Some conditions for the oscillation of second-order differential equations with several mixed delays

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Abstract. In this work, we obtain necessary and sufficient conditions for the oscillation of the solutions to a second-order neutral differential equation with mixed delays. Two examples are provided to show effectiveness and feasibility of main results. Our main tool is the Lebesgue's Dominated Convergence theorem.

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1. Introduction

In this paper, we are interested in the study of certain oscillation properties of second-order differential equations containing mixed several delays.

Nowadays, the study of qualitative properties of ordinary differential equations attracts considerable attention from the scientific community due to numerous applications of them to several contexts, such as Biology, Physics, Chemistry, and Dynamical Systems. For some details related to the recent studies on oscillation and non-oscillation properties, exponential stability, instability, existence of unbounded solutions of the equations under consideration, we refer the reader to the books [1, 2].

It is worthy pointing out that both oscillation and stability criteria are currently used in the studies of non-linear mathematical models with delay for single species and several species with interactions, in logistic models, α -delay models, mathematical models with varying capacity, mathematical models for food-limited population dynamics with periodic coefficients, and diffusive logistic models (for instance, diffusive Malthus-type models with several delays, autonomous diffusive delayed logistic models with Neumann boundary conditions, periodic diffusive logistic Volterra-type models with delays, and so on).

The literature is full of very interesting results related to the oscillation properties for second-order differential equations. Now, we recall some studies that possess a strong connection with the content of this paper. In [3], Baculiková, Li, and Džurina obtained some oscillation criteria for the following second-order neutral differential equations:

$$(p(t)[u(t) + q(t)u(\varsigma(t))]')' + r(t)u(\sigma(t)) + v(t)u(\eta(t)) = 0$$

considering the cases in which the arguments are delayed, advanced, or mixed.

In [4], Arul and Shobha investigated some oscillation properties of the solutions of the following equation:

$$(p(t)z'(t))' + r(t)u(\sigma(t)) = 0, \quad t \geq t_0 \geq 0,$$

being $z(t) = u(t) + a(t)u(t - \tau) + b(t)u(t + \delta)$.

In [5], Thandapani and Rama considered the second-order non-linear neutral differential equations of mixed type. Precisely, the authors studied the following neutral differential equations:

$$\begin{aligned} [(u(t) + au(t - \varsigma_1) - bu(t + \varsigma_2))^\alpha]'' &= q(t)u^\beta(t - \sigma_1) + p(t)u^\beta(t + \sigma_2), \\ [(u(t) - au(t - \varsigma_1) + bu(t + \varsigma_2))^\alpha]'' &= q(t)u^\beta(t - \sigma_1) + p(t)u^\beta(t + \sigma_2), \end{aligned}$$

being α and $\beta \geq 1$ defined as the ratios of odd positive integers. Some generalizations of the results discussed in [5] are contained in [6].

More general results are contained in [7] where Thandapani, Padmavathi, and Pinelas derived oscillation theorems for even-order non-linear neutral differential equations with mixed type with the following form:

$$\left(p(t)(u(t) + bu(t - \varsigma_1) + cu(t + \varsigma_2))^{(n-1)} \right)' + r(t)u^\alpha(t - \sigma_1) + q(t)u^\beta(t + \sigma_2) = 0$$

being $t \geq t_0$, $n \geq 2$ an even integer, $\alpha \geq 1$ and $\beta \geq 1$ ratios of odd positive integers. The case in which n is odd was treated for slightly different equations in [8, 9].

It is interesting to notice that in the aforementioned works, the authors obtained only sufficient conditions that ensure the oscillation of the solutions of the considered equations. A problem worthy of investigations is the study of necessary and sufficient conditions for the oscillation and some satisfactory answers were given in [10, 13].

Finally, we refer the interested reader to the following paper and to the references therein for some recent results on the oscillation theory for ordinary differential equations of several orders [11–18].

In this work, we deal with necessary and sufficient conditions for the oscillation of solutions to a second-order non-linear differential equations of the form

$$\left(p(t)(w'(t))^\alpha \right)' + \sum_{j=1}^m r_j(t)g_j(u(\nu_j(t))) = 0, \quad t \geq t_0, \quad (1.1)$$

where

$$w(t) = u(t) + q(t)u(\varsigma(t)),$$

being the functions $g_j, r_j, p, q, \nu_j, \varsigma$ continuous and such that the following conditions stated below hold:

- (a) $\nu_j \in C([0, \infty), \mathbb{R})$, $\varsigma \in C^2([0, \infty), \mathbb{R})$, if we consider the simple delay, then $\nu_j(t) < t$, $\varsigma(t) < t$, $\lim_{t \rightarrow \infty} \nu_j(t) = \infty$, $\lim_{t \rightarrow \infty} \varsigma(t) = \infty$;
- (b) $\nu_j \in C([0, \infty), \mathbb{R})$, $\varsigma \in C^2([0, \infty), \mathbb{R})$, if we consider the advanced delay, then (a) can be modified by $\nu_j(t) > t$, $\varsigma(t) < t$, $\lim_{t \rightarrow \infty} \nu_j(t) = \infty$, $\lim_{t \rightarrow \infty} \varsigma(t) = \infty$;
- (c) $p \in C^1([0, \infty), \mathbb{R})$, $r_j \in C([0, \infty), \mathbb{R})$; $0 < p(t)$, $0 \leq r_j(t)$ for all $t \geq 0$ and $j = 1, 2, \dots, m$; $\sum r_j(t)$ is not identically zero in any interval $[b, \infty)$;
- (d) $q \in C^2([0, \infty), \mathbb{R}_+)$ with $0 \leq q(t) \leq a < 1$;
- (e) $g_j \in C(\mathbb{R}, \mathbb{R})$ is non-decreasing and $g_j(t)t > 0$ for $t \neq 0$, $j = 1, 2, \dots, m$;
- (f) $\lim_{t \rightarrow \infty} P(t) = \infty$ where $P(t) = \int_0^t p^{-1/\alpha}(s) ds$;
- (g) α is the quotient of two positive odd integers.

For the sake of completeness, we recall some basic definitions. A *solution* of (1.1) is a function $u : [t_u, \infty[\rightarrow \mathbb{R}$, $t_u \geq t_0$, such that $p(t)(w'(t))^\alpha$ and $u(t)$ are continuously differentiable for all $t \in [t_u, \infty[$ and it satisfies (1.1) for all $t \in [t_u, \infty[$. A solution $u(t)$ of (1.1) is said to be *non-oscillatory* if it is eventually positive or eventually negative; otherwise, it is said to be *oscillatory*. Finally, we say that (1.1) is *oscillatory* if all of its solutions are oscillatory.

2. Preliminary results

For the sake of simplicity, we set

$$R_1(t) = \sum_{j=1}^m r_j(t) g_j \left((1-a)w(\nu_j(t)) \right).$$

Lemma 2.1. *Suppose that (a)–(g) hold for $t \geq t_0$ and that u is an eventually positive solution of (1.1). Then, w satisfies*

$$0 < w(t), \quad w'(t) > 0, \quad \text{and} \quad (p(t)(w'(t))^\alpha)' \leq 0 \quad \text{for } t \geq t_1. \quad (2.1)$$

Proof. Let u be an eventually positive solution. Then, $w(t) > 0$ and there exists $t_0 \geq 0$, such that $u(t) > 0$, $u(\nu_j(t)) > 0$, $u(\varsigma(t)) > 0$ for all $t \geq t_0$ and $j = 1, 2, \dots, m$. Then, (1.1) gives that

$$(p(t)(w'(t))^\alpha)' = - \sum_{j=1}^m r_j(t) g_j(u(\nu_j(t))) \leq 0, \quad (2.2)$$

which shows that $p(t)(w'(t))^\alpha$ is non-increasing for $t \geq t_0$. Next, we claim that $p(t)(w'(t))^\alpha$ is positive for $t \geq t_1 > t_0$. If not, there exists a point $t \geq t_1 > t_0$, such that $p(t)(w'(t))^\alpha \leq 0$. Therefore, we can choose $c > 0$, such that

$$p(t)(w'(t))^\alpha \leq -c,$$

that is

$$w'(t) \leq (-c)^{1/\alpha} p^{-1/\alpha}(t).$$

Integrating both sides from t_1 to t , we get

$$w(t) - w(t_1) \leq (-c)^{1/\alpha} (P(t) - P(t_1)).$$

Taking the limit of both sides as $t \rightarrow \infty$, we have $\lim_{t \rightarrow \infty} w(t) \leq -\infty$ which leads to a contradiction to $w(t) > 0$. Hence, $p(t)(w'(t))^\alpha > 0$ for $t \geq t_1$, that is, $w'(t) > 0$ for $t \geq t_1$. This completes the proof. $\square$

Lemma 2.2. *Suppose that (a)–(g) hold for $t \geq t_0$ and that u is an eventually positive solution of (1.1). Then, w satisfies*

$$u(t) \geq (1-a)w(t) \quad \text{for } t \geq t_1. \quad (2.3)$$

Proof. Assume that u be an eventually positive solution of (1.1). Then, $w(t) > 0$ and there exists $t \geq t_1 > t_0$, such that

$$\begin{aligned} u(t) &= w(t) - q(t)u(\varsigma(t)) \\ &\geq w(t) - q(t)w(\varsigma(t)) \\ &\geq w(t) - q(t)w(t) \\ &= (1 - q(t))w(t) \\ &\geq (1 - a)w(t). \end{aligned}$$

Hence, w satisfies (2.3) for $t \geq t_1$. $\square$

3. Main results

In Theorem 3.1, we use a constant β , quotient of two odd positive integers with $\beta > \alpha$, for which

$$\frac{g_j(t)}{t^\beta} \text{ is non-decreasing for } 0 < t, j = 1, 2, \dots, m. \quad (3.1)$$

Existence of such constant can be established by taking $g_j(t) = |y|^\delta \operatorname{sgn}(t)$, with $\beta < \delta$.

Theorem 3.1. *Let (b)–(g) and (3.1) for $t \geq t_0$. Then, every solution of (1.1) is oscillatory if and only if*

$$\int_0^\infty p^{-1/\alpha}(s) \left[\int_s^\infty \sum_{j=1}^m r_j(\psi) d\psi \right]^{1/\alpha} ds = \infty. \quad (3.2)$$

Proof. Let u is an eventually positive solution of (1.1). Then, $w(t) > 0$ and there exists $t_0 \geq 0$, such that $u(t) > 0$, $u(\nu_j(t)) > 0$, $u(\varsigma(t)) > 0$ for all $t \geq t_0$ and $j = 1, 2, \dots, m$. Thus, Lemmas 2.1 and 2.2 hold for $t \geq t_1$. By Lemma 2.1, there exists $t_2 > t_1$, such that $w'(t) > 0$ for all $t \geq t_2$. Then, there exist $t_3 > t_2$ and $c > 0$, such that $w(t) \geq c$ for all $t \geq t_3$. Next, using Lemma 2.2, we get $u(t) \geq (1-a)w(t)$ for all $t \geq t_3$ and (1.1) become

$$\left(p(t)(w'(t))^\alpha \right)' + R_1(t) \leq 0 \quad \text{for } y \neq \phi_k. \quad (3.3)$$

Integrating (3.3) from t to ∞ , we get

$$[p(s)(w'(s))^\alpha]_t^\infty + \int_t^\infty R_1(s) \, ds \leq 0.$$

Since $p(t)(w'(t))^\alpha$ is positive and non-decreasing $\lim_{t \rightarrow \infty} p(t)(w'(t))^\alpha$ finitely exists and positive

$$p(t)(w'(t))^\alpha \geq \int_t^\infty R_1(s) \, ds,$$

that is

$$w'(t) \geq p^{-1/\alpha}(t) \left[\int_t^\infty R_1(s) \, ds \right]^{1/\alpha}. \quad (3.4)$$

Since

$$\begin{aligned} g_j[(1-a)w(\nu_j(t))] &= \frac{g_j[(1-a)w(\nu_j(t))]}{(1-a)^\beta w^\beta(\nu_j(t))} (1-a)^\beta w^\beta(\nu_j(t)) \\ &\geq \frac{g_j[c(1-a)]}{c^\beta(1-a)^\beta} (1-a)^\beta w^\beta(\nu_j(t)) \\ &= \frac{g_j[c(1-a)]}{c^\beta} w^\beta(\nu_j(t)). \end{aligned} \quad (3.5)$$

Then, we use (3.5) in (3.4) to get

$$w'(t) \geq p^{-1/\alpha}(t) \left[\int_t^\infty \sum_{j=1}^m r_j(s) \frac{g_j[c(1-a)]}{c^\beta} w^\beta(\nu_j(s)) \, ds \right]^{1/\alpha}.$$

Next, if we set $K = \frac{g_0[c(1-a)]}{c^\beta}$ where $g_0[c(1-a)] = \min_{1 \leq j \leq m} g_j[c(1-a)]$, the above inequality becomes

$$w'(t) \geq K^{1/\alpha} p^{-1/\alpha}(t) \left[\int_t^\infty \sum_{j=1}^m r_j(s) w^\beta(\nu_j(s)) \, ds \right]^{1/\alpha}.$$

Using (b) and $w(t)$ is non-decreasing, we have

$$w'(t) \geq K^{1/\alpha} p^{-1/\alpha}(t) \left[\int_t^\infty \sum_{j=1}^m r_j(s) \, ds \right]^{1/\alpha} w^{\beta/\alpha}(t),$$

that is

$$\frac{w'(t)}{w^{\beta/\alpha}(t)} \geq K^{1/\alpha} p^{-1/\alpha}(t) \left[\int_t^\infty \sum_{j=1}^m r_j(s) \, ds \right]^{1/\alpha}.$$

Integrating both sides from t_3 to ∞ , we get

$$K^{1/\alpha} \int_{t_3}^\infty p^{-1/\alpha}(s) \left[\int_s^\infty \sum_{j=1}^m r_j(\psi) \, d\psi \right]^{1/\alpha} \, ds \leq \int_{t_3}^\infty \frac{w'(s)}{w^{\beta/\alpha}(s)} \, ds < \infty$$

due to $\beta > \alpha$, which is a contradiction to (3.2), and hence, the sufficiency part of the theorem is proved.

Next, we prove necessary part reasoning by contradiction. If (3.2) does not hold, then for every $\varepsilon > 0$, there exists $t \geq t_0$ for which

$$\int_t^\infty p^{-1/\alpha}(s) \left[\int_s^\infty \sum_{j=1}^m r_j(\psi) d\psi \right]^{1/\alpha} ds < \varepsilon \quad \text{for } t \geq Y,$$

where $2\varepsilon = \left[\max_{\{1 \leq j \leq m\}} g_j\left(\frac{1}{1-a}\right) \right]^{-1/\alpha} > 0$.

Let us define the set

$$V = \left\{ u \in C([0, \infty)) : \frac{1}{2} \leq u(t) \leq \frac{1}{1-a} \text{ for all } t \geq Y \right\}$$

and $\Phi : V \rightarrow V$ as

$$(\Phi u)(t) = \begin{cases} 0 & \text{if } t \leq Y, \\ \frac{1+a}{2(1-a)} - q(t)u(\varsigma(t)) \\ + \int_T^t p^{-1/\alpha}(s) \left[\int_s^\infty \sum_{j=1}^m r_j(\psi) g_j(u(\nu_j(\psi))) d\psi \right]^{1/\alpha} ds & \text{if } t > Y. \end{cases}$$

Now we prove that $(\Phi u)(t) \in V$. For $u(t) \in V$, we have

$$\begin{aligned} (\Phi u)(t) &\leq \frac{1+a}{2(1-a)} + \int_T^t p^{-1/\alpha}(s) \left[\int_s^\infty \sum_{j=1}^m r_j(\psi) g_j\left(\frac{1}{1-a}\right) d\psi \right]^{1/\alpha} ds \\ &\leq \frac{1+a}{2(1-a)} + \left[\max_{1 \leq j \leq m} g_j\left(\frac{1}{1-a}\right) \right]^{1/\alpha} \cdot \varepsilon \\ &= \frac{1+a}{2(1-a)} + \frac{1}{2} = \frac{1}{1-a} \end{aligned}$$

and furthermore, for $u(t) \in V$,

$$(\Phi u)(t) \geq \frac{1+a}{2(1-a)} - q(t) \cdot \frac{1}{1-a} + 0 \geq \frac{1+a}{2(1-a)} - \frac{a}{1-a} = \frac{1}{2}.$$

Hence, Φ maps from V to V . Now, we find a fixed point for Φ in V which will give an eventually positive solution of (1.1). To this end, we define a sequence of functions in V by

$$\begin{aligned} u_0(t) &= 0 \quad \text{for } t \geq 0, \\ u_1(t) &= (\Phi u_0)(t) = \begin{cases} 0 & \text{if } t < Y, \\ \frac{1}{2} & \text{if } t \geq Y, \end{cases} \\ u_{n+1}(t) &= (\Phi u_n)(t) \quad \text{for } n \geq 1, t \geq Y. \end{aligned}$$

We have $u_1(t) \geq u_0(t)$ for each fixed t and $\frac{1}{2} \leq u_{n-1}(t) \leq u_n(t) \leq \frac{1}{1-a}$ $t \geq Y$ for all $n \geq 1$. Thus, $(u_n)_{n \in \mathbb{N}}$ converges pointwise to a function u . By Lebesgue's Dominated Convergence Theorem, u is a fixed point of Φ in V , which shows that there is a non-oscillatory solution. This completes the proof of the theorem. $\square$

In Theorem 3.2, we use a constant β , quotient of two odd positive integers with $\beta < \alpha$, for which

$$\frac{g_j(t)}{t^\beta} \text{ is non-increasing for } 0 < t, j = 1, 2, \dots, m. \quad (3.6)$$

Existence of such constant can be established by taking $g_j(t) = |y|^\delta \operatorname{sgn}(t)$, with $\beta > \delta$. The assumption upon β can be withdrawn by taking $|u|^\beta \operatorname{sgn}(u)$ instead of u^β .

Theorem 3.2. *Let (a), (c)–(g) and (3.6) hold for $t \geq t_0$. Then, every solution of (1.1) is oscillatory if*

$$\frac{1}{(2c)^\beta} \left[\int_0^\infty \sum_{j=1}^m r_j(\psi) g_j[c(1-a)P(\nu_j(\psi))] d\psi \right] = \infty \quad \forall c \neq 0. \quad (3.7)$$

Proof. Let $u(t)$ be an eventually positive solution of (1.1). Then, proceeding as in Theorem 3.1, we have $t_2 > t_1 > t_0$, such that Eq. (3.4) holds for all $t \geq t_2$. Using (e), there exists $t_3 > t_2$ for which $P(t) - P(t_3) \geq \frac{1}{2}P(t)$ for $t \geq t_3$. Integrating (3.4) from t_3 to t , we have

$$\begin{aligned} w(t) - w(t_3) &\geq \int_{t_3}^t p^{-1/\alpha}(s) \left[\int_s^\infty R_1(\kappa) d\kappa \right]^{1/\alpha} ds \\ &\geq \int_{t_3}^t p^{-1/\alpha}(s) \left[\int_t^\infty R_1(\kappa) d\kappa \right]^{1/\alpha} ds, \end{aligned}$$

that is

$$\begin{aligned} w(t) &\geq (P(t) - P(t_3)) \left[\int_t^\infty R_1(\kappa) d\kappa \right]^{1/\alpha} \\ &\geq \frac{1}{2}P(t) \left[\int_t^\infty R_1(\kappa) d\kappa \right]^{1/\alpha}. \end{aligned} \quad (3.8)$$

Since $p(t)(w'(t))^\alpha$ is non-increasing and positive, then there exists $c > 0$ and $t_4 > t_3$, such that $p(t)(w'(t))^\alpha \leq c^\alpha$ for $t \geq t_4$. Integrating the relation $w'(t) \leq cp^{-1/\alpha}(t)$ from t_4 to t , we have

$$w(t) - w(t_4) \leq c(P(t) - P(t_4)),$$

that is

$$w(t) \leq cP(t) \quad \text{for } t \geq t_4. \quad (3.9)$$

Using (3.6) and (3.9), we obtain

$$\begin{aligned} g_j[(1-a)w(\nu_j(t))] &= \frac{g_j[(1-a)w(\nu_j(t))]}{(1-a)^\beta w^\beta(\nu_j(t))} (1-a)^\beta w^\beta(\nu_j(t)) \\ &\geq \frac{g_j[c(1-a)P(\nu_j(t))]}{c^\beta (1-a)^\beta P^\beta(\nu_j(t))} (1-a)^\beta w^\beta(\nu_j(t)) \\ &= \frac{g_j[c(1-a)P(\nu_j(t))]}{c^\beta P^\beta(\nu_j(t))} w^\beta(\nu_j(t)) \quad \forall t \geq t_4. \end{aligned} \quad (3.10)$$

Using (3.10) in (3.8), we obtain

$$w(t) \geq \frac{1}{2}P(t) \left[\int_t^\infty \sum_{j=1}^m r_j(\kappa) \frac{g_j[c(1-a)P(\nu_j(\kappa))]}{c^\beta P^\beta(\nu_j(\kappa))} w^\beta(\nu_j(\kappa)) \, d\kappa \right]^{1/\alpha}.$$

Hence

$$w(t) \geq \frac{1}{2}P(t)U^{1/\alpha}(t) \quad \text{for } t \geq t_4,$$

where

$$U(t) = \frac{1}{c^\beta} \left[\int_t^\infty \sum_{j=1}^m r_j(\kappa) g_j[c(1-a)P(\nu_j(\kappa))] \frac{w^\beta(\nu_j(\kappa))}{P^\beta(\nu_j(\kappa))} \, d\kappa \right].$$

Now

$$U'(t) = -\frac{1}{c^\beta} \sum_{j=1}^m r_j(t) g_j[c(1-a)P(\nu_j(t))] \frac{w^\beta(\nu_j(t))}{P^\beta(\nu_j(t))} \quad (3.11)$$

$$\leq -\frac{1}{(2c)^\beta} \sum_{j=1}^m r_j(t) g_j[c(1-a)P(\nu_j(t))] U^{\beta/\alpha}(\nu_j(t)) \leq 0, \quad (3.12)$$

which shows that $U(t)$ is non-increasing on $[t_4, \infty)$ and $\lim_{t \rightarrow \infty} U(t)$ exists. Using (3.11) and (a), we find

$$\begin{aligned} [U^{1-\beta/\alpha}(t)]' &= (1-\beta/\alpha)U^{-\beta/\alpha}(t)U'(t) \\ &\leq -\frac{1-\beta/\alpha}{(2c)^\beta} \sum_{j=1}^m r_j(t) g_j[c(1-a)P(\nu_j(t))] U^{\beta/\alpha}(\nu_j(t)) U^{-\beta/\alpha}(t) \\ &\leq -\frac{1-\beta/\alpha}{(2c)^\beta} \sum_{j=1}^m r_j(t) g_j[c(1-a)P(\nu_j(t))]. \end{aligned} \quad (3.13)$$

Now, integrating (3.13) from t_4 to t , we have

$$[U^{1-\beta/\alpha}(s)]_{t_4}^t \leq -\frac{1-\beta/\alpha}{(2c)^\beta} \int_{t_4}^t \sum_{j=1}^m r_j(s) g_j[c(1-a)P(\nu_j(s))] \, ds,$$

that is

$$\frac{1-\beta/\alpha}{(2c)^\beta} \int_0^\infty \sum_{j=1}^m r_j(s) g_j[c(1-a)P(\nu_j(s))] \, ds \leq -[U^{1-\beta/\alpha}(s)]_{t_4}^t < U^{1-\beta/\alpha}(t_4) < \infty,$$

which contradicts (3.7). This completes the proof. $\square$

Example 3.1. Consider the neutral differential equation

$$\left(((u(t) + e^{-t}u(\varsigma(t)))')^{1/3} \right)' + t(u(t-2))^{7/3} + (t+1)(u(t-3))^{11/3} = 0. \quad (3.14)$$

Here, $\alpha = 1/3$, $p(t) = 1$, $0 < q(t) = e^{-t} < 1$, $\nu_j(t) = t - (j+1)$, $g_j(t) = t^{(4j+3)/3}$. For $\beta = 5/3$, we have $\delta_j = (4j+3)/3 > \beta = 5/3 > \alpha = 1/3$, and

$g_1(t)/t^\beta = t^{2/3}$ and $g_2(t)/t^\beta = t^2$ which are both increasing functions. Now, we check (3.2). We have

$$\begin{aligned} & \int_{t_0}^{\infty} \left[\frac{1}{p(s)} \left[\int_s^{\infty} \sum_{j=1}^m r_j(\psi) d\psi \right] \right]^{1/\alpha} ds \\ & \geq \int_{t_0}^{\infty} \left[\frac{1}{p(s)} \left[\int_s^{\infty} r_1(\psi) d\psi \right] \right]^{1/\alpha} ds \\ & \geq \int_2^{\infty} \left[\int_s^{\infty} \psi d\psi \right]^3 ds = \infty. \end{aligned}$$

Therefore, all the conditions of Theorem 3.1 hold. Thus, each solution of (3.14) is oscillatory.

Example 3.2. Consider the neutral differential equation

$$\left(e^{-t} \left((u(t) + e^{-t} u(\zeta(t)))' \right)^{11/3} \right)' + \frac{1}{t+1} (u(t-2))^{1/3} + \frac{1}{t+2} (u(t-3))^{5/3} = 0. \quad (3.15)$$

Here, $\alpha = 11/3$, $p(t) = e^{-t}$, $0 < q(t) = e^{-t} < 1$, $\nu_j(t) = t - (j+1)$, $P(t) = \int_0^t e^{3s/11} ds = \frac{11}{3} (e^{3t/11} - 1)$, $g_j(t) = t^{(4j-3)/3}$. For $\beta = 7/3$, we have $\delta_j = (4j-3)/3 < \beta = 7/3 < \alpha = 11/3$, and $g_1(t)/t^\beta = t^{-2}$ and $g_2(t)/t^\beta = t^{-2/3}$ which are both decreasing functions. Now, we check (3.7). We have

$$\begin{aligned} & \frac{1}{(2c)^\beta} \left[\int_0^{\infty} \sum_{j=1}^m r_j(\psi) g_j[c(1-a)P(\nu_j(\psi))] d\psi \right] \\ & \geq \frac{1}{(2c)^{7/3}} \int_0^{\infty} r_1(\psi) g_1[c(1-a)P(\nu_1(\psi))] d\psi \\ & = \frac{1}{(2c)^{7/3}} \int_0^{\infty} \frac{1}{\psi+1} \left[c(1-a) \frac{11}{3} (e^{3(\psi-2)/11} - 1) \right]^{1/3} d\psi = \infty \quad \forall c > 0. \end{aligned}$$

Therefore, all the conditions of Theorem 3.2 hold, and therefore, each solution of (3.15) is oscillatory.

4. Conclusions

In this work, we considered the second-order highly non-linear neutral differential equations with several mixed delays (1.1) and established necessary and sufficient conditions for the oscillation of (1.1) when the neutral coefficient q lies in $[0, 1)$.

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