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Families of Legendrians and Lagrangians with unbounded spectral norm

Georgios Dimitroglou Rizell

Abstract. Viterbo has conjectured that any Lagrangian in the unit co-disc bundle of a torus which is Hamiltonian isotopic to the zero-section satisfies a uniform bound on its spectral norm; a recent result by Shelukhin showed that this is indeed the case. The modest goal of our note is to explore two natural generalisations of this geometric setting in which the bound of the spectral norm fails: first, passing to Legendrian isotopies in the contactisation of the unit co-disc bundle (recall that any Hamiltonian isotopy can be lifted to a Legendrian isotopy) and, second, considering Hamiltonian isotopies but after modifying the co-disc bundle by attaching a critical Weinstein handle.

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Birkhäuser

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1. Introduction and results

Spectral invariants were introduced in Viterbo's seminal work [31]. Since their appearance they have become one of the most fundamental tools of quantitative symplectic topology. We do not intend to give an overview of its development and many applications here; instead we direct the reader to work by Oh [23] for a thorough introduction to the subject from a modern perspective.

Very briefly, spectral invariants in the symplectic case consist of functions from the group of Hamiltonian diffeomorphisms

$$c\colon Ham(X,\omega) \to \mathbf{R}$$

that take values in the real numbers, and which satisfy a list of axioms that will be omitted. The spectral invariants that we consider here are constructed as follows. For a pair of exact Lagrangian submanifolds $L_0, L_1 \subset (X, d\lambda)$ (the symplectic manifold is thus necessarily exact) one can associate the Floer complex $CF(L_0, \phi(L_1))$ to any Hamiltonian diffeomorphism $\phi \in Ham(X,\omega)$ endowed with its canonical action filtration. Spectral invariants are certain real numbers that encode information about this filtered chain complex. To make this precise, we utilise the language of the **barcode** from the theory of persistent homology in topological data analysis, which goes back to work by Carlsson–Zomorodian–Collins–Guibas [8]. This theory has been proven to be very useful in quantitative symplectic topology, where it was introduced by Polterovich–Shelukhin [25] and Usher–Zhang [30]; also see the recent work [24] by Polterovich–Rosen–Samvelyan–Zhang for a systematic introduction. Here we give a quick definition of the barcode of a filtered complex that will suit our needs in Sect. 2.2.

The barcode can be defined for any chain complex $(C, \partial, \mathfrak{a})$ with a filtration by subcomplexes

$$C_*^{<c} := \mathfrak{a}^{-1}[-\infty, c) \subset C_*$$

defined by an "action" function

$$\mathfrak{a}\colon C \to \{-\infty\} \cup \mathbf{R},$$

where $\mathfrak{a}^{-1}(-\infty) = \{0\}$. Phrased in this language, the spectral invariants are the values of the starting points of the semi-infinite bars of the barcode associated to the Floer complex. In fact, the main interest here is not the spectral invariants themselves, but rather the following derived quantities (see Definition 2.7):

- The **spectral range** of a filtered complex, denoted by

$$\rho(C, \partial, \mathfrak{a}) \in \{-\infty\} \cup [0, +\infty].$$

  This quantity is defined as the supremum of the distances between the starting points of two semi-infinite bars in the corresponding barcode. (It takes the value $-\infty$ if and only if there are no semi-infinite bars.)
- The **boundary depth** of a filtered complex, denoted by

$$\beta(C, \partial, \mathfrak{a}) \in \{-\infty\} \cup [0, +\infty].$$

  This quantity is defined as the supremum of the lengths of a finite bar in the corresponding barcode. (It takes the value $-\infty$ if and only if there are no bars of finite length.)

For the Floer complex $CF(L, \phi_H^1(L))$ of a closed embedded exact Lagrangian and its Hamiltonian deformation, the spectral range coincides with a quantity called the **spectral norm**. This can be seen using Leclercq's results from [20, Corollary 1.7], after relating the spectral invariants used in that article to the endpoints of semi-infinite bars in the relevant barcode. Since we will not use any of the particular features satisfied by the spectral norm here, we will gloss over the difference between these two concepts and simply define the spectral norm as

$$\gamma(CF(\Lambda, \phi_H^1(\Lambda))) := \rho(CF(\Lambda, \phi_H^1(\Lambda))),$$

i.e. we prescribe it to be equal to the spectral range.

Remark 1.1. The correct way to define the spectral norm in the setting of Legendrians would be to define γ as the difference of action levels of the classes that correspond to the unit for the cup-product and its image under Poincaré duality. We do not go into details of products and Poincaré duality here, but when Λ is a Legendrian without Reeb chords, we again expect an equality between spectral norm and spectral range. In general there should be an inequality $\gamma \leq \rho$.

We also need a generalisation of the above spectral invariants to contact manifolds. Since we will only consider contact manifolds of a very particular type, namely contactisations

$$(Y, \alpha) = (X \times \mathbf{R}, dz + \lambda)$$

of exact symplectic manifolds $(X, d\lambda)$ (see Sect. 2.1), this can be done by relying on well-established techniques. From our point of view, the spectral invariants of a contact manifold are defined for the group of contactomorphisms which are contact-isotopic to the identity, and yield functions

$$c\colon Cont_0(Y, \alpha) \to \mathbf{R}.$$

Note that the value does depend on the choice of contact form α here, and not just on the contact structure $\ker\alpha \subset TY$. It should be noted that spectral invariants in the contact setting are much less studied and developed than the symplectic version. However, the original formulation of the spectral invariants, which appeared in [31] for symplectic cotangent bundles $(X, \omega) = (T^*M, d(p\,dq))$, admits a straightforward generalisation to the standard contact jet-space

$$(J^1M = T^*M \times \mathbf{R}, dz - p\,dq),$$

as shown by Zapolsky [33]. In fact, the spectral invariants in [31] are based on a version of Floer homology defined using generating families, and this theory can be generalised to invariants of Legendrian isotopies inside jet-spaces by work of Chekanov [7]. Note that jet-spaces are particular cases of contactisations.

The spectral invariants considered here can be defined either by using generating families as in [33], or using a Floer homology constructed using the Chekanov–Eliashberg algebra as first done in [13] by Ekholm–Etnyre–Sabloff; also see work [4] by the author together with Chantraine–Ghiggini–Golovko. The Chekanov–Eliashberg algebra is a Legendrian isotopy invariant in the form of a unital differential graded algebra (DGA) that is freely generated by the Reeb chords on the Legendrian, which are all assumed to be transverse. Given a pair of Legendrians Λ_0 and Λ_1, the spectral invariants that we consider are associated to the barcode of the Floer complex $CF(\Lambda_0, \phi(\Lambda_1))$ where ϕ is a contactomorphism that is contact isotopic to the identity. See Sect. 2.3 for the definition of this Floer complex.

Viterbo conjectured in [32] that the spectral norm $\gamma(CF(0_{\mathbf{T}^n}, \phi(0_{\mathbf{T}^n})))$ of the Floer complex of the zero section $0_{\mathbf{T}^n} \subset T^*\mathbf{T}^n$ satisfies a uniform bound whenever $\phi \in Ham(T^*\mathbf{T}^n)$ maps the zero section $\phi(0_{\mathbf{T}^n}) \subset DT^*\mathbf{T}^n$ into the unit-disc cotangent bundle. In recent work by Shelukhin [28,29] this property was finally shown to be the case, even for a wide range of cotangent bundles beyond the torus case. The main point of our work here is to give examples of geometric settings beyond symplectic co-disc bundles, where the analogous boundedness of the spectral norm fails. It should be stressed that, in the time of writing of this article, there are still many cases of cotangent bundles for which the problem remains open: does the spectral norm of an exact Lagrangian inside DT^*M which is Hamiltonian isotopic to the zero-section satisfy a uniform bound for an arbitrary closed smooth manifold M?

As a first result, in Part (1) of Theorem A, we show that the spectral norm of Legendrians inside the contactisation $D^*S^1 \times \mathbf{R} \subset J^1S^1$ which are Legendrian isotopic to the zero section does not satisfy a uniform bound. Recall that any Hamiltonian isotopy of $0_{S^1} \subset DT^*S^1$ lifts to a Legendrian isotopy of the zero section $j^10 \subset J^1S^1$ (see Lemma 2.1); consequently, one way to formulate Part (1) of Theorem A is by saying that Viterbo's conjecture cannot be generalised to Legendrian isotopies.

Below we denote by

$$F_{\theta_0,z_0} := \{\theta = \theta_0, z = z_0\} \subset J^1S^1$$

a Legendrian lift of the Lagrangian cotangent fibre $T^*_{\theta_0}S^1$.

Theorem A.

(1) There exists a contact isotopy $\phi^t\colon J^1S^1 \to J^1S^1$ that satisfies

$$\phi^t|_{j^10}\colon j^10 \hookrightarrow DT^*S^1 \times \mathbf{R} = (S^1 \times [-1,1]) \times \mathbf{R},$$

and for which $CF(j^10, \phi^t(j^10))$ all are generated by precisely two mixed Reeb chords, whose difference in length grows indefinitely as $t \to +\infty$. In particular, the spectral norm $\gamma(CF(j^10, \phi^t(j^10)))$ becomes arbitrarily large as $t \to +\infty$.

(2) There exists a contact isotopy $\phi^t\colon J^1S^1 \to J^1S^1$ that satisfies

$$\phi^t|_{\Lambda_{\mathrm{st}}}\colon \Lambda_{\mathrm{st}} \hookrightarrow DT^*S^1 \times \mathbf{R}_{>0} = (S^1 \times [-1,1]) \times \mathbf{R}_{>0} \subset J^1S^1,$$

where $\Lambda_{\mathrm{st}} \subset J^1S^1$ is the standard Legendrian unknot shown in Fig. 1, and for which the boundary depth $\beta(CF(\phi^t(\Lambda_{\mathrm{st}}), F_{\theta_0,0}))$ becomes arbitrarily large as $t \to +\infty$. In addition, we may assume that ϕ^t is supported inside some subset $\{z \geq c\}$, where $c > 0$, and for which the inclusion $\{z \geq c\} \cap \Lambda_{\mathrm{st}} \subsetneq \Lambda_{\mathrm{st}}$ is a strict subset.

In recent work [2, Section 6.2] Biran–Cornea showed that a bound $\gamma(CF(0^M, L)) \leq C$ on the spectral norm of the Floer complex of a Lagrangian $L \subset T^*M$, where L is Hamiltonian isotopic to the zero section, implies the bound $\beta(CF(L, T^*_{pt}M)) \leq 2C$ on the boundary depth of the Floer complex of L and a cotangent fibre. The Legendrians produced by Part (2) of Theorem A can be used to show that the analogous result cannot be generalised to Legendrian isotopies. More precisely,

Corollary B. *There exists a contact isotopy $\phi^t\colon J^1S^1 \to J^1S^1$ that satisfies*

$$\phi^t|_{j^10}\colon j^10 \hookrightarrow DT^*S^1 \times \mathbf{R} = (S^1 \times [-1,1]) \times \mathbf{R},$$

and for which the spectral norm $\gamma(CF(j^10, \phi^1(j^10)))$ is uniformly bounded for all $t \geq 0$, while the boundary depth $\beta(CF(\phi^1(0_{S^1}), F_{\theta_0,z_0}))$ becomes arbitrarily large as $t \to +\infty$.

Proof. Take a cusp-connected sum of a C^1-small perturbation of the zero-section j^10 and any unknot Λ^t_{st} from the family produced by Part (2) of Theorem A; the case of Λ_{st} is shown in Fig. 1. We refer to [9] for the definition of cusp-connected sum (also called ambient Legendrian 0-surgery) along a Legendrian arc (the so-called surgery disc). We perform the cusp-connected sum along a Legendrian arc contained inside the region $\{z < c\}$, and which is disjoint from the support of the Legendrian isotopy of the unknots. Note that the Legendrian resulting from the cusp-connected sum is Legendrian isotopic to the zero-section, as shown in Fig. 1. It follows that the same is true for the cusp-connected sum of j^10 and any Legendrian Λ^t_{st} from the family.

Finally, to evaluate the effect of the ambient surgery on the barcodes of the Floer complexes we apply Theorem C. To that end, the following two facts are needed. First, $CF(\Lambda_{\mathrm{st}}, j^10)$ is acyclic, and thus its barcode consists of only finite bars. The acyclicity of the Floer complex is a consequence of the

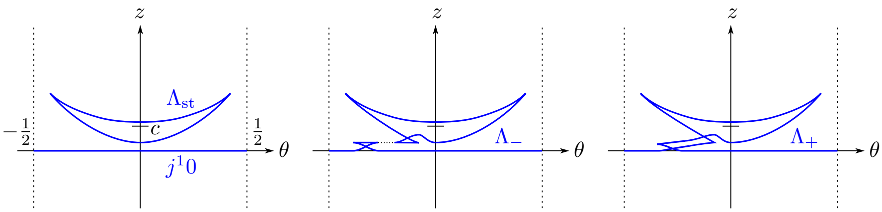

Figure 1. Left: the front projection of the zero section $j^1 0 \subset J^1(\mathbf{R}/\mathbf{Z}) = J^1 S^1$ and a standard Legendrian unknot Λ_{st}. Middle: the result of a Legendrian RI-move on each component, Λ_- denotes the union of the two components. Right: the Legendrian Λ_+ which is the result after a cusp-connected sum along the dotted arc shown in the middle picture. Λ_+ is Legendrian isotopic to the zero section (Λ_+ is obtained by performing two RI-moves on the zero-section)

invariance under Legendrian isotopy. (After a translation of Λ_{st} sufficiently far in the p-coordinate, all generators of the Floer complex disappear.) Second,

$$CF(j^1 f \cup \Lambda^t_{\mathrm{st}}, j^1 0) = CF(j^1 f, j^1 0) \oplus CF(\Lambda^t_{\mathrm{st}}, j^1 0)$$

is a direct sum of complexes. The barcode of the complex on the left-hand side is thus the union of the barcodes of the two complexes in the direct sum on the right-hand side. Here we have suppressed the choices of augmentations, since these Floer complexes do not depend on these choices (up to action preserving automorphism); see Remark 1.2. □

To define Floer homology for a pair of Legendrians Λ_0, Λ_1, it is necessary to also include the data of augmentations $\varepsilon_i \colon \mathcal{A}(\Lambda_i) \to \mathbf{k}$ of their Chekanov–Eliashberg algebras; these are unital DGA-morphisms onto the ground field. We write $CF((\Lambda_0, \varepsilon_0), (\Lambda_1, \varepsilon_1))$ for the induced complex, which in general does depend on the choices of augmentations; we refer to Sect. 2.4 for more details.

Remark 1.2. There are Legendrian isotopy classes for which different choices of augmentations always give rise to Floer complexes that are isomorphic as filtered chain complexes; this can be characterised using the invariance of the augmentation category of Chantraine–Bourgeois from [3]. Cases include the Legendrian isotopy class of the zero section $j^1 0$, the Legendrian fibre $F_{\theta,z}$, and the standard Legendrian unknot. The property is a consequence of the fact that these Legendrian isotopy classes admit representatives for which the Chekanov–Eliashberg algebra admits a *unique* augmentation.

Theorem C. *Let Λ_+ be a Legendrian obtained from Λ_- by a Legendrian ambient surgery. After making the surgery-region sufficiently small, we can assume that there is an action-preserving isomorphism*

$$CF((\Lambda_+, \varepsilon_+), (\Lambda, \varepsilon)) \to CF((\Lambda_-, \varepsilon_-), (\Lambda, \varepsilon))$$

of complexes, where (Λ, ε) is an arbitrary but fixed Legendrian equipped with an augmentation ε of its Chekanov–Eliashberg algebra $\mathcal{A}(\Lambda)$, and where the *augmentation* ε_+ *of* $\mathcal{A}(\Lambda_+)$ *is induced by pulling back the augmentation* ε_- *of* $\mathcal{A}(\Lambda_-)$ *under the unital DGA-morphism induced by the standard Lagrangian handle-attachment cobordism. In particular, the barcodes of the two Floer complexes coincide.*

In the setting of exact Lagrangian cobordisms in the sense of Arnol'd between exact Lagrangian submanifolds similar results were found in [2, Section 5.3] .

Finally we present a Hamiltonian isotopy of a closed exact Lagrangian inside a Liouville domain for which the spectral norm becomes arbitrarily large. The simplest examples of such a Liouville domain is the 2-torus with an open ball removed; we denote this space by $(\Sigma_{1,1}, d\lambda)$ and depict it in Fig. 13. The detailed construction is given in Sect. 2.1.2. As kindly pointed out to the author by the anonymous referee, this fact is not new. The same phenomenon was exhibited in, e.g. Zapolsky's work [34, Lemma 1.10], as well as the more recent [19, Remark 6] by Kislev–Shelukhin.

Theorem D. *There exists a closed exact Lagrangian submanifold* $L \subset (\Sigma_{1,1}, \omega)$ *and a compactly supported Hamiltonian* $H\colon \Sigma_{1,1} \to \mathbf{R}$ *for which the induced compactly supported Hamiltonian isotopy* $\phi_H^t\colon (\Sigma_{1,1}, \omega) \to (\Sigma_{1,1}, \omega)$ *satisfies the property that the spectral norm* $\gamma(CF(L, \phi_H^t(L)))$ *becomes arbitrarily large as* $t \to +\infty$.

1.1. Why the proofs of uniform bounds fail for Legendrians

The techniques that are used in [29] and [2] to prove the results in the case of the cotangent bundle are not yet fully developed in the case of Legendrians in contactisations. This includes the closed-open map, which is a crucial ingredient in [29], and a unital A_∞-structure on the Floer complex with relevant PSS-isomorphisms, which is crucial in [2]. Nevertheless, we still do expect that these operations can be defined also for the Floer homology of Legendrians in contactisations. In fact the A_∞-structure was recently extended to this setting by Legout [21]. Assuming the possibility to define these operations in the Legendrian setting, what goes wrong when one tries to generalise the proofs to the Legendrian case?

First we recall the properties of the Floer homology complex of a Legendrian and itself; see, e.g. [13] for the details. To define $CF(\Lambda, \Lambda)$ one must first make the mixed Reeb chords transverse by a Legendrian perturbation of the second copy of Λ. We do this by replacing Λ with a section $j^1 f$ in its standard contact jet-space neighbourhood, where $f\colon \Lambda \to \mathbf{R}$ is a C^1-small Morse function. In this manner, we obtain

$$CF(\Lambda, \Lambda) = C^{\mathrm{Morse}}(f; \mathbf{k}) \oplus \bigoplus_{c \in \mathcal{Q}(\Lambda)} \mathbf{k} p_c \oplus \mathbf{k} q_c$$

where $\mathcal{Q}(\Lambda)$ denotes the set of Reeb chords on Λ, and $C^{\mathrm{Morse}}(f; \mathbf{k})$ is the Morse homology complex with basis given by the critical points of the function $f\colon \Lambda \to \mathbf{R}$. The action of the former chords are approximately equal to $\mathfrak{a}(p_c) = \ell(c)$ and $\mathfrak{a}(q_c) = -\ell(c)$ while the action of the latter is equal to $\mathfrak{a}(x) = f(x)$. What is important to notice here is that the generators of

$C^{\text{Morse}}(f; \mathbf{k})$ may be assumed to have arbitrarily small action, while this is not the case for the generators that correspond to pure Reeb chords. When Λ is the Legendrian lift of a Lagrangian *embedding*, there are of course only generators of the type $C^{\text{Morse}}(f; \mathbf{k})$. This turns out to be the crucial difference between the symplectic and the contact case.

Example in Part (1) of Theorem A: The proof in [29] uses the closed-open map. More precisely, a crucial ingredient in the proof is the action-preserving property of the operations P'_a on the Floer homology $CF(0_M, \phi^1_H(0_M))$, which are defined using the length-0 part $\phi^0(a)$ and length-1 part $\phi^1(a, \cdot)$ of the closed open map for certain elements $a \in SH(T^*M)$ in symplectic cohomology (by this we mean the unital version of the symplectic invariant, which is contravariant with respect to inclusion of exact Liouville subdomains). In the case when the Legendrian has pure Reeb chords (i.e. it is not the lift of an exact Lagrangian embedding), the chain $\phi^0(a) \in CF(\Lambda, \Lambda)$ may consist of generators whose action does not vanish (since they do not correspond to Morse generators). In this case the action-preserving property of P'_a can no longer be determined from the aciton of $a \in SH(T^*M)$ alone.

Example in Part (2) of Theorem A: The proof in [2, Section 6.2] uses the fact that there are continuation elements $a \in CF(\phi^1_H(0_M), 0_M)$ and $b \in CF(0_M, \phi^1_H(0_M))$ for which $\mu_2(a, b) \in CF(\phi^1_H(0_M), \phi^1_H(0_M))$ is the unique maximum of a suitable Morse function. In the Legendrian case the element $\mu_2(a, b) \in CF(\phi^1(j^1 0), \phi^1(j^1 0))$ is still a homology unit; however, it not necessarily a sum of Morse chords, and can, therefore, have significant action. In particular, multiplication with the element $\mu_2(a, b)$ is not necessarily identity on the chain level, nor is it necessarily homotopic to the identity by a chain homotopy of small action. The geometrically induced chain homotopy $\mu_3(a, b, \cdot)$ between $\mu_2(a, \mu_2(b, \cdot))$ and $\mu_2(\mu_2(a, b), \cdot)$ increases action by at most the spectral norm, and is used in [2] for establishing the bound on the boundary depth. However, this chain homotopy does not do the job any more, since we also need an additional chain-homotopy (of unknown action properties) to take us from the map $\mu_2(\mu_2(a, b), \cdot)$ to the chain level identity.

2. Background

2.1. Contact geometry of jet-spaces and contactisations

An **exact symplectic manifold** is a smooth $2n$-dimensional manifold $(X^{2n}, d\lambda)$ equipped with a choice of a primitive one-form λ for an exact symplectic two-form $\omega = d\lambda$, i.e. ω is skew-symmetric, non-degenerate, and closed. Note that the primitive λ should be considered as part of the data describing the exact symplectic manifold. A compact exact symplectic manifold with boundary $(W, d\lambda)$ is a **Liouville domain** if the **Liouville vector field**, i.e. the vector field ζ given as the symplectic dual of λ via the equation $\iota_\zeta d\lambda = \lambda$, is transverse to the boundary ∂W. The flow generated by ζ is called the **Liouville flow** and satisfies $(\phi^t_\zeta)^*\lambda = e^t\lambda$. An open exact symplectic manifold $(\overline{W}, d\lambda)$ is a **Liouville manifold** if the all critical points of the Liouville vector field are contained inside some compact Liouville domain $W \subset (\overline{W}, d\lambda)$, and if the Liouville flow is complete.

A **Hamiltonian isotopy** is a smooth isotopy of X which is generated by a time-dependent vector field $V_t \in \Gamma(TX)$ that satisfies $\iota_{V_t} d\lambda = -dH_t$ for some smooth time-dependent function

$$H\colon X \times \mathbf{R}_t \to \mathbf{R}$$

which is called the **Hamiltonian**; a diffeomorphism of X which is the time-t flow generated by such a vector field preserves the symplectic form (but not necessarily the primitive) and is denoted by

$$\phi_H^t\colon (X, \omega) \to (X, \omega);$$

we call such a map a **Hamiltonian diffeomorphism**, and the corresponding flow a **Hamiltonian isotopy**. Conversely, any choice of Hamiltonian function induces a **Hamiltonian isotopy** ϕ_H^t in the above manner. Since we consider exact symplectic manifolds, a smooth isotopy $\phi^t\colon X \to X$ is a Hamiltonian isotopy if and only if $(\phi^t)^*\lambda = \lambda + dG_t \in \Omega^1(X)$ holds for some smooth function

$$G\colon X \times \mathbf{R}_t \to \mathbf{R}.$$

Note that the Hamiltonian function that corresponds to a Hamiltonian isotopy is determined only up to the addition of a function that only depends on t.

Any exact $2n$-dimensional symplectic manifold $(X^{2n}, d\lambda)$ gives rise to a $2n+1$-dimensional contact manifold $(X \times \mathbf{R}_z, dz + \lambda)$ called its **contactisation**, which is equipped with the canonical contact one-form $\alpha_{\mathrm{st}} := dz + \lambda$. The contactisations induced by choices of primitives of the symplectic form λ and $\lambda' = \lambda + df$ that differ by the exterior differential of $f\colon X \to \mathbf{R}$ are isomorphic via the coordinate change $z \mapsto z - f$. Recall that the contact condition is equivalent to $d\alpha_{\mathrm{st}}$ being non-degenerate on the contact planes $\ker \alpha_{\mathrm{st}} \subset T(X \times \mathbf{R})$. A **contact isotopy** is a smooth isotopy which preserves the distribution $\ker \alpha_{\mathrm{st}}$ (but not necessarily the contact form). The contraction $\iota_{V_t} \alpha_{\mathrm{st}}$ of the contact form and the infinitesimal generator gives a bijective correspondence between contact isotopies and smooth time-dependent functions on $X \times \mathbf{R}$, the latter are called **contact Hamiltonians**. We refer to [18] for more details.

Lemma 2.1. *A Hamiltonian isotopy $\phi_H^t\colon (X, d\lambda) \to (X, d\lambda)$ with a choice of Hamiltonian $H_t\colon X \to \mathbf{R}$ lifts to a contact isotopy*

$$\begin{aligned} X \times \mathbf{R} &\to X \times \mathbf{R}, \\ (x, z) &\mapsto (\phi_H^t(x), z - G_t(x)), \end{aligned}$$

where the function $G\colon X \times \mathbf{R}_t \to \mathbf{R}$ is defined by

$$G_t(x) = \int_0^t \lambda(V_s(\phi_H^s(x))) - H_s(\phi_H^s(x))ds$$

and satisfies the property

$$(\phi_H^t)^*\lambda = \lambda + d\,G_t.$$

Moreover, this contact isotopy preserves the contact form α_{st} *and is generated by the time-dependent contact Hamiltonian* $H_t \circ \mathrm{pr}_X \colon X \times \mathbf{R}_z \to \mathbf{R}$.

A smooth immersion of an n-dimensional manifold

$$\Lambda \looparrowright (X^{2n} \times \mathbf{R}, dz + \lambda)$$

in the contactisation is **Legendrian** if it is tangent to $\ker \alpha_{\mathrm{st}}$, while a smooth n-dimensional immersion $L \looparrowright (X^{2n}, \lambda)$ in an exact symplectic manifold is **exact Lagrangian** if λ pulls back to an *exact* one-form. The following relation between Legendrians and exact Lagrangians is immediate:

Lemma 2.2. *The canonical projection of a Legendrian immersion to* (X, λ) *is an exact Lagrangian immersion. Conversely, any exact Lagrangian immersion lifts to a Legendrian immersion of the contactisation* $X \times \mathbf{R}$*. Moreover, the lift is uniquely determined by the choice of a primitive* $f \colon L \to \mathbf{R}$ *of the pull-back* $\lambda|_{TL} = df$*, via the formula* $z = -f$*.*

Transverse double points of Lagrangian immersions are stable. On the other hand, generic Legendrian immersions are in fact *embedded*. However, there are stable self-intersections of Legendrians that appear in one-parameter families. Recall the following standard fact; again we refer to, e.g. [18] for details.

Lemma 2.3. *A compactly supported smooth isotopy* $\phi^t(\Lambda) \subset X \times \mathbf{R}$ *through Legendrian embeddings, also called a* **Legendrian isotopy***, can be generated by an ambient contact isotopy.*

2.1.1. The cotangent bundle and jet-space. There is a canonical exact symplectic two-form $-d(p\,dq)$ on any smooth cotangent bundle T^*M, whose primitive $-p\,dq$ is the tautological one-form with a minus sign. The cotangent bundle is a Liouville manifold and any co-disc bundle is a Liouville domain. The zero-section $0_M \subset T^*M$ is obviously an exact Lagrangian embedding.

The contactisation of T^*M is the one-jet space $J^1M = T^*M \times \mathbf{R}_z$ with the canonical contact one-form $dz - p\,dq$. The zero-section in T^*M lifts to the one-jet j^1c of any constant function c (obviously the one-jet j^1f of an arbitrary function $f \colon M \to \mathbf{R}$ is Legendrian isotopic to j^10). For us the most relevant example is actually the two-dimensional symplectic cotangent bundle $T^*S^1 = S^1 \times \mathbf{R}_p$ equipped with the exact symplectic two-form $-d(p\,d\theta)$, and its corresponding contactisation, i.e. the three-dimensional contact manifold

$$(J^1S^1 = T^*S^1 \times \mathbf{R}_z, dz - p\,d\theta)$$

(note the sign convention for the Liouville form).

To describe Legendrians in J^1M we will make use of the **front-projection**, by which one simply means the canonical projection

$$\Pi_F \colon J^1M \to M \times \mathbf{R}_z.$$

A Legendrian immersion can be uniquely determined by its post-composition with the front projection. A generic Legendrian knot in J^1S^1 has a front projection whose singular locus consists of

- non-vertical cubical cusps and

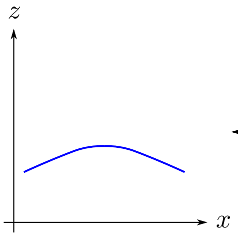

FIGURE 2. RI: the first Legendrian Reidemeister move in the front projection

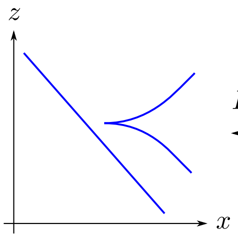

FIGURE 3. RII: the second Legendrian Reidemeister move in the front projection

- transverse self-intersections.

Note that the front projection cannot be tangent to ∂_z by the Legendrian condition (i.e. there are no vertical tangencies).

Two sheets of the front projection that have the same slopes (i.e. p-coordinates) above some given point in the base, project to a double-point inside T^*M. There is a bijection between double points of this projection and Reeb chords, where a Reeb chord is an integral curve of ∂_z with both endpoints on the Legendrian. The difference of z-coordinate of the endpoint and starting point of a Reeb chord c is called its **length** and is denoted by $\ell(c) \geq 0$.

Double-points of the Legendrian immersion itself correspond to self-tangencies of the front projection. This is not a stable phenomenon, and double-points of Legendrians generically arise only in one-parameter families. These double-points can be considered as Reeb chords of length zero.

Two Legendrian knots inside $J^1\mathbf{R}$ or J^1S^1 with generic fronts are Legendrian isotopic if and only if their front projections can be related by a sequence of *Legendrian Reidemeister moves* [26] together with an ambient isotopy of the front inside $S^1 \times \mathbf{R}_z$; see [15] for an introduction to Legendrian knots.

For convenience we will also introduce a composite move that we will make repeated use of; this is the one shown in Fig. 5, which involves taking two cusp edges with different slopes, and making them cross each other (it is important that the cusps have different slopes).

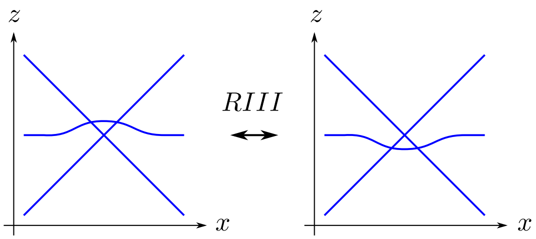

Figure 4. RIII: the third Legendrian Reidemeister move in the front projection

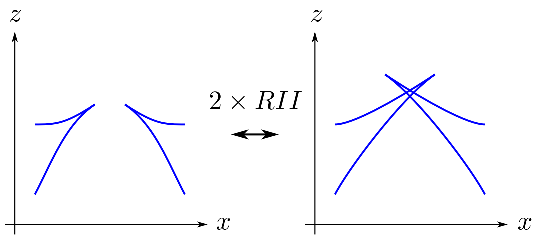

Figure 5. A composite move: the front to the right is obtained by performing two consecutive RII-moves on the front to the left together with an isotopy

2.1.2. The punctured torus. Here we construct an example of a two-dimen sional non-planar Liouville domain: the two torus minus an open ball, which we denote by $(\Sigma_{1,1}, d\lambda)$.

First, consider the primitive

$$\lambda_0 = -\frac{1}{2}(p\,dq - q\,dp)$$

of the standard linear symplectic form $dq \wedge dp$ on $\mathbf{R}^2$. We have the identities

$$\lambda_0 + d\left(\frac{pq}{2}\right) = q\,dp,$$
$$\lambda_0 - d\left(\frac{pq}{2}\right) = -p\,dq.$$

Take a smooth function $\sigma\colon \mathbf{R}^2 \to \mathbf{R}$ which in the standard coordinates labelled by $(p,q) \in \mathbf{R}^2$ is given by

- $\sigma(p,q) = pq/2$ on $\{|q| \leq 1, |p| > 2\}$, while it is of the form $g(p)q/2$ for some smooth function g that satisfies $g(p), g'(p) \geq 0$ on $\{|q| \leq 1, |p| \geq 1\}$;
- $\sigma(p,q) = -pq/2$ on $\{|q| > 2, |p| \leq 1\}$, while it is of the form $-g(q)p/2$ for some smooth function g that satisfies $g(q), g'(q) \geq 0$ on $\{|q| \geq 1, |p| \leq 1\}$;
- $\sigma(p,q) = 0$ on $\{|q| < 1, |p| < 1\}$; and

Consider the exact symplectic manifold $(X, d\lambda)$ which is obtained by taking the cross-shaped domain

$$\{p \in [-2, 2], q \in [-1, 1]\} \cup \{q \in [-2, 2], p \in [-1, 1]\} \subset \mathbf{R}^2$$

and identifying $\{p = 2\}$ with $\{p = -2\}$, and $\{q = -2\}$ with $\{q = 2\}$ in the obvious manner. Topologically the result is a punctured torus. The Liouville form $\lambda_0 + d\sigma$ on $\mathbf{R}^2$ extends to a Liouville form λ on this punctured torus. The punctured torus has a skeleton $Sk \subset X$ which is the image of the cross $\{pq = 0\}$ under the quotient; in other words, $Sk \subset X$ is the union of two smooth Lagrangian circles that intersect transversely in a single point. Note that

$$Sk = \bigcap_{T=1}^{\infty} \phi^{-T}(X).$$

We claim that the sought Liouville domain $(\Sigma_{1,1}, d\lambda)$ can be realised as a suitable subset of this exact symplectic manifold, simply by smoothing its corners; see Fig. 13.

Since $(\Sigma_{1,1}, \lambda)$ is a surface with non-empty boundary, it admits a symplectic trivialisation of its tangent bundle. This implies that the all Lagrangian submanifolds of $\Sigma_{1,1}$ have a well-defined Maslov class; see Sect. 2.5 for more details. We will make heavy use of the fact that the Maslov class depends on the choice of a symplectic trivialisation; in this case, symplectic trivialisations up to homotopy can be identified with homotopy classes of maps

$$\Sigma_{1,1} \to S^1$$

i.e. cohomology classes $H^1(\Sigma_{1,1}; \mathbf{Z})$.

2.2. Barcode of a filtered complex and notions from spectral invariants

A **(strict) filtered complex** over some field $\mathbf{k}$ is a chain complex $(C, \partial, \mathfrak{a})$ in which each element is endowed with an action $\mathfrak{a}(c) \in \mathbf{R} \sqcup \{-\infty\}$ and such that the following properties are satisfied:

- $\mathfrak{a}(c) = -\infty$ if and only if $c = 0$,
- $\mathfrak{a}(r \cdot c) = \mathfrak{a}(c)$ for any $r \in \mathbf{k}^*$,
- $\mathfrak{a}(a + b) \leq \max\{\mathfrak{a}(a), \mathfrak{a}(b)\}$, and
- $\mathfrak{a}(\partial(a)) < \mathfrak{a}(a)$ for any $a \neq 0$.

The subset

$$C^{<a} = \mathfrak{a}^{-1}(\{-\infty\} \cup (-\infty, a))$$

is a $\mathbf{k}$-subspace by the first three bullet points; this subspace is a subcomplex by the last bullet point.

We say that a basis $\{e_i\}$ is **compatible** with the filtration, if the action of a general element $c \in C$ is given by

$$\mathfrak{a}(r_1 e_1 + \cdots + r_n e_n) = \max\{\mathfrak{a}(e_i);\ r_i \neq 0\}, \quad r_i \in \mathbf{k}, \tag{2.1}$$

i.e. the action function is determined by its values on elements in the basis.

Remark 2.4. The non-trivial condition in the definition is the equality "=" in Formula (2.1); for a general basis the above equality gets replaced with an *inequality* "$\leq$".

The existence of a compatible basis for any filtered complex was proven by Barannikov [1]; see [30] for a more general version (in that article they are called "orthogonal bases"), as well as [24]. For proof adapted to the notation used here, see [10, Lemma 2.2].

Given a basis with a specified action on each basis element, one can also use the above formula to *construct* a filtration on the entire complex, under the assumption that the differential decreases action. The Floer complexes described below get endowed with filtrations in precisely this manner, i.e. by specifying an action for each canonical and geometrically induced basis element.

For every filtered complex there is a notion of a **barcode**; we refer to [10, Section 2] for the details of the presentation that we rely on here. The barcode is a set of intervals of the form $[a, b)$ and $[a, +\infty)$, where $a, b \in \mathbf{R}$, and we allow multiplicities. Instead of giving the usual definition of the barcode, we give it the following alternative characterisation.

Lemma 2.5. (Lemma 2.6 in [10]) *The barcode can be recovered from the following data:*

(1) For any basis which is compatible with the action filtration, there is a bijection between the set of actions of basis elements and the union of start and endpoints of bars (counted with multiplicities).
(2) For any two numbers $a < b$, the number of bars of C_* whose endpoints e satisfy $e \in (b, +\infty]$ and starting points s satisfy $s \in [a, b)$ is equal to $\dim H(C^{<b}/C^{<a})$.

Corollary 2.6. *Assume that the barcode contains a finite bar $[a, b)$. Then, for any compatible basis $\{e_i\}$, we can deduce the existence of basis elements e_i and e_j with $\mathfrak{a}(e_i) = b$, $\mathfrak{a}(e_j) = a$, such that $\langle \partial e_i, e_j \rangle \neq 0$.*

Conversely, if there exists a compatible basis $\{e_i\}$ for which $\partial e_i = r e_j$ for some coefficient $r \neq 0$, then the barcode contains the finite bar $[\mathfrak{a}(e_j), \mathfrak{a}(e_i))$.

Note that the barcode considered here is independent of the grading. An efficient way to deduce properties of the barcode is thus to find (possibly several different) gradings for the complex, for which the differential remains an operation of degree -1. The existence of such gradings imposes restrictions on the differential, which in view of the previous corollary imposes restrictions on the barcode. This technique will be used in the proofs given in Sects. 3.1 and 3.3.

For a filtered complex as above we can associate the following important notions.

Definition 2.7.

(1) The *spectral range* $\rho(C, \partial, \mathfrak{a}) \in \{-\infty\} \cup [0, +\infty]$ is the supremum of the distances between starting points of the semi-infinite bars in the barcode.

(2) The *boundary depth* $\beta(C, \partial, \mathfrak{a}) \in \{-\infty\} \cup [0, +\infty]$ is supremum of the lengths of the finite bars in the barcode.

Note that the above quantities automatically are equal to $-\infty$ in the case when the supremum is taken over the empty set (i.e. when there are no semi-infinite and finite bars, respectively).

An important feature of the barcode is that remains invariant under simple bifurcations of the complex, i.e. action preserving handle-slides and birth/deaths. Legendrian isotopies induce one-parameter families of the Floer complex considered here, which undergoes bifurcations of precisely this type; hence the corresponding barcode undergoes continuous deformations under Legendrian isotopies. Since this property will not be needed, we do not give more details here, but instead direct the interested reader to [10].

2.3. Outline of Floer homology and generating family homology for Legendrians

Floer homology for pairs (L_0, L_1) of closed exact Lagrangian submanifolds of cotangent bundles was originally defined by Floer [16]. For any such pair one obtains the Floer chain complex $CF(L_0, L_1)$ with a basis given by the intersections $L_0 \cap L_1$, which here are assumed to be transverse. Floer also showed that the homology of the complex—the so-called Floer homology $HF(L_0, L_1)$—is invariant under Hamiltonian isotopy of either Lagrangian L_i. Moreover, in the case when L_1 is a C^1-small Hamiltonian perturbation of L_0 the Floer complex $CF(L_0, L_1) = C^{\mathrm{Morse}}(f)$ is the Morse complex for a C^1-small Morse function $f\colon L_0 \to \mathbf{R}$ and suitable auxiliary data; see Floer's original computation [17]. (This property might not hold for the Floer homology of a Legendrian, due to additional generators corresponding to Reeb chords; see Sect. 1.1.)

Nowadays there are several different techniques available for constructing Floer homology. Here we will consider the setting of Legendrian submanifolds of contactisations $(\overline{W} \times \mathbf{R}, \alpha_{\mathrm{st}})$ of a Liouville manifold $(\overline{W}, d\lambda)$, in which Floer homology associates a chain complex $CF(\Lambda_0, \Lambda_1)$ to a pair of Legendrian submanifolds equipped with additional data. In this case, the homology of the complex is invariant under Legendrian isotopy of either Legendrian Λ_i. This is the version that we will use also in the case of exact Lagrangian embeddings in $(\overline{W}, d\lambda)$. To that end, recall that exact Lagrangians admit lifts to Legendrians by Lemma 2.2, and that a Hamiltonian isotopy of the Lagrangian induces a Legendrian isotopy of the Legendrian lift by Lemma 2.1.

In the case when $\overline{W} = T^*M$, and thus $\overline{W} \times \mathbf{R} = J^1M$, in [33] Zapolsky relied on generating family homology defined for generating families due to Chekanov [7] to define spectral invariants. Generating family homology is a Hamiltonian isotopy invariant obtained by Morse functions on finite-dimensional spaces, which behaves very similarly to Floer homology. In certain cases these two invariants have even been shown to be equivalent. Since we will work with contactisations that are more general than jet-spaces, we instead follow the techniques from [13] by Ekholm–Etnyre–Sabloff, where the Floer chain complex is constructed as the linearised Legendrian contact-homology complex associated to the Chekanov–Eliashberg algebra [6], [12].

First we outline the general set-up Floer homology in the setting of Legendrians, which applies equally well to either the version used here or the version defined using generating families (when applicable). Given a pair of Legendrians $\Lambda_0, \Lambda_1 \subset \overline{W} \times \mathbf{R}$, equipped with additional data denoted by ε_i to be specified below (in the version defined using generating families, these additional data are simply the choice of a generating family), one obtains a graded (grading is in $\mathbf{Z}$ or $\mathbf{Z}/\mu\mathbf{Z}$ depending on the Maslov class as described in Sect. 2.5) filtered chain complex

$$(CF_*((\Lambda_0, \varepsilon_0), (\Lambda_1, \varepsilon_1)), \partial, \mathfrak{a})$$

with a canonical compatible basis as a $\mathbf{k}$-vector space given by the

- Reeb chords c from Λ_0 to Λ_1 of action $\mathfrak{a}(c) = \ell(c)$ equal to the Reeb chord length; together with the
- Reeb chords c from Λ_1 to Λ_0 of action $\mathfrak{a}(c) = -\ell(c)$ equal to minus the Reeb chord length.

We assume that all Reeb chords are transversely cut out, and hence that they form a discrete subset, which thus is finite whenever the Legendrians are closed. With our conventions the differential is *strictly action decreasing and of degree* -1. In the case of generating family homology, the differential is the Morse homology differential for a Morse function on a finite-dimensional manifold that is constructed using the generating family. Below we give more details of the Floer complex defined via the Chekanov–Eliashberg algebra, for which the differential counts pseudoholomorphic strips in $\overline{W}$ with boundary on the Lagrangian projections $\Pi_{\overline{W}}(\Lambda_i) \subset \overline{W}$ (these are exact Lagrangian immersions with transverse self-intersections). In this case the strips are moreover allowed to have corners that map to the double points of the Lagrangian projections; the strips are then counted with weights given by the value of the augmentations on the corresponding pure Reeb chords. More details are given in Sect. 2.4 below.

The Floer complex satisfies the following important properties; see [13] for details.

- A Legendrian isotopy of the Legendrian Λ_i induces a canonical continuation of the additional data ε_i, and the resulting one-parameter family of Floer complexes undergoes only simple bifurcations, i.e. handle-slides and births/deaths. In particular, the homology of the complex is not changed under such a deformation.
- In the case when $\Lambda \subset \overline{W} \times \mathbf{R}$ has no Reeb chords (i.e. it is the lift of an exact Lagrangian embedding), and when Λ' is a C^1-small Legendrian perturbation, then the induced Floer complex

  $$(CF((\Lambda, \varepsilon), (\Lambda', \varepsilon')), \partial, \mathfrak{a}) = C^{\mathrm{Morse}}(f; \mathbf{k})$$

  is the Morse homology complex of some C^1-small Morse function $f \colon \Lambda \to \mathbf{R}$.

Again we refer to Sect. 1.1 for a description of the complex under the presence of pure Reeb chords; in this case the Morse complex is only realised as a quotient complex of a subcomplex.

2.4. Floer complex as the linearised Chekanov–Eliashberg algebra

Here we present the relevant technical details for the particular construction of Floer homology used in this paper, i.e. relying on the Chekanov–Eliashberg algebra for Legendrians in contactisation from [12]. Using the Chekanov–Eliashberg algebra to define Floer homology for Legendrian submanifolds is not new, it goes back to work [13] by Ekholm–Etnyre–Sabloff; also see [22] by Lanzat–Zapolsky for a nice application of this theory together with a systematic treatment.

Assume that $\Lambda_0, \Lambda_1 \subset \overline{W} \times \mathbf{R}$ are two Legendrian submanifolds. Further, assume that the Chekanov–Eliashberg algebras of Λ_i admit augmentations

$$\varepsilon_i \colon (\mathcal{A}(\Lambda_i), \partial) \to \mathbf{k};$$

recall that the Chekanov–Eliashberg algebra is a unital DGA generated by the Reeb chords of the Legendrian, and that an augmentation is a unital DGA morphism to the ground field. In particular, when the Legendrian Λ_i has no Reeb chords, the Chekanov–Eliashberg algebra takes the simple form $\mathcal{A}(\Lambda_i) = \mathbf{k}$, and there is a canonical augmentation. An important property of augmentations is that they can be pushed forward under a Legendrian isotopy; see, e.g. [4] and [6].

Typically one wants more additional data than just an augmentation. For instance, to use coefficients in a field of characteristic different from two, one also needs to fix the choice of a spin structure on both Legendrians Λ_i. To endow the Floer complex a $\mathbf{Z}$-grading, we need to specify a Maslov potential; we refer to Sect. 2.5 for more details concerning the grading, which will play an important role for us.

The Floer complex

$$CF((\Lambda_0, \varepsilon_0), (\Lambda_1, \varepsilon_1))$$

is generated by the chords that have one endpoint on Λ_0 and one endpoint on Λ_1 (either being a starting point). These Reeb chords on $\Lambda_0 \cup \Lambda_1$ are called the **mixed** Reeb chords. To define the differential, we will identify the above vector space with the underlying vector space linearised Legendrian contact homology complex of the link $\Lambda_0 \cup \phi^T_{\partial_z}(\Lambda_1)$, where the latter is the $\mathbf{k}$-vector space is generated by all Reeb chords that start on Λ_0 and end on the translation $\phi^T_{\partial_z}(\Lambda_1)$ of Λ_1 in the positive z-direction. Note that the mixed chords on $\Lambda_0 \cup \Lambda_1$ are in bijective correspondence with the mixed chords on $\Lambda_0 \cup \phi^T_{\partial_z}(\Lambda_1)$ for any choice of $T \in \mathbf{R}$. In the following we take $T \gg 0$ to be sufficiently large, so that no chord starts on $\phi^T_{\partial_z}(\Lambda_1)$ and ends on Λ_0. Of course, the length of a mixed chord c above depends on the parameter T and will not be equal to the action $\mathfrak{a}(c)$ defined above; the relation between action and length is given by

$$\ell(c) = \mathfrak{a}(c) + T.$$

The remaining Reeb chords on the link $\Lambda_0 \cup \phi^T_{\partial_z}(\Lambda_1)$ have both endpoints either on Λ_0 or $\phi^T_{\partial_z}(\Lambda_1)$, and are called **pure**. Note that the Reeb chords on $\phi^T_{\partial_z}(\Lambda_1)$ are in bijective correspondence with those of Λ_1. In fact, their Chekanov–Eliashberg algebras are even canonically isomorphic.

The differential is the Linearised Legendrian contact homology differential induced by a choice of almost complex structure, together with the augmentations ε_i for the Chekanov–Eliashberg algebras $\mathcal{A}(\Lambda_i)$ generated by the pure chords. This version of a Floer complex defined via the Chekanov–Eliashberg algebra was originally considered in [13]; also see [4] for a more recent realisation. We now give a sketch of the definition of the differential. It is roughly speaking defined by counts of rigid pseudoholomorphic discs in $(\overline{W}, d\lambda)$, for some choice of compatible almost complex structure, where the disc has

- boundary on the exact Lagrangian immersion $\Pi_{\overline{W}}(\Lambda_0 \cup \phi^T_{\partial_z}(\Lambda_1)) \subset (\overline{W}, \lambda)$;
- precisely one positive puncture at a double point which corresponds to a mixed chord—this is the input;
- precisely one negative puncture at a double point which corresponds to a mixed chord—this is the output; and
- several additional negative punctures at double points which correspond to pure chords.

By positive (resp. negative) boundary puncture, one means a point where the boundary of the pseudoholomorphic disc makes a jump that increases (resp. decreases) the z-value of the Legendrian $\Lambda_0 \cup \phi^T_{\partial_z}(\Lambda_1) \subset \overline{W} \times \mathbf{R}_z$ when following the boundary according to the orientation of the disc induced by the almost complex structure. When counting the strip, one weighs the count by the value of the augmentation ε_i on the pure chords from the last point. This is a part of the so-called linearised differential induced by the augmentation, as defined in [6]; also see the notion of the bilinearised Legendrian contact homology as defined by Bourgeois–Chantraine in [3].

From positivity of symplectic area of such pseudoholomorphic discs together with Stokes' theorem one obtains that the Reeb chord length of the input chord must be larger than the Reeb chord of the output. In other words, the complex is strictly filtered in the sense defined in Sect. 2.2, and the Reeb chords constitute a compatible basis.

From the index formula for the expected dimension of the moduli space of pseudoholomorphic discs, it follows that the degree of the input is one greater than the degree of the output; i.e. the differential is of degree -1.

2.5. Maslov potential and grading

The Maslov potential is a useful framework for introducing gradings in Lagrangian Floer homology which originally is due to Seidel [27]. The choice of a Maslov potential gives a well-defined grading in $\mathbf{Z}$. In general the potential is only well-defined modulo the Maslov number $\mu \in \mathbf{Z}$ (the positive generator of the subgroup of $\mathbf{Z}$ which is the image of the Maslov class); in that case the grading is only defined in $\mathbf{Z}/\mu\mathbf{Z}$. Here we describe a grading for which the differential of the Floer complex considered above becomes a map of degree -1, i.e. it decreases the degree.

Assume that $\overline{W}$ has vanishing first Chern class; this is, e.g. the case when $\overline{W}$ has a symplectic trivialisation, which is automatic when $\dim_{\mathbf{R}} \overline{W} = 2$. The $\mathbf{Z}$–grading of the generators is defined as follows.

Consider the determinant bundle

$$\mathbf{C}^* \to \det_{\mathbf{C}} T\overline{W} \to \overline{W}$$

induced by some choice of a compatible almost complex structure. The quotient

$$\mathbf{C}^*/\mathbf{R}^* = (\mathbf{R}^2\backslash\{0\})/\mathbf{R}^* = \mathbf{R}P^1 = \mathbf{R}/\pi\mathbf{Z}$$

gives rise to an induced $\mathbf{R}P^1$-bundle that we denote by

$$\mathcal{L} = (\det_{\mathbf{C}} T\overline{W})/\mathbf{R}^* \to \overline{W}.$$

Note that the bundle $\mathcal{L}$ is trivial when $\overline{W}$ has vanishing first Chern class (actually, the first Chern class being two-torsion is sufficient). In this case there might be several choices of homotopy classes of trivialisations.

First, we make the choice of a trivialisation of the above determinant bundle. This choice gives rise to a trivialisation $\mathcal{L} = \mathbf{R}P^1 \times \overline{W} \to \overline{W}$ of the $\mathbf{R}P^1$-bundle as well. Then, taking the fibre-wise universal cover of this trivial $\mathbf{R}P^1$-bundle, we obtain the affine $\mathbf{R}$-bundle $\tilde{\mathcal{L}} = \mathbf{R} \times \overline{W} \to \overline{W}$. The fibre of this bundle is thus the choice of an $\mathbf{R}$-lift of the angle in $\mathbf{R}P^1 = \mathbf{R}/\pi\mathbf{Z}$ of an unoriented real line.

Second, one makes the choice of a **Maslov potential** for each of the Legendrians Λ_i. This is the lift of the canonically defined section

$$(\det_{\mathbf{R}} T\Pi_{\overline{W}}(\Lambda_i))/\mathbf{R}^* \subset (\det_{\mathbf{C}} T\overline{W})/\mathbf{R}^*$$

along Λ_i of the above $\mathbf{R}P^1$-bundle $\mathcal{L}$ to the associated $\mathbf{R}$-bundle $\tilde{\mathcal{L}}$. Recall that a non-zero Maslov class is the obstruction to the existence of such a lift. When a Maslov potential exists and the Legendrian is connected, there is a natural free and transitive $\mathbf{Z}$-action on its Maslov potentials.

Given choices of Maslov potentials, the grading of a generator $c \in CF_*((\Lambda_0, \varepsilon_0), (\Lambda_1, \varepsilon_1))$ is finally obtained in the following manner. Denote by $\tilde{\varphi}_i \in \tilde{\mathcal{L}}_c$ the $\mathbf{R}$-lift of the angle of the real determinant line

$$(\det_{\mathbf{R}} T_c\Pi_{\overline{W}}(\Lambda_i))/\mathbf{R}^* \subset (\det_{\mathbf{C}} T_c\overline{W})/\mathbf{R}^*$$

specified by the choices of Maslov potentials. Consider a compatible almost complex structure J on $T_c\overline{W}$ for which $J \cdot T_c\Pi_{\overline{W}}(\Lambda_0) = T_c\Pi_{\overline{W}}(\Lambda_1)$ together with the induced family of Lagrangian planes $e^{it}T_c\Pi_{\overline{W}}(\Lambda_0) \in T_c\overline{W}$, $t \in [0, \pi/2]$, that joins $T_c\Pi_{\overline{W}}(\Lambda_0)$ to

$$e^{i\pi/2}T_c\Pi_{\overline{W}}(\Lambda_0) = J \cdot T_c\Pi_{\overline{W}}(\Lambda_0) = T_c\Pi_{\overline{W}}(\Lambda_1).$$

There is a continuous path of real determinant lines $\varphi_0^t \in \mathcal{L}_c$; denote by $\tilde{\varphi}_0^t \in \tilde{\mathcal{L}}_c$ the continuous lift to the fibre-wise universal cover $\mathbf{R} \to \mathbb{R}P^1$, where $\tilde{\varphi}_0^0 = \tilde{\varphi}_0$. In particular, $\tilde{\varphi}_0^{\pi/2}$ is a lift of the determinant line φ_1. The **degree of the generator** c is finally defined by

$$|c| = (\tilde{\varphi}_0^{\pi/2} - \tilde{\varphi}_1)/\pi \in \mathbf{Z}.$$

In the below examples we provide some useful techniques for specifying Maslov potentials and computing degrees in the cases that we are interested in here.

Example 2.8.

(1) In the case of $\overline{W} = T^*M$ there is a trivialisation of $\det_{\mathbf{C}} T(T^*M)$ in which the tangent planes to the zero section all coincide with the real part $\mathbf{R}^* \subset \mathbf{C}^*$ of the fibres. The zero-section $\Lambda_0 = j^1 0$ can be induced with the Maslov potential $\tilde{\varphi}_0$ which is zero in each $\mathbf{R}$-fibre of $\tilde{\mathcal{L}}$.

(2) A choice of Maslov potential for a general Legendrian $\Lambda_1 \subset J^1M$ for the trivialisation from Part (1) above (if it exists) can be described by comparing it to the canonical Maslov potential for the zero section $j^1 0$. More precisely, the difference between the Maslov potential at a point $x \in \Lambda_1$ for which $T_x\Pi_{\overline{W}}(\Lambda_1)$ is transverse to the Lagrangian fibre of T^*M and the canonical Maslov potential for $j^1 0 = \Lambda_0$ at the point $p(x) \in \Lambda_0$, where $p\colon J^1M \to M$ is the bundle projection, can be described by an integer $m(x) \in \mathbf{Z}$ in the following manner.
The fibre-wise rescaling of J^1M induces an isotopy of Legendrian tangent planes that isotopes any tangent plane $T_x\Lambda_1$ which is transverse to the fibre to the tangent plane $T_{p(x)}j^1 0$ of the zero section. There is an induced continuous path of determinant lines $\varphi_1^t \in \mathcal{L}$ where

$$\varphi_1^0 = (\det_{\mathbf{R}} T_x\Pi_{\overline{W}}(\Lambda_1))/\mathbf{R}^* \subset (\det_{\mathbf{C}} T_x(T^*M))/\mathbf{R}^*,$$

and φ_1^1 is the determinant line of the zero-section at the point $p(x)$. Consider the continuous choice of lifts $\tilde{\varphi}_1^t \in \tilde{\mathcal{L}}$ that extend the choice of Maslov potential for Λ_1. The difference

$$m(x) = (\tilde{\varphi}_1^1 - \tilde{\varphi}_0)/\pi \in \mathbf{Z}$$

is an integer that uniquely recovers the choice of Maslov potential at $x \in \Lambda_1$ (for x where the Lagrangian projection is transverse to the fibre).
The integer $m(x)$ is locally constant in the open subsets of Λ_1 for which the Lagrangian projection is transverse to the fibres, and changes by $+1$ as one traverses a cusp-edge in the direction of decreasing z-value. This is illustrated in Fig. 6.

(3) In the case when $\det_{\mathbf{C}} T\overline{W}$ is trivial, the homotopy classes of trivialisations of $\det_{\mathbf{C}} T\overline{W}$ are in bijection with homotopy classes of maps $\overline{W} \to \mathbf{C}^*$, which is the same as classes in $H^1(\overline{W}; \mathbf{Z})$. In the particular case $\overline{W} = T^*S^1$, the description of the Maslov potential given in Part (2) is only valid above a simply connected subset, e.g. $T^*(-\pi, \pi) \subset T^*S^1$. The Maslov potential for a general Legendrian in this setting can be described by the choice of numbers $m(x)$ as above, that, however, satisfy the additional property that they make a jump by a fixed value $l \in 2\mathbf{Z}$ when traversing the hypersurface $\{\theta = \pi\}$ in the direction of increasing θ-value. (The case $l = 0$ corresponds to the canonical trivialisation for which the zero-section admits a Maslov potential.) This is illustrated in the top of Fig. 7.

(4) Let $\Lambda_i \subset J^1M$, $i = 0, 1$, be two Legendrians with choices of Maslov potentials that have the form $j^1 f_i$ over some subset in M (i.e. the Lagrangian projections are transverse to the fibre there), where the Maslov potentials are determined by integers $m_i \in \mathbf{Z}$ in the manner described above. If $f_1 - f_0$ has a non-degenerate critical point at $p \in M$ (i.e. there is a transverse Reeb chord c there), then the above degree formula becomes

$$|c| = \mathrm{index}_p^{\mathrm{Morse}}(f_1 - f_0) + m_0 - m_1$$

where the first term on the right-hand side is the Morse index of the critical point. We refer to [11, Lemma 3.4] for the computation.

Lemma 2.9.

(1) Let $\phi^1 : \overline{W} \times \mathbf{R} \to \overline{W} \times \mathbf{R}$ be the time-one map of a compactly supported contact isotopy. For any choice of Maslov potential on the Legendrian Λ there an induced Maslov potential on its image $\phi^1(\Lambda) \subset \overline{W} \times \mathbf{R}$ uniquely defined by the property that the Maslov potentials extend over the exact Lagrangian cobordism from Λ to $\phi^1(\Lambda)$ induced by the isotopy.

(2) If ϕ^1 is a generic C^1-small contact isotopy, then the small chords of $\Lambda \cup \phi^1(\Lambda)$ are in bijective correspondence with the critical points of a C^1-small Morse function $f : \Lambda \to \mathbf{R}$, and the above grading coincides with the Morse index, if $\phi^1(\Lambda)$ is endowed with the Maslov potential induced from Λ via the isotopy ϕ^t as in Part (1).

Proof. (1) The trace of the Legendrian isotopy can be made into a Lagrangian cylinder inside the symplectisation

$$(\mathbf{R}_t \times \overline{W} \times \mathbf{R}_z, d(e^t \alpha_{\mathrm{st}}))$$

with cylindrical ends over the initial and final Legendrian; see work [5] by Chantraine. The Maslov potential of Λ induces a Maslov potential on the negative end of this cobordism. This Maslov potential can be extended to the entire cobordism by elementary topology (it is a Lagrangian cylinder). The induced Maslov potential on the positive end is the sought Maslov potential on $\phi^1(\Lambda)$.

(2) This computation is standard, and can be performed in a small neighbourhood of Λ. In particular, for a small perturbation of the zero-section $j^1 0 \subset J^1M$ by a section $j^1 f$, this is an immediate consequence of Part (4) of Example 1. In general, recall that any Legendrian Λ has a standard neighbourhood which is contactomorphic to a neighbourhood of the zero section $j^1 0 \subset J^1\Lambda$, under which Λ, moreover, is identified with $j^1 0$; see [18]. The perturbation can be assumed to be given by the one-jet $j^1 f$ of some C^1-small smooth function $f : \Lambda \to \mathbf{R}$ in the same neighbourhood. □

3. Examples that exhibit unbounded spectral norms

The following basic auxiliary results facilitate our computations, and will be invoked repeatedly.

Lemma 3.1.

(1) *Let* $\phi^t \colon \Lambda_0 \hookrightarrow \overline{W} \times \mathbf{R}$ *be a Legendrian isotopy of a closed Legendrian* Λ_0 *that admits a Maslov potential, and endow* $\phi^1(\Lambda_0)$ *with the Maslov potential induced from* Λ_0 *via the isotopy, as described in Part (1) of Lemma 2.9. Further assume that* Λ_0 *has no Reeb chords. If the complex* $CF(\Lambda_0, \phi^1(\Lambda_0))$ *in degrees* 0 *and* $\dim \Lambda_0$ *consists of unique Reeb chord generators* c *and* d*, then the spectral range satisfies*

$$\rho(CF(\Lambda_0, \phi^1(\Lambda_0))) \geq |\ell(c) - \ell(d)|.$$

(In fact, it is even true that the spectral range is equal *to* $\ell(c) - \ell(d)$*, where this quantity, moreover, is positive, but we will not show this.)*

(2) *Consider a Floer complex* $CF(\Lambda_0, \Lambda_1)$ *which is* $\mathbf{Z}$*-graded and acyclic. Furthermore, assume that there is a choice of symplectic trivialisation and Maslov potential for which there are no generators in degrees* $i+1$ *or* $i-2$*, while there are unique Reeb chords* c, d *in the degrees* $|c| = i$ *and* $|d| = i-1$*. Then the boundary depth satisfies the bound*

$$\beta(CF(\Lambda_0, \Lambda_1)) \geq \ell(c) - \ell(d).$$

Proof. (1): This follows from invariance properties of the Floer homology. Note that the homology of $CF(\Lambda_0, \Lambda_0)$ has unique generators in degrees 0 and $\dim \Lambda$ which represent the point class and fundamental class in Morse homology. It follows by degree reasons that the Reeb chord generators c and d must both be cycles which are not boundaries. The two corresponding semi-infinite bars in the barcode have endpoints that are separated by precisely $|\ell(c) - \ell(d)|$ as sought.

(2): Acyclicity together with the degree assumptions implies that $\partial c = d$. The statement then follows by the second part of Corollary 2.6 since the Reeb chords form a compatible basis. □

3.1. Legendrian isotopy of the unknot (Proof of Part (2) of Theorem A)

Consider the contact manifold $J^1\mathbf{R} = \mathbf{R}_q \times \mathbf{R}_p \times \mathbf{R}_z$ with coordinates q, p, z and contact form $dz - p\,dq$. Under the quotient $\mathbf{R}_q \to \mathbf{R}/2\pi\mathbf{Z} = S^1$ we obtain the angular coordinate θ induced by $\theta \equiv q \mod 2\pi$. In other words, the aforementioned contact manifold $J^1\mathbf{R}$ is the universal cover of the contact manifold $J^1S^1 = S^1 \times \mathbf{R}_p \times \mathbf{R}_z$ equipped with the standard contact form $dz - p\,d\theta$.

First consider the standard Legendrian unknot $\Lambda_{\mathrm{st}} \subset J^1S^1$ with front projection as shown in Fig. 6, which thus is contained inside the subset $J^1(-\pi, \pi) \subset J^1S^1$. The p-coordinate of this particular representative can be seen to be estimated in terms of the ratio of a and b, which yields

$$\Lambda_{\mathrm{st}} \subset \{|p| \leq 2a/b\}.$$

Recall the well-known fact that Λ_{st} has vanishing Maslov class and hence admits a Maslov potential; see Fig. 6. Further, this Legendrian has a unique transverse Reeb chord and its Chekanov–Eliashberg algebra is equal to the polynomial algebra in one variable of degree 1 with no differential (either for $\mathbf{k} = \mathbf{Z}_2$ or for arbitrary $\mathbf{k}$ and the choice of bounding spin structure);

see [14]. In particular, its Chekanov–Eliashberg algebra admits the trivial augmentation.

We also fix a Legendrian fibre

$$F = F_{(\pi/4,0)} = \{\pi/2\} \times \mathbf{R}_p \times \{0\} \subset J^1(-\pi,\pi) \subset J^1S^1.$$

Note that the Reeb chords between any Legendrian Λ and F are in bijective correspondence with the intersection points of Λ and the hypersurface $\{\theta = \pi/4\}$. Note that the image of F under the front projection is given by the point $\{(\pi/4,0)\}$; Reeb chords correspond to lines contained inside $\{\theta = \pi/4\}$ in the front projection that have one endpoint on $\{(\pi/4,0\})$ and one endpoint on the projection of Λ. These chords are depicted in Fig. 6.

Since F that has no Reeb chords, its Chekanov–Eliashberg algebra trivially admits an augmentation. We can thus define the Floer homology complex $CF(\Lambda_{\mathrm{st}},F)$ which is generated by two Reeb chords c and d, where $0>\ell(c)>\ell(d)$ and $|c|=|d|+1$. Note that $CF(\Lambda_{\mathrm{st}},F)$ is an acyclic complex by invariance under Legendrian isotopy; after shrinking the unknot sufficiently, all mixed chords disappear.

The goal is to construct a Legendrian isotopy $\Lambda^t_{\mathrm{st}} \subset J^1S^1$ of the unknot confined to the subset

$$\{|p| \leq 2a/b\} \subset J^1S^1$$

for which the boundary depth of $CF(\Lambda^T_{\mathrm{st}},F)$ becomes arbitrarily large as $t \to +\infty$. This isotopy will be constructed as the projection of an isotopy $\tilde{\Lambda}^t_{\mathrm{st}} \subset J^1\mathbf{R}$ of the unknot inside the universal cover $J^1\mathbf{R} \to J^1S^1$. In fact, the Legendrian isotopy $\tilde{\Lambda}^t_{\mathrm{st}}$ is very simple; it is the rescaling of

$$\tilde{\Lambda}_{\mathrm{st}} = \Lambda_{\mathrm{st}} \subset J^1(-\pi,\pi) \subset J^1\mathbf{R}$$

under the contact isotopy $(q,p,z) \mapsto (e^t \cdot q, p, e^t \cdot z)$ defined on the universal cover; note that this contact isotopy simply rescales the front projection.

It is easy to check that $CF(\tilde{\Lambda}^t_{\mathrm{st}},F)$ satisfies the property that the boundary depth goes to $+\infty$ as $t \to +\infty$. Indeed, these complexes are generated by the two unique transversely cut out Reeb chords c_t and d_t between $\tilde{\Lambda}^t_{\mathrm{st}}$ and F for all values $t>0$. These chords, moreover, satisfy the property that $\ell(c_t)-\ell(d_t)$ becomes arbitrarily large as $t \to +\infty$; c.f. Part (2) of Lemma 3.1.

What remains to prove is the following two claims for the projection $\Lambda^t_{\mathrm{st}} \subset J^1S^1$ of the Legendrian rescaling $\tilde{\Lambda}_t \subset J^1\mathbf{R}$. First, we claim that Λ^t_{st} indeed is a Legendrian isotopy. Second, we show that the boundary depth of $CF(\Lambda^t_{\mathrm{st}},F)$ goes to $+\infty$ as $t \to +\infty$

The fact that Λ^t_{st} is a Legendrian isotopy can be seen by considering the sequence of front projections; see Figs. 7 and 8. Except for an isotopy of the front, the front also undergoes a sequence *RIII*-moves together with the composite move shown in Fig. 5. The Lagrangian projection of $\tilde{\Lambda}_2$ is shown in Fig. 9.

Then we need to estimate the boundary depth of the sequence of Floer complexes $CF(\Lambda^t_{\mathrm{st}},F)$. In addition to Reeb chords c_t and d_t, which correspond to the Reeb mixed Reeb chords on the lift and have exactly the same

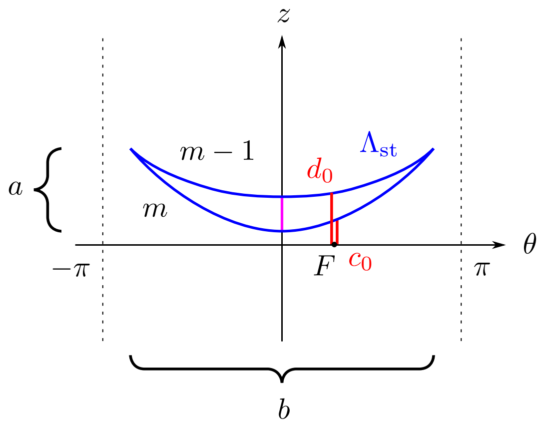

Figure 6. The standard Legendrian unknot $\Lambda_{\rm st}$ and the Legendrian fibre F. Note that there are precisely two transverse Reeb chords c_0, d_0 between F and $\Lambda_{\rm st}$. The choice of $m \in \mathbf{Z}$ determines a Maslov potential on $\Lambda_{\rm st}$ as described in Part (2) of Example 1

actions, there are additional Reeb chords between $\Lambda^t_{\rm st}$ and F that appear as $t \to +\infty$. Nevertheless, we claim that the boundary depth of $CF(\Lambda^t_{\rm st}, F)$ still is bounded from below by the boundary depth $\beta(CF(\tilde{\Lambda}^t_{\rm st}, F))$.

To see the last claim, we will consider different gradings of the complexes $CF(\Lambda^t_{\rm st}, F)$, obtained by changing the symplectic trivialisation of T^*S^1. Note that $\Lambda_{\rm st}$ is null-homotopic inside J^1S^1 and thus has a vanishing Maslov class independently of the choice of symplectic trivialisation. Moreover, the chords c_t and d_t always satisfy $|c_t| - |d_t| = 1$ regardless of the choice of Maslov potential and symplectic trivialisation; see the top of Fig. 7.

We claim that, after changing the symplectic trivialisation of T^*S^1 by introducing a sufficiently large number $l/2 \gg 0$ of full rotations of the standard symplectic frame as one traverses the hypersurface $\{\theta = \pi\}$ in the direction of increasing θ-coordinate, all generators c' in the complex except different from c_t and d_t acquire degrees that satisfy

$$|c'| - |c_t| \notin [-10, 10].$$

To see this, we note that the Maslov potential of these sheets acquire an additional term kl where $k \in \mathbf{Z}\backslash\{0\}$; see Parts (3) and (4) of Example 1.

Since these degree properties can be achieved, the statement now follows directly by Part (2) of Lemma 3.1. □

3.2. Legendrian isotopy of the zero-section (Proof of Part (1) of Theorem A)

We use the same coordinates as in the above Sect. 3.1. In fact, the sought Legendrian isotopy is also constructed in a manner similar to the construction of Λ_t given there, by performing a rescaling of a part of the front inside the universal cover $J^1\mathbf{R}$ (and then projecting back to J^1S^1). The isotopy is shown in Figs. 10 and 11. One starts by considering a Legendrian perturbation

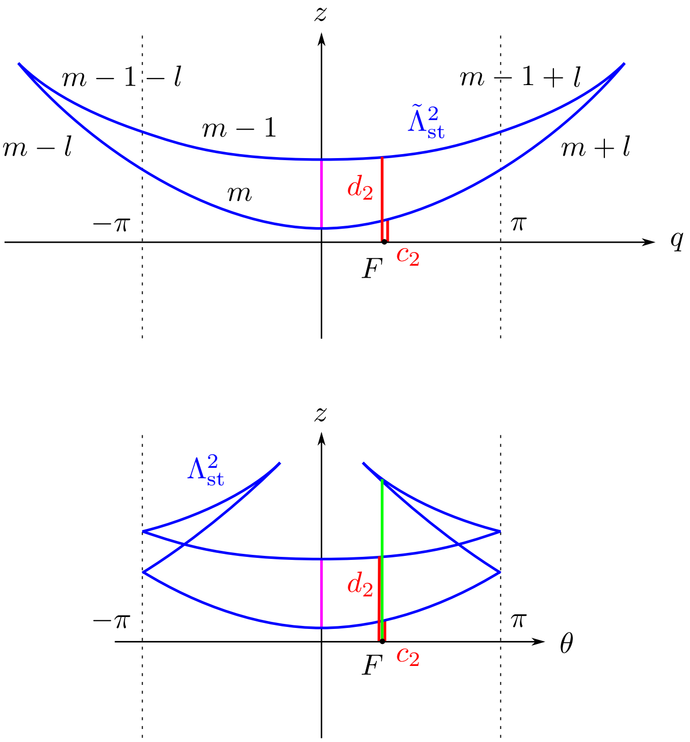

FIGURE 7. Above: $\tilde{\Lambda}^2_{\text{st}}$ has a front which is a linear rescaling of the front of Λ_{st} inside $J^1\mathbf{R}$. The number m defines a choice of Maslov potential for Λ^2_{st}, where $l \in 2\mathbf{Z}$ depends on the homotopy class of the trivialisation of $\det_{\mathbf{C}}(T(T^*S^1))$. Below: Λ^2_{st} is the projection of $\tilde{\Lambda}^2_{\text{st}}$ inside J^1S^1. Except for the mixed chords c_t and d_t that exist for the lift, there are now additional mixed chords

$j^1 f$ of $j^1 0$ which has precisely two chords. Then one performs a RII-move. Rescaling the front of the Legendrian introduced by the RII-move in the universal cover $\mathbf{R}^2$ and then projecting back to $S^1 \times \mathbf{R}$ is again a Legendrian isotopy. In Fig. 11 one sees that there are exactly two chords between $j^1 0$ and the produced Legendrians, while the difference in action between these two generators grows indefinitely as $t \to +\infty$. □

3.3. Hamiltonian isotopy on the punctured torus (Proof of Theorem D)

Here we consider the exact Lagrangian embedding $L \subset (\Sigma_{1,1}, d\lambda)$ of S^1 which is given as the image of $\{p = 0\} \subset \mathbf{R}^2$ under the quotient construction in Sect. 2.1.2; see Fig. 13. We perform a Hamiltonian perturbation L' that intersects the original Lagrangian transversely in precisely two points c and d. The spectral norm is thus $\gamma(CF(L, L')) = \ell(c) - \ell(d)$.

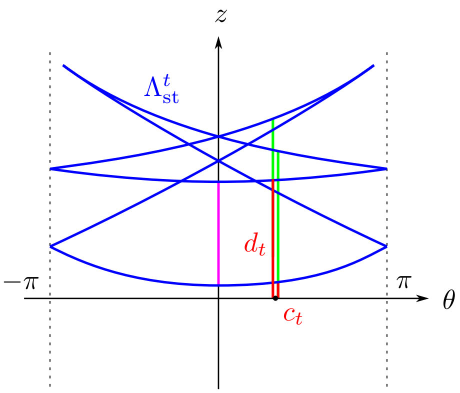

FIGURE 8. This shows the projection Λ^t_{st} of the rescaling $\tilde{\Lambda}^t_{\mathrm{st}}$ under the universal cover $J^1\mathbf{R} \to J^1S^1$

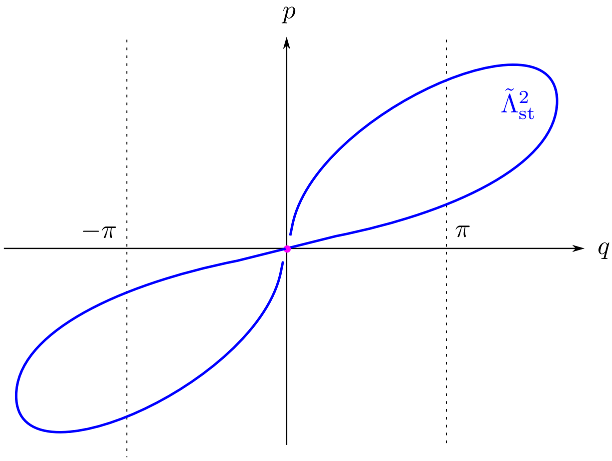

FIGURE 9. The figure depicts the Lagrangian projection of $\tilde{\Lambda}^2_{\mathrm{st}}$ to $T^*\mathbf{R}$. The Lagrangian projection of Λ^2_{st} to T^*S^1 is induced by the quotient projection $\mathbf{R} \to S^1$. The Lagrangian projection of $\tilde{\Lambda}^t_{\mathrm{st}}$ is obtained by rescaling the q-coordinate of $T^*\mathbf{R}$ followed by the canonical projection to T^*S^1

Then consider the autonomous Hamiltonian

$$\rho\colon \Sigma_{1,1} \to \mathbf{R}_{\leq 0}$$

with support inside $\{q \in [-\delta, \delta]\}$ for some small $\delta > 0$, and which is equal to the smooth bump-function $\rho(q) \leq 0$ in one variable of the form

- $\rho(q) \equiv -1$ in a neighbourhood of $q = 0$;
- $\rho(q) = \rho(-q)$;
- and $\rho'(q) \leq 0$ for $q < 0$.

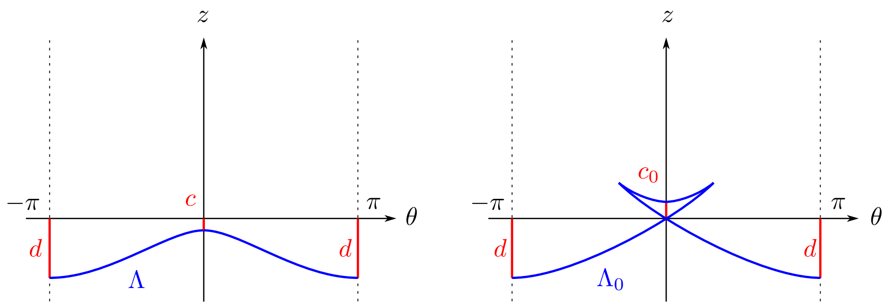

FIGURE 10. Left: a Legendrian perturbation of the zero section. The vertical chords denote the two Reeb chords between the zero-section $j^1 0$ and the perturbation. Right: the perturbed version of the zero-section after a suitable Legendrian RI-move

The Hamiltonian isotopy ϕ^t_ρ wraps the region $q \in (-\delta, 0)$ in the negative p-direction, while it wraps the region $q \in (0, \delta)$ in the positive p-direction.

We claim that $CF(L, \phi^t_\rho(L'))$ has a spectral norm which becomes arbitrarily large as $t \to +\infty$. What is clear is that $\ell(c) - \ell(d) \to +\infty$ as $t \to +\infty$. (Use, e.g. Lemma 2.1.) Again there are additional generators that appear as $t \to +\infty$, so knowing that $\ell(c) - \ell(d) \to +\infty$ is not sufficient.

As in Sect. 3.1 a change of symplectic trivialisation can again give us what we need. First consider the canonical symplectic trivialisation, induced by the trivialisation of $\mathbf{R}^2$ and the quotient projection. Then deform this trivialisation by adding a number $l/2 \gg 0$ of full rotations of the standard symplectic frame (relative the constant one) as one traverses the $\{p = 1\}$. Note that the Lagrangian corresponding to $\{p = 0\}$ still has a Maslov potential after this change of trivialisation. Similarly to the computation in Sect. 3.1, it is now readily seen that all generators c' different from c and d satisfy the property that

$$|c'| - |c| \notin [-10, 10],$$

after we have chosen $l \gg 0$ sufficiently large. In the meantime, $|c| - |d| = 1$ is always satisfied.

The spectral norm can now finally be computed by invoking Part (1) of Lemma 3.1.

4. Proof of Theorem C

By definition, our two Floer complexes are the linearised Legendrian contact homology complexes generated as a $\mathbf{k}$-vector space by the mixed Reeb chords on the Legendrian link

$$\Lambda_\pm \cup \phi^T_{\partial_z}(\Lambda).$$

Here $T \gg 0$ is fixed but sufficiently large.

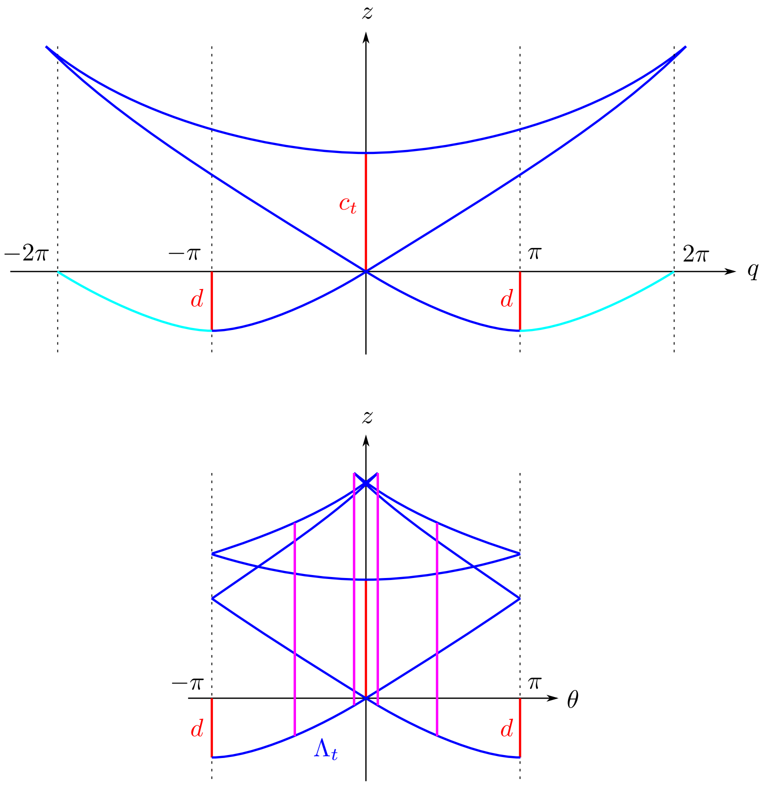

FIGURE 11. Λ_t is obtained from Λ_0 by a linear rescaling of the front inside $\{z \geq 0\}$ in the universal cover $J^1\mathbf{R}^2$ followed by the canonical projection $J^1\mathbf{R} \to J^1S^1$. The front of Λ_t undergoes the composite move shown in Fig. 5 consisting of two consecutive RII-moves along with $RIII$-moves

The cusp-connected sum performed on $\Lambda_- \cup \phi^T_{\partial_z}(\Lambda)$ produces $\Lambda_+ \cup \phi^T_{\partial_z}(\Lambda)$ (of course, only the first component is affected). There is an associated exact standard Lagrangian handle-attachment cobordism

$$\mathcal{L} \subset (\mathbf{R}_t \times \overline{W} \times \mathbf{R}_z, d(e^t\alpha_{\mathrm{st}}))$$

inside the symplectisation as constructed in [9]. This is a cobordism with cylindrical ends from

$$\Lambda_- \cup \phi^T_{\partial_z}(\Lambda) \;\; \text{to} \;\; \Lambda_+ \cup \phi^T_{\partial_z}(\Lambda),$$

i.e. from the Legendrian link before surgery (at the concave end) to the link after surgery (at the convex end). One component of this cobordism is simply the trivial cylinder $\mathbf{R} \times \phi^T_{\partial_z}(\Lambda)$. This Lagrangian cobordism induces a unital DGA-morphism

$$\Phi_{\mathcal{L}} \colon \mathcal{A}(\Lambda_+ \cup \phi^T_{\partial_z}(\Lambda)) \to \mathcal{A}(\Lambda_- \cup \phi^T_{\partial_z}(\Lambda))$$

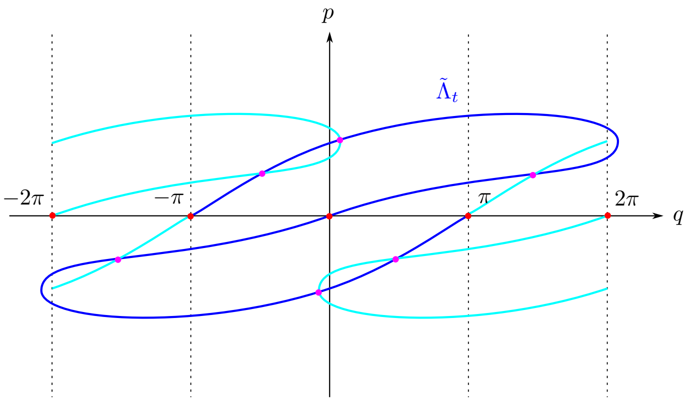

Figure 12. The Lagrangian projection in $T^*\mathbf{R}$ of the universal cover $\tilde{\Lambda} \subset J^1\mathbf{R}$ of $\Lambda_t \subset J^1S^1$ shown in Fig. 11, where $\tilde{\Lambda}_t \cong \mathbf{R}$. The interval shown in dark blue is a fundamental domain for $\tilde{\Lambda}_t$

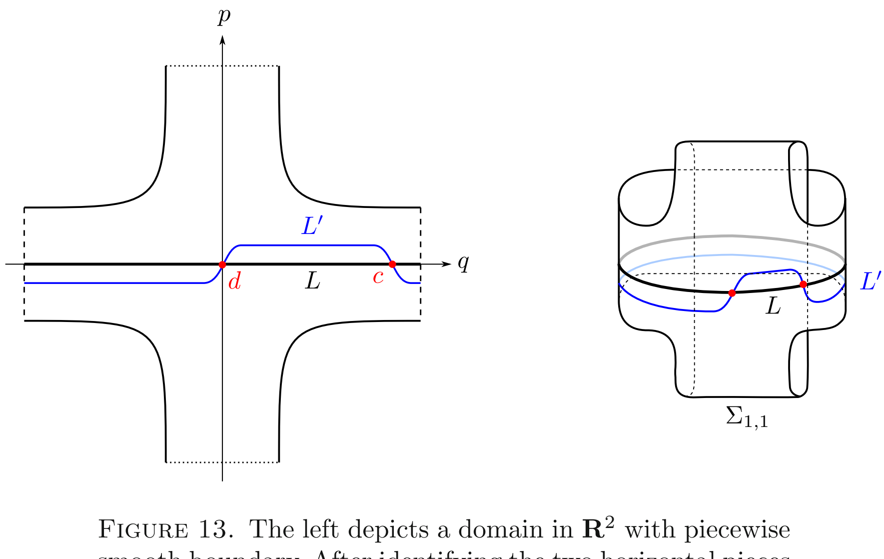

Figure 13. The left depicts a domain in $\mathbf{R}^2$ with piecewise smooth boundary. After identifying the two horizontal pieces of the boundary, as well as the two vertical pieces, one obtains the Liouville domain shown on the right, with Liouville form described in Sect. 2.1.2. The closed exact Lagrangian L is the image of $\{p = 0\}$ and L' is a small Hamiltonian perturbation of L

of the Chekanov–Eliashberg algebras. In particular, the choice of augmentation ε_- of the Chekanov–Eliashberg algebra of Λ_- pulls back to an augmentation $\varepsilon_+ = \varepsilon_- \circ \Phi_{\mathcal{L}}$ of the Chekanov–Eliashberg algebra of Λ_+.

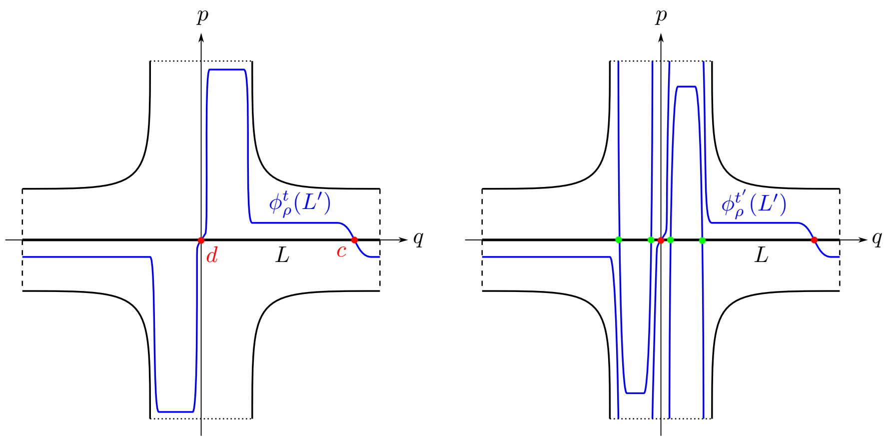

Figure 14. A Hamiltonian isotopy that wraps the Lagrangian $\{p = 0\}$ around the one-handle with core $\{q = 0\}$, while fixing a neighbourhood of the latter core. Note that the Hamiltonian function is positive but constant near $q = 0$. Here $t' > t$

The above DGA-morphism $\Phi_{\mathcal{L}}$ of the Chekanov–Eliashberg algebras after and before the surgery was computed in [9, Theorem 1.1] under the assumption that the handle-attachment is sufficiently small. This computations in particular shows that the mixed chords c on $\Lambda_+ \cup \phi^T_{\partial_z}(\Lambda)$ are mapped to

$$\Phi_{\mathcal{L}}(c) = c + \sum_i r_i \mathbf{d}_i, \;\; r_i \in \mathbf{k},$$

where $\mathbf{d}_i$ are words of Reeb chords that each contain an *odd* number of mixed chords of $\Lambda_- \cup \phi^T_{\partial z}(\Lambda)$, and in which every mixed chord, moreover, is of length strictly less than $\ell(c)$. It now follows by pure algebraic considerations that the map

$$CF_*((\Lambda_+, \varepsilon_+), (\Lambda, \varepsilon)) \to CF_*((\Lambda_-, \varepsilon_-), (\Lambda, \varepsilon))$$

induced by linearising the DGA-morphism $\Phi_{\mathcal{L}}$ using the augmentations ε, ε_+, and ε_- (see [3] and [4]) is an action-preserving isomorphism of the Floer complexes as claimed. □

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Georgios Dimitroglou Rizell
Department of Mathematics
Uppsala University
Box 480751 06 Uppsala
Sweden
e-mail: georgios.dimitroglou@math.uu.se

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