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α-Firmly nonexpansive operators on metric spaces

Arian Bërdëllima, Florian Lauster and D. Russell Luke

Abstract. We extend to p-uniformly convex spaces tools from the analysis of fixed point iterations in linear spaces. This study is restricted to an appropriate generalization of single-valued, pointwise averaged mappings. Our main contribution is establishing a calculus for these mappings in p-uniformly convex spaces, showing in particular how the property is preserved under compositions and convex combinations. This is of central importance to splitting algorithms that are built by such convex combinations and compositions, and reduces the convergence analysis to simply verifying that the individual components have the required regularity pointwise at fixed points of the splitting algorithms. Our convergence analysis differs from what can be found in the previous literature in that the regularity assumptions are only with respect to fixed points. Indeed we show that, if the fixed point mapping is pointwise nonexpansive at all cluster points, then these cluster points are in fact fixed points, and convergence of the sequence follows. Additionally, we provide a quantitative convergence analysis built on the notion of gauge metric subregularity, which we show is *necessary* for quantifiable convergence estimates. This allows one for the first time to prove convergence of a tremendous variety of splitting algorithms in spaces with curvature bounded from above.

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1. Introduction

Our focus is on the extension to p-uniformly convex spaces of tools from the analysis of fixed point iterations in linear spaces. We are indebted to the works of Kuwae [26] and Ariza-Ruiz, Leuştean, López-Acedo, and Nicolae [2,3] who studied firmly nonexpansive mappings in nonlinear spaces, though the asymptotic behavior of averaged mappings in uniformly convex Banach spaces was already studied by Baillon, Bruck and Reich in [6]. Reich and Shafrir established an approach to the study of convex combinations of nonexpansive mappings in *hyperbolic spaces* [41], the foundations for which were developed in [15]. Building on this, we follow the framework for nonconvex optimization established in [35] which is predicated on only two fundamental elements in a Euclidean setting: pointwise almost α-averaging [35, Definition 2.2] and metric subregularity [19, Definition 2.1b]. Almost averaged mappings are, in general, set-valued. In nonlinear metric spaces, there are several difficulties that arise: first, there is no straight-forward generalization of the averaging property since addition is not defined on general metric spaces; and second, multivaluedness, which comes with allowing mappings to be expansive. The issue of multivaluedness introduces technical overhead, but does not, at this early stage, seem to present any conceptual difficulties. The issue of violations of conventional regularity notions is more fundamental. We show that such violations are unavoidable if one wants to work with resolvents. The foundations for working with these difficulties are established here. A direct study of resolvents on spaces with curvature bounded from above appears in [29].

We therefore restrict our attention to an appropriate generalization of single-valued, pointwise α-averaged mappings. This generalization leads to a definition of firmly nonexpansive mappings that is less restrictive than notions with the same name studied in [3,5,15,40,41], though, we show that our notion is implied by the previously studied objects. The progenitors of these operators have been studied in [9,13,38,39]. Our main contribution is establishing a calculus for these mappings in p-uniformly convex spaces, showing in particular how the property is preserved under compositions and convex combinations. This is of central importance to splitting algorithms that are built by such convex combinations and compositions, and reduces the convergence analysis to simply verifying that the individual components of the splitting algorithms satisfy the required regularity. Our convergence analysis also differs from what can be found in the previous literature in that we assume only that the fixed point mapping has the required regularity at fixed points. Indeed we show (Theorem 27) that, if the fixed point mapping is pointwise nonexpansive at the asymptotic centers of all subsequences, then all asymptotic centers are fixed points and weak (precisely, Δ-) convergence of the fixed point sequence is guaranteed. Additionally, we provide a quantitative convergence analysis built on the notion of gauge metric subregularity, which we show is in fact *necessary* for quantifiable convergence estimates. This allows one to prove convergence of a tremendous variety of splitting algorithms for the first time in spaces with curvature bounded from above.

After introducing notation, we begin in Sect. 2.1 with pointwise almost α-firmly nonexpansive mappings, the central objects of this study (Definition 1). Section 3 is devoted to developing elementary properties and the calculus of these mappings. Proposition 4 and Lemma 7 in Sect. 3.1 establish asymptotic regularity. The calculus of pointwise nonexpansive mappings (2) in various settings is established in Proposition 8 of Sect. 3.2. The calculus of quasi α-firmly nonexpansive mappings is established in Theorem 11 (compositions) of Sect. 3.3 and Theorem 21 (convex combinations) of Sect. 3.4. Convergence of fixed point iterations of α-firmly nonexpansive mappings is studied in Sect. 4 where convergence without rates is established for mappings that are only pointwise nonexpansive at the asymptotic centers of all subsequences (Theorem 27) and quantitative convergence in Theorem 30 under the additional assumption of (gauge) metric subregularity (Definition 29). We show in Theorem 32 that metric subregularity with some gauge is in fact necessary to guarantee quantitative convergence estimates. Some basic applications and examples are presented in Sect. 5.

2. Notation and foundations

Throughout, (G, d) denotes a metric space. A geodesic path emanating from a point $x \in G$ and extending to the point $y \in G$ is a mapping $\gamma : [0, l] \to G$ with $\gamma(0) = x$, $\gamma(l) = y$ and $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$ whenever $t_1, t_2 \in [0, l]$. When there is only one geodesic path joining any two points x and y, we use the notation $z = (1-t)x \oplus ty$ where $t = d(z, x)/d(x, y)$ to denote the point on the geodesic connecting x and y such that $d(z, x) = td(x, y)$. A geodesic space is a metric space (G, d) for which every pair of points in G is joined by a geodesic. If each pair of points is joined by one and only one geodesic, the metric space is uniquely geodesic. A convex set $C \subset G$ is a set containing all geodesics joining any two points in C. Following [3] we focus on *p-uniformly convex spaces with parameter c* [36]: for $p \in (1, \infty)$, a metric space (G, d) is p-uniformly convex with constant $c > 0$ whenever it is a geodesic space, and

$$(\forall t \in [0,1])(\forall x, y, z \in G) \quad d(z, (1-t)x \oplus ty)^p \\ \leq (1-t)d(z, x)^p + td(z, y)^p - \tfrac{c}{2}t(1-t)d(x, y)^p. \tag{1}$$

Examples of p-uniformly convex spaces include L_p spaces, and $CAT(\kappa)$ spaces with sufficiently small diameter if $\kappa > 0$ (see Alexandrov [1] and Gromov [16]). $CAT(0)$ spaces can be defined by (1) with $p = 2$ and $c = 2$. $CAT(\kappa)$ spaces for $\kappa > 0$ are relevant for the study of phase retrieval and source localization [33]. When the diameter of the space, $\operatorname{diam} G$, is bounded above by $\pi/(2\sqrt{\kappa})$, then the corresponding $CAT(\kappa)$ space is 2-uniformly convex with constant $c = (\pi - 2\sqrt{\kappa}\epsilon)\tan(\epsilon\sqrt{\kappa})$ for $\epsilon \in (0, \pi/(2\sqrt{\kappa}) - \operatorname{diam} G]$ (see [37]). Kuwae has established bounds for the constants p and c, illustrating their interdependence [26, Proposition 2.5]. In particular, we note that if $c = 2$, then $p = 2$. For all other $p \in (1, +\infty)$ the constant c lies in the open interval $(0, 2)$. There is a connection with the modulus of convexity of a Banach space $(Y, \|\cdot\|)$ given by $\delta(\epsilon) := \inf\left\{1 - \left\|\frac{x+y}{2}\right\| \mid x, y \in Y, \|x\| = \|y\| = 1, \|x - y\| \geq \epsilon\right\}$, i.e. Y

will be p-uniformly convex with constant $c > 0$ if $\delta(\epsilon) \geq c\epsilon^p$ [26, Remark 2.7]. Finally, we will use the notation (H, d) to indicate a Hadamard space—a *complete* $CAT(0)$ space—and $\mathcal{H}$ will indicate a Hilbert space.

The distance of a point x to a set D with respect to the metric d is denoted $d(x, D) := \inf_{z \in D} d(x, z)$ and when this distance is attained at some point $\overline{x} \in D$ we call this point a *projection* of x onto D. The mapping of a point x to its set of projections is called the *projector* and is denoted $P_D(x) := \{\overline{x} \in D \mid d(x, \overline{x}) = d(x, D)\}$.

A standard approach to showing the convergence of fixed point sequences is to show that the residual of the fixed point operator vanishes. More precisely, a self-mapping $T : G \to G$ is *asymptotically regular* at a point $x \in G$ whenever $\lim_{k \to \infty} d(T^k x, T^{k+1} x) = 0$. The mapping is said to be asymptotically regular on $D \subset G$ if it is asymptotically regular at all points on D. A sequence $(x_k)_{k \in \mathbb{N}}$ is said to be asymptotically regular whenever $\lim_{k \to \infty} d(x_k, x_{k+1}) = 0$.

One of the fundamental regularities of mappings is pointwise Lipschitz continuity with Lipschitz constant 1. Mappings $T : D \to G$ $(D \subset G)$ are said to be *pointwise nonexpansive at* $y \in D$ *on* D whenever

$$d(Tx, Ty) \leq d(x, y) \quad \forall x \in D. \tag{2}$$

Firmly nonexpansive mappings, as the name suggests, satisfy a condition that, in Hadamard spaces, implies that the mappings are nonexpansive; in general metric spaces, however, this correspondence fails. We postpone a precise definition until later, but this type of regularity is central to convergence of fixed point iterations. In a Banach space setting, firmly nonexpansive mappings possessing fixed points are asymptotically regular, and sequences of fixed point iterations converge weakly to a fixed point [40]. This is also true on p-uniformly convex (nonlinear) spaces [3]. We extend these results to the generalization of averaged mappings, what we call α-firmly nonexpansive mappings, that possess fixed points in Theorem 27. Rates of convergence of the iterates are achieved in Theorem 30 under the additional assumption that the fixed point mapping admits an *error bound*. The notion of α-firmly nonexpansive operators greatly simplifies the analysis of algorithms, and opens the door to a study of *expansive* operators [35] where convexity/monotonicity plays no role.

2.1. α-Firmly nonexpansive operators in uniformly convex spaces

Extending Bruck's original definition of firmly nonexpansive mappings in uniformly convex Banach spaces [12], Ariza-Ruiz, Leuştean and López-Acedo [2] defined λ-firmly nonexpansive operators on subsets D of W-hyperbolic spaces, as those operators satisfying

$$\exists \lambda \in (0,1): \quad d(Tx, Ty) \leq d((1-\lambda)x \oplus \lambda Tx, (1-\lambda)y \oplus \lambda Ty) \quad \forall x, y \in D. \tag{3}$$

If (3) holds for all $\lambda \in (0, 1)$ the mapping T is called firmly nonexpansive in [2,3]. The analog to α-averaged mappings [35, Definition 2.2] is problematic since it requires the extension of geodesics beyond the point Tx (i.e. $\alpha < 1/2$ and $\lambda \in \left(0, \frac{1-\alpha}{\alpha}\right)$).

Another notion of regularity in the context of Hadamard spaces that is equivalent to (3) for an operator $T : H \to H$ and $x, y \in H$ uses

$$\phi_T(t) := d((1-t)x \oplus tTx, (1-t)y \oplus tTy), \quad \text{for } t \in [0,1]. \tag{4}$$

In [15, Chapter 24] an operator $T : H \to H$ is called firmly nonexpansive whenever ϕ_T is nonincreasing on $[0,1]$ (see also [5, Definition 2.1.13]).

For reasons that will become apparent in Sect. 4.2 we define the regularity of T in terms of an auxiliary function that accounts for how T deforms the parallelogram with corners at x, y, Tx and Ty. Define

$$\begin{aligned}\psi_T^{(p,c)}(x,y) := &\tfrac{c}{2}\left(d(Tx,x)^p + d(Ty,y)^p\right.\\ &\left.+ d(Tx,Ty)^p + d(x,y)^p - d(Tx,y)^p - d(x,Ty)^p\right).\end{aligned} \tag{5}$$

We call this the *transport discrepancy*. In a Hilbert space setting this is recognizable as

$$\|(\operatorname{Id} - T)x - (\operatorname{Id} - T)y\|^2 = \psi_T^{(2,2)}(x,y).$$

The next definition generalizes firmly nonexpansive mappings to those that may violate the defining inequality in a manner analogous to such mappings studied in [32,35]. We do not fully develop the potential of this extension here, but will use it in a result about proximal mappings in Corollary 23.

Definition 1. Let (G,d) be a p-uniformly convex metric space with constant c. The operator $T : G \to G$ is pointwise almost α-firmly nonexpansive at $y \in D \subset G$ on D if

$$\exists \alpha \in (0,1), \epsilon \geq 0 : \quad d(Tx,Ty)^p \leq (1+\epsilon)d(x,y)^p - \tfrac{1-\alpha}{\alpha}\psi_T^{(p,c)}(x,y) \quad \forall x \in D. \tag{6}$$

A value of ϵ for which (6) holds is called a *violation*. If (6) holds with $\epsilon = 0$, then T is pointwise α-firmly nonexpansive at $y \in D \subset G$ on D. If (6) holds at all $y \in D$ with the same constant α, then T is said to be (almost) α-firmly nonexpansive on D. If $D = G$ the mapping T is simply said to be (almost) α-firmly nonexpansive. If $D \supset \operatorname{Fix} T \neq \emptyset$ and (6) holds at all $y \in \operatorname{Fix} T$ with the same constant α then T is said to be (almost) quasi α-firmly nonexpansive.

The transport discrepancy $\psi_T^{(p,c)}$ is closely related to another object used by Berg and Nikolaev [11] in the study of a $CAT(0)$ space (G,d). In p-uniformly convex spaces with constant c this takes the form:

$$\Delta^{(p,c)}(x,y,u,v) := \frac{c}{4}(d(x,v)^p + d(y,u)^p - d(x,u)^p - d(y,v)^p). \tag{7}$$

Specializing to a Hilbert space $\mathcal{H}$, this is identifiable with the inner product:

$$\begin{aligned}\langle x-y, u-v\rangle &= \frac{1}{2}(\|x-v\|^2 + \|y-u\|^2 - \|x-u\|^2 - \|y-v\|^2)\\ &= \Delta^{(2,2)}(x,y,u,v).\end{aligned} \tag{8}$$

In the context of characterizing the regularity of a mapping T we take $u = Tx$ and $v = Ty$ and it is convenient to denote $\Delta_T^{(p,c)}(x,y) := \Delta^{(p,c)}(x,y,Tx,Ty)$.

This object was introduced in [10, Chapter 7] in the context of Hadamard spaces ($p = c = 2$) where it was called the discrepancy mapping. In particular, note that

$$\psi_T^{(p,c)}(x, y) = \tfrac{c}{2}\left(d(Tx, Ty)^p + d(x, y)^p\right) - 2\Delta_T^{(p,c)}(x, y). \tag{9}$$

This leads to the following equivalent characterization of pointwise α-firmly nonexpansive mappings.

Proposition 2. *Let (G, d) be a p-uniformly convex space with constant $c > 0$ and let $T : D \to G$ for $D \subset G$. The mapping T is pointwise almost α-firmly nonexpansive at $y \in D$ with constant α and violation ϵ on D if and only if*

$$\begin{aligned}&\left(\alpha + (1-\alpha)\tfrac{c}{2}\right) d(Tx, Ty)^p + \left(\alpha(1+\epsilon) - (1-\alpha)\tfrac{c}{2}\right) d(x, y)^p \\ &\quad \le 2(1-\alpha)\Delta_T^{(p,c)}(x, y), \quad \forall x \in D.\end{aligned} \tag{10}$$

Proof. The result follows from the definition, by expanding the transport discrepancy (5) and using the relation (9). □

When $p = 2$ and $c = 2$, we show below in Proposition 4(v) that mappings T satisfying (3) also satisfy

$$d(Tx, Ty)^2 \le \Delta_T^{(2,2)}(x, y) \qquad \forall x, y \in G. \tag{11}$$

Indeed, in this case, (11) is precisely Property P_2 of [3]. When (11) for general p and c holds only pointwise at y on a neighborhood $D \subset G$ of y we write

$$d(Tx, Ty)^p \le \Delta_T^{(p,c)}(x, y) \qquad \forall x \in D. \tag{12}$$

This provides for a natural extension of monotonicity: an operator $T : G \to G$ is monotone whenever

$$\Delta_T^{(p,c)}(x, y) \geqslant 0, \qquad \forall x, y \in G. \tag{13}$$

From these definitions it follows that if T satisfies (3) for all $\lambda \in (0, 1)$ then it is monotone.

In a similar vein, in Hadamard space settings, when a mapping is firmly nonexpansive, it is easy to see that it is also nonexpansive. But this implication is a consequence of Busemann convexity [2] and does not hold in general metric spaces. Nevertheless, the implication is recovered for p-uniformly convex spaces for *pointwise* firmly nonexpansive mappings at their fixed points, since $\psi_T^{(p,c)}$ is non-negative in this case.

Remark 3. It is clear from the definition that $T : G \to G$ satisfies (3) if and only if ϕ_T defined by (4) is a nonincreasing function on $[0, 1]$ for all $x, y \in G$. Banert [7, Remark pp.658] shows that any mapping T satisfying (3) for all $\lambda \in (0, 1]$ is α-firmly nonexpansive with constant $\alpha = 1/2$. Since the argument will be useful later, we show this here. Note that $\phi_T(1) \le \phi_T(t)$ for all $t \in [0, 1]$ whenever $\phi_T(t)$ is a nonincreasing function on $[0, 1]$ for all $x, y \in G$. On the other hand from applying (1) with $p = 2$ and $c = 2$ twice we obtain

$$\begin{aligned}\phi_T^2(t) &\le (1-t)^2 d(x, y)^2 + t^2 d(Tx, Ty)^2 \\ &\qquad + t(1-t)[d(x, Ty)^2 + d(y, Tx)^2 - d(x, Tx)^2 - d(y, Ty)^2].\end{aligned}$$

Hence

$$\phi_T^2(1) = d(Tx, Ty)^2 \leq (1-t)^2 d(x,y)^2 + t^2 d(Tx,Ty)^2 + 2t(1-t)\Delta_T^{(2,2)}(x,y),$$

or equivalently

$$(1-t^2) d(Tx,Ty)^2 \leq (1-t)^2 d(x,y)^2 + 2t(1-t)\Delta_T^{(2,2)}(x,y). \tag{14}$$

Dividing by $1-t$ and letting $t \uparrow 1$ yields $d(Tx,Ty)^2 \leq \Delta_T^{(2,2)}(x,y)$. By Proposition 2 this shows that T is α-firmly nonexpansive with constant $\alpha = 1/2$ as claimed.

3. Properties of pointwise nonexpansive and α-firmly nonexpansive mappings in metric spaces

Before developing the calculus of α-firmly nonexpansive mappings, we begin with some elementary properties of pointwise α-firmly nonexpansive mappings.

3.1. Elementary properties of α-firmly nonexpansive operators

Proposition 4. *Let (G, d) be a p-uniformly convex space with constant $c > 0$ and let $T : D \to G$ for $D \subset G$.*

(i) *Whenever $y \in \operatorname{Fix} T$*

$$\psi_T^{(p,c)}(x,y) = \tfrac{c}{2} d(Tx,x)^p \quad \forall x \in D. \tag{15}$$

For fixed $y \in \operatorname{Fix} T$ the function $\psi_T^{(p,c)}(\cdot, y)$ is non-negative on D and $\psi_T^{(p,c)}(x,y) = 0$ only when $x \in \operatorname{Fix} T$.

(ii) *Let $y \in \operatorname{Fix} T$. T is pointwise α-firmly nonexpansive at y on D if and only if*

$$\exists \alpha \in [0,1): \quad d(Tx,y)^p \leq d(x,y)^p - \tfrac{1-\alpha}{\alpha}\tfrac{c}{2} d(Tx,x)^p \quad \forall x \in D. \tag{16}$$

In particular, T is quasi α-firmly nonexpansive on D whenever T possesses fixed points and (16) holds at all $y \in \operatorname{Fix} T$ with the same constant $\alpha \in [0,1)$.

(iii) *If T is pointwise α-firmly nonexpansive at $y \in \operatorname{Fix} T$ on D with constant $\underline{\alpha} \in [0,1)$ then it is pointwise α-firmly nonexpansive at y on D with constant α taking any value in the interval $[\underline{\alpha}, 1]$. In particular, if T is pointwise α-firmly nonexpansive at $y \in \operatorname{Fix} T$ on D, then it is pointwise nonexpansive at y on D (see (2)).*

(iv) *If at $y \in \operatorname{Fix} T$*

$$\exists \lambda \in (0,1): \quad d(Tx,y) \leq d((1-\lambda)x \oplus \lambda Tx, y) \quad \forall x \in D, \tag{17}$$

then T is pointwise α-firmly nonexpansive at y with constant $\alpha = 1/(\lambda + 1)$ on D.

(v) *Let $p = 2$ and $c = 2$ (that is, (G,d) is a $CAT(0)$ space). Then*
   (a) *$\Delta_T^{(2,2)}(x,y) \leq d(x,y) d(Tx,Ty)$ for all $x, y \in G$;*
   (b) *$\psi_T^{(2,2)}(x,y) \geq 0$ for all $x, y \in G$;*
   (c) *the following are equivalent for any $\underline{\alpha} \in (0,1)$:*

(1) *T is pointwise α-firmly nonexpansive at y with constant $\underline{\alpha}$ on D;*
(2) *T is pointwise α-firmly nonexpansive at y with constant α taking any value in the interval $[\underline{\alpha}, 1]$ on D;*
(3)

$$(1+\lambda)d(Tx,Ty)^2 \leq (1-\lambda)d(x,y)^2 + 2\lambda\Delta_T^{(2,2)}(x,y) \quad \forall x \in D, \forall \lambda \in [0, \tfrac{1-\underline{\alpha}}{\underline{\alpha}}];$$

(4) *T satisfying equation (3) is α-firmly nonexpansive with constant $\alpha = \frac{1}{1+\lambda}$ on D.*

Proof. (i). Fix $y \in \operatorname{Fix} T$. Equation (15) follows directly from (5), from which the rest of the claim is immediate.

(ii). This is immediate from the definition and (i).

(iii). This follows immediately from (i) and (ii).

(iv) Starting with (17), by the characterization of p-uniformly convex spaces (1)

$$\begin{aligned} d(Tx,y)^p &\leq d((1-\lambda)x \oplus \lambda Tx, y)^p \\ &\leq (1-\lambda)d(x,y)^p + \lambda d(Tx,y)^p - (1-\lambda)\lambda\tfrac{c}{2}d(Tx,x)^p \end{aligned}$$

for all $x \in D$ and some $\lambda \in (0,1)$. Rearranging terms yields

$$\exists \lambda \in (0,1): \quad d(Tx,y)^p \leq d(x,y)^p - \lambda\tfrac{c}{2}d(Tx,x)^p \quad \forall x \in D.$$

When $y \in \operatorname{Fix} T$, by (ii), this is equivalent to T being pointwise α-firmly nonexpansive at y with constant $\alpha = \frac{1}{\lambda+1}$ on D.

(v)(a) This is a direct consequence of the inequality

$$\Delta^{(2,2)}(x,y,u,v) \leq d(x,y)d(u,v) \tag{18}$$

(see [23, Theorem 2.3.1] or [28, Lemma 2.1]).

(v)(b) By (9) and (v)(a)

$$\begin{aligned} \psi_T^{(2,2)}(x,y) &= d(x,y)^2 + d(Tx,Ty)^2 - 2\Delta_T^{(2,2)}(x,y) \\ &\geq d(x,y)^2 + d(Tx,Ty)^2 - 2d(x,y)d(Tx,Ty) \\ &= (d(x,y) - d(Tx,Ty))^2 \geq 0 \end{aligned}$$

for all $x, y \in G$, as claimed.

(v)(c) By (v)(b) if T is pointwise α-firmly nonexpansive at y with constant $\underline{\alpha}$ on D then

$$d(Tx,Ty)^2 \leq d(x,y)^2 - \tfrac{1-\underline{\alpha}}{\underline{\alpha}}\psi_T^{(2,2)}(x,y) \quad \forall x \in D$$
$$\iff$$
$$d(Tx,Ty)^2 \leq d(x,y)^2 - \tfrac{1-\alpha}{\alpha}\psi_T^{(2,2)}(x,y) \quad \forall x \in D, \forall \alpha \in [\underline{\alpha}, 1]. \tag{19}$$

Consequently, T is pointwise α-firmly nonexpansive at y with constant $\underline{\alpha}$ on D if and only if it is pointwise α-firmly nonexpansive at y with constant α tanking any value in the interval $[\underline{\alpha}, 1]$ on D. A change of variables in (19) yields

$$d(Tx,Ty)^2 \leq d(x,y)^2 - \lambda\psi_T^{(2,2)}(x,y) \quad \forall x \in D, \forall \lambda \in [0, \tfrac{1-\underline{\alpha}}{\underline{\alpha}}]$$

which, from (9), is equivalent to

$$(1+\lambda)d(Tx,Ty)^2 \leq (1-\lambda)d(x,y)^2 + 2\lambda\Delta_T^{(2,2)}(x,y) \quad \forall x \in D, \forall \lambda \in [0, \tfrac{1-\alpha}{\alpha}].$$

(v)(d) For fixed $\lambda \in [0,1)$ (14) in Remark 3 yields

$$(1+\lambda)d(Tx,Ty)^2 \leq (1-\lambda)d(x,y)^2 + 2\lambda\Delta_T^{(2,2)}(x,y), \quad \forall x,y \in D.$$

By (v)(c), this implies that T is α-firmly nonexpansive at all $y \in D$ for any constant $\alpha \in [\frac{1}{1+\lambda}, 1]$ on D. This completes the proof. □

Remark 5. Property (16) is a specialization of Property (P1) of [3] to α-firmly nonexpansive mappings on p-uniformly convex spaces.

Closedness and convexity of the set of fixed points of nonexpansive mappings is easily established. Note, however, that convexity of the fixed point set depends on convexity of the domain. In Sect. 4.2 we will not require convexity of the domain.

Lemma 6. *Let (G,d) be a p-uniformly convex metric space with constant $c > 0$ and let $D \subseteq G$ be closed and convex. Let $T : G \to G$ be pointwise nonexpansive at all $y \in \operatorname{Fix} T \cap D \neq \emptyset$ on D (see (2)). Then $\operatorname{Fix} T \cap D$ is a closed and convex set.*

Proof. This statement for T a nonexpansive (not pointwise) mapping on a uniquely geodesic space is in [2, Lemma 6.2]. Their proof also works for pointwise nonexpansive mappings. □

In [3] the central property of *asymptotic regularity* of a mapping T at its fixed points hinges on (i) existence of fixed points, and (ii) the validity of inequality (16) at all $y \in \operatorname{Fix} T$. Proposition 4(ii) shows that these two requirements are *equivalent* to T being quasi α-firmly nonexpansive. We show in Theorem 27 that, as a consequence of the next theorem on asymptotic regularity, the assumption that T is pointwise α-firmly nonexpansive at reasonable subsets of fixed points is all that is needed to achieve weak convergence of fixed point iterations.

Lemma 7. *Let (G,d) be a p-uniformly convex space, let $D \subset G$, and let $T : G \to G$ with $\operatorname{Fix} T \cap D$ nonempty and $T(D) \subseteq D$. Suppose further that T is pointwise α-firmly nonexpansive at all $y \in \operatorname{Fix} T \cap D$ on D. Then given any starting point $x_0 \in D$ the sequence $(x_k)_{k\in\mathbb{N}}$ defined by $x_{k+1} = Tx_k$ is asymptotically regular on D.*

Proof. By Proposition 4(ii) and Remark 5, the statement is a specialization of [3, Theorem 3.1] to the case of just a single operator. □

We show below that compositions and convex combinations of (quasi) α-firmly nonexpansive mappings are quasi α-firmly nonexpansive. Therefore, by the theorem above, fixed point iterations of such compositions and convex combinations are asymptotically regular.

3.2. Calculus of nonexpansive operators

Compositions and, with some restrictions, convex combinations of pointwise nonexpansive mappings defined by (2) are also pointwise nonexpansive, as the next result shows.

Proposition 8. *Let $D \subset G$ where (G, d) is a p-uniformly convex space with constant $c > 0$ and let $T_1, T_2 : D \to G$.*

(i) *If $\operatorname{Fix} T_2 \cap \operatorname{Fix} T_1 \neq \emptyset$ then any convex combination of T_2 and T_1 is pointwise nonexpansive at $y \in \operatorname{Fix} T_2 \cap \operatorname{Fix} T_1$ on D whenever T_2 and T_1 are pointwise nonexpansive there.*

(ii) *When (G, d) is a $CAT(0)$ space, (that is, $p = 2$, and $c = 2$), then any convex combination of T_2 and T_1 is pointwise nonexpansive at $y \in D$ on D whenever T_2 and T_1 are pointwise nonexpansive there.*

(iii) *For $D_1 := \{z \mid z = T_1 y, y \in D\}$ let $T_2 : D_1 \to G$ be pointwise nonexpansive at $T_1 y \in D_1$ on D_1 and let T_1 be pointwise nonexpansive at $y \in D$. Then the composition $T_2 \circ T_1$ is pointwise nonexpansive at y on D.*

In particular, the set of all nonexpansive operators in $CAT(0)$ spaces is closed under compositions and convex combinations.

Proof. (i) Let $\lambda \in (0, 1)$ and define $T_\lambda := (1-\lambda)T_2 \oplus \lambda T_1$. Applying (1) first to $T_\lambda y$ and then to $T_\lambda x$ yields

$$\begin{aligned} d(T_\lambda x, T_\lambda y)^p &\leq (1-\lambda)d(T_\lambda x, T_2 y)^p + \lambda d(T_\lambda x, T_1 y)^p - \tfrac{c}{2}\lambda(1-\lambda)d(T_2 y, T_1 y)^p \\ &\leq (1-\lambda)^2 d(T_2 x, T_2 y)^p + \lambda^2 d(T_1 x, T_1 y)^p \\ &\quad + \lambda(1-\lambda)\left(d(T_1 x, T_2 y)^p + d(T_2 x, T_1 y)^p\right) \\ &\quad - \frac{\lambda(1-\lambda)c}{2}\left(d(T_2 x, T_1 x)^p + d(T_2 y, T_1 y)^p\right). \end{aligned} \tag{20}$$

For $y \in \operatorname{Fix} T_2 \cap \operatorname{Fix} T_1$ this yields

$$\begin{aligned} d(T_\lambda x, T_\lambda y)^p &\leq (1-\lambda)d(T_2 x, y)^p + \lambda d(T_1 x, y)^p \\ &\leq (1-\lambda)d(x, y)^p + \lambda d(x, y)^p = d(x, y)^p \quad \forall x \in D, \end{aligned}$$

where the last inequality uses the assumed regularity of T_1 and T_2 at y. Therefore $(1-\lambda)T_2 \oplus \lambda T_1$ is pointwise nonexpansive at $y \in \operatorname{Fix} T_2 \cap \operatorname{Fix} T_1$ on D for all $\lambda \in [0, 1]$, as claimed.

(ii) Let $\lambda \in (0, 1)$ and define $T_\lambda := (1-\lambda)T_2 \oplus \lambda T_1$. Applying (20) with $p = 2$ and $c = 2$ yields

$$\begin{aligned} d(T_\lambda x, T_\lambda y)^2 &\leq (1-\lambda)^2 d(T_2 x, T_2 y)^2 + \lambda^2 d(T_1 x, T_1 y)^2 \\ &\quad + (1-\lambda)\lambda(d(T_2 x, T_1 y)^2 + d(T_1 x, T_2 y)^2 \\ &\quad - d(T_2 y, T_1 y)^2 - d(T_2 x, T_1 x)^2) \\ &= (1-\lambda)^2 d(T_2 x, T_2 y)^2 + \lambda^2 d(T_1 x, T_1 y)^2 \\ &\quad + (1-\lambda)\lambda\Delta^{(2,2)}(T_2 x, T_2 y, T_1 x, T_1 y) \end{aligned}$$

for any $x \in D$, where $\Delta^{(2,2)}$ is defined by (7). On the other hand (G, d) is a $CAT(0)$ space, so (18) holds, and in particular,

$$\Delta^{(2,2)}(T_2 x, T_2 y, T_1 x, T_1 y) \leq 2d(T_2 x, T_2 y)d(T_1 x, T_1 y)$$

for any $x, y \in G$. Therefore

$$d(T_\lambda x, T_\lambda y)^2 \leq \left((1-\lambda)d(T_2x, T_2y) + \lambda d(T_1x, T_1y)\right)^2 \quad \forall x \in D.$$

By assumption both T_2 and T_1 are pointwise nonexpansive at y on D, so

$$d(T_\lambda x, T_\lambda y)^2 \leq ((1-\lambda)d(x,y) + \lambda d(x,y))^2 = d(x,y)^2 \quad \forall x \in D$$

and hence $d(T_\lambda x, T_\lambda y) \leq d(x,y)$ for all $x \in D$ as claimed.

(iii). This is well-known and a simple calculation. □

3.3. Compositions of α-firmly nonexpansive operators

In this section, we show how the composition of two α-firmly nonexpansive operators is again α-firmly nonexpansive. In general this does not hold, but the property does hold pointwise at fixed points of the composite operator, and for many applications this is all that is needed. The next lemma relates the fixed points of compositions of α-firmly nonexpansive mappings to the intersection of the fixed points of the individual mappings.

Lemma 9. *Let (G,d) be a metric space.*

(i) *Let $T_2, T_1 : G \to G$ satisfy $\operatorname{Fix} T_2 \cap \operatorname{Fix} T_1 \neq \emptyset$. If T_2 is pointwise nonexpansive at all $y \in \operatorname{Fix} T_2 \cap \operatorname{Fix} T_1 \neq \emptyset$ on $D \subset G$ where $\operatorname{Fix} T_2 \cap \operatorname{Fix} T_1 \subset D$, and T_1 is pointwise α-firmly nonexpansive at all $y \in \operatorname{Fix} T_2 \cap \operatorname{Fix} T_1$ on D, then $\operatorname{Fix} T_2T_1 = \operatorname{Fix} T_2 \cap \operatorname{Fix} T_1$.*

(ii) *Let $\{T_1, T_2, \ldots, T_m\}$ be a collection of quasi α-firmly nonexpansive mappings, each with respective constants α_j on $D \supset \cap_{j=1}^m \operatorname{Fix} T_j \neq \emptyset$. Then $\operatorname{Fix}(T_m \circ T_{m-1} \circ \cdots \circ T_1) = \cap_{j=1}^m \operatorname{Fix} T_j$.*

Proof. (i) The inclusion $\operatorname{Fix} T_2 \cap \operatorname{Fix} T_1 \subseteq \operatorname{Fix} T_2T_1$ is obvious. Now let x be any point in $\operatorname{Fix} T_2T_1$ and y any point in $\operatorname{Fix} T_2 \cap \operatorname{Fix} T_1$. Then

$$\begin{aligned}\exists \alpha \in (0,1) \ : \ d(x,y)^p &= d(T_2T_1x, T_2y)^p \leq d(T_1x, y)^p = d(T_1x, T_1y)^p \\ &\leq d(x,y)^p - \frac{1-\alpha}{\alpha}\frac{c}{2} d(x, T_1x)^p,\end{aligned}$$

where the first inequality follows from the assumption that T_2 is pointwise nonexpansive at $y \in \operatorname{Fix} T_2 \cap \operatorname{Fix} T_1$ on D, and the second inequality follows from the assumption that T_1 is pointwise α-firmly nonexpansive at $y \in \operatorname{Fix} T_2 \cap \operatorname{Fix} T_1$ on D and Proposition 4(ii). If $x \notin \operatorname{Fix} T_1$, then the equality above yields $d(x,y)^p > d(x,y)^p$, which is absurd, so the only alternative is that $x \in \operatorname{Fix} T_1$. It follows that $x = T_2T_1x = T_2x$, thus $x \in \operatorname{Fix} T_1 \cap T_2$ and $\operatorname{Fix} T_2T_1 \subseteq \operatorname{Fix} T_2 \cap \operatorname{Fix} T_1$ which completes the proof.

(ii) In light of Remark 5, this follows immediately from [3, Proposition 2.1]. □

Lemma 10. *Let (G,d) be a p-uniformly convex space with constant c, let $D \subset G$ and fix $y \in D$. Let $T_1 : D \to G$ be pointwise α-firmly nonexpansive at y on D with constant α_1 and let $T_2 : D_1 \to G$ be pointwise α-firmly nonexpansive at T_1y on D_1 with constant α_2 where $D_1 := \{T_1x \mid x \in D\}$. Then the composition $\overline{T} := T_2 \circ T_1$ is pointwise α-firmly nonexpansive at y on D whenever*

$$\exists\, \overline{\alpha} \in (0,1) : \quad \frac{1-\alpha_1}{\alpha_1}\psi_{T_1}^{(p,c)}(x,y) + \frac{1-\alpha_2}{\alpha_2}\psi_{T_2}^{(p,c)}(T_1x, T_1y)$$

$$\geq \frac{1-\overline{\alpha}}{\overline{\alpha}}\psi_{\overline{T}}^{(p,c)}(x,y) \quad \forall x \in D. \tag{21}$$

Proof. The proof is a straightforward application of Definition 1 and (21). □

Theorem 11. *Let (G,d) be a p-uniformly convex space. Let $T_1 : D \to G$ for $D \subset G$, $T_2 : D_1 \to G$ for $D_1 := \{T_1 x \mid x \in D\}$, define $\overline{T} := T_2 \circ T_1$ and let $\operatorname{Fix}\overline{T} \subset D$ and $\operatorname{Fix} T_1 \cap \operatorname{Fix} T_2$ both be nonempty. If T_1 is pointwise α-firmly nonexpansive at all $y \in \operatorname{Fix}\overline{T}$ with constant α_1 on D, and if T_2 is pointwise α-firmly nonexpansive at all $y \in \operatorname{Fix}\overline{T}$ with constant α_2 on D_1, then the composite operator $\overline{T}$ is quasi α-firmly nonexpansive on D with constant*

$$\overline{\alpha} = \tfrac{\alpha_1+\alpha_2-2\alpha_1\alpha_2}{\frac{c}{2}(1-\alpha_1-\alpha_2+\alpha_1\alpha_2)+\alpha_1+\alpha_2-2\alpha_1\alpha_2}. \tag{22}$$

Proof. By Lemma 10, it suffices to show (21) at all points $y \in \operatorname{Fix}\overline{T}$. First, note that by Lemma 9, $\operatorname{Fix}\overline{T} = \operatorname{Fix} T_2 \cap \operatorname{Fix} T_1$, so by (15) it holds that, for any $y \in \operatorname{Fix}\overline{T}$ and for all $x \in D$, $\psi_{T_1}^{(p.c)}(x,y) = \frac{c}{2}d(x,T_1x)^p$, $\psi_{T_2}^{(p.c)}(T_1x,T_1y) = \frac{c}{2}d(T_1x,\overline{T}x)^p$, and $\psi_{\overline{T}}^{(p.c)}(x,y) = \frac{c}{2}d(x,\overline{T}x)^p$. Then whenever $y \in \operatorname{Fix}\overline{T}$ the inequality (21) simplifies to

$$\exists \overline{\kappa} > 0 : \quad \kappa_1 d(x,T_1x)^p + \kappa_2 d(T_1x,\overline{T}x)^p \geq \overline{\kappa} d(x,\overline{T}x)^p \qquad \forall x \in D, \tag{23}$$

where $\kappa_1 := \frac{1-\alpha_1}{\alpha_1}$, $\kappa_2 := \frac{1-\alpha_2}{\alpha_2}$ and $\overline{\kappa} := \frac{1-\overline{\alpha}}{\overline{\alpha}}$ with $\overline{\alpha} \in (0,1)$. By (1), we have

$$\begin{aligned}\tfrac{c}{2}t(1-t)d(x,\overline{T}x)^p &\leq \tfrac{c}{2}t(1-t)d(x,\overline{T}x)^p + d(T_1x,(1-t)x \oplus t\overline{T}x)^p \\ &\leq (1-t)d(T_1x,x)^p + td(T_1x,\overline{T}x)^p \quad \forall x \in D, \forall t \in (0,1)\end{aligned} \tag{24}$$

Letting $t = \frac{\kappa_2}{\kappa_1+\kappa_2}$ yields $(1-t) = \frac{\kappa_1}{\kappa_1+\kappa_2}$, so that (24) becomes

$$\tfrac{c}{2}\tfrac{\kappa_1\kappa_2}{\kappa_1+\kappa_2}d(x,\overline{T}x)^p \leq \kappa_1 d(T_1x,x)^p + \kappa_2 d(T_1x,\overline{T}x)^p \quad \forall x \in D. \tag{25}$$

It follows that (23) holds for any $\overline{\kappa} \in (0, \frac{c\kappa_1\kappa_2}{2(\kappa_1+\kappa_2)}]$. We conclude that the composition $\overline{T}$ is quasi α-firmly nonexpansive with constant

$$\overline{\alpha} = \frac{\kappa_1+\kappa_2}{\frac{c}{2}\kappa_1\kappa_2+\kappa_1+\kappa_2}.$$

A short calculation shows that this is the same as (22), which completes the proof. □

Remark 12. The fact that inequality (21) or, more specifically (23), implies that $\overline{T}$ is quasi α-firmly nonexpansive is a consequence of the assumed regularity of the individual operators T_2 and T_1. Whether or not this inequality holds is a property of the *space* and is entirely independent of the operators. Also note that the constant $\overline{\alpha}$ given in (22) corresponds exactly to the constant found in [8, Proposition 4.44] for mappings on Hilbert spaces.

Corollary 13. (finite compositions of quasi α-firmly nonexpansive operators are quasi α-firmly nonexpansive) *Let (G,d) be a p-uniformly convex space. Let $T_1 : D_1 \to G$ where $D_1 \subset G$ and for $j = 2,3,\ldots,m$ let $T_j : D_j \to G$ for $D_j := \{T_{j-1}x \mid x \in D_{j-1}\}$. If the mappings T_j are quasi α-firmly nonexpansive with constant α_j on D_j $(j = 1,2,\ldots,m)$ and $\operatorname{Fix}(T_m \circ T_{m-1} \circ \cdots \circ T_1) \subset D_1$ is* *nonempty, then the composite operator* $\overline{T}_m := T_m \circ T_{m-1} \circ \cdots \circ T_1$ *is quasi* α*-firmly nonexpansive on* D_1 *with constant given recursively by*

$$\overline{\alpha}_m := \frac{\overline{\kappa}_{m-1} + \kappa_m}{\frac{c}{2}\overline{\kappa}_{m-1}\kappa_m + \overline{\kappa}_{m-1} + \kappa_m} \quad (m \geq 3), \tag{26a}$$

where

$$\overline{\kappa}_j := \frac{1 - \overline{\alpha}_j}{\overline{\alpha}_j} \quad (j \geq 2), \tag{26b}$$

$$\kappa_j := \frac{1 - \alpha_j}{\alpha_j} \quad (j \geq 1), \text{ and} \tag{26c}$$

$$\overline{\alpha}_2 := \frac{\kappa_1 + \kappa_2}{\frac{c}{2}\kappa_1\kappa_2 + \kappa_1 + \kappa_2}. \tag{26d}$$

Proof. The result follows from Theorem 11 and an elementary induction argument. □

Remark 14. It is well known that the composition of two firmly nonexpansive mappings (for instance, projectors) in a Hilbert space ($\alpha = 1/2$, $p = 2$, and $c = 2$) is α-firmly nonexpansive with constant $\alpha = \frac{2}{3}$. Theorem 11 yields this as a special case.

3.4. Convex combinations of α-firmly nonexpansive operators

In this subsection, we see that the property of being α-firmly nonexpansive is preserved under p-convex combinations of operators. To prove this we use the concept of p-uniformly convex functions.

Definition 15. Let (G, d) be a p-uniformly convex space. A function $f\colon G \to \mathbb{R}$ is said to be p-uniformly convex with constant $m > 0$ if

$$\begin{aligned} f(tx \oplus (1-t)y) \leq & tf(x) + (1-t)f(y) \\ & - \frac{1}{2}mt(1-t)d(x,y)^p \quad \forall x, y \in G, \ \forall t \in [0,1]. \end{aligned}$$

Remark 16. It is obvious from the definition that the sum of two p-uniformly convex functions with constants m_1 and m_2 is p-uniformly convex with constant $m = m_1 + m_2$. For any $z \in G$ the function $x \mapsto d(z,x)^p$ is a p-uniformly convex function with constant $m = c$ if (G, d) is a p-uniformly convex space with constant $c > 0$.

Lemma 17. *Let* $f\colon G \to \mathbb{R}$ *be* p*-uniformly convex with constant* $m > 0$ *and* $x \in \operatorname{argmin} f \neq \emptyset$*. Then*

$$f(y) \geq f(x) + \frac{m}{2}d(x,y)^p \quad \forall y \in G.$$

Proof. Let $x \in \operatorname{argmin} f$ and f be p-uniformly convex with constant m. Then

$$\begin{aligned} (1-t)f(y) &\geq f(tx \oplus (1-t)y) - tf(x) + \frac{m}{2}t(1-t)d(x,y)^p \\ &\geq (1-t)f(x) + \frac{m}{2}t(1-t)d(x,y)^p \end{aligned}$$

by the definition of p-uniformly convex functions and the fact that $x \in \operatorname{argmin} f$. Now divide by $1-t$ and take the limit $t \to 1$ to obtain the claim. □

The p-convex combination of n points $x_1, \ldots, x_n$ with weights $\omega_1, \omega_2, \ldots, \omega_n \in [0,1]$ such that $\sum_{i=1}^n \omega_i = 1$ is denoted ${}_p\bigoplus_i^n \omega_i x_i$ where

$$ {}_p\bigoplus_i^n \omega_i x_i := \operatorname{argmin}_y \sum_{i=1}^n \omega_i d(y, x_i)^p. \tag{27} $$

Convex combinations of operators T_i are defined accordingly by

$$ \mathscr{T}x := {}_p\bigoplus_i^n \omega_i T_i x := \operatorname{argmin}_y \sum_{i=1}^n \omega_i d(y, T_i x)^p. \tag{28} $$

The following proposition shows that p-convex combinations exist and are unique in complete p-uniformly convex spaces; this is a special case of existence and uniqueness of p-barycenters in p-uniformly convex spaces proved by Kuwae.

Proposition 18. (Lemma 3.5 of [26]) *Let (G, d) be a complete p-uniformly convex space with constant $c > 0$. Then the* argmin *in (27) exists and is unique.*

Definition 19. [26] Let (G, d) be a geodesic space. Let γ and η be two geodesics through p. Then γ is said to be perpendicular to η at point p denoted by $\gamma \perp_p \eta$ if

$$ d(x, p) \leq d(x, y) \quad \forall x \in \gamma, y \in \eta. $$

A space is said to be symmetric perpendicular if for all geodesics γ and η with common point p we have

$$ \gamma \perp_p \eta \Leftrightarrow \eta \perp_p \gamma. $$

Examples of symmetric perpendicular spaces are $CAT(0)$ spaces and $CAT(\kappa)$ spaces for $\kappa > 0$ with diameter strictly less than $\frac{\pi}{2\sqrt{\kappa}}$ [26, Theorem 2.11].

Theorem 20. *Let (G, d) be a complete, p-uniformly convex space and for $i = 1, 2, \ldots, n$ let the mappings $T_i : G \to G$ be pointwise α-firmly nonexpansive on $\operatorname{Fix} T_i$ with respective constant α_i. Then for $\mathscr{T}$ defined by (28), $\cap_{i \in \{1,\ldots,n\}} \operatorname{Fix} T_i \subset \operatorname{Fix} \mathscr{T}$. Suppose in addition that $\cap_{i \in \{1,\ldots,n\}} \operatorname{Fix} T_i \neq \emptyset$ and G is symmetric perpendicular, then $\operatorname{Fix} \mathscr{T} = \cap_{i \in \{1,\ldots,n\}} \operatorname{Fix} T_i$.*

Proof. The inclusion $\operatorname{Fix} \mathscr{T} \supset \cap_{i=1}^n \operatorname{Fix} T_i$ is clear. To see the converse inclusion when the intersection $\cap_{i \in \{1,\ldots,n\}} \operatorname{Fix} T_i \neq \emptyset$ and G is symmetric perpendicular, let $x \notin \cap_{i=1}^n \operatorname{Fix} T_i$ and $y \in \cap_{i=1}^n \operatorname{Fix} T_i$. For at least one $j \in \{1, \ldots, n\}$ we have $x \notin \operatorname{Fix} T_j$. We use a contradiction to prove $P_{[x,y]}(T_j x) \neq x$. Therefore assume that $P_{[x,y]}(T_j x) = x$. Then $[x, T_j x] \perp_x [x, y]$ and by symmetric perpendicularity $[x, y] \perp_x [x, T_j x]$. Hence $d(y, x) \leq d(y, T_j x)$ this contradicts $d^p(y, T_j x) \leq d^p(x, y) - \frac{1-\alpha_i}{\alpha_j} \frac{c}{2} d^p(T_j x, x) < d^p(y, x)$. Therefore $t = 0$ is not a minimum of the convex function $t \mapsto g_j(t) := d(T_j x, ty \oplus (1-t)x)^p$ on the interval $[0, 1]$ and the right side derivative $d^+ g_j(0) < 0$ for all j with $T_j x \neq x$. For i with $T_i x = x$ we have $g_i(t) = t^p$ and hence $d^+ g_i(0) = 0$. So the function

$$ g(t) := \sum_{i=1}^n \omega_i d(ty \oplus (1-t)x, T_i x)^p = \sum_{i=1}^n \omega_i g_i(t) $$

has $d^+g(0) < 0$, and hence x can not be a minimum of

$$z \mapsto \sum_{i=1}^{n} \omega_i d(z, T_i x)^p.$$

This shows that $\mathscr{T}x \neq x$ and completes the proof. □

Theorem 21. (convex combinations of pointwise α-firmly nonexpansive mappings are pointwise α-firmly nonexpansive) *Let (G, d) be a p-uniformly convex space with constant $c > 0$ that is symmetric perpendicular. Let T_i be pointwise α-firmly nonexpansive with constant α_i $(i = 1, 2, \ldots, n)$ at all points in $\cap_{i=1}^n \operatorname{Fix} T_i \neq \emptyset$ on D, and $\omega_i \in [0, 1]$ with $\sum_{i=1}^n \omega_i = 1$. Then $\mathscr{T}$ defined by (28) is pointwise α-firmly nonexpansive at all $y \in \operatorname{Fix} \mathscr{T}$ on D with constant*

$$\alpha = \max_i \alpha_i.$$

Proof. Let $x \in D$. By convexity of $d(\cdot, y)^p$ and Jensen's inequality [26, Theorem 4.1] for p-uniformly convex spaces with the symmetric perpendicular property we have

$$\begin{aligned}
d(\mathscr{T}x, \mathscr{T}y)^p &= d\left({}_p\oplus_i^n\, \omega_i T_i x, y\right)^p && \text{(29a)}\\
&\leq {}_p\oplus_i^n\, \omega_i d(T_i x, y)^p && \text{(29b)}\\
&= \operatorname{argmin}_{t\in\mathbb{R}} \sum_{i=1}^n \omega_i |t - d(T_i x, y)^p|^p && \text{(29c)}\\
&\leq \operatorname{argmin}_{t\in\mathbb{R}} \sum_{i=1}^n \omega_i \left| t - \left(d(x,y)^p - \frac{1-\alpha_i}{\alpha_i}\frac{c}{2} d(x, T_i x)^p \right) \right|^p && \text{(29d)}\\
&\leq \operatorname{argmin}_{t\in\mathbb{R}} \sum_{i=1}^n \omega_i \left| t - \left(d(x,y)^p - \frac{1-\alpha}{\alpha}\frac{c}{2} d(x, T_i x)^p \right) \right|^p && \text{(29e)}\\
&= d(x,y)^p - \operatorname{argmin}_{t\in\mathbb{R}} \sum_{i=1}^n \omega_i \left| t - \frac{1-\alpha}{\alpha}\frac{c}{2} d(x, T_i x)^p \right|^p && \text{(29f)}\\
&\leq d(y,x)^p - \frac{1-\alpha}{\alpha}\frac{c}{2} d\left(x, {}_p\oplus_i^n\, \omega_i T_i x\right)^p && \text{(29g)}\\
&= d(y,x)^p - \frac{1-\alpha}{\alpha}\frac{c}{2} d(x, \mathscr{T}x)^p. && \text{(29h)}
\end{aligned}$$

For the estimation in (29d) and (29e) we used the property that $\operatorname{argmin}_{t\in\mathbb{R}} \sum_{i=1}^n \omega_i |t - \lambda_i|^p$ is increasing in every constant λ_i. This can be easily concluded since $\sum_{i=1}^n \omega_i |t - \lambda_i|^p$ is a convex function and

$$\partial_t \sum_{i=1}^n \omega_i |t - \lambda_i|^p = \sum_{i=1}^n \omega_i p |t - \lambda_i|^{p-1} sgn(t - \lambda)$$

is decreasing in every λ_j for fixed t. □

3.5. Constructing α-firmly nonexpansive operators

In a complete p-uniformly convex space the p-proximal mapping of a proper lower semicontinuous function f is defined by

$$\operatorname{prox}^p_{f,\lambda}(x) := \operatorname{argmin}_{y\in G}\left\{ f(y) + \frac{1}{p\lambda^{p-1}} d(x,y)^p \right\}. \tag{30}$$

The argmin in (30) exists and is unique if f is proper, lsc and convex [21, Proposition 2.7]. This is a very natural definition of the proximal mapping, as the corresponding Moreau–Yosida envelope given by

$$e^p_{f,\lambda}(x) := \inf_{y\in G}\left\{ f(y) + \frac{1}{p\lambda^{p-1}} d(x,y)^p \right\}$$

satisfies the semigroup property $e^p_{(e^p_{f,\lambda}),\mu} = e^p_{f,\lambda+\mu}$ (see [22,27]).

Proposition 22. ([21, Lemma 2.8]) *Let (G,d) be a p-uniformly convex space with parameter $c>0$, $\lambda>0$ and $f\colon G\to(-\infty,+\infty]$ be a proper, convex and lower semicontinuous function. Then for all $x,y\in G$ we have*

$$\begin{aligned} d(\operatorname{prox}^p_{f,\lambda}(x), \operatorname{prox}^p_{f,\lambda}(y))^p &\le \frac{1}{c}[d(v,y)^p + d(x,w)^p - d(x,v)^p - d(y,w)^p] \\ &= \Delta^{(p,\frac{4}{c})}_{\operatorname{prox}^p_{f,\lambda}}(x,y) \end{aligned}$$

for $v=\operatorname{prox}^p_{f,\lambda}(x)$ and $w=\operatorname{prox}^p_{f,\lambda}(y)$.

Proof. This follows directly from [21, Lemma 2.8] with $\mu = \frac{p\lambda}{2}$. □

Corollary 23. (proximal mappings are almost α-firmly nonexpansive) *Let (G,d) be a p-uniformly convex space with parameter $c\in(1,2]$, let $\lambda>0$, and let $f\colon G\to(-\infty,+\infty]$ be a proper, convex and lsc function. Then $\operatorname{prox}^p_{f,\lambda}$ is almost α-firmly nonexpansive with constant $\alpha_c = \frac{c^2-c}{c^2-c+2}$ and violation $\epsilon_c = \frac{2-c}{c-1}$.*

Proof. Let $x\in\operatorname{Fix}\operatorname{prox}^p_{f,\lambda}$, $y\in G$ and $w=\operatorname{prox}^p_{f,\lambda}(y)$. Then by Proposition 22 and elementary calculations

$$\begin{aligned} d(x,w)^p \le \frac{1}{c-1}(d(x,y)^p - d(y,w)^p) &= (1+\epsilon_c)d(x,y)^p - \frac{1}{c-1}d(y,w)^p \\ &= (1+\epsilon_c)d(x,y)^p - \frac{c}{2}\frac{1-\alpha_c}{\alpha_c}d(y,w)^p \end{aligned}$$

where

$$\alpha_c = \frac{c(c-1)}{c(c-1)+2}.$$

□

Remark 24. In the special case $c=2$ and hence $p=2$ the violation is $\epsilon_2=0$ and $\operatorname{prox}^2_{f,\lambda}$ is quasi α-firmly nonexpansive with constant $\alpha=\frac{1}{2}$.

Proposition 25. (projectors are pointwise firmly nonexpansive) *Let (G,d) be a complete, symmetric perpendicular p-uniformly convex space, $C\subset G$ a closed convex subset. The metric projection onto the set C, denoted P_C, is pointwise α-firmly nonexpansive at any $y\in C$ with constant $\alpha=\frac{1}{2}$.*

Proof. First note that $[x, P_C x] \perp_{P_C x} [y, P_C x]$ since P_C is the metric projector. Then $[y, P_C x] \perp_{P_C x} [x, P_C x]$ by symmetric perpendicularity of the space. Hence $t = 0$ is a minimum of the function $t \mapsto d(tx \oplus (1-t)P_C x, y)^p$ on the interval $[0, 1]$ and

$$d(tx \oplus (1-t)P_C x, y)^p \le td(x,y)^p + (1-t)d(P_C x, y)^p - \frac{c}{2}t(1-t)d(x, P_C x)^p,$$

with equality at $t = 0$. Now $t = 0$ has to be a minimum of the right hand side and

$$\begin{aligned} 0 \le \ & \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} td(x,y)^p + (1-t)d(P_C x, y)^p - \frac{c}{2}t(1-t)d(x, P_C x)^p \\ &= d(x,y)^p - d(P_C x, y)^p - \frac{c}{2}d(x, P_C x)^p, \end{aligned}$$

which yields the claim. □

Proposition 26. (Krasnoselsky–Mann relaxations) *Let* (G, d) *be a* p*-uniformly convex space and* $T\colon G \to G$ *be pointwise nonexpansive at all* $y \in \operatorname{Fix} T$*. Then* $T_\lambda{:=}\lambda T \oplus (1-\lambda)Id$ *is pointwise* α*-firmly nonexpansive at all* $y \in \operatorname{Fix} T$ *with constant* $\alpha_\lambda = \frac{\lambda^{p-1}}{1-\lambda+\lambda^{p-1}}$.

Proof. Clearly $\operatorname{Fix} T = \operatorname{Fix} T_\lambda$ and $d(x, T_\lambda x)^p = \lambda^p d(x, Tx)^p$. Let $y \in FixT_\lambda$ then

$$\begin{aligned} d(y, T_\lambda x)^p &= d(y, \lambda Tx \oplus (1-\lambda)x)^p \\ &\le \lambda d(y, Tx)^p + (1-\lambda)d(y,x)^p - \frac{c}{2}\lambda(1-\lambda)d(x, Tx)^p \\ &\le d(x,y)^p - \frac{c}{2}\frac{1-\lambda}{\lambda^{p-1}}d(x, T_\lambda x)^p. \end{aligned}$$

Solving $\frac{1-\lambda}{\lambda^{p-1}} = \frac{1-\alpha_\lambda}{\alpha_\lambda}$ for α_λ yields the claim. □

4. Convergence of iterated α-firmly nonexpansive mappings

The *asymptotic center* [14] of a bounded sequence $(x_k)_{k\in\mathbb{N}}$ in a metric space (G, d) is the set

$$A((x_k)_{k\in\mathbb{N}}) := \left\{ x \in G \,\middle|\, \limsup_{k\to\infty} d(x, x_k) = r((x_k)_{k\in\mathbb{N}}) \right\} \tag{31}$$

where

$$r((x_k)_{k\in\mathbb{N}}) := \inf \left\{ \limsup_{k\to\infty} d(y, x_k) \,\middle|\, y \in G \right\}. \tag{32}$$

Following [31], a sequence $(x_k)_{k\in\mathbb{N}}$ is said to Δ-converge to $\overline{x} \in G$ whenever $\overline{x}$ is the unique asymptotic center of every subsequence of $(x_k)_{k\in\mathbb{N}}$. In this case $\overline{x}$ is said to be the Δ-limit of the sequence and we write $x_k \xrightarrow{\Delta} \overline{x}$.

4.1. Convergence: no rate

The next theorem is a slight, but important generalization of analogous results that can be found elsewhere in the literature. There are two main differences: namely, that we assume only that the fixed point operator is *quasi* α-firmly nonexpansive, and secondly, the operator is only required to be nonexpansive at the asymptotic centers of all subsequences.

Theorem 27. *Let (G, d) be a p-uniformly convex space, let $D \subseteq G$ be convex, and let $T : G \to G$ with $T(D) \subseteq D$ be pointwise α-firmly nonexpansive at all $y \in \operatorname{Fix} T \cap D$ on D. Define the sequence $(x_k)_{k\in\mathbb{N}}$ by $x_{k+1} = Tx_k$ with $x_0 \in D$. Let $\mathcal{C}$ denote the set of asymptotic centers of all subsequences of $(x_k)_{k\in\mathbb{N}}$. If T is pointwise nonexpansive on D at all points in $\mathcal{C}$, then*

(i) $\mathcal{C} \subset \operatorname{Fix} T$,
(ii) *$\mathcal{C}$ is a singleton, and*
(iii) $x_k \overset{\Delta}{\to} \overline{x} \in \mathcal{C}$.

If, in addition, $T(D)$ is a boundedly compact subset of G, then $x_k \to \overline{x} \in \operatorname{Fix} T \cap D$. In particular, if T is nonexpansive on D and $T(D)$ is boundedly compact, then $x_k \to \overline{x} \in \operatorname{Fix} T \cap D$ for any $x_0 \in D$.

Our proof is nearly identical to the proof of [3, Theorem 4.1], but the stronger assumptions of the theorem of that work obscure the relationship between the regularity of fixed point operators at asymptotic centers and Δ-convergence. In both [2] and [3], the assumption that the fixed point mapping is α-firmly nonexpansive implies that it is nonexpansive, which is not the case here. More importantly, it is far too restrictive to require that the mapping is α-firmly nonexpansive everywhere when the property is really only required at its fixed points where there is still hope that the property enjoys a reasonable calculus. An example illustrates how our result differs from [3, Theorem 4.1], which is the specialization of Theorem 27 to the case of α-firmly nonexpansive mappings that are also nonexpansive.

Example 1. (Δ-convergence of a pointwise nonexpansive mapping). For the sets $A := (\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R})$ and $B = \{(x, x) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$ define the mapping $T := \widetilde{P}_A P_B$ where P_B is the Euclidean projection onto B and $\widetilde{P}_A : B \to \mathbb{R}^2$ is defined by

$$\widetilde{P}_A(x, x) := \begin{cases} (x, 0) & \text{when } x \in \mathbb{Q}, \\ (0, x) & \text{when } x \notin \mathbb{Q}, \end{cases}$$

where $\mathbb{Q} \subset \mathbb{R}$ is the set of rational numbers. The mapping T is everywhere single-valued and not even almost nonexpansive (consider a point $(x, x) \in B$ with x irrational and any arbitrarily close rational point (y, y)). However, T is *pointwise* nonexpansive at the origin on all of $\mathbb{R}^2$, and every fixed point sequence initialized anywhere on $\mathbb{R}^2$ converges strongly to the origin. The tools introduced introduced in the next section were used in a Euclidean space setting in [18, Theorem III.8] to show that convergence is actually linear. Theorem 4.1 of [3] cannot be applied here, while Theorem 27 correctly identifies the relevant properties guaranteeing convergence. The awkward definition of $\widetilde{P}_A$ is an artifact of requiring T to be single-valued; had we allowed multi-valued mappings, we could have simply taken the Euclidean projector on to A in the definition of T and the results still would hold. The restriction of this study to single-valued mappings is mostly to keep the notation as simple as possible.

Proof of Theorem 27. Let $\mathcal{N}$ denote any infinite subset of $\mathbb{N}$ and consider the corresponding subsequence $(x_k)_{k\in\mathcal{N}}$. This subsequence is bounded since T is a self-mapping on D ($T(D) \subseteq D$) and pointwise α-firmly nonexpansive—and hence by Proposition 4(iii) nonexpansive—at all $y \in \operatorname{Fix} T \cap D$ on D. Since D is convex, this subsequence therefore possesses a unique asymptotic center [30], which we denote by $\overline{x}_{\mathcal{N}}$. Since T is pointwise nonexpansive at $\overline{x}_{\mathcal{N}}$ on D, we have

$$\forall k \in \mathcal{N} \qquad d(T\overline{x}_{\mathcal{N}}, x_k) \le d(T\overline{x}_{\mathcal{N}}, Tx_k) + d(Tx_k, x_k) \\ \le d(\overline{x}_{\mathcal{N}}, x_k) + d(Tx_k, x_k).$$

Again, since T is pointwise α-firmly nonexpansive at all $y \in \operatorname{Fix} T \cap D$, by Lemma 7 we have $d(Tx_k, x_k) \to 0$ as $k \to \infty$. Therefore by [2, Lemma 2.11] (see also [30]), this implies that $T\overline{x}_{\mathcal{N}} = \overline{x}_{\mathcal{N}}$, that is, $\overline{x}_{\mathcal{N}} \in \operatorname{Fix} T$. This establishes part (i) of the claim.

To see part (ii) denote the unique asymptotic center of the entire sequence $(x_k)_{k\in\mathbb{N}}$ by $\overline{x}$. Then

$$\begin{aligned}\limsup_{k \underset{\mathcal{N}}{\to} \infty} d(x_k, \overline{x}_{\mathcal{N}}) &\le \limsup_{k \underset{\mathcal{N}}{\to} \infty} d(x_k, \overline{x}) \\ &\le \limsup_{k \underset{\mathbb{N}}{\to} \infty} d(x_k, \overline{x}) \\ &\le \limsup_{k \underset{\mathbb{N}}{\to} \infty} d(x_k, \overline{x}_{\mathcal{N}}) \\ &= \lim_{k \underset{\mathbb{N}}{\to} \infty} d(x_k, \overline{x}_{\mathcal{N}}) = \lim_{k \underset{\mathcal{N}}{\to} \infty} d(x_k, \overline{x}_{\mathcal{N}}),\end{aligned}$$

where the first equality follows from the fact that the sequence of distances is monotone decreasing and bounded below. Therefore $\overline{x}_{\mathcal{N}} = \overline{x}$.

Since $\mathcal{N}$ was an arbitrary infinite subset of $\mathbb{N}$, this establishes Δ-convergence of (x_k), part (iii). To see strong convergence when $T(D)$ is boundedly compact, since $(x_k)_{k\in\mathbb{N}}$ is a bounded sequence in $T(D)$, it has a convergent subsequence with limit $\overline{x}$. Whenever $(d(x_k, \overline{x}))_{k\in\mathbb{N}}$ converges, we can conclude that $x_k \to \overline{x}$.

If T is in fact nonexpansive on D, not just pointwise, then it is pointwise nonexpansive at all asymptotic centers of all fixed point sequences and subsequences initialized on D, and so the above results apply for any fixed point sequence on D. □

4.2. Quantitative convergence: error bounds

Our analysis of the convergence of fixed point iterations follows the same pattern developed in [17,34,35]. In addition to the regularity properties developed above, we use the notions of *gauge monotonicity* of sequences and *metric subregularity*. What we are calling gauge monotone sequences were first introduced in [34] where they are called μ-monotone.

Definition 28. (gauge monotonicity [34]). Let (G,d) be a metric space, let $(x_k)_{k\in\mathbb{N}}$ be a sequence on G, let $D \subset G$ be nonempty and let the continuous mapping $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy $\mu(0) = 0$ and

$$\mu(t_1) < \mu(t_2) \leq t_2 \quad \text{whenever} \quad 0 \leq t_1 < t_2.$$

(i) $(x_k)_{k\in\mathbb{N}}$ is said to be *gauge monotone with respect to D with rate μ* whenever

$$d(x_{k+1}, D) \leq \mu\left(d(x_k, D)\right) \quad \forall k \in \mathbb{N}. \tag{33}$$

(ii) $(x_k)_{k\in\mathbb{N}}$ is said to be *linearly monotone with respect to D* with rate r if (33) is satisfied for $\mu(t) = r\cdot t$ for all $t \in \mathbb{R}_+$ and some constant $r \in [0,1]$.

A sequence $(x_k)_{k\in\mathbb{N}}$ is said to converge *gauge monotonically* to some element $x^* \in G$ with rate $s_k(t) := \sum_{j=k}^{\infty} \mu^{(j)}(t)$ whenever it is gauge monotone with gauge μ satisfying $\sum_{j=1}^{\infty} \mu^{(j)}(t) < \infty$ $\forall t \geq 0$, and there exists a constant $a > 0$ such that $d(x_k, x^*) \leq a s_k(t)$ for all $k \in \mathbb{N}$.

All Fejér monotone sequences are linearly monotone (with constant $r = 1$) but the converse does not hold (see Proposition 1 and Example 1 of [34]). Gauge-monotonic convergence for a linear gauge in the definition above is just R-linear convergence.

The definition of metric subregularity below is modeled mainly after [19, Definition 2.1b)] and [20, Definition 1 b)]. Recall that $\rho : [0,\infty) \to [0,\infty)$ is a *gauge function* if ρ is continuous, strictly increasing with $\rho(0) = 0$, and $\lim_{t\to\infty} \rho(t) = \infty$.

Definition 29. (metric regularity on a set). Let (G_1, d_1) and (G_2, d_2) be metric spaces and let $\mathcal{T} : G_1 \to G_2$, $U_1 \subset G_1$, $U_2 \subset G_2$. For $\Lambda \subset G_1$, the mapping $\mathcal{T}$ is called *metrically regular on $U_1 \times U_2$ relative to Λ with gauge ρ* whenever

$$d_1\left(x, \mathcal{T}^{-1}(y) \cap \Lambda\right) \leq \rho(d_2\left(y, \mathcal{T}(x)\right)) \tag{34}$$

holds for all $x \in U_1 \cap \Lambda$ and $y \in U_2$ with $0 < \rho(d_2\left(y, \mathcal{T}(x)\right))$ where $\mathcal{T}^{-1}(y) := \{z \mid \mathcal{T}(z) = y\}$. When the set U_2 consists of a single point, $U_2 = \{\bar{y}\}$, then $\mathcal{T}$ is said to be *metrically subregular for $\bar{y}$ on U_1 relative to Λ with gauge ρ*.

The usual definition of metric subregularity is in the case where the gauge is just a linear function: $\rho(t) = rt$. The "relative to" part of the definition is also not common in the literature, but allows one to isolate the regularity to subsets (mostly manifolds) where the iterates of algorithms are naturally confined. See [4, Example 1.8] for a concrete example. In [24, Example 3.9] this is placed in a context of the *modulus of regularity* of a mapping with respect to its zeros. For our purposes, the easiest way to understand metric subregularity is as one-sided Lipschitz continuity of the (set-valued) inverse mapping $\mathcal{T}^{-1}$. We will refer to the case when the gauge is linear to *linear metric subregularity*.

We construct ρ implicitly from another non-negative function $\theta : [0,\infty) \to [0,\infty)$ satisfying

$$\text{(i) } \theta(0) = 0; \quad \text{(ii) } 0 < \theta(t) < t \ \forall t > 0; \quad \text{(iii) } \sum_{j=1}^{\infty} \theta^{(j)}(t) < \infty \ \forall t \geq 0. \tag{35}$$

The gauge we will use satisfies

$$\rho\left(\left(\frac{t^p - (\theta(t))^p}{\tau}\right)^{1/p}\right) = t \quad \iff \quad \theta(t) = \left(t^p - \tau\left(\rho^{-1}(t)\right)^p\right)^{1/p} \tag{36}$$

for $\tau > 0$ fixed and θ satisfying (35).

In the case of linear metric subregularity on a 2-uniformly convex space (think Hilbert space) we have

$$\rho(t) = rt \quad \iff \quad \theta(t) = \left(1 - \frac{\tau}{r^2}\right)^{1/2} t \quad (r \geq \sqrt{\tau}).$$

The condition $r \geq \sqrt{\tau}$ is spurious since, if (34) is satisfied for some $r' > 0$, then it is satisfied for all $r \geq r'$.

From the transport discrepancy $\psi_T^{(p,c)}$ defined in (5) and a subset $S \subset G$ we construct the following surrogate mapping $\mathcal{T}_S : G \to \mathbb{R}_+ \cup \{+\infty\}$ by

$$\mathcal{T}_S(x) := \left(\tfrac{2}{c} \inf_{y \in S} \psi_T^{(p,c)}(x,y)\right)^{1/p}. \tag{37}$$

If $S = \emptyset$ then, by definition, $\mathcal{T}_S(x) := +\infty$ for all x. When $S \subseteq \operatorname{Fix} T$, then by Proposition 4(i)

$$\mathcal{T}_S(x) = \sqrt[p]{\tfrac{2}{c}} d(Tx, x) > 0 \quad (S \neq \emptyset). \tag{38}$$

This function is thus proper (finite at least at one point, and does not take the value $-\infty$) when $S \subseteq \operatorname{Fix} T$ is nonempty. This can be interpreted as the pointwise transport discrepancy relative to the fixed points and will be used to characterize the regularity of the mapping T at fixed points.

For the remainder of this section, it will be assumed that (G, d) is a p-uniformly convex space with constant c, $D \subset G$, where the self-mapping $T : G \to G$ satisfies $T(D) \subseteq D$ and $S := \operatorname{Fix} T \cap D$ is nonempty.

Theorem 30. (quantitative convergence) *In addition to the standing assumptions, let $T(D)$ be boundedly compact. Assume*

(i) *T is pointwise α-firmly nonexpansive at all points $y \in S := \operatorname{Fix} T \cap D$ with the same constant α on D;*
(ii) *$\mathcal{T}_S$ defined by (37) is metrically subregular for 0 relative to D on D with gauge ρ given by (36) for $\tau = c(1-\alpha)/(2\alpha)$, that is,*

$$d(x, S) \leq \rho(d(Tx, x)), \quad \forall x \in D. \tag{39}$$

Then for any $x_0 \in D$, the sequence $(x_k)_{k \in \mathbb{N}}$ defined by $x_{k+1} := Tx_k$ satisfies

$$d(x_{k+1}, S) \leq \theta(d(x_k, S)) \quad \forall k \in \mathbb{N}, \tag{40}$$

where θ given implicitly by (36) satisfies (35). Moreover, the sequence $(x_k)_{k\in\mathbb{N}}$ converges gauge monotonically to some $x^* \in S$ with rate $O(s_k(t_0))$ where $s_k(t) := \sum_{j=k}^{\infty} \theta^{(j)}(t)$ and $t_0 := d(x_0, S)$.

Before proving the result, we establish convergence of gauge monotone sequences.

Lemma 31. (α-firmly nonexpansive mappings with gauge monotone fixed point sequences converge to fixed points) *In addition to the standing assumptions, let $T(D)$ be boundedly compact, and assume that T is pointwise α-firmly nonexpansive at all $y \in S$ with the same constant α on D. If the sequence $(x_k)_{k\in\mathbb{N}}$ defined by $x_{k+1} = Tx_k$ and initialized in D is gauge monotone relative to S with rate θ satisfying (35), then $(x_k)_{k\in\mathbb{N}}$ converges gauge monotonically to some $x^* \in S$ with rate $O(s_k(t_0))$ where $s_k(t) := \sum_{j=k}^{\infty} \theta^{(j)}(t)$ and $t_0 := d(x_0, S)$.*

Proof. By (15), the assumption that T is pointwise α-firmly nonexpansive at all $y \in S \subset \operatorname{Fix} T$ with constant α on D yields

$$d(Tx, y)^p \leq d(x, y)^p - \tfrac{c(1-\alpha)}{2\alpha} d(x, Tx)^p, \quad \forall x \in D, \forall y \in S. \tag{41}$$

Let $x_0 \in D$ and define the sequence $x_{k+1} := Tx_k$ for all $k \in \mathbb{N}$. Since $T(D)$ is boundedly compact and T is pointwise α-firmly nonexpansive at all points in S on D, by Proposition 4(iii) and Lemma 6, $P_S x_k$ is nonempty (though possibly set-valued) for all k; denote any selection by $\bar{x}_k \in P_S x_k$ for each $k \in \mathbb{N}$. Then (41) yields

$$d(x_{k+1}, \bar{x}_k)^p \leq d(x_k, \bar{x}_k)^p - \tfrac{c(1-\alpha)}{2\alpha} d(x_k, x_{k+1})^p, \quad \forall k \in \mathbb{N},$$

which implies that

$$d(x_k, x_{k+1}) \leq \left(\tfrac{c(1-\alpha)}{2\alpha}\right)^{-1/p} d(x_k, \bar{x}_k), \quad \forall k \in \mathbb{N}.$$

On the other hand $d(x_k, \bar{x}_k) = d(x_k, S) \leq \theta\left(d(x_{k-1}, S)\right)$ since $(x_k)_{k\in\mathbb{N}}$ is gauge monotone relative to S with rate θ. Therefore an iterative application of gauge monotonicity yields

$$d(x_k, x_{k+1}) \leq \left(\tfrac{c(1-\alpha)}{2\alpha}\right)^{-1/p} \theta^{(k)}\left(d(x_0, S)\right), \quad \forall k \in \mathbb{N}.$$

Let $t_0 = d(x_0, S)$. For any given natural numbers k, l with $k < l$ an iterative application of the triangle inequality yields the upper estimate

$$\begin{aligned} d(x_k, x_l) &\leq d(x_k, x_{k+1}) + d(x_{k+1}, x_{k+2}) + \dots + d(x_{l-1}, x_l) \\ &\leq \left(\tfrac{c(1-\alpha)}{2\alpha}\right)^{-1/p} \left(\theta^{(k)}(t_0) + \theta^{(k+1)}(t_0) + \cdots + \theta^{(l-1)}(t_0)\right) \\ &< \left(\tfrac{c(1-\alpha)}{2\alpha}\right)^{-1/p} s_k(t_0), \end{aligned}$$

where $s_k(t_0) := \sum_{j=k}^{\infty} \theta^{(j)}(t_0) < \infty$ for θ satisfying (35). Since $(\theta^{(k)}(t_0))_{k\in\mathbb{N}}$ is a summable sequence of non-negative numbers, the sequence of partial sums $s_k(t_0) \to 0$ monotonically as $k \to \infty$ and hence $(x_k)_{k\in\mathbb{N}}$ is a Cauchy sequence.

Because (G, d) is a complete metric space we conclude that $x_k \to x^*$ for some $x^* \in G$. Letting $l \to +\infty$ yields

$$\lim_{l\to+\infty} d(x_k, x_l) = d(x_k, x^*) \le a s_k(t_0), \quad a := \left(\frac{c(1-\alpha)}{2\alpha}\right)^{-1/p}.$$

Therefore $(x_k)_{k\in\mathbb{N}}$ converges gauge monotonically to x^* with rate $O(s_k(t_0))$.

It remains to show that $x^* \in S$. Note that for each $k \in \mathbb{N}$ we have

$$d(x_k, \bar{x}_k) = d(x_k, S) \le \theta^{(k)}(t_0),$$

which yields $\lim_k d(x_k, \bar{x}_k) = 0$. But by the triangle inequality

$$d(\bar{x}_k, x^*) \le d(x_k, \bar{x}_k) + d(x_k, x^*),$$

so $\lim_k d(\bar{x}_k, x^*) = 0$. By construction $(\bar{x}_k)_{k\in\mathbb{N}} \subseteq S$ and by Lemma 6 S is closed, hence $x^* \in S$. □

Proof of Theorem 30. Since $S \subset \operatorname{Fix} T$, by Proposition 4(i) we have $\psi_T^{(p,c)}(x, y) = \frac{c}{2} d(Tx, x)^p$ for all $y \in \operatorname{Fix} T$, so in fact $\mathcal{T}_S(x) = d(Tx, x)$. Also by Proposition 4(i) $\mathcal{T}_S$ takes the value 0 only on $\operatorname{Fix} T$, that is, $\mathcal{T}_S^{-1}(0) = \operatorname{Fix} T$. So by assumption (ii) and the definition of metric subregularity (Definition 29)

$$\begin{aligned} d(x, S) &= d(x, \mathcal{T}_S^{-1}(0) \cap D) \\ &\le \rho\left(|\mathcal{T}_S(x)|\right) = \rho(d(Tx, x)) \quad \forall x \in D. \end{aligned}$$

In other words,

$$\tfrac{1-\alpha}{\alpha}\tfrac{c}{2}\left(\rho^{-1}\left(d(x, S)\right)\right)^p \le \tfrac{1-\alpha}{\alpha}\tfrac{c}{2} d(Tx, x)^p \quad \forall x \in D. \tag{42}$$

On the other hand, by assumption (i) we have

$$\tfrac{1-\alpha}{\alpha}\tfrac{c}{2} d(Tx, x)^p \le d(x, y)^p - d(Tx, y)^p \quad \forall y \in S, \forall x \in D. \tag{43}$$

Incorporating (42) into (43) and rearranging the inequality yields

$$d(Tx, y)^p \le d(x, y)^p - \tfrac{1-\alpha}{\alpha}\tfrac{c}{2}\left(\rho^{-1}\left(d(x, S)\right)\right)^p \quad \forall y \in S, \forall x \in D. \tag{44}$$

Since this holds at *any* $x \in D$, it certainly holds at the iterates x_k with initial point $x_0 \in D$ since T is a self-mapping on D ($T(D) \subset D$). Therefore

$$d\left(x_{k+1}, y\right) \le \sqrt[p]{d\left(x_k, y\right)^p - \frac{1-\alpha}{\alpha}\tfrac{c}{2}\left(\rho^{-1}\left(d\left(x_k, S\right)\right)\right)^p} \quad \forall y \in S, \ \forall k \in \mathbb{N}. \tag{45}$$

Equation (45) simplifies. Indeed, by Lemma 6, S is closed. Moreover, since $T(D)$ is assumed to be boundedly compact, for every $k \in \mathbb{N}$ the distance $d(x_k, S)$ is attained at some $y_k \in S$ yielding

$$\begin{aligned} d(x_{k+1}, y_{k+1})^p &\le d(x_{k+1}, y_k)^p \\ &\le d(x_k, y_k)^p - \tfrac{1-\alpha}{\alpha}\tfrac{c}{2}\left(\rho^{-1}\left(d(x_k, y_k)\right)\right)^p \quad \forall k \in \mathbb{N}. \end{aligned} \tag{46}$$

Taking the p-th root and recalling (36) yields (40).

This establishes also that the sequence $(x_k)_{k\in\mathbb{N}}$ is gauge monotone relative to S with rate θ satisfying Eq.(35). By Lemma 31 we conclude that the sequence $(x_k)_{k\in\mathbb{N}}$ converges gauge monotonically to $x^* \in S$ with the rate $O(s_k(d(x_0, S)))$ where $s_k(t) := \sum_{j=k}^{\infty} \theta^{(j)}(t)$. □

In [34, Theorem 2] it is shown that if every fixed point sequence initialized on $D \subset G$ is linearly monotone with respect to $\operatorname{Fix} T \cap D$ with rate $c < 1$ then the surrogate mapping Ψ is linearly metrically subregular for 0 relative to D on D. From this they establish that linear metric subregularity is in fact necessary for linear convergence of fixed point sequences generated by almost α-firmly nonexpansive mappings [34, Corollary 1]. We show that this extends more generally to fixed point iterations in p-uniform metric spaces of quasi α-firmly nonexpansive mappings where the iterates converge at a rate characterized by θ.

Theorem 32. (necessity of metric subregularity for monotone sequences) *Suppose all sequences $(x_k)_{k\in\mathbb{N}}$ defined by $x_{k+1} = Tx_k$ and initialized in D are gauge monotone relative to $S := \operatorname{Fix} T \cap D$ with rate θ satisfying (35). Suppose, in addition, that $(\operatorname{Id} - \theta)^{-1}(\cdot)$ is continuous on $\mathbb{R}_+$, strictly increasing, and $(\operatorname{Id} - \theta)^{-1}(0) = 0$. Then $\mathcal{T}_S$ defined by (37) is metrically subregular for 0 relative to D on D with gauge $\rho(\cdot) = (\operatorname{Id} - \theta)^{-1}(\cdot)$.*

Proof. If the fixed point sequence is gauge monotone relative to S with rate θ satisfying (35) then by the triangle inequality

$$\begin{aligned} d(x_{k+1}, x_k) &\geq d(x_k, S) - d(x_{k+1}, S) \\ &\geq d(x_k, S) - \theta\left(d(x_k, S)\right) \quad \forall k \in \mathbb{N}. \end{aligned} \tag{47}$$

On the other hand, as shown in the proof of Theorem 30

$$\begin{aligned} \mathcal{T}_S^{-1}(0) &= \operatorname{Fix} T, \\ d(0, \mathcal{T}_S(x_k)) &= d(x_{k+1}, x_k) \end{aligned} \tag{48}$$

Combining (47) and (48) yields

$$d(0, \mathcal{T}_S(x_k)) \geq d(x_k, \mathcal{T}_S^{-1}(0) \cap D) - \theta\left(d(x_k, \mathcal{T}_S^{-1}(0) \cap D)\right) \quad \forall k \in \mathbb{N} \tag{49}$$

By assumption $(\operatorname{Id} - \theta)^{-1}(\cdot)$ is continuous on $\mathbb{R}_+$, strictly increasing, and $(\operatorname{Id} - \theta)^{-1}(0) = 0$, so

$$(\operatorname{Id} - \theta)^{-1}\left(d(0, \mathcal{T}_S(x_k))\right) \geq d(x_k, \mathcal{T}_S^{-1}(0) \cap D) \quad \forall k \in \mathbb{N}. \tag{50}$$

Since this holds for *any* sequence $(x_k)_{k\in\mathbb{N}}$ initialized in D, we conclude that $\mathcal{T}_S$ is metrically subregular for 0 on D with gauge $\rho = (\operatorname{Id} - \theta)^{-1}$. □

The next corollary is an immediate consequence of Lemma 31 and Theorem 32.

Corollary 33. (necessity of metric subregularity for gauge monotone convergence) *In addition to the standing assumptions, let $T(D)$ be boundedly compact and assume that T is α-firmly nonexpansive at all $y \in S$ with the same constant α on D. Suppose that all sequences $(x_k)_{k\in\mathbb{N}}$ defined by $x_{k+1} = Tx_k$ and initialized in D are gauge monotone relative to S with rate θ satisfying (35). Suppose, in addition, that $(\operatorname{Id} - \theta)^{-1}(\cdot)$ is continuous on $\mathbb{R}_+$, strictly increasing, and $(\operatorname{Id} - \theta)^{-1}(0) = 0$. Then all sequences initialized on D converge gauge monotonically to some $\overline{x} \in S$ with rate $O(s_k(t_0))$ where $s_k(t) := \sum_{j=k}^{\infty} \theta^{(j)}(t)$ and $t_0 := d(x_0, S)$. Moreover, $\mathcal{T}_S$ defined by (37) is metrically subregular for 0 relative to D on D with gauge $\rho(\cdot) = (\operatorname{Id} - \theta)^{-1}(\cdot)$.*

5. Examples

Most of the concrete examples provided here are for p-uniformly convex spaces with $p = c = 2$, i.e. $CAT(0)$ spaces, and these are mostly known. We hint at a path beyond this setting and in the case of cyclic projections obtain an extension of [3, Proposition 4.1] to complete, symmetric perpendicular, p-uniformly convex spaces.

5.1. Proximal splitting

Let (H, d) be a Hadamard space, $f_i : H \to H$ be proper lsc convex functions for $i = 1, 2, \ldots N$. Consider the problem

$$\inf_{x\in H} \sum_{i=1}^{N} f_i(x). \tag{51}$$

In a Hadamard space the p-proximal mapping of a function f defined by (30) simplifies to

$$\operatorname{prox}^2_{f,\lambda}(x) := \operatorname{argmin}_{y\in H} \left\{ f(y) + \tfrac{1}{2\lambda} d(x, y)^2 \right\}. \tag{52}$$

This has been studied in $CAT(0)$ spaces in [2,7,23] and in the Hilbert ball in [25]. To reduce notational clutter, we drop the superscript 2. In these earlier works it was already known that resolvents of lsc convex functions are (everywhere) α-firmly nonexpansive with $\alpha = 1/2$. The specialization of Corollary 23 to the case $p = c = 2$ confirms this. Applying *backward-backward splitting* to this problem yields Algorithm 1. We are certainly not the first to study this algorithm. Indeed, convergence has been established already in [3, Theorem 4.1]. This conclusion also follows immediately from Theorem 27 upon application of Corollary 13 which shows that the composition of quasi α-firmly nonexpansive prox mappings, $\operatorname{prox}_{f_i,\lambda_i}$, is quasi α-firmly nonexpansive on H with constant $\overline{\alpha}_m$ given recursively by (26). If on a neighborhood of $\operatorname{Fix} \overline{T}_m$, denoted by D, the mapping $\mathcal{T}_{\operatorname{Fix} \overline{T}_m \cap D}$ defined by (37)—which by Proposition 4(i) simplifies to (38)—satisfies

$$d(x, \operatorname{Fix} \overline{T}_m \cap D) \le \rho(d(\overline{T}_m x, x)) \quad \forall x \in D \tag{53}$$

where ρ is a gauge given by (36) for $\tau = \frac{1-\overline{\alpha}_m}{\overline{\alpha}_m}$, then by Theorem 30 the sequence (x_k) converges gauge monotonically to some $x^* \in \operatorname{Fix} \overline{T}_m \cap D$ with rate $O(s_k(t_0))$ where $s_k(t) := \sum_{j=k}^{\infty} \theta^{(j)}(t)$ and $t_0 := d(x_0, \operatorname{Fix} \overline{T}_m \cap D)$ for θ given implicitly by (36).

Algorithm 1: Proximal splitting

Parameters: Functions $f_1 \ldots, f_m$ and $\lambda_i > 0$ $(i = 1, 2, \ldots, m)$.
Initialization: Choose $x_0 \in H$.
for $k = 0, 1, 2, \ldots$ **do**

$$x_{k+1} = \overline{T}_m x_k := \left(\operatorname{prox}_{f_m,\lambda_m} \circ \cdots \circ \operatorname{prox}_{f_2,\lambda_2} \circ \operatorname{prox}_{f_1,\lambda_1}\right)(x_k)$$

By Corollary 23, on spaces with curvature bounded above, the p-proximal mapping is only *almost* α-firmly nonexpansive, which then yields that the composition of p-proximal mappings is also only almost α-firmly nonexpansive. However, the violation is given by $\epsilon_c = \frac{2-c}{c-1}$, where c is the constant of curvature of the space. This constant can be made arbitrarily small by choosing a small enough domain. In this way, the violation can also be made arbitrarily small. As shown in [32, Theorem 4.2] in the context of Euclidean spaces, if $\mathcal{T}_{\operatorname{Fix}\overline{T}_m}$ is metrically subregular, then the violation of an α-firmly nonexpansive mapping can be overcome to yield quantifiable (e.g. linear) convergence on neighborhoods of $\operatorname{Fix}\overline{T}_m$. Based on the foundations established here, this idea has been used in [29, Theorem 25] [1] to show for the first time convergence of proximal splitting algorithms on spaces with positive curvature.

5.2. Projected gradients

Here we specialize problem (51) to the case $N = 2$ and $f_2 = \iota_C$, the indicator function of some closed convex set $C \subset H$. Recall that in a Hadamard space the Moreau–Yosida envelope of f is defined by

$$e_{f,\lambda}(x) := \inf_{y\in H} \left(f(y) + \tfrac{1}{2\lambda} d(x,y)^2\right).$$

In a Hilbert space setting, the proximal mapping of a convex function f and the resolvent of its subdifferential are one and the same. Moreover, $e_{f,\lambda}$ is continuously differentiable with $\nabla e_{f,\lambda} = \frac{1}{\lambda}\left(\operatorname{Id} - \operatorname{prox}_{f,\lambda}\right)$. A step of length τ in the direction of steepest descent of the Moreau–Yosida envelope of f takes the form

$$x - \tau\nabla e_{f,\lambda}(x) = \left((1-\tau)\operatorname{Id} + \tau \operatorname{prox}_{f,\lambda}\right)(x).$$

Formally transposing this to a $CAT(0)$ space yields the nonlinear analog to the direction of steepest descent for $e_{f,\lambda}$:

$$(1-\tau)x \oplus \tau \operatorname{prox}_{f,\lambda}(x). \tag{54}$$

This leads to Algorithm 2, the analog to projected gradients in $CAT(0)$ space, which is nothing more than a projected resolvent/ projected proximal iteration. Theorem 21 establishes that the mapping $x \mapsto \left((1-\tau)\operatorname{Id} \oplus \tau \operatorname{prox}_{f,\lambda}\right)$ is α-firmly nonexpansive with constant $\alpha = 1/2$. Therefore, by Theorem 11 the operator T_{PG} is α-firmly nonexpansive on H with constant $\alpha_{PG} = \frac{2}{3}$. Theorem 27 then guarantees that the sequence (x_k) is Δ-convergent to some $x^* \in \operatorname{Fix} T_{PG}$, with strong convergence whenever T_{PG} is boundedly compact. If in addition (53) is satisfied with T replaced by T_{PG} and with gauge ρ given by (36) for $\tau = 1/2$, then, again, by Theorem 30 the sequence (x_k) converges gauge monotonically to some $x^* \in \operatorname{Fix} T$ with rate $O(s_k(t_0))$ where $s_k(t) := \sum_{j=k}^{\infty} \theta^{(j)}(t)$ and $t_0 := d(x_0, \operatorname{Fix} T)$ for θ given implicitly by (36).

[1] In fact, [29] was completed *after* this study, though the publication date does not reflect this.

Algorithm 2: Metric projected gradients

Parameters: $f : H \to \mathbb{R}$, the closed set $C \subset H$, $\lambda > 0$ and $\tau \in (0,1)$.
Initialization: Choose $x_0 \in H$.
for $k = 0, 1, 2, \ldots$ **do**

$$x_{k+1} = T_{PG}(x_k) := P_C\left((1-\tau)\,\mathrm{Id} \oplus \tau \,\mathrm{prox}_{f,\lambda}\right)(x_k)$$

5.3. Cyclic projections in *p*-uniformly convex spaces

For compositions of projectors we are not confined to Hadamard spaces. We consider Algorithm 1 when the functions $f_i := \iota_{C_i}$, the indicator functions of closed convex sets $C_i \subset G$, where (G, d) is a complete, symmetric perpendicular p-uniformly convex space with constant c. The p-proximal mappings of indicator functions are metric projectors and so by Proposition 25 these are pointwise α-firmly nonexpansive at all points in $\cap_i C_i$ (assuming, of course, that this is nonempty). By Lemma 10, when the intersection is nonempty the cyclic projections mapping

$$T_{CP} := P_{C_m} \cdots P_{C_2} P_{C_1} \tag{55}$$

is pointwise α-firmly nonexpansive at all points in $\cap_i C_i = \mathrm{Fix}\, T_{CP}$ with constant $\alpha_{CP} = \frac{m-1}{m}$ on G. The only asymptotic centers of subsequences of cyclic projections are points in this intersection, and here the projectors, and hence the cyclic projections mapping, are pointwise nonexpansive. So by Theorem 27 the cyclic projections sequence Δ-converges to a point in $\cap_i C_i$ whenever this is nonempty, and converges strongly whenever at least one of the sets C_i is compact. This generalizes [3, Proposition 4.1] which is limited to $CAT(\kappa)$ spaces (i.e. $p = 2$, $c < 2$ small enough).

If in addition

$$d(x, \cap_i C_i) \leq \rho(d(T_{CP}x, x)) \quad \forall x \in G \tag{56}$$

where ρ is a gauge given by (36) for $\tau = \frac{1}{m-1}$, then by Theorem 30 the sequence (x_k) converges gauge monotonically to some $x^* \in \mathrm{Fix}\, T$ with rate $O(s_k(t_0))$ where $s_k(t) := \sum_{j=k}^{\infty} \theta^{(j)}(t)$ and $t_0 := d(x_0, \mathrm{Fix}\, T)$ for θ given implicitly by (36).

6. Open problems

The property of being nonexpansive is fairly robust and carries over to compositions and convex combinations of mappings without requiring that those operators share fixed points. Our notion of α-firmly nonexpansive mappings appears to be much more demanding. Our development begs the question: is the regularity of (pointwise) α-firmly nonexpansive mappings preserved (with possibly different constants) under compositions and convex compositions when these mappings do not share common fixed points? The answer to this question has immediate bearing on the analysis of simple algorithms like cyclic projections for inconsistent feasibility or coordinate descents in nonlinear spaces.

Another open problem is to find other characterizations for gauge metric subregularity that make it easier to verify. In particular, are there easily verifiable situations in nonlinear settings—like polyhedrality or semi-algebraicity in Euclidean settings—where metric subregularity comes for free?

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Arian Bërdëllima
Institute of Mathematics
Technische Universität Berlin
Straße des 17. Juni 136
10623 Berlin
Germany
e-mail: berdellima@math.tu-berlin.de

Florian Lauster
Institute for Numerical and Applied Mathematics
University of Göttingen
Lotzestraße 16-18
37083 Göttingen
Germany
e-mail: f.lauster@math.uni-goettingen.de

D. Russell Luke
Institute for Numerical and Applied Mathematics
University of Goettingen
Lotzestraße 16-18
37083 Göttingen
Germany
e-mail: r.luke@math.uni-goettingen.de

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