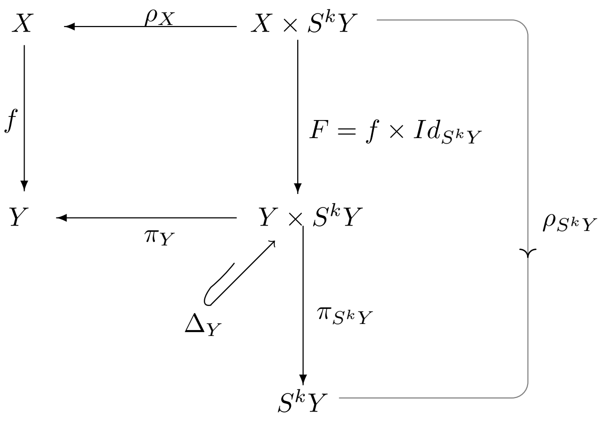
We will also consider the universal divisor Δ_{pk} of $S^{pk}X$ and the projections

$\pi_X : X \times S^iX \to X$,
$\mathsf{p}_X : X \times S^{pk}X \to X$ and
$\rho_X : X \times S^kY \to X$.

The diagram (6.18) involves the projection ρ_{S^kY} which is decomposed as $\pi_{S^kY} \circ F$, where $F = f \times Id_{S^kY}$ and π_{S^kY} is the projection $Y \times S^kY \mapsto S^kY$. We will often use the line bundle

$\bar{\xi} = \rho_X^* \xi$ on $X \times S^k Y$.
For example, we have

$$(Id_X \times \iota_D)^* (\xi^{-1}(2\Delta_i)) \cong \bar{\xi}^{-1}(2\overline{\Delta}). \tag{6.17}$$



(6.18)

Lemma 6.1. *Let Δ_Y be the universal divisor of $S^k Y$. Consider the line bundles $\bar{\xi} = {\rho_X}^* \xi$ on $X \times S^k Y$ and $\mathscr{L}_{s,j,n} = {\pi_Y}^* \lambda_{s,j,n}$ on $Y \times S^k Y$. Then we have for $\ell \geq 0$*

$$R^\ell \rho_{S^k Y *}(\bar{\xi}^s(-n\overline{\Delta})) \cong \bigoplus_{j=0}^{p-1} R^\ell \pi_{S^k Y *}(\mathscr{L}_{s,j,n}(-n\Delta_Y)). \tag{6.19}$$

In particular, the ν^j-eigenbundles of $R^\ell \rho_{S^k Y *}(\bar{\xi}^s(-n\overline{\Delta}))$ and $R^0_{F*}\{\bar{\xi}^s(-n\overline{\Delta})\}$ are given by

$$R^\ell \rho_{S^k Y *}(\bar{\xi}^s(-n\overline{\Delta}))(\nu^j) \cong R^\ell \pi_{S^k Y *}(\mathscr{L}_{s,j,n}(-n\Delta_Y)) \text{ and} \tag{6.20}$$

$$R^0_{F*}\{\bar{\xi}^s(-n\overline{\Delta})\}(\nu^j) \cong \mathscr{L}_{s,j,n}(-n\Delta_Y). \tag{6.21}$$

Proof. Since $\rho_{S^k Y} = \pi_{S^k Y} \circ F$, ($F = f \times Id_{S^k Y}$) and F has finite fibres one can write

$$R^\ell \rho_{S^k Y *} \bar{\xi}^s(-n\overline{\Delta}) \cong R^\ell_{\pi_{S^k Y} *}(R^0_{F*}\{\bar{\xi}^s(-n\overline{\Delta})\}). \tag{6.22}$$

Next, we shall write $\overline{\Delta}$ in terms of the universal divisor Δ_{pk} of $S^{pk} X$ and of the divisor D (see formula (6.24) below). We have (by definition of $\overline{\Delta}$ and equation (6.15))

$$\overline{\Delta} = (Id_X \times \iota_D)^* \Delta_i = (Id_X \times \iota)^* (Id_X \times \mathscr{A}_D)^* (\Delta_i) \tag{6.23}$$

and by the universal property of Δ_i applied to the right-hand side square of the diagram (6.16) one has that $(Id_X \times \mathscr{A}_D)^*(\Delta_i)$ is the relative divisor of degree i inducing $\mathscr{A}_D$. So we have

$$(Id_X \times \mathscr{A}_D)^*(\Delta_i) = \Delta_{pk} + \mathsf{p}_X^* D$$

and using this in (6.23), we get

$$\overline{\Delta} = (\rho_X)^* D + (Id_X \times \iota)^* \Delta_{pk}. \tag{6.24}$$

Now, using (6.24) in (6.22) and the fact that

$$(Id_X \times \iota)^* \Delta_{pk} = F^* \Delta_Y$$

we have

$$\begin{aligned} R^\ell \rho_{S^kY*} \bar{\xi}^s(-n\overline{\Delta}) &\cong R^\ell_{\pi_{S^kY}*}(R^0_{F*}\{\bar{\xi}^s(-n\rho_X^* D - nF^*\Delta_Y)\}) \\ &= R^\ell_{\pi_{S^kY}*}(R^0_{F*}\{\bar{\xi}^s(-n\rho_X^* D) \otimes \mathscr{O}_{X\times S^kY}(-nF^*\Delta_Y)\}). \end{aligned}$$

Applying the projection formula to the direct image R^0_{F*} one gets

$$R^\ell \rho_{S^kY*} \bar{\xi}^s(-n\overline{\Delta}) \cong R^\ell_{\pi_{S^kY}*}(\mathscr{O}_{Y\times S^kY}(-n\Delta_Y) \otimes R^0_{F*}\{\rho_X^*(\xi^s(-nD)\}).$$

Now we use the following base change isomorphism induced from the left-hand side square of (6.18)

$$R^0_{F*}\{\rho_X^*(\xi^s(-nD))\} \cong \pi_Y^* f_*(\xi^s(-nD)) \tag{6.25}$$

to get

$$R^\ell \rho_{S^kY*} \bar{\xi}^s(-n\overline{\Delta}) \cong R^\ell_{\pi_{S^kY}*}(\mathscr{O}_{Y\times S^kY}(-n\Delta_Y) \otimes \pi_Y^* f_*(\xi^s(-nD)))$$

and using the decomposition into eigenbundles (6.9) we get

$$\begin{aligned} R^\ell \rho_{S^kY*} \bar{\xi}^s(-n\overline{\Delta}) &\cong R^\ell_{\pi_{S^kY}*}(\mathscr{O}_{Y\times S^kY}(-n\Delta_Y) \otimes \pi_Y^*(\bigoplus_{j=0}^{p-1} \lambda_{s,j,n})) \\ &= R^\ell_{\pi_{S^kY}*}(\bigoplus_{j=0}^{p-1} \mathscr{O}_{Y\times S^kY}(-n\Delta_Y) \otimes \mathscr{L}_{s,j,n}) \\ &= R^\ell_{\pi_{S^kY}*}(\bigoplus_{j=0}^{p-1} \mathscr{L}_{s,j,n}(-n\Delta_Y)). \end{aligned}$$

Notice that given $f : X \to Y$ with X Noetherian, the higher direct images commute with direct sums namely $R^j f_*(\bigoplus_i \mathscr{F}_i) = \bigoplus_i R^j f_*(\mathscr{F}_i)$. So we have

$$R^\ell_{\pi_{S^kY}*}(\bigoplus_{j=0}^{p-1} \mathscr{L}_{s,j,n}(-n\Delta_Y)) = \bigoplus_{j=0}^{p-1} R^\ell_{\pi_{S^kY}*}(\mathscr{L}_{s,j,n}(-n\Delta_Y)), \tag{6.26}$$

from which the Lemma follows. □

Lemma 6.2. *Let Y be an irreducible non singular projective curve of genus g_Y. Consider the projection $\pi_{S^kY} : Y \times S^kY \to S^kY$ in (6.18) and the universal divisor $\Delta_Y \subset Y \times S^kY$. For any line bundle M on Y and any $m \in \mathbb{Z}$ we have the following Chern character*

$$ch\ \left((\pi_{S^kY})_! \pi_Y^* M(m\Delta_Y)\right) = ((deg\ M + mk + 1 - g_Y) - m^2\theta)e^{mx}, \tag{6.27}$$

where $\pi_Y^* M(m\Delta_Y) = (\pi_Y^* M) \otimes \mathscr{O}_{Y\times S^kY}(m\Delta_Y)$.

Proof. This is (7.4) in [32] setting $X = Y$, $g = g_Y$, $i = k$, $\pi = \pi_{S^kY}$ (for Thaddeus $\pi : X\times S^iX \to S^iX$), $k = m \in \mathbb{Z}$ and also recall that for Thaddeus $\sigma = \theta$ and $\eta = x$. □

Proof of Theorem 6.1. **(a)** We first notice that

$$\iota_D^* W_i^+ \cong R^1_{\rho_{S^kY}*}(Id_X \times \iota_D)^*(\xi^{-1}(2\Delta_i)). \tag{6.28}$$

By definition (equation (4.2)) we have $\iota_D^* W_i^+ = \iota_D^*(R^1_{\pi_{S^iX}})_*\xi^{-1}(2\Delta_i)$ then (6.28) follows because under our conditions the natural base change morphism

$$\iota_D^*(R^j_{\pi_{S^iX}})_*\xi^{-1}(2\Delta_i) \mapsto R^j_{\rho_{S^kY}*}(Id_X \times \iota_D)^*(\xi^{-1}(2\Delta_i)), \tag{6.29}$$

induced from diagram (6.14), is an isomorphism $\forall\, j \geq 0$. That is, from Corollary 2 pg. 50 in [26] we see that the higher direct images $(R^j_{\pi_{S^iX}})_*\xi^{-1}(2\Delta_i)$, $j \geq 0$, are locally free sheaves and that for any $y \in S^iX$ the natural maps

$$\phi_y^j : (R^j_{\pi_{S^iX}})_*\xi^{-1}(2\Delta_i) \otimes k(y) \mapsto H^j((X \times S^iX)_y, \xi^{-1}(2\Delta_i)_y)$$

are isomorphisms (in fact, the only ones that are non-zero are $(R^1_{\pi_{S^iX}})_*\xi^{-1}(2\Delta_i)$ and ϕ_y^1). So the isomorphism in (6.29) follows as a particular case of Corollary 6.9.9.2 in [18] or from Theorem 2.1 in [11].
Since (by equation (6.17))

$$R^1_{\rho_{S^kY}*}(Id_X \times \iota_D)^*(\xi^{-1}(2\Delta_i)) \cong R^1_{\rho_{S^kY}*}(\bar{\xi}^{-1}(2\overline{\Delta})),$$

we have for the ν^j-eigenbundles

$$R^1_{\rho_{S^kY}*}(Id_X \times \iota_D)^*(\xi^{-1}(2\Delta_i))(\nu^j) \cong R^1_{\rho_{S^kY}*}(\bar{\xi}^{-1}(2\overline{\Delta}))(\nu^j). \tag{6.30}$$

Now by equation (6.28) the left-hand side of (6.30) is $\iota_D^* W_i^+(\nu^j)$ and the right-hand side is $R^1_{\pi_{S^kY}*}(\mathscr{L}_{-1,j,-2}(2\Delta_Y))$ by Lemma 6.1 (equation (6.20) taking n=-2, s=-1) . That is

$$\iota_D^* W_i^+(\nu^j) \cong R^1_{\pi_{S^kY}*}(\mathscr{L}_{-1,j,-2}(2\Delta_Y)), \tag{6.31}$$

so that using the Grothendieck-Riemann-Roch Theorem one can compute the Chern characters $ch(\iota_D^* W^+(\nu^j))$. In fact, $ch(\iota_D^* W^+(\nu^j))$ can be derived from Lemma 6.2, that is, consider $\mathscr{L}_{s,j,n}(2\Delta_Y) = \pi_Y^*\lambda_{s,j,n} \otimes \mathscr{O}_{Y\times S^kY}(2\Delta_Y)$ so taking $M = \lambda_{s,j,n}$ we have $deg\ M = deg\ \lambda_{s,j,n}$ and in our case $s = -1$ and $n = -2$ so $deg\ M = m'_{j,-2}$ ($m'_{j,n}$ as defined in the statement of Theorem 6.1) and so taking $m = 2$ Lemma 6.2 tell us that

$$\begin{aligned} &ch(R^0_{\pi_{S^kY}*}(\mathscr{L}_{-1,j,-2}(2\Delta_Y)) - R^1_{\pi_{S^kY}*}(\mathscr{L}_{-1,j,-2}(2\Delta_Y))) \\ &\quad = ((m'_{j,-2} + 2k + 1 - g_Y) - 4\theta)e^{2x}, \end{aligned}$$

we will see below that $R^0_{\pi_{S^kY}*}(\mathscr{O}_{Y\times S^kY}(2\Delta_Y) \otimes \pi_Y^*\lambda_{-1,j,-2}) = 0$ so we have that

$$ch(R^1_{\pi_{S^kY}*}(\mathscr{L}_{-1,j,-2}(2\Delta_Y)))) = -e^{2x}(1 + m'_{j,-2} - (-2k + g_Y + 4\theta)),$$

that is

$$ch(\iota_D^* W_i^+(\nu^j)) = -e^{2x}(1 + m'_{j,-2} - (-2k + g_Y + 4\theta)). \tag{6.32}$$

Now $R^0_{\pi_{S^kY}*}(\mathscr{O}_{Y\times S^kY}(2\Delta_Y)\otimes \pi_Y^*\lambda_{-1,j,-2}) = 0$ because if

$$0 = R^0_{\rho_{S^kY*}}(\bar{\xi}^s(-n\overline{\Delta}))$$

then $R^0_{\pi_{S^kY}*}(\mathscr{O}_{Y\times S^kY}(2\Delta_Y)\otimes \pi_Y^*\lambda_{-1,j,-2}) = 0$ because by Lemma 6.1 we know that

$$R^0_{\rho_{S^kY*}}(\bar{\xi}^s(-n\overline{\Delta})) \cong \bigoplus_{j=0}^{p-1} R^0_{\pi_{S^kY*}}(\mathscr{L}_{s,j,n}(-n\Delta_Y)).$$

Now we have (again by equation (6.17)) that

$$R^0_{\rho_{S^kY}*}(Id_X\times\iota_D)^*(\xi^{-1}(2\Delta_i)) \cong R^0_{\rho_{S^kY}*}(\bar{\xi}^{-1}(2\overline{\Delta})),$$

and as we mention in the proof of (6.29)

$$R^0_{\rho_{S^kY}*}(Id_X\times\iota_D)^*(\xi^{-1}(2\Delta_i)) = 0.$$

On the other hand, using (6.7) one has the factorization

$$\frac{e^{\frac{4\theta}{1+2x}}}{(1+2x)^{(1+m'_{j,-2}+2k-g_Y)}} = (1+2x)^{(r''-g_Y)}\cdot\prod_{i=1}^{g_Y}(1+4\sigma_i+2x), \tag{6.33}$$

where $r'' = -1 - m_{j,-2} - 2k + g_Y$. So (6.33) can be seen as the Chern class $c(L^{\oplus(r''-g_Y)}\oplus(L\otimes E))$, where L is a line bundle with Chern class $1+2x$ and E is a rank g_Y vector bundle with $c(E) = e^{4\theta}$. Now we see, by properties of the Chern character and (6.32), that

$$\begin{aligned} ch(L^{\oplus(r''-g_Y)}\oplus(L\otimes E)) &= ch(L^{\oplus(r''-g_Y)}) + ch(L)ch(E)\\ &= e^{2x}(r''-g_Y) + e^{2x}(g_Y+4\theta)\\ &= ch(\iota_D^*W_i^+(\nu^j)).\end{aligned}$$

So we have $c(L^{\oplus(r''-g_Y)}\oplus(L\otimes E)) = c(\iota_D^*W_i^+(\nu^j))$.
proof of b):

Using the definition of W_i^- (equation (4.1)), we have

$$\iota_D^*W_i^- = \iota_D^*(R^0_{\pi_{S^iX}})_*\mathscr{E}$$

where

$$\mathscr{E} = \{j'_*\mathscr{O}_{\Delta_i}\}\otimes\pi_X^*(\xi)\otimes\mathscr{O}_{X\times S^iX}(-\Delta_i)$$

is a flat sheaf on $X\times S^iX$ because $(R^0_{\pi_{S^iX}})_*\mathscr{E} = W_i^-$ is a locally free sheaf on S^iX. One can argue similarly as in the proof of a) to get the following base change isomorphism induced by the diagram (6.14)

$$\iota_D^*(R^0_{\pi_{S^iX}})_*\mathscr{E} \cong (R^0_{\rho_{S^kY}})_*(Id_X\times\iota_D)^*\mathscr{E}$$

that is

$$\iota_D^*W_i^- \cong (R^0_{\rho_{S^kY}})_*(Id_X\times\iota_D)^*\mathscr{E}.$$

One can see that

$$(Id_X \times \iota_D)^* \mathscr{E} \cong \beta'_* \{\mathscr{O}_{\bar{\Delta}} \bar{\xi}(-\overline{\Delta})\} \tag{6.34}$$

(by using the base change isomorphism $(Id_X \times \iota_D)^* j'_* \mathscr{O}_{\Delta_i} \cong \beta'_* \kappa^* \mathscr{O}_{\Delta_i}$ induced by the square diagram obtained on the top of the diagram (6.14) by adding the natural projection $\kappa : \overline{\Delta} \to \Delta_i$). Then

$$\begin{aligned} \iota_D^* W_i^- &\cong (R^0_{\rho_{S^kY}})_* \beta'_* \{\mathscr{O}_{\bar{\Delta}} \bar{\xi}(-\overline{\Delta})\} \\ &\quad \text{using the projection formula the last is} \\ &\cong (R^0_{\rho_{S^kY}})_* \{\{\beta'_* \mathscr{O}_{\bar{\Delta}}\} \otimes \bar{\xi}(-\overline{\Delta})\} \\ &\quad \text{using } \ (R^0 \rho_{S^kY})_* = (R^0 \pi_{S^kY})_* (R^0_F)_* \quad \text{the last becomes} \\ &\cong R^0_{\pi_{S^kY}*} (R^0_{F*} \{\{\beta'_* \mathscr{O}_{\bar{\Delta}}\} \otimes \bar{\xi}(-\overline{\Delta})\}). \end{aligned} \tag{6.35}$$

Consider the exact sequence of sheaves on $X \times S^kY$

$$0 \to \bar{\xi}(-2\overline{\Delta}) \to \bar{\xi}(-\overline{\Delta}) \to \bar{\xi} \otimes \beta'_* \mathscr{O}_{\bar{\Delta}}(-\overline{\Delta}) \to 0. \tag{6.36}$$

Since F has finite fibres we get an exact sequence of sheaves on $Y \times S^kY$

$$0 \to R^0_{F*} \bar{\xi}(-2\overline{\Delta}) \to R^0_{F*} \bar{\xi}(-\overline{\Delta}) \to R^0_{F*}(\bar{\xi} \otimes \beta'_* \mathscr{O}_{\bar{\Delta}}(-\overline{\Delta})) \to 0, \tag{6.37}$$

which induces, for each $j = 0, 1 \cdots p-1$, an exact sequence of ν^j-eigen sheaves

$$0 \to R^0_{F*} \bar{\xi}(-2\overline{\Delta})(\nu^j) \to R^0_{F*} \bar{\xi}(-\overline{\Delta})(\nu^j) \to \{R^0_{F*}(\bar{\xi} \otimes \beta'_* \mathscr{O}_{\bar{\Delta}}(-\overline{\Delta}))\}(\nu^j) \to 0. \tag{6.38}$$

We continue by making the following observations.

(1) Let

$$\mathscr{F} = R^0_{F*} \{\{\beta'_* \mathscr{O}_{\bar{\Delta}}\} \otimes \bar{\xi}(-\overline{\Delta})\}.$$

Notice that $\mathscr{F} \cong R^0_{F*}(\bar{\xi} \otimes \beta'_* \mathscr{O}_{\bar{\Delta}}(-\overline{\Delta}))$ (by the projection formula) is the non-zero term on the right-hand side of the exact sequence of (6.37).

(2) We have $\mathscr{F} = \bigoplus_j \mathscr{F}(\nu^j)$ and then $R^0_{\pi_{S^kY}*} \mathscr{F} = \bigoplus_j R^0_{\pi_{S^kY}*} \{\mathscr{F}(\nu^j)\}$. Since $R^0_{\pi_{S^kY}*} \{\mathscr{F}(\nu^j)\}$ inherits the action of h from $\mathscr{F}(\nu^j)$ we see that

$$\{R^0_{\pi_{S^kY}*} \mathscr{F}\}(\nu^j) = R^0_{\pi_{S^kY}*} \{\mathscr{F}(\nu^j)\}.$$

(3) From (6.35) $R^0_{\pi_{S^kY}*} \{\mathscr{F}\} \cong \iota_D^* W_i^-$ so we get that

$$\iota_D^* W_i^-(\nu^j) \cong R^0_{\pi_{S^kY}*} \{\mathscr{F}(\nu^j)\}$$

for $j = 0, \cdots, p-1$.

(4) Also notice that $\mathscr{F}(\nu^j) \cong \{R^0_{F*}(\bar{\xi} \otimes \beta'_* \mathscr{O}_{\bar{\Delta}}(-\overline{\Delta}))\}(\nu^j)$ is the non-zero term on the right-hand side of the exact sequence of (6.38).

(5) From (6.38) we have the following identity in the Grothendieck ring $K(Y \times S^kY)$

$$\mathscr{F}(\nu^j) = R^0_{F*} \bar{\xi}(-\overline{\Delta})(\nu^j) - R^0_{F*} \bar{\xi}(-2\overline{\Delta})(\nu^j).$$

So applying $\pi_{S^kY!} : K(Y \times S^kY) \to K(S^kY)$ followed by the Chern character ch one has that

$$\begin{aligned} &ch(\pi_{S^kY!} \{\mathscr{F}(\nu^j)\}) \\ &\quad = ch(\pi_{S^kY!} \{[R^0_{F*} \bar{\xi}(-\overline{\Delta})](\nu^j)\}) - ch(\pi_{S^kY!} \{[R^0_{F*} \bar{\xi}(-2\overline{\Delta})](\nu^j)\}). \end{aligned} \tag{6.39}$$

(6) For $s\geq 1$ we have

$$0=(R^s_{\rho_{S^kY}})_*\beta'_*\{\mathscr{O}_{\bar{\Delta}}\bar{\xi}(-\overline{\Delta})\}$$

similarly to how we arrive to (6.35) this is

$$\cong R^s_{\pi_{S^kY}*}(R^0_{F*}\{\{\beta'_*\mathscr{O}_{\bar{\Delta}}\}\otimes\bar{\xi}(-\overline{\Delta})\})$$
$$=R^s_{\pi_{S^kY}*}\{\mathscr{F}\}\cong R^s_{\pi_{S^kY}*}\{\bigoplus_j\mathscr{F}(\nu^j)\}\cong\bigoplus_j R^s_{\pi_{S^kY}*}\{\mathscr{F}(\nu^j)\},$$

and consequently $R^s_{\pi_{S^kY}*}\{\mathscr{F}(\nu^j)\}=0$.

With (3) and (6) in (6.39) above we see that

$$ch(\iota_D^*W_i^-(\nu^j))=ch((\pi_{S^kY})_!\{[R^0_{F*}\bar{\xi}(-\overline{\Delta})](\nu^j)\})-ch((\pi_{S^kY})_!\{[R^0_{F*}\bar{\xi}(-2\overline{\Delta})](\nu^j)\}),$$
and using Lemma 6.1 the last is
$$=ch((\pi_{S^kY})_!\{\mathscr{L}_{1,j,1}(-\Delta_Y)\})-ch((\pi_{S^kY})_!\{\mathscr{L}_{1,j,2}(-2\Delta_Y)\}). \tag{6.40}$$

Applying Lemma 6.2 to each character in (6.40) with $M=\lambda_{1,j,1}$ for the first one and $M=\lambda_{1,j,2}$ for the second one we have

$$\begin{aligned}ch(i_D^*W_i^-(\nu^j))=e^{-x}(1+m_{j,1}-(k+g_Y+\theta))\\ -e^{-2x}(1+m_{j,2}-(2k+g_Y+4\theta)).\end{aligned} \tag{6.41}$$

(7) For the calculation of the Chern class we write

$$ch(i_D^*W_i^-(\nu^j))=e^{-x}(r-\theta))+e^{-2x}(r'+4\theta), \tag{6.42}$$

where $r=1+m_{j,1}-k-g_Y$ and $r'=-1-m_{j,2}+2k+g_Y$. Assume $r,r'\geq g_Y$. Then

$$ch(i_D^*W_i^-(\nu^j))=ch((L_1\otimes E_1)\oplus(L_2\otimes E_2)),$$

where L_1 and L_2 are line bundles with Chern classes $1-x$ and $1-2x$ respectively and E_1 and E_2 are vector bundles with Chern characters $r-\theta$ and $r'+4\theta$ respectively. So,

$$c(i_D^*W_i^-(\nu^j))=c(L_1\otimes E_1)\cdot c(L_2\otimes E_2).$$

From (6.4), one can assume that the non-zero Chern roots of E_1 and E_2 are $-\sigma_1,\dots,-\sigma_{g_Y}$ and $4\sigma_1,\dots,4\sigma_{g_Y}$ respectively, so that

$$\begin{aligned}c(i_D^*W_i^-(\nu^j))=((1-x)^{r-g_Y}\prod_{i=1}^{g_Y}(1-x-\sigma_i))\\ \cdot((1-2x)^{r'-g_Y}\prod_{i=1}^{g_Y}(1-2x+4\sigma_i))\end{aligned} \tag{6.43}$$

and using (6.7) the last is

$$=(1-x)^r e^{\frac{-\theta}{1-x}}(1-2x)^{r'}e^{\frac{4\theta}{1-2x}}. \tag{6.44}$$

In case $r,r'\ngeq g_Y$ one argues similarly by choosing integers s,s' so that $r+s,r'+s'\geq g_Y$, then adding $e^{-x}s+e^{2x}s'$ to both sides of (6.42) one has

$$ch(i_D^*W_i^-(\nu^j))+e^{-x}(s)+e^{2x}(s')=e^{-x}(r+s-\theta))+e^{-2x}(r'+s'+4\theta),$$

which can be seen as

$$ch(i_D^*W_i^-(\nu^j)\oplus L_1^{\oplus s}\oplus L_2^{\oplus s'}) = ch((L_1\otimes F_1)\oplus(L_2\otimes F_2)),$$

where F_1, F_2 have ranks $r+s$ and $r+s'$ respectively. So

$$c(i_D^*W_i^-(\nu^j)\oplus L_1^{\oplus s}\oplus L_2^{\oplus s'}) = c((L_1\otimes F_1)\oplus(L_2\otimes F_2)),$$

and by (6.44) we can compute the right-hand side, namely

$$c(i_D^*W_i^-(\nu^j)\oplus L_1^{\oplus s}\oplus L_2^{\oplus s'}) = (1-x)^{r+s}e^{\frac{-\theta}{1-x}}(1-2x)^{r'+s'}e^{\frac{4\theta}{1-2x}},$$

and we also get

$$\begin{aligned} c(i_D^*W_i^-(\nu^j)) &= (1-x)^{r+s}e^{\frac{-\theta}{1-x}}(1-2x)^{r'+s'}e^{\frac{4\theta}{1-2x}}(c(L_1^{\oplus s}))^{-1}(c((L_2^{\oplus s'}))^{-1} \\ &= (1-x)^{r+s}e^{\frac{-\theta}{1-x}}(1-2x)^{r'+s'}e^{\frac{4\theta}{1-2x}}(1-x)^{-s}(1-2x)^{-s'} \quad (6.45) \\ &= (1-x)^{r}e^{\frac{-\theta}{1-x}}(1-2x)^{r'}e^{\frac{4\theta}{1-2x}}. \end{aligned}$$

□

7. Stable characteristic classes

Theorem 7.1 below is a generalization of Theorem 2.3 in [24] where the case $D=0$ is considered. The proof, which we have omitted here, can be done using similar arguments to those in the proof of Theorem 6.1 above.

Theorem 7.1. *Let Z_D be a k-dimensional component of fixed points of h in S^iX. Let n_j and n'_j be the degrees of the line bundles $\lambda_{0,j,-1}$ and $\lambda_{0,j,0}$ in formula (6.9) respectively. Then*

(a) $$ch(N_{Z_D/S^iX}(\nu^j)) = -(1+n'_j-g_Y)+e^x(1+n_j+k-g_Y-\theta), \quad (7.1)$$

(b) $$c(N_{Z_D/S^iX}(\nu^j)) = (1+x)^{1+n_j+k-g_Y}e^{-\frac{\theta}{1+x}}, \quad (7.2)$$

(c) $$U_j(N_{Z_D/S^iX}(\nu^j)) = \left(1-\frac{1}{\nu^j}\right)^{A}\left(1-\frac{e^{-x}}{\nu^j}\right)^{-A} exp\left(\frac{\theta e^{-x}}{\nu^j-e^{-x}}\right) \cdot\left(\frac{1-e^{-x}/\nu^j}{1-\nu^{-j}}\right)^{-n_j}, \quad (7.3)$$

(d) $$\prod_{j=1}^{p-1} U_j(N_{Z_D/S^iX}(\nu^j)) = p^A m(e^{-x})^{-A}e^{\theta q(e^{-x})}\prod_{j=1}^{p-1}\left(\frac{1-e^{-x}/\nu^j}{1-\nu^{-j}}\right)^{-n_j}, \quad (7.4)$$

where $A=k+1-g_Y$, $m(z)=\sum_{j=0}^{p-1}z^i$ and $q(z)=\frac{-zm'(z)}{m(z)}$.

In particular, we have

$$\det(Id-h_{|}N^{\vee}_{Z_D/S^iX}) = (1-\nu^{p-1})^{a_1}\cdots(1-\nu)^{a_{p-1}} \quad (7.5)$$

where

$$a_j = \mathrm{rank}(N_{Z_D/S^iX}(\nu^{p-j})) = (n_{p-j}-n'_{p-j}+k) \text{ by part a).}$$

8. The generalized Chern character for $B_{i,m,n}$

Let Z_D be a component of fixed points of the automorphism h. Let E and F be $h-$linearized vector bundles on Z_D. The generalized Chern character of E is given by

$$ch_h(E) = \sum_{j=0}^{p-1} \nu^j ch[E(\nu^j)],$$

where $ch[E(\nu^j)]$ is the Chern character of the eigenbundle $E(\nu^j)$. As in the case of the usual Chern character one has that

$$ch_h(E \otimes F) = ch_h(E) ch_h(F).$$

So, from equation (4.8) we have

$$ch_h(\iota_D^* B_{i,m,n}) = ch_h(\iota_D^* L_i^m) ch_h(\wedge^i \iota_D^* W_i^-) ch_h(\iota_D^* S^{q_i-(i)} U_i). \tag{8.1}$$

The factors on the right-hand side of (8.1) are given in the following result.

Theorem 8.1. *Let Z_D be a k-dimensional component of fixed points of h in S^iX. Let g_Y be the genus of the quotient curve Y. Let $d_k = i - pk$ be the degree of D. Let ν^l and $\nu^{l'}$ be the eigenvalues corresponding to the action of h on the line-bundles $\iota_D^* L_i$ and $\wedge^i \iota_D^* W_i^-$ respectively.*
We have the following

$$(a)\, ch_h(\iota_D^* L_i^m) = \nu^{lm} \cdot e^{m(d-2i)x+2mp\theta}. \tag{8.2}$$

$$(b)\, ch_h(\wedge^i \iota_D^* W_i^-) = \nu^{l'} \cdot e^{(d-3i+1-g_X)x+3p\theta}. \tag{8.3}$$

(c) Let $m_{j,n}$ and $m'_{j,n}$ denote respectively the degrees of the bundles $\lambda_{1,j,n}$ and $\lambda_{-1,j,n}$ in formula (6.9) .Then

$$\begin{aligned} &ch_h(\iota_D^* S^{q_i-i} U_i) = \\ &= \underset{t^{q_i-i}}{coef} \left[exp\left(\frac{-pt^p\theta}{e^{px}-t^p}\right) \cdot \frac{(1-t^p e^{-px})^{k+g_Y-1}}{(1-t^p e^{-2px})^{2g_Y-2}} \right. \\ &\left. \cdot \prod_{j=0}^{p-1} \left\{ \frac{(1-\nu^j t e^{-2x})^{m_{j,2}+m'_{p-j,-2}}}{(1-\nu^j t e^{-x})^{m_{j,1}}} \right\} \right]. \end{aligned} \tag{8.4}$$

Proof. Parts a) and b) follow from (7.5) in [32] and the restriction rules $\iota_D^*\theta = p\theta$ and $\iota_D^* x = x$, where we are using the same notation for the cohomology classes of the symmetric products S^iX and $S^kY \cong Z_D$ as introduced in Sect. 6 above (recall that for Thaddeus $\sigma = \theta$ and $\eta = x$).
For (c), let E be a rank r_E vector bundle on Z_D and let

$$P(E,t) := \sum_{l=0}^{\infty} ch[S^l(E)] \cdot t^l. \tag{8.5}$$

One has (see proof of (7.6) in [32])

$$P(E,t) = \prod_{\text{Chern roots } \alpha \text{ of } E,} \frac{1}{1-te^{\alpha}}. \tag{8.6}$$

Let F be an h-linearized vector bundle on Z_D and let

$$Q_h(F,t)=\sum_{l=0}^{\infty} ch_h(S^lF)\cdot t^l. \tag{8.7}$$

Since $S^lF=\bigoplus_{j=0}^{p-1}(S^lF)(\nu^j)$, a Chern root γ of S^lF is a Chern root of $(S^lF)(\nu^j)$ for some j, say $\gamma=\sum_{i=1}^{s}\beta_i\alpha_i$ where $\beta_i\geq 0$, $\sum_{i=1}^{s}\beta_i=l$ and α_i is a Chern root of $F(\nu^{j_i})$ for some integer j_i. Then $\nu^je^{\gamma}=m(\nu^{j_1}e^{\alpha_1},\ldots,\nu^{j_s}e^{\alpha_s})$ where m is the degree l monomial $m(x_1,\ldots,x_s)=\prod_{i=1}^{s}x_i^{\beta_i}$. So one has that

$$Q_h(F,t)=\prod_{\substack{\text{Chern roots } \alpha \text{ of } F(\nu^j),\\ j=0,\ldots,p-1.}}\frac{1}{1-\nu^jte^{\alpha}}, \tag{8.8}$$

from which one sees that

$$Q_h(F,t)=\prod_{j=0}^{p-1}P(F(\nu^j),\nu^jt). \tag{8.9}$$

Now we shall assume that $F=\iota_D^*U_i$ and compute $Q_h(F,t)$ using (8.9). Using (4.6) and taking Chern class, we have that

$$c(F(\nu^j))=c(\iota_D^*W_i^-(\nu^j))\cdot c((\iota_D^*W_i^+)^{\vee}(\nu^j)). \tag{8.10}$$

From equations (6.43) and (6.33), one has the following factorizations

$$\begin{aligned}
&c(\iota_D^*W_i^-(\nu^j))=\\
&=(1-x)^{(r-g_Y)}\cdot(1-2x)^{(r'-g_Y)}\prod_{i=1}^{g_Y}(1-\sigma_i-x)\cdot\prod_{i=1}^{g_Y}(1+4\sigma_i-2x)\\
&=\prod_{\text{Chern roots } \alpha \text{ of } \iota_D^*W_i^-(\nu^j)}(1+\alpha).
\end{aligned} \tag{8.11}$$

and

$$\begin{aligned}
c((\iota_D^*W_i^+)^{\vee}(\nu^j))&=(1-2x)^{(r''-g_Y)}\cdot\prod_{i=1}^{g_Y}(1-4\sigma_i-2x),\\
&=\prod_{\text{Chern roots } \alpha \text{ of } (\iota_D^*W_i^+)^{\vee}(\nu^j)}(1+\alpha).
\end{aligned} \tag{8.12}$$

where

$$\begin{aligned}
r&=1+m_{j,1}-k-g_Y,\\
r'&=-1-m_{j,2}+2k+g_Y \quad \text{and}\\
r''&=-1-m'_{p-j,-2}-2k+g_Y.
\end{aligned} \tag{8.13}$$

So from (8.10), (8.11) and (8.12) we have the Chern roots of $F(\nu^j)$ and we use them in (8.6) to compute

$$P(F(\nu^j),t) = \left(\frac{1}{1-te^{-x}}\right)^{r-g_Y} \cdot \prod_{i=1}^{g_Y}\left(\frac{1}{1-te^{-\sigma_i - x}}\right) \times$$
$$\times \left(\frac{1}{1-te^{-2x}}\right)^{r'-g_Y} \cdot \prod_{i=1}^{g_Y}\left(\frac{1}{1-te^{4\sigma_i-2x}}\right)\cdot\left(\frac{1}{1-te^{-2x}}\right)^{r''-g_Y}$$
$$\cdot \prod_{i=1}^{g_Y}\left(\frac{1}{1-te^{-4\sigma_i-2x}}\right).$$

Let $h(z) := \frac{1}{1-te^{-z}}$. Expanding the following around $\sigma_i = 0$ and using $\sigma_i^2 = 0$ one has, as the reader may check, that

$$h(\sigma_i + x) = \frac{1}{1-te^{-\sigma_i-x}} = h(x)\left(1+\sigma_i\frac{h'(x)}{h(x)}\right), \tag{8.14}$$

$$h(-4\sigma_i + 2x) = \frac{1}{1-te^{4\sigma_i-2x}} = h(2x)\left(1-4\sigma_i\frac{h'(2x)}{h(2x)}\right) \tag{8.15}$$

and

$$h(4\sigma_i + 2x) = \frac{1}{1-te^{-4\sigma_i-2x}} = h(2x)\left(1+4\sigma_i\frac{h'(2x)}{h(2x)}\right). \tag{8.16}$$

Now we represent $P(F(\nu^j),t)$ as the product $G_1G_2G_3$ of 3 factors defined below which we will modify using (8.14), (8.15) and (8.16):

$$G_1 := \left(\frac{1}{1-te^{-x}}\right)^{r-g_Y} \cdot \prod_{i=1}^{g_Y}\left(\frac{1}{1-te^{-\sigma_i-x}}\right)$$
$$= \left(\frac{1}{1-te^{-x}}\right)^{r} \cdot \prod_{i=1}^{g_Y}\left(1+\sigma_i\frac{h'(x)}{h(x)}\right),$$

using (6.7) one has that $\prod_{i=1}^{g_Y}\left(1+\sigma_i\frac{h'(x)}{h(x)}\right) = e^{\left(\theta\frac{h'(x)}{h(x)}\right)}$. Also notice that $\frac{h'(x)}{h(x)} = -\frac{te^{-x}}{1-te^{-x}} = -\frac{t}{e^x-t}$ so

$$G_1 = \left(\frac{1}{1-te^{-x}}\right)^{r} \cdot e^{\left(\theta\frac{h'(x)}{h(x)}\right)} = (1-te^{-x})^{-r}\cdot exp\left(\frac{-t\theta}{e^x-t}\right). \tag{8.17}$$

In a similar way, we compute

$$G_2 := \left(\frac{1}{1-te^{-2x}}\right)^{r'-g_Y} \cdot \prod_{i=1}^{g_Y}\left(\frac{1}{1-te^{4\sigma_i-2x}}\right)$$
$$= \left(\frac{1}{1-te^{-2x}}\right)^{r'} \cdot \prod_{i=1}^{g_Y}\left(1-4\sigma_i\frac{h'(2x)}{h(2x)}\right)$$
$$= \left(\frac{1}{1-te^{-2x}}\right)^{r'} \cdot e^{\left(-4\theta\frac{h'(2x)}{h(2x)}\right)} = (1-te^{-2x})^{-r'}\cdot exp\left(\frac{4t\theta}{e^{2x}-t}\right) \tag{8.18}$$

and

$$\begin{aligned} G_3 &:= \left(\frac{1}{1-te^{-2x}}\right)^{r''-g_Y} \cdot \prod_{i=1}^{g_Y}\left(\frac{1}{1-te^{-4\sigma_i-2x}}\right) \\ &= \left(\frac{1}{1-te^{-2x}}\right)^{r''} \cdot \prod_{i=1}^{g_Y}\left(1+4\sigma_i\frac{h'(2x)}{h(2x)}\right) \\ &= \left(\frac{1}{1-te^{-2x}}\right)^{r''} \cdot e^{\left(4\theta\frac{h'(2x)}{h(2x)}\right)} = (1-te^{-2x})^{-r''}\cdot exp\left(\frac{-4t\theta}{e^{2x}-t}\right). \end{aligned} \tag{8.19}$$

Multiplying (8.17), (8.18) and (8.19) we get, using the values for r, r' and r'' given in (8.13), that

$$\begin{aligned} P(F(\nu^j),t) &= (1-te^{-x})^{-1-m_{j,1}+k+g_Y} \\ &\quad\cdot exp\left(\frac{-t\theta}{e^x-t}\right)(1-te^{-2x})^{2+m_{j,2}+m'_{p-j,-2}-2g_Y} \end{aligned} \tag{8.20}$$

and replacing t by $\nu^j t$

$$\begin{aligned} P(F(\nu^j),\nu^j t) &= (1-\nu^j te^{-x})^{-1-m_{j,1}+k+g_Y} \\ &\quad\cdot exp\left(\frac{-\nu^j t\theta}{e^x-\nu^j t}\right)(1-\nu^j te^{-2x})^{2+m_{j,2}+m'_{p-j,-2}-2g_Y}. \end{aligned} \tag{8.21}$$

Now we use (8.21) in (8.9) and recalling that $\nu = e^{2i\pi/p}$ one can verify that

$$\begin{aligned} Q_h(F,t) &= exp\left(\frac{-pt^p\theta}{e^{px}-t^p}\right)\cdot\frac{(1-t^pe^{-px})^{k+g_Y-1}}{(1-t^pe^{-2px})^{2g_Y-2}} \\ &\quad\cdot\prod_{j=0}^{p-1}\left\{\frac{(1-\nu^j te^{-2x})^{m_{j,2}+m'_{p-j,-2}}}{(1-\nu^j te^{-x})^{m_{j,1}}}\right\}. \end{aligned} \tag{8.22}$$

Therefore,

$$\begin{aligned} ch_h(\iota_D^* S^l U_i) &= \underset{t^l}{coef}(Q_h(F,t)) \\ &= \underset{t^l}{coef}\left[\exp\left(\frac{-pt^p\theta}{e^{px}-t^p}\right)\cdot\frac{(1-t^pe^{-px})^{k+g_Y-1}}{(1-t^pe^{-2px})^{2g_Y-2}}\right. \\ &\quad\left.\cdot\prod_{j=0}^{p-1}\left\{\frac{(1-\nu^j te^{-2x})^{m_{j,2}+m'_{p-j,-2}}}{(1-\nu^j te^{-x})^{m_{j,1}}}\right\}\right]. \end{aligned}$$

In particular, if $l = q_i - i$, (q_i as in equation (4.8)), one has

$$ch_h(\iota_D^* S^{q_i-(i)} U_i) = \underset{t^{q_i-(i)}}{coef}(Q_h(F,t)) \tag{8.23}$$

□

9. The involution of a hyperelliptic curve

Putting all data available to us so far in formula (1.2) the contribution of a component Z_D of fixed points in S^iX of an automorphism h of order p to the number $N_i(h)$ is given by the following (see details in Section 11):

$$
\begin{aligned}
C_{i,Z_D}(h) = {} & \frac{p^A\nu^{l'+lm}}{(1-\nu^{p-1})^{a_1}\cdots(1-\nu)^{a_{p-1}}} \underset{t^{q_i-(i)}}{Coef}\ \underset{x=0}{Res}\Bigg\{[m(e^{-x})]^{-A}\prod_{j=1}^{p-1}\left(\frac{1-\frac{e^{-x}}{\nu^j}}{1-\nu^{-j}}\right)^{-n_j} \\
& \times e^{[d(1+m)-i(3+2m)+1-g_X]x}\cdot\frac{(1-t^pe^{-px})^{k+g_Y-1}}{(1-t^pe^{-2px})^{2g_Y-2}} \\
& \cdot\prod_{j=0}^{p-1}\left\{\frac{(1-\nu^jte^{-2x})^{m_{j,2}+m'_{p-j,-2}}}{(1-\nu^jte^{-x})^{m_{j,1}}}\right\} \\
& \times\left(\frac{x}{1-e^{-x}}\right)^{k-g_Y+1} \\
& \cdot\frac{\left(1+x\left[q(e^{-x})+p(3+2m)-\frac{pt^p}{e^{px}-t^p}+\left(\frac{1}{e^x-1}-\frac{1}{x}\right)\right]\right)^{g_Y}}{x^{k+1}}dx\Bigg\},
\end{aligned} \tag{9.1}
$$

where $A=k+1-g_Y$,

$$a_j = rank(N_{Z_D/S^iX}(\nu^{p-j})) = (n_{p-j}-n'_{p-j}+k).$$

The constants $l, l', n_j, n'_j, m_{j,2}, m_{j,1}, m'_{p-j,-2}$ (for their definitions see Theorem 8.1, Theorem 7.1 and Theorem 6.1) appearing in formula (9.1) depend on the particular situation (the curve X, the automorphism h, the line bundle ξ) and in this section, we will compute them for the case where X is a hyperelliptic curve of genus $g=g_X$, the automorphism h is the hyperelliptic involution and $\xi=K_X^2$ (see Lemmas 9.1, 9.2 and 9.3 below).

For the involution of a hyperelliptic curve, the contribution $C_{i,Z_D}(h)$ to the Lefschetz number $N_i(h)$ does not depend on D but only on the dimension of Z_D, the dimension of S^iX and the genus g_X of X. So we write

$$C_{i,k,g_X}(h) = C_{i,Z_D}(h) \tag{9.2}$$

for Z_D a $k-$dimensional component. There are $2g_X+2$ fixed points of h in the curve X and there are $\binom{2g_X+2}{i-2k}$ $k-$dimensional components Z_D of fixed points of h in S^iX each one corresponding to a divisor D of degree $i-2k$ supported on $i-2k$ distinct fixed points of h. Notice the maximal dimension of a component of fixed points in S^iX is $k_{max}=[i/2]$. If we use (3.6) to compute the Lefschetz numbers $N_i(h)$ then (4.9) becomes

$$Trace(h_{|V_{m,n}}) = N_0(h) + \sum_{i=1}^{w}\sum_{k=0}^{[i/2]}(-1)^i\binom{2g_X+2}{i-2k}C_{i,k,g_X}(h). \tag{9.3}$$

Let $f_*\xi^s(-nD) = \bigoplus_{j=0}^{1}\lambda_{s,j,n}$.

In Sect. 10 we shall use (9.3) to compute the Verlinde traces of a hyperelliptic curve of genus $g_X=2$ and to compute the contributions $C_{i,k,g_X}(h)$ it remains to compute the constants $l, l', n_j, n'_j, m_{j,2}, m_{j,1}, m'_{p-j,-2}$. From their definitions in Theorem 6.1 and Theorem 7.1 one has:

$$m_{j,1} = deg\lambda_{1,j,1},$$

$m_{j,2} = deg\lambda_{1,j,2}$,
$m'_{j,n} = deg\lambda_{-1,j,n}$,
$n_j = deg\lambda_{0,j,-1}$,
$n'_j = deg\lambda_{0,j,0}$.

In order to compute the degrees of the line bundles $\lambda_{s,j,n}$, consider the virtual representation

$$W = H^0(X, \xi^s(-nD)) - H^1(X, \xi^s(-nD)),$$

then the virtual dimensions of its eigenspaces are given by

$$dimW(\nu^1) = \frac{1}{2}[L(h^0, \xi^s(-nD)) - L(h, \xi^s(-nD))] \tag{9.4}$$

and

$$dimW(\nu^0) = \frac{1}{2}[L(h^0, \xi^s(-nD)) + L(h, \xi^s(-nD))]. \tag{9.5}$$

By Riemann-Roch Theorem we have:

$$L(h^0, \xi^s(-nD)) = sd - n(i-2k) - g_X + 1. \tag{9.6}$$

Next we use the Atiyah-Bott formula (3.7) to compute $L(h, \xi^s(-nD))$ with $\xi = K_X^2$ (see (9.7) below). Setting $E = \xi^s(-nD)$ we have

$$L(h, \xi^s(-nD)) = \sum_{l=1}^{2g_X+2} \frac{trz\ h|E_{p_l}}{det(Id - h|T^\vee_{X,p_l})}.$$

Now h acts as multiplication by -1 on $T^\vee_{X,p_l}$, so $det(Id - h|T^\vee_{X,p_l}) = 2$. For fixed points $p_l, p_j \in X^h$ the action of h in the fibre $\mathscr{O}(p_j)_{p_l}$ is multiplication by $(-1)^{\delta_{j,l}}$, where $\delta_{j,l}$ is the Kronecker delta. To see this, one first notice that the action of h on the fibre $\mathscr{O}_X(p_l)_{p_l} = \mathscr{O}_{p_l}(p_l)$ is multiplication by -1 because $\mathscr{O}_{p_l}(p_l)$ can be identified with the normal bundle of the point embedding $p_l \hookrightarrow X$ which in this case is T_{X,p_l} since p_l has dimension zero. If $j \neq l$ then when one considers the stalks at p_l of the following exact sequence of sheaves induced by the point embedding $p_j \hookrightarrow X$

$$0 \to \mathscr{O}_X \to \mathscr{O}_X(p_j) \to \mathscr{O}_{p_j}(p_j) \to 0$$

one gets an isomorphism on the fibres $\mathscr{O}_{X\,p_l} \cong \mathscr{O}_X(p_j)_{p_l}$ and h acts trivially on the fibres $\mathscr{O}_{X\,p_l}$ of the trivial line bundle $\mathscr{O}_X$.

Now one can compute the action of h on $E_{p_l} = (\xi^s \otimes \mathscr{O}_X(-nD))_{p_l}$. First one notice that the action on $(\xi^s)_{p_l}$ is trivial because we are taking $\xi = K_X^2 = T^\vee_X \otimes T^\vee_X$. As we mentioned at the beginning of this section the divisors D are supported at $\deg D$ distinct points in X^h then if p_l is a point in the support of D the action of h on $\mathscr{O}_X(D)_{p_l}$ is multiplication by -1 and therefore the action on $\mathscr{O}_X(-nD)_{p_l}$ is multiplication by $(-1)^n$. If p_l is not a point in the support of D then one has that the action of h on $\mathscr{O}_X(-nD)_{p_l}$ is multiplication by $(1)^n$. Then when we apply Atiyah-Bott we get

$$\begin{aligned} L(h, \xi^s(-nD)) &= (-1)^n Deg(D)/2 + (1)^n(2g_X + 2 - Deg(D))/2 \\ &= (-1)^n(i-2k)/2 + (2g_X + 2 - (i-2k))/2. \end{aligned} \tag{9.7}$$

Next we use that

$$H^{\ell}(X,\xi^s(-nD))(\nu^j)\cong H^{\ell}(Y,\lambda_{s,j,n}),$$

where Y is the quotient curve $X/<h>=\mathbb{P}^1$. We have
that the Euler characteristics of the eigenbundles $\lambda_{s,j,n}$ are given by

$$\chi(Y,\lambda_{s,j,n})=dimW(\nu^j)=deg\lambda_{s,j,n}+1$$

that is,

$$deg\,\lambda_{s,j,n}=dimW(\nu^j)-1. \tag{9.8}$$

In particular, we have the following

Lemma 9.1. *Let h be the involution of a hyperelliptic curve of genus g_X and let $\xi=K_X^2$ then*

$$\begin{aligned}
&m_{1,1}=g_X-3,\\
&m_{0,1}=2g_X-2+2k-i,\\
&m_{1,2}=g_X-3-i+2k,\\
&m_{0,2}=m_{0,1},\\
&m'_{1,-2}=-2k+i-3g_X+1,\\
&m'_{2,-2}=m'_{0,-2}=-2k+i-2g_X+2,\\
&n_1=i-2k-g_X-1,\\
&n'_1=-g_X-1.
\end{aligned}$$

In the next two lemmas, we will use the composition ι_D of equation (6.3).

Lemma 9.2. *The action of h on $\wedge^i\iota_D^*W_i^-$ is multiplication by $(-1)^{i+k}$.*

Proof. Consider the decomposition into eigenbundles

$$\iota_D^*W_i^-=\iota_D^*W_i^-(\nu^0)\oplus\iota_D^*W_i^-(\nu^1).$$

Let d_0,d_1 be the ranks of the eigenbundles $\iota_D^*W_i^-(\nu^0)$ and $\iota_D^*W_i^-(\nu^1)$ respectively. Then

$$\wedge^i\iota_D^*W_i^-=\wedge^{d_0}\iota_D^*W_i^-(\nu^0)\otimes\wedge^{d_1}\iota_D^*W_i^-(\nu^1)$$

and the action of the involution h on $\wedge^i\iota_D^*W_i^-$ is given by

$$\nu^{d_1},$$

that is,

$$\wedge^i\iota_D^*W_i^-=\wedge^i\iota_D^*W_i^-(\nu^{d_1}).$$

To compute d_1 it is enough to compute degree 0 part of $ch(\iota_D^*W_i^-(\nu^1))$ (see the expansion of $ch(\mathscr{E})$ in [19] pg. 432). So from Theorem 6.1 part b) we have

$$d_1=m_{1,1}-m_{1,2}+k, \tag{9.9}$$

then using Lemma 9.1, the result follows. □

Lemma 9.3. *The action of h on $\iota_D^*L_i^m$ is multiplication by $(-1)^{mi}$.*

Proof. Let $p \in S^kY$, it will be enough to compute the action on the fibre $(L_i)_{\iota_D(p)}$. First one notice that

$$\begin{aligned} det((\pi_{S^iX})_!\mathcal{O}_{X\times S^iX}(\Delta))_{\iota_D(p)} &= det(\{R^0_{\pi_{S^iX}*}\mathcal{O}_{X\times S^iX}(\Delta)\}_{\iota_D(p)} - \{R^1_{\pi_{S^iX}*}\mathcal{O}_{X\times S^iX}(\Delta)\}_{\iota_D(p)}) \\ &= det(H^0(X,l(D)) - H^1(X,l(D))) \\ &= det(H^0(X,l(D))) \otimes det(H^1(X,l(D)))^{-1}, \end{aligned}$$

where $l = \mathcal{O}_X(\iota(p)) = \mathcal{O}_X(f^*p)$, ι_D and ι as defined in (6.3), in particular one has $l(D) = \mathcal{O}_X(\iota(p) + D)$.
If we take the virtual representation
$W = H^0(X, l(D)) - H^1(X, l(D))$, then
$det\ W = det(W(\nu^0)) \otimes \det(W(\nu^1))$ and if we consider the virtual dimensions $d_i = dim\ W(\nu^i)$ then $det(W(\nu^i)) = (\nu^i)^{d_i}$. So the action of h on $det\ W$ is given by $(-1)^{d_1}$. These dimensions d_i can be computed as explained before *Lemma* 9.1, that is, we use (9.4) taking $s = 1$, $n = -1$, $\xi = l = \mathcal{O}_X(f^*p)$. One has

$$\begin{aligned} L(h^0, l(D)) &= \deg(l) + \deg(D) - g_X + 1 \\ &= 2k + (i - 2k) - g_X + 1. \end{aligned}$$

As for $L(h^1, l(D))$, we can use (9.7), because the action on the fibres l_{p_j} of the fixed points is trivial(the action on the curve Y is trivial and $l = f^*\mathcal{O}_Y(p)$ is the pull-back of a line bundle on Y) exactly as it happens with $\xi = K_X^2$. Then we obtain

$$L(h^1, l(D)) = (-1)(i - 2k)/2 + [2g_X + 2 - (i - 2k)]/2.$$

Therefore

$$d_1 = k + (i - 2k - g_x) \equiv i + g_X + k \mod 2.$$

Similarly for $\xi = K_X^2$:

$$det((\pi_{S^iX})_!\xi\mathcal{O}_{X\times S^iX}(-\Delta))_{\iota_D(p)} = det(H^0(X, \xi l^{-1}(-D)) - H^1(X, \xi l^{-1}(-D))),$$

and the action of h on $det((\pi_{S^iX})_!\xi\mathcal{O}_{X\times S^iX}(-\Delta))_{\iota_D(p)}$ is given by

$$(-1)^{g_X+k}.$$

Therefore, the action of h on $\iota_D^* L_i^m$ is given by $(-1)^{m(d_1+g_X+k)} = (-1)^{mi}$. □

10. A hyperelliptic curve of genus $g_X = 2$

Let X be a hyperelliptic curve of genus $g_X = 2$, let h be its hyperelliptic involution and take $\xi = K_X^2$. We have the embedding $X \overset{\xi K_X}{\hookrightarrow} \mathbb{P}^4$ and we will see that

$$Trace(h_| V_{l,l(d/2-1)}) = dim\ H^0(\mathbb{P}^3, \mathcal{O}(l)) \qquad (10.1)$$

for each integer $l \geq 0$. Notice from equation (1.1) that these are the Verlinde traces. Since ξ has degree $d = 4g_X - 4 = 4$ we have that $w = 1$ (recall that

$w=[(d-1)/2]$, see Theorem 4.1), then by (9.3) we have that (taking $m=l$ and $n=l(d/2-1)=l$)

$$Trace(h_{|}V_{l,l(d/2-1)})=Trace(h_{|}V_{l,l})=N_0(h)+\sum_{i=1}^{1}\sum_{k=0}^{[i/2]}(-1)^i\binom{6}{i-2k}C_{i,k,2}(h)$$
$$=N_0(h)-\binom{6}{1}C_{1,0,2}(h),$$

one has $m+n=2l$ so using (4.11) the last becomes

$$=\underset{t^{2l}}{coef}\left[\frac{1}{\det(I-t\cdot h_{|}H^0(X,K_X\xi))}\right]-\binom{6}{1}C_{1,0,2}(h). \tag{10.2}$$

To compute $\det(I-t\cdot h_{|}H^0(X,K_X\xi))$ we compute the dimensions of the eigenspaces of $h_{|}H^0(X,K_X\xi)$ (this is similar to the calculation of (9.4) and (9.5) in the previous section) and we have

$$dimH^0(X,K_X\xi)(\nu^1)=3g_X-2=4$$

and

$$dimH^0(X,K_X\xi)(\nu^0)=2g_X-3=1.$$

So we have

$$N_0(h)=\underset{t^{2l}}{Coef}\left(\frac{1}{(1+t)^4(1-t)}\right). \tag{10.3}$$

Now, for $C_{1,0,2}(h)$ we have that $i=1$, $q_i-i=n-1=l-1$ and $k=0$. By (9.2) and (9.1)

$$C_{1,0,2}(h)=(-1)^{l+1}\underset{t^{l-1}}{Coef}\cdot\underset{x=0}{Res}\left\{\frac{1}{4}\frac{1}{(1-te^{-x})^2}\left(\frac{1+e^{-x}}{1-e^{-x}}\right)e^{2lx}\frac{(1-te^{-2x})^2}{(1+te^{-2x})^4}dx\right\}, \tag{10.4}$$

denoting e^{-x} by λ the residue above becomes

$$\underset{\lambda=1}{Res}\left\{-\frac{1}{4}\frac{1}{(1-t\lambda)^2}\left(\frac{1+\lambda}{1-\lambda}\right)\frac{1}{\lambda^{2l+1}}\frac{(1-t\lambda^2)^2}{(1+t\lambda^2)^4}d\lambda\right\}. \tag{10.5}$$

Notice that if we denote the function inside braces by $F(\lambda)$ it has a pole of order $n=1$ on $\lambda=1$, then (10.5) is the coefficient of $(\lambda-1)^{n-1}$ in the Taylor expansion of $\{(\lambda-1)^nF(\lambda)\}$ about $\lambda=1$, that is,

$$\lim_{\lambda\to1}\left[\frac{1}{(n-1)!}\frac{d^{n-1}}{d\lambda^{n-1}}\{(\lambda-1)^nF(\lambda)\}\right].$$

This limit is equal to $\frac{1}{2(1+t)^4}$, in consequence

$$C_{1,0,2}(h)=(-1)^{l+1}\underset{t^{l-1}}{Coef}\left(\frac{1}{2(1+t)^4}\right)$$

and

$$N_1(h)=\binom{6}{1}C_{1,0,2}(h)=(-1)^{l+1}3\cdot\underset{t^{l-1}}{Coef}\left(\frac{1}{(1+t)^4}\right).$$

We have accomplished

$$\begin{aligned} Trace(h_| V_{l,l}) &= \underset{t^{2l}}{Coef}\left(\frac{1}{(1+t)^4(1-t)}\right) + (-1)^{l+2} 3\cdot \underset{t^{l-1}}{Coef}\left(\frac{1}{(1+t)^4}\right) \\ &= \underset{t^{2l}}{Coef}\left(\frac{(1-3t+3t^2-t^3}{(1-t^2)^4}\right) + (-1)^{l+2} 3\cdot \underset{t^{l-1}}{Coef}\left(\frac{1}{(1+t)^4}\right). \end{aligned}$$

Using the Hilbert series of the ring $K[x_0,\dots,x_n]$, namely

$$\sum_{d\geq 0}\binom{n+d}{n} t^d = \frac{1}{(1-t)^{n+1}}, \tag{10.6}$$

one obtains then

$$Trace(h_| V_{l,l}) = \binom{3+l}{3}.$$

11. Contribution simplification

Here, we derive formula (9.1). Using formula (1.2) and the data obtained in Sections 6–8, the contribution of a k-dimensional component Z_D of $(S^iX)^h$ corresponding to the divisor D is given by

$$\begin{aligned} C_{i,Z_D}(h) = \int_{S^kY} \Bigg\{ & \nu^{lm}\cdot e^{m(d-2i)x+2mp\theta} \\ & \times \nu^{l'}\cdot e^{(d-3i+1-g_X)x+3p\theta} \\ & \times \underset{t^{q_i-i}}{coef}\left[exp\left(\frac{-pt^p\theta}{e^{px}-t^p}\right)\cdot \frac{(1-t^pe^{-px})^{k+g_Y-1}}{(1-t^pe^{-2px})^{2g_Y-2}} \right. \\ & \left. \cdot \prod_{j=0}^{p-1}\left\{\frac{(1-\nu^j te^{-2x})^{m_{j,2}+m'_{p-j,-2}}}{(1-\nu^j te^{-x})^{m_{j,1}}}\right\}\right] \\ & \times p^A m(e^{-x})^{-A} e^{\theta q(e^{-x})}\prod_{j=1}^{p-1}\left(\frac{1-e^{-x}/\nu^j}{1-\nu^{-j}}\right)^{-n_j} \\ & \times \left(\frac{x}{1-e^{-x}}\right)^{k-g_Y+1} exp\left(\frac{\theta}{e^x-1}-\frac{\theta}{x}\right) \\ & \times \frac{1}{(1-\nu^{p-1})^{a_1}\cdots(1-\nu)^{a_{p-1}}}\Bigg\}, \end{aligned}$$

where the first 4 lines of the integrand correspond to the generalized Chern character (Theorem 8.1), the fifth line is the product of stable characteristic classes of the eigenbundles of the normal bundle N_{Z_D/S^iX} (Theorem 7.1, (d)), the sixth is the Todd class of the component $Z_D \cong S^kY$ (formula (6.8)) and the seventh line is the inverse of $\det(Id - h_| N^{\vee}_{Z_D/S^iX})$.

Now, one can remove θ from the expression for $C_{i,Z_D}(h)$ by using the following formula (this is (7.2) from [32])

$$\int_{S^kY}\alpha(x)exp(\beta(x)\theta)=Res_{x=0}\left\{\frac{\alpha(x)(1+x\beta(x))^{g_Y}dx}{x^{k+1}}\right\}.$$

Let $\alpha(x),\beta(x)$ be as defined in (11.1) and (11.2) below.

$$\begin{aligned}\alpha(x)=\Bigg\{&[m(e^{-x})]^{-A}\prod_{j=1}^{p-1}\left(\frac{1-\frac{e^{-x}}{\nu^j}}{1-\nu^{-j}}\right)^{-n_j}\times\\&\times e^{[d(1+m)-i(3+2m)+1-g_X]x}\cdot\frac{(1-t^pe^{-px})^{k+g_Y-1}}{(1-t^pe^{-2px})^{2g_Y-2}}\times\\&\times\prod_{j=0}^{p-1}\left\{\frac{(1-\nu^jte^{-2x})^{m_{j,2}+m'_{p-j,-2}}}{(1-\nu^jte^{-x})^{m_{j,1}}}\right\}\left(\frac{x}{1-e^{-x}}\right)^{k-g_Y+1}\Bigg\}\end{aligned}\tag{11.1}$$

and

$$\beta(x)=\left[q(e^{-x})+p(3+2m)-\frac{pt^p}{e^{px}-t^p}+\left(\frac{1}{e^x-1}-\frac{1}{x}\right)\right].\tag{11.2}$$

It is not hard to verify that

$$C_{i,Z_D}(h)=\frac{p^A\nu^{l'+lm}}{(1-\nu^{p-1})^{a_1}\cdots(1-\nu)^{a_{p-1}}}\int_{S^iX}\underset{t^{q_i-i}}{coef}\Bigg[\alpha(x)exp(\beta(x)\theta)\Bigg].$$

Then

$$\begin{aligned}C_{i,Z_D}(h)=&\frac{p^A\nu^{l'+lm}}{(1-\nu^{p-1})^{a_1}\cdots(1-\nu)^{a_{p-1}}}\\&\times\underset{t^{q_i-i}}{coef}\left[Res_{x=0}\left\{\frac{\alpha(x)(1+x\beta(x))^{g_Y}dx}{x^{k+1}}\right\}\right]\end{aligned}$$

from which we obtained (9.1).

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