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Axiomatization of the degree of Fitzpatrick, Pejsachowicz and Rabier

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Abstract. In this paper, we prove an analogue of the uniqueness theorems of Führer [15] and Amann and Weiss [1] to cover the degree of Fredholm operators of index zero constructed by Fitzpatrick, Pejsachowicz and Rabier [13], whose range of applicability is substantially wider than for the most classical degrees of Brouwer [5] and Leray–Schauder [22]. A crucial step towards the axiomatization of this degree is provided by the generalized algebraic multiplicity of Esquinas and López-Gómez [8,9,25], χ, and the axiomatization theorem of Mora-Corral [28,32]. The latest result facilitates the axiomatization of the parity of Fitzpatrick and Pejsachowicz [12], $\sigma(\cdot,[a,b])$, which provides the key step for establishing the uniqueness of the degree for Fredholm maps.

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1. Introduction

The Leray–Schauder degree was introduced in [22] to get some rather pioneering existence results on Nonlinear Partial Differential Equations. It refines, very substantially, the finite-dimensional degree introduced by Brouwer [5] to prove his celebrated fixed point theorem. The Leray–Schauder degree is a generalized topological counter of the number of zeros that a continuous map, f, can have on an open bounded subset, Ω, of a real Banach space, X. To be defined, f must be a compact perturbation of the identity map. Although this always occurs in finite-dimensional settings, it fails to be true

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in many important applications where the involved operators are not compact perturbations of the identity map but Fredholm operators of index zero between two real Banach spaces X and Y. For Fredholm maps, it is available the degree of Fredholm maps of Fitzpatrick, Pejsachowicz and Rabier [13], a refinement of the Elworthy and Tromba degree, [7], based on the topological concepts of parity and orientation discussed by Fitzpatrick and Pejsachowicz in [12]. Very recently, the authors of this article established in [29] the hidden relationships between the degree for Fredholm maps of [13] and the concept of generalized algebraic multiplicity of Esquinas and López-Gómez in [8,9,25], in a similar manner as the Leray–Schauder formula relates the Leray–Schauder degree to the classic algebraic multiplicity. The main goal of this paper is axiomatizing the Fitzpatrick–Pejsachowicz–Rabier degree in the same vein as the Brouwer and Leray–Schauder degrees were axiomatized by Führer [15] and Amann and Weiss [1], respectively. In other words, we will give a minimal set of properties that characterize the topological degree of Fitzpatrick, Pejsachowicz and Rabier.

Throughout this paper, for any given pair of real Banach spaces X, Y with $X \subset Y$, we denote by $\mathcal{L}_c(X,Y)$ the set of linear and continuous operators, $L \in \mathcal{L}(X,Y)$, that are a compact perturbation of the identity map, $L = I_X - K$. Then, the *linear group*, $GL(X,Y)$ is defined as the set of linear isomorphisms $L \in \mathcal{L}(X,Y)$. Similarly, the *compact linear group*, $GL_c(X,Y)$, is defined as $GL(X,Y) \cap \mathcal{L}_c(X,Y)$. For any $L \in \mathcal{L}(X,Y)$, the sets $N[L]$ and $R[L]$ stand for the null space (kernel) and the range (image) of L, respectively. An operator $L \in \mathcal{L}(X,Y)$ is said to be a Fredholm operator if

$$\dim N[L] < \infty \quad \text{and} \quad \operatorname{codim} R[T] < \infty.$$

In such case, $R[L]$ must be closed and the index of L is defined by

$$\operatorname{ind} L := \dim N[L] - \operatorname{codim} R[L].$$

In this paper, the set of Fredholm operators of index zero, $L \in \mathcal{L}(X,Y)$, is denoted by $\Phi_0(X,Y)$, and we set $\Phi_0(X) := \Phi_0(X,X)$. Moreover, a map $f : \overline{\Omega} \subset X \to Y$ is said to be compact if it sends bounded subsets of $\overline{\Omega}$ into relatively compact sets of Y.

In the context of the Leray–Schauder degree, for any real Banach space X, any open and bounded domain $\Omega \subset X$ and any map $f : \overline{\Omega} \subset X \to X$, it is said that (f, Ω) is an *admissible pair* if

(i) $f \in \mathcal{C}(\overline{\Omega}, X)$;
(ii) f is a compact perturbation of the identity map I_X;
(iii) $0 \notin f(\partial\Omega)$.

The class of admissible pairs will be denoted by $\mathscr{A}_{LS}$. Note that $(I_X, \Omega) \in \mathscr{A}_{LS}$ for every open and bounded subset $\Omega \subset X$, such that $0 \notin \partial\Omega$. Actually, $(I_X, \Omega) \in \mathscr{A}_{LS,GL_c}$, where $\mathscr{A}_{LS,GL_c}$ stands for the set of admissible pairs $(L, \Omega) \in \mathscr{A}_{LS}$, such that $L \in GL_c(X)$. A homotopy $H \in \mathcal{C}([0,1] \times \overline{\Omega}, X)$ is said to be admissible if $0 \notin H([0,1] \times \partial\Omega)$ and $H(t,x) = x - C(t,x)$, where $C : [0,1] \times \Omega \to X$ is a compact map. The class of admissible homotopies (H, Ω) will be denoted by $\mathscr{H}_{LS}$. The next fundamental theorem establishes the existence and the uniqueness of the Leray–Schauder degree. The existence goes back to Leray and Schauder [22] and the uniqueness is attributable to Amann and Weiss [1], though Führer [15] had already proven the uniqueness of the Brouwer degree when [1] was published.

Theorem 1.1. *For any real Banach space X, there exists a unique integer-valued map, $\deg_{LS} : \mathscr{A}_{LS} \to \mathbb{Z}$, satisfying the following properties:*

(N) ***Normalization:*** $\deg_{LS}(I_X, \Omega) = 1$ *if* $0 \in \Omega$.

(A) ***Additivity:*** *For every* $(f, \Omega) \in \mathscr{A}_{LS}$ *and any pair of open disjoint subsets,* Ω_1 *and* Ω_2*, of* Ω*, such that* $0 \notin f(\overline{\Omega} \backslash (\Omega_1 \uplus \Omega_2))$

$$\deg_{LS}(f, \Omega) = \deg_{LS}(f, \Omega_1) + \deg_{LS}(f, \Omega_2). \tag{1.1}$$

(H) ***Homotopy Invariance:*** *For every admissible homotopy* $(H, \Omega) \in \mathscr{H}_{LS}$

$$\deg_{LS}(H(0, \cdot), \Omega) = \deg_{LS}(H(1, \cdot), \Omega).$$

Moreover, for every $(L, \Omega) \in \mathscr{A}_{LS,GL_c}$ *with* $0 \in \Omega$

$$\deg_{LS}(L, \Omega) = (-1)^{\sum_{i=1}^{q} \mathfrak{m}_{\mathrm{alg}}[I_X - L; \mu_i]} \tag{1.2}$$

where

$$\mathrm{Spec}(I_X - L) \cap (1, \infty) = \{\mu_1, \mu_2, ..., \mu_q\}, \qquad \mu_i \neq \mu_j \quad \textit{if } i \neq j.$$

The map $\deg_{LS}$ is refereed to as the *Leray–Schauder degree*. In (1.2), setting $K := I_X - L$, for any eigenvalue $\mu \in \mathrm{Spec}(K)$, we have denoted by $\mathfrak{m}_{\mathrm{alg}}[K; \mu]$ the classical algebraic multiplicity of μ, that is

$$\mathfrak{m}_{\mathrm{alg}}[K; \mu] = \dim \mathrm{Ker}[(\mu I_X - K)^{\nu(\mu)}],$$

where $\nu(\mu)$ is the *algebraic ascent* of μ, i.e., the minimal integer, $\nu \geq 1$, such that

$$\mathrm{Ker}[(\mu I_X - K)^{\nu}] = \mathrm{Ker}[(\mu I_X - K)^{\nu+1}].$$

In Theorem 1.1, the axiom (N) is called the *normalization property*, because, for every $n \in \mathbb{Z}$, the map $n \deg_{LS}$ also satisfies the axioms (A) and (H), though not (N). Thus, the axiom (N) normalizes the degree so that, for the identity map, it provides us with its exact number of zeroes. The axiom (A) packages three basic properties of the Leray–Schauder degree. Indeed, by choosing $\Omega = \Omega_1 = \Omega_2 = \emptyset$, it becomes apparent that

$$\deg_{LS}(f, \emptyset) = 0, \tag{1.3}$$

so establishing that no continuous map can admit a zero in the empty set. Moreover, in the special case, when $\Omega = \Omega_1 \uplus \Omega_2$, (1.2) establishes the *additivity property* of the degree. Finally, in the special case, when $\Omega_2 = \emptyset$, it follows from (1.2) and (1.3) that:

$$\deg_{LS}(f, \Omega) = \deg_{LS}(f, \Omega_1),$$

which is usually refereed to as the *excision property* of the degree. If, in addition, also $\Omega_1 = \emptyset$, then

$$\deg_{LS}(f, \Omega) = 0 \quad \text{if } \; f^{-1}(0) \cap \overline{\Omega} = \emptyset.$$

Therefore, for every $(f,\Omega)\in\mathscr{A}_{LS}$, such that $\deg_{LS}(f,\Omega)\neq 0$, the equation $f(x)=0$ admits, at least, a solution in Ω. This key property is refereed to as the *fundamental, or solution, property* of the degree.

The axiom (H) establishes the *invariance by homotopy* of the degree. It allows to calculate the degree in the practical situations of interest from the point of view of the applications. Not surprisingly, when dealing with analytic maps, f, in $\mathbb{C}$, it provides us with the exact number of zeroes of f, counting orders, in Ω (see, e.g., Chapter 11 of [24]).

From a geometrical point of view, the construction of the Leray–Schauder degree relies on the concept of orientation, that is, on the fact that $GL_c(X)$ consists of two path connected components. Let $\mathfrak{L}\in\mathcal{C}([0,1],GL_c(X))$ be a continuous operator curve on $GL_c(X)$. Since $\mathfrak{L}$ can be regarded as the admissible homotopy $H\in\mathcal{C}([0,1]\times\overline{\Omega},X)$ with $(H(t,\cdot),\Omega)\in\mathscr{A}_{LS,GL_c}$ for each $t\in[0,1]$ defined by $H(t,\cdot):=\mathfrak{L}(t)$, $t\in[0,1]$, by the axiom (H), the integer $\deg_{LS}(\mathfrak{L}(t),\Omega)$ is constant for all $t\in[0,1]$. This introduces an equivalence relation between the operators of $GL_c(X)$. Indeed, for every pair of operators L_0, $L_1\in GL_c(X)$, it is said that $L_0\sim L_1$ if L_0 and L_1 are homotopic in $\mathscr{A}_{LS,GL_c}$ in the sense that $L_0=\mathfrak{L}(0)$ and $L_1=\mathfrak{L}(1)$ for some curve $\mathfrak{L}\in\mathcal{C}([0,1],GL_c(X))$. This equivalence relation divides $GL_c(X)$ into two path-connected components, $GL_c^+(X)$ and $GL_c^-(X)$, separated away by $\mathcal{S}(X)\cap GL_c(X)$, where

$$\mathcal{S}(X):=\mathcal{L}(X)\setminus GL(X).$$

This allows us to define a map

$$\deg_{LS}(L,\Omega):=\begin{cases}1 & \text{if } L\in GL_c^+(X) \ \text{ and } \ 0\in\Omega,\\ -1 & \text{if } L\in GL_c^-(X) \ \text{ and } \ 0\in\Omega,\\ 0 & \text{if } L\in GL_c(X) \ \text{ and } \ 0\notin\Omega,\end{cases} \tag{1.4}$$

verifying the three axioms of the Leray–Schauder degree in the class $\mathscr{A}_{LS,GL_c}$ and, in particular, the homotopy invariance. Once defined the degree in $\mathscr{A}_{LS,GL_c}$, one can extend this restricted concept of degree to the regular pairs (f,Ω) through the identity

$$\deg_{LS}(f,\Omega)=\sum_{x\in f^{-1}(0)\cap\Omega}\deg_{LS}(Df(x),\Omega).$$

A pair (f,Ω) is said to be regular if 0 is a regular value of $f:\overline{\Omega}\subset X\to X$, i.e., if $Df(x)\in GL_c(X)$ for each $x\in f^{-1}(0)\cap\Omega$. Finally, according to the Sard–Smale theorem and the homotopy invariance property, it can be extended to be defined for general admissible pairs, $(f,\Omega)\in\mathscr{A}_{LS}$. A crucial feature that facilitates this construction of the degree is the fact that the space $GL_c(X)$ consists of two path-connected components. This fails to be true in the general context of Fredholm operators of index zero, which makes the mathematical analysis of this paper much more sophisticated technically.

The main goal of this paper is establishing an analogous of Theorem 1.1 for Fredholm Operators of index zero within the context of the degree for Fredholm maps of Fitzpatrick, Pejsachowicz and Rabier [13]. Let Ω be an open and bounded subset of a real Banach space X. Then, an operator $f:\overline{\Omega}\subset X\to Y$ is said to be $\mathcal{C}^1$-*Fredholm of index zero* if

$$f\in\mathcal{C}^1(\overline{\Omega},Y)\quad\text{and}\quad Df\in\mathcal{C}(\Omega,\Phi_0(X,Y)).$$

In this paper, the set of all these operators is denoted by $\mathscr{F}_0^1(\Omega,Y)$. A given operator $f\in\mathscr{F}_0^1(\Omega,Y)$ is said to be *orientable* when $Df:\Omega\to\Phi_0(X,Y)$ is an orientable map (see Sect. 3 for the concept of orientability of maps). Moreover, for any open and bounded subset, Ω, of X and any map $f:\overline{\Omega}\subset X\to Y$ satisfying

(1) $f\in\mathscr{F}_0^1(\Omega,Y)$ is *orientable* with orientation ε,
(2) f is *proper* in $\overline{\Omega}$, i.e., $f^{-1}(K)$ is compact for every compact subset $K\subset Y$,
(3) $0\notin f(\partial\Omega)$,

it will be said that (f,Ω,ε) is a *Fredholm admissible triple*. The set of all Fredholm admissible triples in the context of Fitzpatrick, Pejsachowicz and Rabier [13] is denoted by $\mathscr{A}$. Given $(f,\Omega,\varepsilon)\in\mathscr{A}$, it is said that (f,Ω,ε) is a regular triple if 0 is a regular value of f, i.e., $Df(x)\in GL(X,Y)$ for all $x\in f^{-1}(0)$. The set of regular triples is denoted by $\mathscr{R}$. Finally, a map $H\in\mathcal{C}^1([0,1]\times\overline{\Omega},Y)$ is said to be a $\mathcal{C}^1$-*Fredholm homotopy* if $D_xH(t,x)\in\Phi_0(X,Y)$ for each $(t,x)\in[0,1]\times\Omega$, and it is called *orientable* if $D_xH:[0,1]\times\Omega\to\Phi_0(X,Y)$ is an orientable map. The main theorem of this paper reads as follows.

Theorem 1.2. *There exists a unique integer-valued map* $\deg:\mathscr{A}\to\mathbb{Z}$ *satisfying the next properties:*

(N) ***Normalization:*** $\deg(L,\Omega,\varepsilon)=\varepsilon(0)$ *for all* $L\in GL(X,Y)$ *if* $0\in\Omega$.
(A) ***Additivity:*** *For every* $(f,\Omega,\varepsilon)\in\mathscr{A}$ *and any pair of disjoint open subsets* Ω_1 *and* Ω_2 *of* Ω *with* $0\notin f(\Omega\backslash(\Omega_1\uplus\Omega_2))$

$$\deg(f,\Omega,\varepsilon)=\deg(f,\Omega_1,\varepsilon)+\deg(f,\Omega_2,\varepsilon).$$

(H) ***Homotopy Invariance:*** *For each proper* $\mathcal{C}^1$*-Fredholm homotopy* $H\in\mathcal{C}^1([0,1]\times\overline{\Omega},Y)$ *with orientation* ε *and* $(H(t,\cdot),\Omega,\varepsilon_t)\in\mathscr{A}$ *for each* $t\in[0,1]$

$$\deg(H(0,\cdot),\Omega,\varepsilon_0)=\deg(H(1,\cdot),\Omega,\varepsilon_1).$$

Moreover, given $(f,\Omega,\varepsilon)\in\mathscr{R}$ *with* Ω *connected and* $\mathcal{R}_{Df}\neq\emptyset$*, for each* $p\in\mathcal{R}_{Df}$

$$\deg(f,\Omega,\varepsilon)=\varepsilon(p)\cdot\sum_{x\in f^{-1}(0)\cap\Omega}(-1)^{\chi[\mathfrak{L}_{\omega,x},[a,b]]},\tag{1.5}$$

where $\mathfrak{L}_{\omega,x}\in\mathscr{C}^\omega([a,b],\Phi_0(X,Y))$ *is any analytic curve* $\mathcal{A}$*-homotopic to* $Df\circ\gamma$ *(see Sect. 3 for the precise meaning), for some* $\gamma\in\mathcal{C}([a,b],\Omega)$*, such that* $\gamma(a)=p$*,* $\gamma(b)=x$*, and*

$$\chi[\mathfrak{L}_{\omega,x},[a,b]]:=\sum_{\lambda_x\in\Sigma(\mathfrak{L}_{\omega,x})\cap[a,b]}\chi[\mathfrak{L}_{\omega,x},\lambda_x],$$

where χ *is the generalized algebraic multiplicity introduced by Esquinas and López-Gómez in [8,9,25] (see Sect. 2 for its definition and main properties).*

As in the context of the Leray–Schauder degree, the axiom (A) packages three fundamental properties of the degree. Namely, the additivity and excision properties, as well as the existence property, that is, whenever $(f,\Omega,\varepsilon)\in\mathscr{A}$ satisfies $\deg(f,\Omega,\varepsilon)\neq 0$, there exists $x\in\Omega$, such that $f(x)=0$.

The existence of the map deg was established by Fitzpatrick, Pejsachowicz and Rabier in [13] in the $\mathcal{C}^2$ case based on the concept of orientability introduced by Fitzpatrick and Pejsachowicz [12] and later generalized to cover the $\mathcal{C}^1$ setting by Pejsachowicz and Rabier in [35]. The identity (1.5) is a substantial sharpening of the classical Leray–Schauder formula in the context of the degree for Fredholm maps; it was proven by the authors in [29] and, by the sake of completeness, it will be proved in this article again in Sect. 5. Thus, the main novelty of Theorem 1.2 is establishing the uniqueness of deg as a direct consequence of (1.5); so, establishing an analogue of Theorem 1.1.

Benevieri and Furi [4] have established the uniqueness of another formulation of the topological degree for Fredholm operators [2,3]. In particular, using different techniques, they introduced another concept of orientability for continuous maps $h:\Lambda\to\Phi_0(X,Y)$ on a topological space Λ. When h has a regular point, that is

$$\mathcal{R}_h:=\{p\in\Lambda : h(p)\in GL(X,Y)\}\neq\emptyset, \tag{1.6}$$

the two notions coincide in the sense that $h:\Lambda\to\Phi_0(X,Y)$ is orientable in the Benivieri–Furi sense (BF-orientable for short) if and only if it is Fitzpatrick–Pejsachowicz orientable (FP-orientable for short). However, when $\mathcal{R}_h=\emptyset$, these two concepts are different. Although the singular maps (with $\mathcal{R}_h=\emptyset$) are orientable adopting the FP-orientation, there are examples of singular h's that are not BF-orientable. More precisely, given a Banach space X of Kuiper type, i.e., such that $GL(X)$ is contractible, consider the map defined by

$$\mathfrak{S}:\mathbb{S}^1\longrightarrow\Phi_0(X\times\mathbb{R}),\quad \mathfrak{S}(t):=\begin{pmatrix}\mathfrak{L}(t) & 0\\ 0 & 0\end{pmatrix}, \tag{1.7}$$

where $\mathbb{S}^1$ stands for the unit circle, the matrix decomposition is given through the canonical projections

$$P_1: X\times\mathbb{R}\to X,\quad P_1(x,\lambda)=x,$$
$$P_2: X\times\mathbb{R}\to\mathbb{R},\quad P_2(x,\lambda)=\lambda,$$

and $\mathfrak{L}:\mathbb{S}^1\to\Phi_0(X)$ is some BF-nonorientable map, whose existence is guaranteed by [4, Th. 3.15]. Then, clearly, $\mathfrak{S}$ is singular, i.e., $\mathcal{R}_{\mathfrak{S}}=\emptyset$, and hence, it is FP-orientable, thought, owing to [4, Pr. 3.8], $\mathfrak{S}$ is not BF-orientable.

Based on this fact, the degree constructed by Benevieri and Furi does not coincide with the degree of Fitzpatrick, Pejsachowicz and Rabier, because there are admissible triples (f,Ω,ε), such that $Df:\Omega\to\Phi_0(X,Y)$ is not BF-orientable. Thus, although Benevieri and Furi proved in [4] an uniqueness result for their degree, our Theorem 1.2 here is independent of their main uniqueness result. Actually, both uniqueness results are independent in the sense that no one implies the other, though in some important applications, both degrees coincide. However, since the algebraic multiplicity χ is defined for Fredholm operator curves $\mathfrak{L} : [a, b] \to \Phi_0(X, Y)$ and the orientability notion of Fitzpatrick and Pejsachowicz is defined through the use of this type of curves by means of their notion of parity, we see far more natural the degree of Fitzpatrick, Pejsachowicz and Rabier for delivering an analogue of the uniqueness theorem of Amann and Weiss through (1.5), within the same vein as in the classical context of the Leray–Schauder degree.

This paper is organized as follows. Section 2 contains some necessary preliminaries on the Leray–Schauder degree and the generalized algebraic multiplicity, χ, used in the generalized Leray–Schauder formula (1.5). Section 3 introduces the concepts of parity and orientation of Fitzpatrick and Pejsachowicz [12] and collects some of the findings of the authors in [29], where the Fitzpatrick–Pejsachowicz parity, σ, was calculated through the generalized algebraic multiplicity χ. These results are needed for axiomatizing the parity σ in Sect. 4. The main result of Sect. 4 is Theorem 4.2, which characterizes σ through a normalization property, a product formula, and its invariance by homotopy, by means of the algebraic multiplicity χ. This result is reminiscent of the uniqueness theorem of Mora-Corral [32] for the multiplicity χ (see also Chapter 6 of [28]). Finally, based on these results, the proof of Theorem 1.2 is delivered in Sect. 5 after revisiting, very shortly, the main concepts of the Fitzpatrick–Pejsachowicz–Rabier degree.

2. Generalized algebraic multiplicity

As the generalized algebraic multiplicity introduced by Esquinas and López-Gómez in [8,9,25] is a pivotal technical device in the proof of Theorem 1.2 through the formula (1.5), we will collect some of its most fundamental properties, among them the uniqueness theorem of Mora-Corral [28,32].

Given two Banach spaces, X and Y, by a *Fredholm path, or curve,* it is meant any map $\mathfrak{L} \in \mathcal{C}([a, b], \Phi_0(X, Y))$. Given a Fredholm path, $\mathfrak{L}$, it is said that $\lambda \in [a, b]$ is an *eigenvalue* of $\mathfrak{L}$ if $\mathfrak{L}(\lambda) \notin GL(X, Y)$. Then, the *spectrum* of $\mathfrak{L}$, $\Sigma(\mathfrak{L})$, consists of the set of all these eigenvalues, that is

$$\Sigma(\mathfrak{L}) := \{\lambda \in [a, b] : \mathfrak{L}(\lambda) \notin GL(X, Y)\}.$$

According to Lemma 6.1.1 of [25], $\Sigma(\mathfrak{L})$ is a compact subset of $[a, b]$, though, in general, one cannot say anything more about it, because for any given compact subset of $[a, b]$, J, there exists a continuous function $\mathfrak{L} : [a, b] \to \mathbb{R}$, such that $J = \mathfrak{L}^{-1}(0)$. Next, we will deliver a concept introduced in [25] to characterize whether, or not, the algebraic multiplicity of Esquinas and López-Gómez [8,9,25] is well defined. Let $\mathfrak{L} \in \mathcal{C}([a, b], \Phi_0(X, Y))$ and $k \in \mathbb{N}$. An eigenvalue $\lambda_0 \in \Sigma(\mathfrak{L})$ is said to be a *k-algebraic eigenvalue* if there exits $\varepsilon > 0$, such that

(a) $\mathfrak{L}(\lambda) \in GL(X, Y)$ if $0 < |\lambda - \lambda_0| < \varepsilon$;

(b) There exits $C > 0$, such that

$$\|\mathfrak{L}^{-1}(\lambda)\| < \frac{C}{|\lambda - \lambda_0|^k} \quad \text{if } 0 < |\lambda - \lambda_0| < \varepsilon; \tag{2.1}$$

(c) k is the least positive integer for which (2.1) holds.

The set of algebraic eigenvalues of $\mathfrak{L}$ or order k will be denoted by $\mathrm{Alg}_k(\mathfrak{L})$. Thus, the set of *algebraic eigenvalues* can be defined by

$$\mathrm{Alg}(\mathfrak{L}) := \biguplus_{k\in\mathbb{N}} \mathrm{Alg}_k(\mathfrak{L}).$$

By Theorems 4.4.1 and 4.4.4 of [25], when $\mathfrak{L}(\lambda)$ is real analytic in $[a,b]$, i.e., $\mathfrak{L} \in \mathcal{C}^\omega([a,b],\Phi_0(X,Y))$, then either $\Sigma(\mathfrak{L}) = [a,b]$, or $\Sigma(\mathfrak{L})$ is finite and $\Sigma(\mathfrak{L}) \subset \mathrm{Alg}(\mathfrak{L})$. According to [28, Ch. 7], $\lambda_0 \in \mathrm{Alg}(\mathfrak{L})$ if, and only if, the lengths of all Jordan chains of $\mathfrak{L}$ at λ_0 are uniformly bounded above, which allows to characterize whether, or not, $\mathfrak{L}(\lambda)$ admits a local Smith form at λ_0 (see Rabier [28,36]). The next concept allows to introduce a generalized algebraic multiplicity, $\chi[\mathfrak{L},\lambda_0]$, in a rather natural manner. It goes back to [9]. Subsequently, we will denote

$$\mathfrak{L}_j := \frac{1}{j!}\mathfrak{L}^{(j)}(\lambda_0), \quad 1 \leq j \leq r$$

if these derivatives exist. Given a path $\mathfrak{L} \in \mathcal{C}^r([a,b],\Phi_0(X,Y))$ and an integer $1 \leq k \leq r$, a given eigenvalue $\lambda_0 \in \Sigma(\mathfrak{L})$ is said to be a *k-transversal eigenvalue* of $\mathfrak{L}$ if

$$\bigoplus_{j=1}^{k} \mathfrak{L}_j \left(\bigcap_{i=0}^{j-1} \mathrm{Ker}(\mathfrak{L}_i) \right) \oplus R(\mathfrak{L}_0) = Y \;\text{ with }\; \mathfrak{L}_k \left(\bigcap_{i=0}^{k-1} \mathrm{Ker}(\mathfrak{L}_i) \right) \neq \{0\}.$$

For these eigenvalues, the *algebraic multiplicity of* $\mathfrak{L}$ *at* λ_0, $\chi[\mathfrak{L},\lambda_0]$, is defined through

$$\chi[\mathfrak{L};\lambda_0] := \sum_{j=1}^{k} j \cdot \dim \mathfrak{L}_j \left(\bigcap_{i=0}^{j-1} \mathrm{Ker}(\mathfrak{L}_i) \right). \tag{2.2}$$

By Theorems 4.3.2 and 5.3.3 of [25], for every $\mathfrak{L} \in \mathcal{C}^r([a,b],\Phi_0(X,Y))$, $k \in \{1,2,...,r\}$ and $\lambda_0 \in \mathrm{Alg}_k(\mathfrak{L})$, there exists a polynomial $\Phi : \mathbb{R} \to \mathcal{L}(X)$ with $\Phi(\lambda_0) = I_X$, such that λ_0 is a k-transversal eigenvalue of the path

$$\mathfrak{L}^\Phi := \mathfrak{L} \circ \Phi \in \mathcal{C}^r([a,b],\Phi_0(X,Y)). \tag{2.3}$$

Moreover, $\chi[\mathfrak{L}^\Phi;\lambda_0]$ is independent of the curve of *trasversalizing local isomorphisms* Φ chosen to transversalize $\mathfrak{L}$ at λ_0 through (2.3), regardless Φ is a polynomial or not. Therefore, the next generalized concept of algebraic multiplicity is consistent

$$\chi[\mathfrak{L};\lambda_0] := \chi[\mathfrak{L}^\Phi;\lambda_0].$$

This concept of algebraic multiplicity can be easily extended by setting

$$\chi[\mathfrak{L};\lambda_0] = 0 \quad \text{if } \; \lambda_0 \notin \Sigma(\mathfrak{L})$$

and

$$\chi[\mathfrak{L};\lambda_0] = +\infty \quad \text{if } \; \lambda_0 \in \Sigma(\mathfrak{L}) \setminus \mathrm{Alg}(\mathfrak{L}) \; \text{ and } \; r = +\infty.$$

Thus, $\chi[\mathfrak{L};\lambda]$ is well defined for all $\lambda \in (a,b)$ of any smooth path $\mathfrak{L} \in \mathcal{C}^\infty([a,b],\Phi_0(X,Y))$ and, in particular, for any analytical curve $\mathfrak{L} \in$

$\mathcal{C}^\omega([a,b],\Phi_0(X,Y))$. In other words, χ can be viewed for each $\lambda \in (a,b)$ as a map

$$\chi[\cdot,\lambda] : \mathcal{C}^\infty([a,b],\Phi_0(X,Y)) \longrightarrow [0,\infty].$$

The next uniqueness result goes back to Mora-Corral [32] and [28, Ch. 6].

Theorem 2.1. *Let U be a non-zero real Banach space, $\lambda_0 \in \mathbb{R}$, and let $\mathfrak{I}(U)$ be a set of Banach spaces isomorphic to U, such that $U \in \mathfrak{I}(U)$. Then, for every $\varepsilon > 0$, the algebraic multiplicity χ is the unique map*

$$\chi[\cdot;\lambda_0] : \bigcup_{X,Y\in\mathfrak{I}(U)} \mathcal{C}^\infty((\lambda_0-\varepsilon,\lambda_0+\varepsilon),\Phi_0(X,Y)) \longrightarrow [0,\infty]$$

satisfying the next two axioms

(P) *If $X,Y,Z \in \mathfrak{I}(U)$ with*

$$\mathfrak{L} \in \mathcal{C}^\infty((\lambda_0-\varepsilon,\lambda_0+\varepsilon),\Phi_0(X,Y)), \quad \mathfrak{M} \in \mathcal{C}^\infty((\lambda_0-\varepsilon,\lambda_0+\varepsilon);\Phi_0(Y,Z))$$

then, the next product formula holds

$$\chi[\mathfrak{M}\circ\mathfrak{L};\lambda_0] = \chi[\mathfrak{L};\lambda_0] + \chi[\mathfrak{M};\lambda_0].$$

(N) *There exits a rank one projection $P_0 \in \mathcal{L}(U)$, such that*

$$\chi[(\lambda-\lambda_0)P_0 + I_U - P_0;\lambda_0] = 1.$$

The axiom (P) is the *product formula* and the axiom (N) is a *normalization property* for establishing the uniqueness of the algebraic multiplicity. From these axioms, one can derive all the remaining properties of the generalized algebraic multiplicity χ. Among them, that it equals the classical algebraic multiplicity when

$$\mathfrak{L}(\lambda) = \lambda I_X - K$$

for some compact operator K. Indeed, according to [25] and [28], for every smooth path $\mathfrak{L} \in \mathcal{C}^\infty((\lambda_0-\varepsilon,\lambda_0+\varepsilon),\Phi_0(X,Y))$, the following properties hold:

- $\chi[\mathfrak{L};\lambda_0] \in \mathbb{N} \uplus \{+\infty\}$;
- $\chi[\mathfrak{L};\lambda_0] = 0$ if, and only if, $\mathfrak{L}(\lambda_0) \in GL(X,Y)$;
- $\chi[\mathfrak{L};\lambda_0] < \infty$ if, and only if, $\lambda_0 \in \mathrm{Alg}(\mathfrak{L})$.
- If $X = Y = \mathbb{R}^N$, then, in any basis

$$\chi[\mathfrak{L};\lambda_0] = \mathrm{ord}_{\lambda_0} \det \mathfrak{L}(\lambda).$$

- Let $L \in \mathcal{L}(X)$ be such that $\lambda I_X - L \in \Phi_0(X)$. Then, for every $\lambda_0 \in \mathrm{Spec}(L)$, there exists $k \geq 1$, such that

$$\begin{aligned} \chi[\lambda I_X - L;\lambda_0] &= \sup_{n\in\mathbb{N}} \dim \mathrm{Ker}[(\lambda_0 I_X - L)^n] \\ &= \dim \mathrm{Ker}[(\lambda_0 I_X - L)^k] = \mathfrak{m}_{\mathrm{alg}}[L;\lambda_0]. \end{aligned} \tag{2.4}$$

Therefore, χ extends, very substantially, the classical concept of algebraic multiplicity.

3. Parity and orientability

This section collects some very recent findings of the authors in [29] in connection with the concepts of *parity* and *orientability* introduced by Fitzpatrick and Pejsachowicz in [12]. We begin by recalling some important features concerning the structure of the space of linear Fredholm operators of index zero, $\Phi_0(X,Y)$, which is an open path-connected subset of $\mathcal{L}(X,Y)$; in general, $\Phi_0(X,Y)$ is not linear. Subsequently, for every $n\in\mathbb{N}$, we denote by $\mathcal{S}_n(X,Y)$ the set of *singular operators of order* n

$$\mathcal{S}_n(X,Y) := \{L\in\Phi_0(X,Y): \ \dim N[L]=n\}.$$

Thus, the set of *singular operators* is given through

$$\mathcal{S}(X,Y) := \Phi_0(X,Y)\backslash GL(X,Y) = \biguplus_{n\in\mathbb{N}} \mathcal{S}_n(X,Y).$$

According to [11], for every $n\in\mathbb{N}$, $\mathcal{S}_n(X,Y)$ is a Banach submanifold of $\Phi_0(X,Y)$ of codimension n^2. This feature allows us to view $\mathcal{S}(X,Y)$ as a stratified analytic set of codimension 1 of $\Phi_0(X,Y)$. By Theorem 1 of Kuiper [19], the space of isomorphisms, $GL(H)$, of any separable infinite-dimensional Hilbert space, H, is path connected. Thus, it is not possible to introduce an orientation in $GL(X,Y)$ for general Banach spaces X,Y, since, in general, $GL(X,Y)$ is path connected. This fact reveals a fundamental difference between finite- and infinite-dimensional normed spaces, because, for every $N\in\mathbb{N}$, the space $GL(\mathbb{R}^N)$ is divided into two path connected components, $GL^{\pm}(\mathbb{R}^N)$.

A key technical tool to overcome this difficulty to define a degree in $\Phi_0(X,Y)$ is provided by the concept of *parity* introduced by Fitzpatrick and Pejsachowicz [12]. The parity is a generalized local detector of the change of orientability of a given *admissible path*. Subsequently, a Fredholm path $\mathfrak{L}\in\mathcal{C}([a,b],\Phi_0(X,Y))$ is said to be *admissible* if $\mathfrak{L}(a),\mathfrak{L}(b)\in GL(X,Y)$, and we denote by $\mathscr{C}([a,b],\Phi_0(X,Y))$ the set of admissible paths. Moreover, for every $r\in\mathbb{N}\uplus\{+\infty,\omega\}$, we set

$$\mathscr{C}^r([a,b],\Phi_0(X,Y)) := \mathcal{C}^r([a,b],\Phi_0(X,Y))\cap\mathscr{C}([a,b],\Phi_0(X,Y)).$$

The fastest way to introduce the notion of parity consists in defining it for $\mathscr{C}$-transversal paths and then for general admissible curves through the density of $\mathscr{C}$-transversal paths in $\mathscr{C}([a,b],\Phi_0(X,Y))$, already established in [11]. A Fredholm path, $\mathfrak{L}\in\mathcal{C}([a,b],\Phi_0(X,Y))$, is said to be $\mathscr{C}$*-transversal* if

(i) $\mathfrak{L}\in\mathscr{C}^1([a,b],\Phi_0(X,Y))$;
(ii) $\mathfrak{L}([a,b])\cap\mathcal{S}(X,Y)\subset\mathcal{S}_1(X,Y)$ and it is finite;
(iii) $\mathfrak{L}$ is transversal to $\mathcal{S}_1(X,Y)$ at each point of $\mathfrak{L}([a,b])\cap\mathcal{S}(X,Y)$.

The path $\mathfrak{L}\in\mathcal{C}^1([a,b],\Phi_0(X,Y))$ is said to be transversal to $\mathcal{S}_1(X,Y)$ at λ_0 if

$$\mathfrak{L}'(\lambda_0)+T_{\mathfrak{L}(\lambda_0)}\mathcal{S}_1(X,Y)=\mathcal{L}(X,Y),$$

where $T_{\mathfrak{L}(\lambda_0)}\mathcal{S}_1(X,Y)$ stands for the tangent space to the manifold $\mathcal{S}_1(X,Y)$ at $\mathfrak{L}(\lambda_0)$.

When $\mathfrak{L}$ is $\mathscr{C}$-transversal, the *parity* of $\mathfrak{L}$ in $[a,b]$ is defined by

$$\sigma(\mathfrak{L},[a,b]) := (-1)^k,$$

where $k \in \mathbb{N}$ equals the cardinal of $\mathfrak{L}([a,b]) \cap \mathcal{S}(X,Y)$. Thus, the parity of a $\mathscr{C}$-transversal path, $\mathfrak{L}(\lambda)$, is the number of times, mod 2, that $\mathfrak{L}(\lambda)$ intersects transversally the stratified analytic set $\mathcal{S}(X,Y)$.

The fact that the $\mathscr{C}$-transversal paths are dense in the set of all admissible paths, together with the next stability property: for any $\mathscr{C}$-transversal path $\mathfrak{L} \in \mathcal{C}([a,b],\Phi_0(X,Y))$, there exists $\varepsilon > 0$, such that

$$\sigma(\mathfrak{L},[a,b]) = \sigma(\tilde{\mathfrak{L}},[a,b])$$

for all $\mathscr{C}$-transversal path $\tilde{\mathfrak{L}} \in \mathcal{C}([a,b],\Phi_0(X,Y))$ with $\|\mathfrak{L}-\tilde{\mathfrak{L}}\|_\infty < \varepsilon$ (see [11]); allows us to define the parity for a general admissible path $\mathfrak{L} \in \mathscr{C}([a,b],\Phi_0(X,Y))$ through

$$\sigma(\mathfrak{L},[a,b]) := \sigma(\tilde{\mathfrak{L}},[a,b]),$$

where $\tilde{\mathfrak{L}}$ is any $\mathscr{C}$-transversal curve satisfying $\|\mathfrak{L}-\tilde{\mathfrak{L}}\|_\infty < \varepsilon$ for sufficiently small $\varepsilon > 0$.

Subsequently, a given homotopy $H \in \mathcal{C}([0,1]\times[a,b],\Phi_0(X,Y))$ is said to be *admissible* if $H([0,1]\times\{a,b\}) \subset GL(X,Y)$. Moreover, two given paths, $\mathfrak{L}_1$ and $\mathfrak{L}_2$, are said to be $\mathcal{A}$-*homotopic* if they are homotopic through an admissible homotopy. A fundamental property of the parity is its invariance under admissible homotopies, which was established in [12].

The next result, proven by the authors in [29], establishes that, as soon as the Fredholm path $\mathfrak{L}(\lambda)$ is defined in $\mathcal{L}_c(X)$, every transversal intersection with $\mathcal{S}(X)$ induces a change of orientation, i.e., a change of path-connected component.

Theorem 3.1. *Let $\mathfrak{L} \in \mathscr{C}([a,b],\mathcal{L}_c(X))$ be an admissible curve with values in $\mathcal{L}_c(X)$. Then, $\sigma(\mathfrak{L},[a,b]) = -1$ if, and only if, $\mathfrak{L}(a)$ and $\mathfrak{L}(b)$ lie in different path-connected components of $GL_c(X)$.*

Theorem 3.1 motivates the geometrical interpretation of the parity as a local detector of the change of orientation of the operators of a Fredholm path. As illustrated by Fig. 1, each transversal intersection of the path $\mathfrak{L}(\lambda)$ with $\mathcal{S}(X)$ can be viewed as a change of path-connected component.

The next result, proven by the authors in [29], shows how the parity of any admissible Fredholm path $\mathfrak{L} \in \mathscr{C}([a,b],\Phi_0(X,Y))$ can be computed through the algebraic multiplicity χ. This result is important from the point of view of the applications.

Theorem 3.2. *Any continuous path $\mathfrak{L} \in \mathscr{C}([a,b],\Phi_0(X,Y))$ is $\mathcal{A}$-homotopic to an analytic Fredholm curve $\mathfrak{L}_\omega \in \mathscr{C}^\omega([a,b],\Phi_0(X,Y))$. Moreover, for any of these analytic paths*

$$\sigma(\mathfrak{L},[a,b]) = (-1)^{\sum_{i=1}^n \chi[\mathfrak{L}_\omega;\lambda_i]},$$

where

$$\Sigma(\mathfrak{L}_\omega) = \{\lambda_1,\lambda_2,...,\lambda_n\}.$$

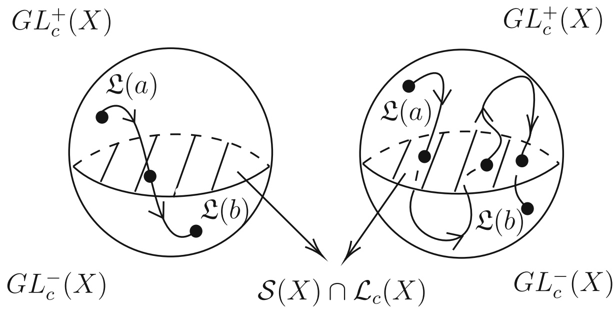

Figure 1. Geometrical interpretation of the parity on $\mathcal{L}_c(X)$

Subsequently, we consider $\mathfrak{L}\in\mathcal{C}([a,b],\Phi_0(X,Y))$ and an isolated eigenvalue $\lambda_0\in\Sigma(\mathfrak{L})$. Then, the *localized parity* of $\mathfrak{L}$ at λ_0 is defined through

$$\sigma(\mathfrak{L},\lambda_0):=\lim_{\eta\downarrow 0}\sigma(\mathfrak{L},[\lambda_0-\eta,\lambda_0+\eta]).$$

As a consequence of Theorem 3.2, the next result, going back to [29], holds.

Corollary 3.3. *Let* $\mathfrak{L}\in\mathcal{C}^r([a,b],\Phi_0(X,Y))$ *with* $r\in\mathbb{N}\uplus\{\infty,\omega\}$ *and* $\lambda_0\in\mathrm{Alg}_k(\mathfrak{L})$ *for some* $1\leq k\leq r$*. Then*

$$\sigma(\mathfrak{L},\lambda_0)=(-1)^{\chi[\mathfrak{L};\lambda_0]}. \tag{3.1}$$

The identity (3.1) establishes the precise relationship between the topological notion of parity and the algebraic concept of multiplicity. The importance of Corollary 3.3 relies on the fact that, since the localized parity detects any change of orientation, (3.1) makes intrinsic to the concept of algebraic multiplicity any change of the local degree.

As the principal difficulty to introduce a topological degree for Fredholm operators of index zero is the absence of orientation in the space of linear isomorphisms $GL(X,Y)\subset\Phi_0(X,Y)$, the notion introduced in the next definition, going back to Fitzpatrick, Pejsachowicz and Rabier [13], restricts the admissible maps to the ones where is possible to introduce a notion of *orientability*. In the sequel, the notation Λ stands for a fixed topological space, and $\mathcal{R}_h$ is defined through (1.6).

Definition 3.4. A continuous map $h:\Lambda\to\Phi_0(X,Y)$ is said to be orientable if there exists a map $\varepsilon:\mathcal{R}_h\to\mathbb{Z}_2$, called orientation, such that

$$\sigma(h\circ\gamma,[a,b])=\varepsilon(\gamma(a))\cdot\varepsilon(\gamma(b)) \tag{3.2}$$

for each curve $\gamma\in\mathcal{C}([a,b],\Lambda)$ with $\gamma(a),\gamma(b)\in\mathcal{R}_h$. A subset $\mathcal{O}\subset\Phi_0(X,Y)$ is said to be orientable if the inclusion map $i:\mathcal{O}\hookrightarrow\Phi_0(X,Y)$ is orientable, i.e., if there exists a map $\varepsilon:\mathcal{O}\cap GL(X,Y)\to\mathbb{Z}_2$, such that

$$\sigma(\mathfrak{L},[a,b])=\varepsilon(\mathfrak{L}(a))\cdot\varepsilon(\mathfrak{L}(b))\quad\text{for all }\ \mathfrak{L}\in\mathscr{C}([a,b],\mathcal{O}). \tag{3.3}$$

Observe that if $\mathcal{R}_h=\emptyset$, then h is trivially orientable. Since the parity of a Fredholm curve $\mathfrak{L}$ can be regarded as a generalized local detector of any change of orientation, it is natural to define an orientation ε of a subset $\mathcal{O}$ of $\Phi_0(X,Y)$ as a map satisfying (3.3). Indeed, owing to (3.3), $\sigma(\mathfrak{L},[a,b])=-1$ if $\varepsilon(\mathfrak{L}(a))$ and $\varepsilon(\mathfrak{L}(b))$ have contrary sign. Also, note that if $\mathcal{O}$ is an orientable subset of $\Phi_0(X,Y)$ with orientation ε, then ε is locally constant, i.e., ε is constant on each path connected component of $\mathcal{O}\cap GL(X,Y)$. This is a rather natural property of an orientation. The same is true for maps h; the map $\varepsilon:\mathcal{R}_h\to\mathbb{Z}_2$ is constant in each path-connected component of $\mathcal{R}_h$.

An orientable map $h:\Lambda\to\Phi_0(X,Y)$ with Λ path connected and $\mathcal{R}_h\neq\emptyset$, admits, exactly, two different orientations. Precisely, given $p\in\mathcal{R}_h$, the two orientations of h are defined through

$$\varepsilon^{\pm}:\mathcal{R}_h\longrightarrow\mathbb{Z}_2,\quad q\mapsto\pm\sigma(h\circ\gamma,[a,b]), \tag{3.4}$$

where $\gamma\in\mathcal{C}([a,b],\Lambda)$ is an arbitrary path linking p with q, and the sign $\pm$ determines the orientation of the path-connected component of p in $\mathcal{R}_h$, i.e., if we choose ε^+, then the orientation of the path connected component of p is 1, whereas it is -1 if ε^- is chosen. Finally, note that if Λ' is any subspace of Λ, then the restriction of an orientation to $\mathcal{R}_h\cap\Lambda'$ gives an orientation for $h|_{\Lambda'}$.

According to [13], if Λ is simply connected, any $h:\Lambda\to\Phi_0(X,Y)$ is orientable. Therefore, the set of orientable maps is really large. More generally, $h:\Lambda\to\Phi_0(X,Y)$ is orientable if the $\mathbb{Z}_2$-cohomology group $H^1(\Lambda,\mathbb{Z}_2)$ is trivial.

The next result, going back to [29], justifies the geometrical interpretation of the parity as a local detector of change of orientation for the operators of a Fredholm path.

Proposition 3.5. *Let $\mathcal{O}$ be an orientable subset of $\Phi_0(X,Y)$ and $\mathfrak{L}\in\mathscr{C}([a,b],\mathcal{O})$. Then, $\sigma(\mathfrak{L},[a,b])=-1$ if, and only if, $\mathfrak{L}(a)$ and $\mathfrak{L}(b)$ lye in different path connected components of $\mathcal{O}\cap GL(X,Y)$ with opposite orientations.*

Finally, the next result, going back as well to [29], reduces the problem of detecting any change of orientation to the problem of the computation of the local multiplicity. It allows to interpret the algebraic multiplicity as a local detector of change of orientation for the operators of a Fredholm path. Given $\mathfrak{L}\in\mathcal{C}([a,b],\Phi_0(X,Y))$ and $\delta>0$, an isolated eigenvalue $\lambda_0\in\Sigma(\mathfrak{L})$ is said to be δ-*isolated* if

$$\Sigma(\mathfrak{L})\cap[\lambda_0-\delta,\lambda_0+\delta]=\{\lambda_0\}.$$

Theorem 3.6. *Suppose that $\mathcal{O}\subset\Phi_0(X,Y)$ is an orientable subset, $\mathfrak{L}\in\mathcal{C}([a,b],\mathcal{O})$ is a Fredholm curve and $\lambda_0\in\Sigma(\mathfrak{L})$ a δ-isolated eigenvalue. Then, the next assertions are equivalent:*

(a) $\sum_{\lambda\in\Sigma(\mathfrak{L}_\omega)}\chi[\mathfrak{L}_\omega;\lambda_0]$ *is odd for any analytical path $\mathfrak{L}_\omega\in\mathscr{C}^\omega([\lambda_0-\delta,\lambda_0+\delta],\Phi_0(X,Y))$, such that $\mathfrak{L}|_{[\lambda_0-\delta,\lambda_0+\delta]}$ and $\mathfrak{L}_\omega$ are $\mathcal{A}$-homotopic.*

(b) *$\mathfrak{L}(\lambda_0-\delta)$ and $\mathfrak{L}(\lambda_0+\delta)$ live in different path-connected components of $\mathcal{O}\cap GL(X,Y)$ with opposite orientations.*

4. Axiomatization and uniqueness of the parity

The aim of this section is axiomatizing the concept of *parity*. Therefore, establishing its uniqueness. Our axiomatization is based on Theorem 2.1 and Corollary 3.3. Thanks to this axiomatization, we are establishing the uniqueness of a local detector of change of orientability. We will begin by axiomatizing the parity as a local object. Then, we will do it in a global setting.

Subsequently, for any interval $\mathcal{I} \subset \mathbb{R}$ and $\lambda_0 \in \operatorname{Int}\mathcal{I}$, we will denote by $\mathcal{C}^\omega_{\lambda_0}(\mathcal{I}, \Phi_0(X,Y))$ the space of all the analytic paths $\mathfrak{L} \in \mathcal{C}^\omega(\mathcal{I}, \Phi_0(X,Y))$, such that $\mathfrak{L}(\lambda) \in GL(X,Y)$ for all $\lambda \in \mathcal{I} \setminus \{\lambda_0\}$.

Theorem 4.1. *For every $\varepsilon > 0$ and $\lambda_0 \in \mathbb{R}$, there exists a unique $\mathbb{Z}_2$-valued map*

$$\sigma(\cdot, \lambda_0): \ \mathcal{C}^\omega_{\lambda_0} \equiv \mathcal{C}^\omega_{\lambda_0}((\lambda_0 - \varepsilon, \lambda_0 + \varepsilon), \Phi_0(X)) \longrightarrow \mathbb{Z}_2,$$

such that

(N) ***Normalization:*** *$\sigma(\mathfrak{L}, \lambda_0) = 1$ if $\mathfrak{L}(\lambda_0) \in GL(X)$, and there exists a rank one projection $P_0 \in \mathcal{L}(X)$, such that $\sigma(\mathfrak{E}, \lambda_0) = -1$ where*

$$\mathfrak{E}(\lambda) := (\lambda - \lambda_0)P_0 + I_X - P_0.$$

(P) ***Product Formula:*** *For every $\mathfrak{L}, \mathfrak{M} \in \mathcal{C}^\omega_{\lambda_0}$,*

$$\sigma(\mathfrak{L} \circ \mathfrak{M}, \lambda_0) = \sigma(\mathfrak{L}, \lambda_0) \cdot \sigma(\mathfrak{M}, \lambda_0).$$

Moreover, for every $\mathfrak{L} \in \mathcal{C}^\omega_{\lambda_0}$, the parity map is given by

$$\sigma(\mathfrak{L}, \lambda_0) = (-1)^{\chi[\mathfrak{L};\lambda_0]}.$$

Proof. First, we will prove that, for every rank one projection $P \in \mathcal{L}(X)$, setting

$$\mathfrak{F}(\lambda) = (\lambda - \lambda_0)P + I_X - P,$$

one has that

$$\sigma(\mathfrak{F}, \lambda_0) = -1. \tag{4.1}$$

Indeed, by Lemma 6.1.1 of [28], there exists $T \in GL(X)$, such that $P = T^{-1}P_0T$. Thus

$$\mathfrak{F}(\lambda) = T^{-1}[(\lambda - \lambda_0)P_0 + I_X - P_0]T = T^{-1}\mathfrak{E}(\lambda)T,$$

and hence, by axioms (P) and (N)

$$\sigma(\mathfrak{L}, \lambda_0) = \sigma(T^{-1}, \lambda_0) \cdot \sigma(\mathfrak{E}(\lambda), \lambda_0) \cdot \sigma(T, \lambda_0) = -1.$$

On the other hand, for any given $\mathfrak{L} \in \mathcal{C}^\omega_{\lambda_0}$, by Corollary 5.3.2(b) of [28], which goes back to the proof of Theorem 5.3.1 of [25], there exist k finite-rank projections $\Pi_0, \Pi_2, \ldots, \Pi_{k-1} \in \mathcal{L}(X)$ and a (globally invertible) path $\mathfrak{I} \in \mathcal{C}^\omega((\lambda_0 - \varepsilon, \lambda_0 + \varepsilon), GL(X))$, such that setting

$$\mathfrak{E}_{\Pi_i}(t) := t\,\Pi_i + I_X - \Pi_i, \qquad i \in \{0,1,...,k-1\}, \;\; t \in \mathbb{R},$$

one has that

$$\mathfrak{L}(\lambda) = \mathfrak{I}(\lambda) \circ \mathfrak{E}_{\Pi_0}(\lambda - \lambda_0) \circ \mathfrak{E}_{\Pi_1}(\lambda - \lambda_0) \circ \cdots \circ \mathfrak{E}_{\Pi_{k-1}}(\lambda - \lambda_0).$$

Moreover, for every $i \in \{0, 1, ..., k-1\}$, there are $r_i = \operatorname{rank}\Pi_i$ projections of rank one, $P_{j,i}$, $1 \leq j \leq r_i$, such that

$$\mathfrak{C}_{\Pi_i} = \mathfrak{C}_{P_{1,i}} \circ \mathfrak{C}_{P_{2,i}} \circ \cdots \circ \mathfrak{C}_{P_{r_i,i}}.$$

Consequently, we find from the axiom (P) and (4.1) that

$$\sigma(\mathfrak{C}_{\Pi_i}, \lambda_0) = \sigma(\mathfrak{C}_{P_{1,i}}, \lambda_0) \cdots \sigma(\mathfrak{C}_{P_{r_i,i}}, \lambda_0) = (-1)^{r_i} = (-1)^{\operatorname{rank}\Pi_i}$$

and, therefore, applying again the axiom (P) yields

$$\sigma(\mathfrak{L}, \lambda_0) = \sigma(\mathfrak{J}, \lambda_0) \cdot \sigma(\mathfrak{C}_{\Pi_0}, \lambda_0) \cdots \sigma(\mathfrak{C}_{\Pi_{k-1}}, \lambda_0) = (-1)^{\sum_{i=0}^{k-1} \operatorname{rank}\Pi_i}.$$

Finally, since owing to Corollary 5.3.2 of [28], we have that

$$\chi[\mathfrak{L}; \lambda_0] = \sum_{i=0}^{k-1} \operatorname{rank}\Pi_i,$$

it becomes apparent that

$$\sigma(\mathfrak{L}, \lambda_0) = (-1)^{\chi[\mathfrak{L};\lambda_0]}.$$

This concludes the proof. □

Once the local parity is determined, we will give the global axiomatization. A pair $(\mathfrak{L}, [a, b])$ is said to be *admissible* if $\mathfrak{L} \in \mathscr{C}([a, b], \Phi_0(X))$. The set of admissible pairs will be denoted by $\mathscr{A}$.

Theorem 4.2. *There exists a unique $\mathbb{Z}_2$-valued map $\sigma : \mathscr{A} \to \mathbb{Z}_2$, such that*

(N) ***Normalization:*** *For every $\mathfrak{L} \in \mathcal{C}^{\omega}_{\lambda_0}([\lambda_0 - \eta, \lambda_0 + \eta], \Phi_0(X))$,*

$$\sigma(\mathfrak{L}, [\lambda_0 - \eta, \lambda_0 + \eta]) = (-1)^{\chi[\mathfrak{L};\lambda_0]}.$$

(P) ***Product Formula:*** *For every $(\mathfrak{L}, [a, b]) \in \mathscr{A}$ and $c \in (a, b)$, such that $c \notin \Sigma(\mathfrak{L})$*

$$\sigma(\mathfrak{L}, [a, b]) = \sigma(\mathfrak{L}, [a, c]) \cdot \sigma(\mathfrak{L}, [c, b]).$$

(H) ***Homotopy Invariance:*** *For every homotopy $H \in \mathcal{C}([0, 1]\times[a, b], \Phi_0(X))$ such that $(H(t, \cdot), [a, b]) \in \mathscr{A}$ for all $t \in [0, 1]$*

$$\sigma(H(0, \cdot), [a, b]) = \sigma(H(1, \cdot), [a, b]).$$

Moreover, $\sigma(\mathfrak{L}, [a, b])$ equals the parity map of Fitzpatrick and Pejsachowitz [12].

Proof. Pick $(\mathfrak{L}, [a, b]) \in \mathscr{A}$. By Theorem 3.2, we already know that there exists an analytic curve $\mathfrak{L}_\omega \in \mathscr{C}^\omega([a, b], \Phi_0(X))$ $\mathcal{A}$-homotopic to $\mathfrak{L}$, i.e., there exists $H \in \mathcal{C}([0, 1] \times [a, b], \Phi_0(X))$, such that $H([0, 1] \times \{a, b\}) \subset GL(X)$

$$H(0, \lambda) = \mathfrak{L}(\lambda) \quad \text{and} \quad H(1, \lambda) = \mathfrak{L}_\omega(\lambda) \quad \text{for all } \lambda \in [a, b].$$

Then, by the axiom (H)

$$\sigma(\mathfrak{L}, [a, b]) = \sigma(\mathfrak{L}_\omega, [a, b]). \tag{4.2}$$

Suppose that $\Sigma(\mathfrak{L}_\omega) \cap [a, b] = \emptyset$ and pick any $\lambda_0 \in (a, b)$. Then, since $\chi[\mathfrak{L}_\omega; \lambda_0] = 0$, it follows from (N) that:

$$\sigma(\mathfrak{L}_\omega, [a, b]) = (-1)^{\chi[\mathfrak{L}_\omega;\lambda_0]} = 1.$$

Therefore, (4.2) implies that

$$\sigma(\mathfrak{L}, [a, b]) = 1.$$

Now, suppose that $\Sigma(\mathfrak{L}_\omega) \neq \emptyset$. Since $\mathfrak{L}_\omega(a) \in GL(X)$, it follows from Theorems 4.4.1 and 4.4.4 of [25] that $\Sigma(\mathfrak{L}_\omega)$ is discrete. Thus

$$\Sigma(\mathfrak{L}_\omega) = \{\lambda_1, \lambda_2, ..., \lambda_n\}$$

for some

$$a < \lambda_1 < \lambda_2 < \cdots < \lambda_n < b.$$

Let $\varepsilon > 0$ be sufficiently small, so that λ_i is ε-isolated for all $i \in \{1, ..., n\}$. Then, by the axioms (P) and (N), we find that

$$\begin{aligned}\sigma(\mathfrak{L}_\omega, [a, b]) &= \prod_{i=1}^{n} \sigma(\mathfrak{L}_\omega, [\lambda_i - \varepsilon, \lambda_i + \varepsilon]) \\ &= \prod_{i=1}^{n} (-1)^{\chi[\mathfrak{L}_\omega, \lambda_i]} = (-1)^{\sum_{i=1}^{n} \chi[\mathfrak{L}_\omega; \lambda_i]}.\end{aligned}$$

Therefore, by (4.2), it becomes apparent that

$$\sigma(\mathfrak{L}, [a, b]) = (-1)^{\sum_{i=1}^{n} \chi[\mathfrak{L}_\omega; \lambda_i]}.$$

Consequently, owing to Theorem 3.2, the map $\sigma : \mathscr{A} \to \mathbb{Z}_2$ is the parity defined by Fitzpatrick and Pejsachowitz in [12]. This concludes the proof. □

Note that the normalization property (N) in Theorem 4.2 is determined by the local uniqueness of the parity provided by Theorem 4.1.

5. Axiomatization and uniqueness of the topological degree

The aim of this section is delivering the proof of Theorem 1.2. We begin by recalling our main theorem. As already discussed in Sect. 1, for any open and bounded subset, Ω, of a Banach space X, an operator $f : \overline{\Omega} \subset X \to Y$ is said to be $\mathcal{C}^1$-*Fredholm of index zero* if $f \in \mathcal{C}^1(\overline{\Omega}, Y)$ and $Df \in \mathcal{C}(\Omega, \Phi_0(X, Y))$, and the set of all these operators is denoted by $\mathscr{F}_0^1(\Omega, Y)$. An operator $f \in \mathscr{F}_0^1(\Omega, Y)$ is said to be *orientable* when $Df : \Omega \to \Phi_0(X, Y)$ is an orientable map. Moreover, for any open and bounded subset, Ω, of a Banach space X and any operator $f : \overline{\Omega} \subset X \to Y$ satisfying

(1) $f \in \mathscr{F}_0^1(\Omega, Y)$ is *orientable* with orientation $\varepsilon : \mathcal{R}_{Df} \to \mathbb{Z}_2$,
(2) f is *proper* in $\overline{\Omega}$, i.e., $f^{-1}(K)$ is compact for every compact subset $K \subset Y$,
(3) $0 \notin f(\partial\Omega)$,

it is said that (f, Ω, ε) is a *Fredholm admissible triple*. The set of all Fredholm admissible triples is denoted by $\mathscr{A}$. Given $(f, \Omega, \varepsilon) \in \mathscr{A}$, it is said that (f, Ω, ε) is a regular triple if 0 is a regular value of f, i.e., $Df(x) \in GL(X, Y)$ for all $x \in f^{-1}(0)$. The set of regular triples is denoted by $\mathscr{R}$. Finally, a map $H \in \mathcal{C}^1([0, 1] \times \overline{\Omega}, Y)$ is said to be $\mathcal{C}^1$-*Fredholm homotopy* if $D_x H(t, x) \in \Phi_0(X, Y)$ for each $(t, x) \in [0, 1] \times \Omega$ and it is called *orientable* if $D_x H : [0, 1] \times \Omega \to$

$\Phi_0(X,Y)$ is an orientable map. Henceforth, the notation ε_t stands for the restriction

$$\varepsilon_t : \mathcal{R}_{H_t} \longrightarrow \mathbb{Z}_2, \quad \varepsilon_t(x) := \varepsilon(t,x)$$

for each $t \in [0,1]$, where $H_t(\cdot) = H(t,\cdot)$. Theorem 1.2 reads as follows.

Theorem 5.1. *There exists a unique integer-valued map* $\deg : \mathscr{A} \to \mathbb{Z}$ *satisfying the next properties*

(N) ***Normalization:*** $\deg(L,\Omega,\varepsilon) = \varepsilon(0)$ *for all* $L \in GL(X,Y)$ *if* $0 \in \Omega$.

(A) ***Additivity:*** *For every* $(f,\Omega,\varepsilon) \in \mathscr{A}$ *and any pair of disjoint open subsets* Ω_1 *and* Ω_2 *of* Ω *with* $0 \notin f(\Omega\backslash(\Omega_1 \uplus \Omega_2))$,

$$\deg(f,\Omega,\varepsilon) = \deg(f,\Omega_1,\varepsilon) + \deg(f,\Omega_2,\varepsilon).$$

(H) ***Homotopy Invariance:*** *For each proper* $\mathcal{C}^1$*-Fredholm homotopy* $H \in \mathcal{C}^1([0,1]\times\overline{\Omega},Y)$ *with orientation* ε *and* $(H(t,\cdot),\Omega,\varepsilon_t) \in \mathscr{A}$ *for each* $t \in [0,1]$

$$\deg(H(0,\cdot),\Omega,\varepsilon_0) = \deg(H(1,\cdot),\Omega,\varepsilon_1).$$

Moreover, given $(f,\Omega,\varepsilon) \in \mathscr{R}$ *with* Ω *connected and* $\mathcal{R}_{Df} \neq \emptyset$, *for each* $p \in \mathcal{R}_{Df}$

$$\deg(f,\Omega,\varepsilon) = \varepsilon(p)\cdot \sum_{x\in f^{-1}(0)\cap\Omega} (-1)^{\chi[\mathfrak{L}_{\omega,x},[a,b]]}, \tag{5.1}$$

where $\mathfrak{L}_{\omega,x} \in \mathscr{C}^{\omega}([a,b],\Phi_0(X,Y))$ *is any analytic curve* $\mathcal{A}$*-homotopic to* $Df \circ \gamma$, *for some* $\gamma \in \mathcal{C}([a,b],\Omega)$, *such that* $\gamma(a) = p$ *and* $\gamma(b) = x$, *and*

$$\chi[\mathfrak{L}_{\omega,x},[a,b]] := \sum_{\lambda_x\in\Sigma(\mathfrak{L}_{\omega,x})\cap[a,b]} \chi[\mathfrak{L}_{\omega,x},\lambda_x].$$

Observe that the right-hand side of (5.1) is well defined, because every open and connected set, Ω, in a locally convex topological space, X, is path connected. Thus, it always exists a path $\mathfrak{L}_x \in \mathcal{C}([a,b],\Omega)$ joining p and x. The existence of the analytic $\mathcal{A}$-homotopic curve was established in [29]. The right-hand side of (5.1) is taken as zero if $f^{-1}(0)\cap\Omega = \emptyset$.

Applying axiom (A) with $\Omega_1 = \Omega_2 = \Omega = \emptyset$, it becomes apparent that $\deg(f,\emptyset,\varepsilon) = 0$. If $(f,\Omega,\varepsilon) \in \mathscr{A}$ and $f^{-1}(0)\cap\Omega = \emptyset$. Applying again (A) with $\Omega_1 = \Omega_2 = \emptyset$, we find that $\deg(f,\Omega,\varepsilon) = 0$. Equivalently, f admits a zero in Ω if $\deg(f,\Omega,\varepsilon) \neq 0$.

As already commented in Sect. 1, since the existence of deg goes back to Fitzpatrick, Pejsachowicz and Rabier [13], and the formula (5.1) was proven by the authors in [29], Theorem 5.1 actually establishes the uniqueness of deg.

To prove the uniqueness, it is appropriate to sketch briefly the construction of deg carried over in [13] and [35]. Let $(f,\Omega,\varepsilon) \in \mathscr{A}$. By definition, $f \in \mathscr{F}^1_0(\Omega,Y)$ is $\mathcal{C}^1$-Fredhom of index zero and it is ε-orientable, i.e., $Df : \Omega \to \Phi_0(X,Y)$ is an orientable map with orientation

$$\varepsilon : \mathcal{R}_{Df} \longrightarrow \mathbb{Z}_2.$$

Once an orientation has been introduced, the degree deg can be defined as the Leray–Schauder degree $\deg_{LS}$ as soon as 0 is a regular value of f, because since in such case, $f^{-1}(0)\cap\Omega$ is finite, possibly empty, one can define, in complete agreement with the axioms (N), (A), and (H)

$$\deg(f,\Omega,\varepsilon) := \sum_{x\in f^{-1}(0)\cap\Omega} \varepsilon(x). \tag{5.2}$$

If $f^{-1}(0)\cap\Omega=\emptyset$, as we already mentioned, $\deg(f,\Omega,\varepsilon)=0$. When 0 is not a regular value, then, by definition

$$\deg(f,\Omega,\varepsilon) := \deg(f-x_0,\Omega,\varepsilon),$$

where x_0 is any regular value of f belonging to a sufficiently small neighborhood of 0. The existence of such regular values is guaranteed by a theorem of Quinn and Sard [34], a version of the Sard–Smale Theorem, [40], not requiring the separability of the involved Banach spaces.

Once introduced the Leray–Schauder degree, many experts generalized it to cover more general operators than compact perturbations of the identity. It is worth mentioning that the degree of Fitzpatrick, Pejsachowicz and Rabier covers most of them under the notion of the degree for $\mathcal{F}$-maps, where $\mathcal{F}$ is a fixed orientable subset of $\Phi_0(X,Y)$. A map $f\in\mathscr{F}^1_0(\Omega,Y)$ is called an $\mathcal{F}$-map if $Df(\Omega)\subset\mathcal{F}$. Clearly, an $\mathcal{F}$-map $f:\Omega\subset X\to Y$ inherits a unique orientation induced by the given orientation on $\mathcal{F}$. Indeed, if $\varepsilon:\mathcal{F}\cap GL(X,Y)\to\mathbb{Z}_2$ denotes the orientation of $\mathcal{F}$, then

$$\varepsilon_f:\mathcal{R}_{Df}\longrightarrow\mathbb{Z}_2,\quad \varepsilon_f(x)=\varepsilon(Df(x)),$$

defines an orientation for f. Many of the existing degrees can be viewed as special cases of the degree for $\mathcal{F}$-maps. For instance, deg extends $\deg_{LS}$ to this more general setting if we restrict ourselves to consider Leray–Schauder admissible pairs of class $\mathcal{C}^1$. Indeed, $\mathcal{L}_c(X)$ is simply connected and, hence, according to [13], orientable. Choose $\mathcal{F}=\mathcal{L}_c(X)$ and the orientation $\varepsilon: GL_c(X)\to\mathbb{Z}_2$ defined by

$$\varepsilon(L)=\deg_{LS}(L,\Omega), \tag{5.3}$$

where the right-hand side of (5.3) is defined by (1.4). Then, for every $\mathcal{C}^1$ Leray–Schauder regular pair (f,Ω), one has that $(f,\Omega,\varepsilon_f)\in\mathscr{R}$ and, thanks to (5.2) and (5.3)

$$\begin{aligned}\deg_{LS}(f,\Omega) &= \sum_{x\in f^{-1}(0)\cap\Omega}\deg_{LS}(Df(x),\Omega)\\ &= \sum_{x\in f^{-1}(0)\cap\Omega}\varepsilon(Df(x)) = \sum_{x\in f^{-1}(0)\cap\Omega}\varepsilon_f(x)=\deg(f,\Omega,\varepsilon_f).\end{aligned}$$

Therefore

$$\deg(f,\Omega,\varepsilon_f)=\deg_{LS}(f,\Omega).$$

Many others, like the Nussbaum–Sadovkii degree, [33,39], the Laloux–Mawhin coincidence degree [20,21,31], the Tromba degree for Röthe maps [41], the Isnard degree [17], and the Fenske degree [10], can be also regarded as special cases of the degree for $\mathcal{F}$-maps for a suitable choice of $\mathcal{F}$. The interested reader is sent to Sect. 2 of Fitzpatrick, Pejsachowicz and Rabier [13] for any further details.

Before proving the uniqueness, it is convenient to illustrate the theory by establishing the generalized Schauder formula (5.1), as it was done in [29].

Proof of (5.1). Take $(f,\Omega,\varepsilon)\in\mathscr{R}$ with Ω connected and $\mathcal{R}_{Df}\neq\emptyset$ and choose $p\in\mathcal{R}_{Df}$. By (5.2), it follows that:

$$\deg(f,\Omega,\varepsilon):=\sum_{x\in f^{-1}(0)\cap\Omega}\varepsilon(x). \tag{5.4}$$

Fix $x\in f^{-1}(0)\cap\Omega$. According to (3.2)

$$\varepsilon(x)=\varepsilon(p)\cdot\sigma(Df\circ\gamma,[a,b]),$$

where $\gamma\in\mathcal{C}([a,b],\Omega)$ is a path linking x with p. By Theorem 3.2, for any analytic curve $\mathfrak{L}_{\omega,x}\in\mathscr{C}^\omega([a,b],\Phi_0(X,Y))$ $\mathcal{A}$-homotopic to $Df\circ\gamma$, we have that

$$\varepsilon(x)=\varepsilon(p)\cdot\sigma(Df\circ\gamma,[a,b])=\varepsilon(p)\cdot(-1)^{\chi[\mathfrak{L}_{\omega,x},[a,b]]},$$

where

$$\chi[\mathfrak{L}_{\omega,x},[a,b]]=\sum_{\lambda_x\in\Sigma(\mathfrak{L}_{\omega,x})\cap[a,b]}\chi[\mathfrak{L}_{\omega,x},\lambda_x].$$

Therefore, by (5.4)

$$\deg(f,\Omega,\varepsilon)=\varepsilon(p)\cdot\sum_{x\in f^{-1}(0)\cap\Omega}(-1)^{\chi[\mathfrak{L}_{\omega,x},[a,b]]},$$

which ends the proof. □

We have all necessary ingredients to prove Theorem 5.1. Naturally, it suffices to prove the uniqueness.

Proof of the uniqueness. We first prove that, for every $(f,\Omega,\varepsilon)\in\mathscr{R}$, the topological degree is given by (5.1) in each connected component of Ω, if $\mathcal{R}_{Df}\neq\emptyset$ and $\deg(f,\Omega,\varepsilon)=0$ if $\mathcal{R}_{Df}=\emptyset$. Pick $(f,\Omega,\varepsilon)\in\mathscr{R}$. Then, $f^{-1}(0)\cap\Omega$ is finite, possibly empty. If it is empty, then, applying axiom (A) with $\Omega_1=\Omega_2=\Omega=\emptyset$, it becomes apparent that $\deg(f,\emptyset,\varepsilon)=0$. Thus, applying again (A) with $\Omega_1=\Omega_2=\emptyset$, we find that

$$\deg(f,\Omega,\varepsilon)=0.$$

If $\mathcal{R}_{Df}=\emptyset$, necessarily $f^{-1}(0)\cap\Omega=\emptyset$ and, therefore, $\deg(f,\Omega,\varepsilon)=0$ as required. Now, suppose that, for some $n\geq 1$ and $x_i\in\Omega$, $i\in\{1,...,n\}$

$$f^{-1}(0)\cap\Omega=\{x_1,x_2,...,x_n\}.$$

By the finiteness of $f^{-1}(0)\cap\Omega$ and the additivity property (A), we can suppose that Ω is connected. Indeed, let us denote by $\mathscr{O}:=\{D_j\}_{j=1}^m$, $m\leq n$, the connected components of Ω with $f^{-1}(0)\cap D_j\neq\emptyset$. Since $0\notin f(\Omega\setminus\uplus_{j=1}^m$

D_j), where $\uplus$ stands for the disjoint union, applying (A) with $\Omega_1 = \uplus_{j=1}^m D_j$ and $\Omega_2 = \emptyset$, we can infer that

$$\deg(f, \Omega, \varepsilon) = \deg(f, \uplus_{j=1}^m D_j, \varepsilon) = \sum_{j=1}^m \deg(f, D_j, \varepsilon),$$

where the second equality follows by an inductive application of the additivity property (A). Thus, without loss of generality, we can assume that Ω is connected. Since 0 is a regular value of f, by the inverse function theorem, $f|_{B_{\eta_i}(x_i)}$ is a diffeomorphism for each $i \in \{1, 2, ..., n\}$ for sufficiently small $\eta_i > 0$ and therefore by axiom (A)

$$\deg(f, \Omega, \varepsilon) = \sum_{i=1}^n \deg(f, B_{\eta_i}(x_i), \varepsilon). \tag{5.5}$$

Subsequently, we fix $i \in \{1, 2, ..., n\}$ and consider the homotopy H_i defined by

$$\begin{aligned} H_i : [0,1] \times B_{\eta_i}(x_i) &\longrightarrow Y \\ (t, x) &\mapsto tf(x) + (1-t)Df(x_i)(x - x_i). \end{aligned}$$

The next result of technical nature holds. □

Lemma 5.2. $H_i \in \mathcal{C}^1([0,1] \times \overline{B_{\tau_i}(x_i)}, Y)$ *and it is proper for sufficiently small* $\tau_i > 0$.

Proof. Obviously, $H_i \in \mathcal{C}^1([0,1] \times B_{\eta_i}(x_i), Y)$ and

$$D_x H_i(t, \cdot) = tDf(\cdot) + (1-t)Df(x_i) = Df(x_i) + t(Df(\cdot) - Df(x_i)).$$

Since $Df \in \mathcal{C}(\Omega, \Phi_0(X, Y))$, $Df(x_i) \in GL(X, Y)$, and $GL(X, Y)$ is open, we have that, for sufficiently small $\eta_i > 0$

$$D_x H_i(t, B_{\eta_i}(x_i)) \subset GL(X, Y) \quad \text{for all } t \in [0,1]. \tag{5.6}$$

In particular, $D_x H_i(t, x) \in \Phi_0(X, Y)$ for all $(t, x) \in [0,1] \times B_{\eta_i}(x_i)$. Thus, by definition, $H_i(t, \cdot) \in \mathscr{F}_0^1(\Omega, Y)$ for all $t \in [0,1]$. This also entails that DH_i is a Fredholm operator of index one from $\mathbb{R} \times X$ to Y. Thus, by Theorem 1.6 of Smale [40], H_i is locally proper, i.e., for every $t \in [0,1]$, there exists an open interval containing t, $\mathcal{I}(t) \subset [0,1]$, and an open ball centered in x_i with radius δ_t, $B_{\delta_t}(x_i)$, such that H_i is proper in $\overline{\mathcal{I}(t)} \times \overline{B_{\delta_t}(x_i)}$. In particular

$$[0,1] \times \{x_i\} \subset \bigcup_{t \in [0,1]} \mathcal{I}(t) \times B_{\delta_t}(x_i).$$

Since $[0,1] \times \{x_i\}$ is compact, there exists a finite subset $\{t_1, t_2, ..., t_n\} \subset [0,1]$, such that

$$[0,1] \times \{x_i\} \subset \bigcup_{j=1}^n \mathcal{I}(t_j) \times B_{\delta_{t_j}}(x_i),$$

as illustrated in Fig. 2. Let

$$\delta_i := \min\{\delta_{t_1}, \delta_{t_2}, ..., \delta_{t_n}\}.$$

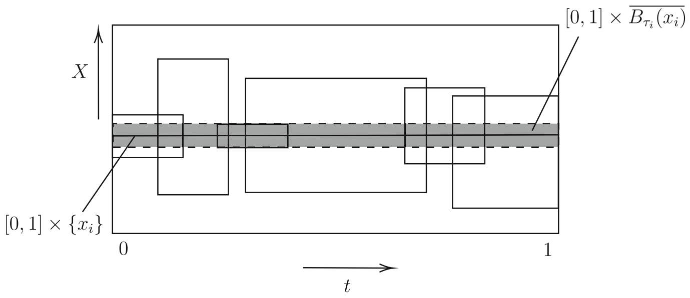

FIGURE 2. Scheme of the construction

Then

$$[0,1]\times\{x_i\}\subset[0,1]\times B_{\delta_i}(x_i).$$

Let $\tau_i<\min\{\eta_i,\delta_i\}$. Then, H_i is proper in $\overline{\mathcal{I}(t)}\times\overline{B_{\tau_i}(x_i)}$, since the restriction of a proper map to a closed subset is also proper. On the other hand, since

$$[0,1]\times\overline{B_{\tau_i}(x_i)}=\bigcup_{i=1}^{n}\overline{\mathcal{I}(t_i)}\times\overline{B_{\tau_i}(x_i)}$$

and each $\overline{\mathcal{I}(t_i)}\times\overline{B_{\tau_i}(x_i)}$ is closed, necessarily H_i is proper on $[0,1]\times\overline{B_{\tau_i}(x_i)}$. Therefore

$$H_i\in\mathcal{C}^1([0,1]\times\overline{B_{\tau_i}(x_i)},Y),$$

and it is proper. The proof is complete. □

Lemma 5.3. *$0\notin H_i(t,\partial B_{\tau_i}(x_i))$ for each $t\in[0,1]$ and sufficiently small $\tau_i>0$.*

Proof. On the contrary, assume that $0\in H_i(t,\partial B_{\tau_i}(x_i))$ for some $t\in[0,1]$ and $\tau_i<\eta_i$, i.e., there exists $x\in\partial B_{\tau_i}(x_i)$, such that $H_i(t,x)=0$. Thus

$$t[f(x)-Df(x_i)(x-x_i)]+Df(x_i)(x-x_i)=0. \tag{5.7}$$

Necessarily, $t>0$, because $t=0$ implies $Df(x_i)(x-x_i)=0$, and in such case, $Df(x_i)$ cannot be an isomorphism. Therefore, dividing (5.7) by t yields

$$f(x)-Df(x_i)(x-x_i)=-\frac{1}{t}Df(x_i)(x-x_i).$$

Taking norms and dividing by $\|x-x_i\|>0$, we obtain that

$$\frac{\|f(x)-Df(x_i)(x-x_i)\|}{\|x-x_i\|}=\frac{1}{t}\Big\|Df(x_i)\Big(\frac{x-x_i}{\|x-x_i\|}\Big)\Big\|\geq\frac{1}{t}\inf_{\|x\|=1}\|Df(x_i)(x)\|. \tag{5.8}$$

Since $Df(x_i)\in GL(X,Y)$ and $\partial B_1(x_i)$ is closed, $Df(x_i)(\partial B_1(x_i))$ is closed. Hence

$$m\equiv\inf_{\|x\|=1}\|Df(x_i)(x)\|$$

is attained and, since $0 \notin Df(x_i)(\partial B_1(x_i))$, we find that $m > 0$, and therefore, it follows from (5.8) that:

$$\frac{\|f(x) - Df(x_i)(x - x_i)\|}{\|x - x_i\|} \geq \frac{m}{t} > 0. \tag{5.9}$$

On the other hand, since f is differentiable at x_i and $f(x_i) = 0$

$$\frac{\|f(x) - Df(x_i)(x - x_i)\|}{\|x - x_i\|} \xrightarrow[x \to x_i]{} 0.$$

Thus, for sufficiently small $\tau_i > 0$, we have that

$$\frac{\|f(x) - Df(x_i)(x - x_i)\|}{\|x - x_i\|} < \frac{m}{t},$$

which contradicts (5.9) and ends the proof. □

By construction, $f|_{B_{\eta_i}(x_i)}$ is a diffeomorphism. Thus, $Df(B_{\eta_i}(x_i))$ is a path connected subset of $GL(X, Y)$, and hence, $B_{\eta_i}(x_i) \subset \mathcal{R}_{Df}$. Since the orientation

$$\varepsilon|_{B_{\eta_i}(x_i)} : \ B_{\eta_i}(x_i) \longrightarrow \mathbb{Z}_2 \tag{5.10}$$

is always constant in each path-connected component of its domain, it is actually constant. Denote its constant value by ε_0. Subsequently, we consider the map

$$\begin{aligned} \varepsilon^{H_i} : [0,1] \times B_{\eta_i}(x_i) &\longrightarrow \mathbb{Z}_2 \\ (t, x) &\mapsto \varepsilon_0. \end{aligned}$$

Note that, thanks to (5.6), for each $t \in [0, 1]$

$$D_x H_i(t, B_{\eta_i}(x_i)) \subset GL(X, Y).$$

Consequently, $D_x H_i([0, 1] \times B_{\eta_i}(x_i))$ is a path connected subset of $GL(X, Y)$ and $[0, 1] \times B_{\eta_i}(x_i) = \mathcal{R}_{D_x H_i}$. Hence, ε^{H_i} provides us with an orientation of H_i. Therefore, thanks to Lemmas 5.2 and 5.3, it becomes apparent that H_i is a proper $\mathcal{C}^1$-Fredholm homotopy with orientation ε^{H_i} and $0 \notin H_i([0, 1] \times \partial B_{\tau_i}(x_i))$ for sufficiently small $\tau_i > 0$. Moreover, $\varepsilon_t^{H_i} (\equiv \varepsilon_0)$ provides us with an orientation of the section $H_i(t, \cdot)$ and, therefore, $(H_i(t, \cdot), \Omega, \varepsilon_t^{H_i}) \in \mathscr{A}$ for each $t \in [0, 1]$. By the axiom (H), and taking into account that $\varepsilon_j^{H_i} = \varepsilon|_{B_{\eta_i}(x_i)} (\equiv \varepsilon_0)$ for each $j \in \{0, 1\}$

$$\deg(f, B_{\tau_i}(x_i), \varepsilon) = \deg(Df(x_i)(\cdot - x_i), B_{\tau_i}(x_i), \varepsilon). \tag{5.11}$$

To remove the affine term in (5.11), we consider the homotopy

$$\begin{aligned} G_i : [0,1] \times \overline{\Pi} &\longrightarrow Y \\ (t, x) &\mapsto Df(x_i)(x - t x_i), \end{aligned}$$

where $\Pi = \bigcup_{t \in [0,1]} B_{\tau_i}(t x_i)$. Obviously, $G_i \in \mathcal{C}^1([0, 1] \times \overline{\Pi}, Y)$. Moreover, since, for every $t \in [0, 1]$, $G_i(t, \cdot) = Df(x_i)(\cdot - t x_i)$ is a diffeomorphism, we have that $G_i(t, \cdot)$ is proper for each $t \in [0, 1]$. Therefore, since G_i is uniformly continuous in t, it follows from Theorem 3.9.2 of [42] that G_i is proper.

As $G_i(t,\cdot) = Df(x_i)(\cdot - tx_i)$ is a diffeomorphism for each $t \in [0,1]$ and $G_i(t, tx_i) = 0$, it is obvious that

$$0 \notin G_i(t, \partial\Pi) \quad \text{for all} \;\; t \in [0,1].$$

Moreover, since, for every $t \in [0,1]$

$$D_x G_i(t,\cdot) = Df(x_i) \in GL(X,Y),$$

it is apparent that

$$D_x G_i([0,1] \times \Pi) = \{Df(x_i)\} \subset GL(X,Y)$$

and if we choose the orientation

$$\varepsilon^{G_i} : [0,1] \times \Pi \longrightarrow \mathbb{Z}_2$$

to be $\varepsilon^{G_i} \equiv \varepsilon_0$, G_i is a proper $\mathcal{C}^1$-Fredholm homotopy with orientation ε^{G_i}, such that $0 \notin G_i([0,1] \times \partial\Pi)$. Moreover, since $D_x G_i(\{t\} \times \Pi) \cap GL(X,Y) = \{Df(x_i)\} \neq \emptyset$ for each $t \in [0,1]$, $\varepsilon_t^{G_i}(\equiv \varepsilon_0)$ provides us with an orientation of the section $G_i(t,\cdot)$ and, therefore, $(G_i(t,\cdot), \Pi, \varepsilon_t^{G_i}) \in \mathscr{A}$ for each $t \in [0,1]$. Thanks to the axiom (H), we find that

$$\begin{aligned}\deg(Df(x_i), \Pi, \varepsilon_0^{G_i}) &= \deg(G_i(0,\cdot), \Pi, \varepsilon_0^{G_i}) = \deg(G_i(1,\cdot), \Pi, \varepsilon_1^{G_i}) \\ &= \deg(Df(x_i)(\cdot - x_i), \Pi, \varepsilon_1^{G_i}). \end{aligned} \tag{5.12}$$

Since

$$F_t(x) = Df(x_i)(x - tx_i), \quad x \in \Pi,$$

is a diffeomorphism for each $t \in \{0,1\}$ and $F_t(tx_i) = 0$, we have that $0 \notin F_t(\Pi \backslash B_{\tau_i}(tx_i))$. Thus, applying the axiom (A) with $\Omega = \Pi$, $\Omega_1 = B_{\tau_i}(tx_i)$ and $\Omega_2 = \emptyset$, it becomes apparent that

$$\deg(Df(x_i), \Pi, \varepsilon_0^{G_i}) = \deg(Df(x_i), B_{\tau_i}(0), \varepsilon_0^{G_i}|_{B_{\tau_i}(0)})$$

and

$$\deg(Df(x_i)(\cdot - x_i), \Pi, \varepsilon_1^{G_i}) = \deg(Df(x_i)(\cdot - x_i), B_{\tau_i}(x_i), \varepsilon_1^{G_i}|_{B_{\tau_i}(x_i)}).$$

Therefore, by (5.12), we infer that

$$\deg(Df(x_i)(\cdot - x_i), B_{\tau_i}(x_i), \varepsilon_0^{G_i}|_{B_{\tau_i}(0)}) = \deg(Df(x_i), B_{\tau_i}(0), \varepsilon_1^{G_i}|_{B_{\tau_i}(x_i)}).$$

Observe that $\varepsilon_1^{G_i}|_{B_{\tau_i}(x_i)} = \varepsilon|_{B_{\tau_i}(x_i)}$ and that

$$\varepsilon_0^{G_i}|_{B_{\tau_i}(0)} = \varepsilon|_{B_{\tau_i}(x_i)} \circ T,$$

where $T : B_{\tau_i}(0) \to B_{\tau_i}(x_i)$ is the translation given by $T(x) = x + x_i$. Consequently, by the axiom (N)

$$\deg(Df(x_i)(\cdot - x_i), B_{\tau_i}(x_i), \varepsilon) = \deg(Df(x_i), B_{\tau_i}(0), \varepsilon|_{B_{\tau_i}(x_i)} \circ T) = \varepsilon(T(0)) = \varepsilon(x_i).$$

Thus, combining the last identity with (5.11) yields

$$\deg(f, B_{\tau_i}(x_i), \varepsilon) = \varepsilon(x_i),$$

and therefore, by (5.5)

$$\deg(f, \Omega, \varepsilon) = \sum_{i=1}^{n} \varepsilon(x_i). \tag{5.13}$$

Now, since $f^{-1}(0) \cap \Omega \neq \emptyset$ and 0 is a regular value, necessarily $\mathcal{R}_{Df} \neq \emptyset$. Take $p \in \mathcal{R}_{Df}$. Then, according to (3.2), we have

$$\varepsilon(x_i) = \varepsilon(p) \cdot \sigma(Df \circ \gamma_{x_i}, [a,b]),$$

where $\gamma_{x_i} \in \mathcal{C}([a,b],\Omega)$ is a path linking x_i with p. By Theorem 3.2, for any analytical curve $\mathfrak{L}_{\omega,x_i} \in \mathscr{C}^\omega([a,b],\Phi_0(X,Y))$ $\mathcal{A}$-homotopic to $Df \circ \gamma_{x_i}$, we have

$$\varepsilon(x_i) = \varepsilon(p) \cdot \sigma(Df \circ \gamma_{x_i}, [a,b]) = \varepsilon(p) \cdot (-1)^{\chi[\mathfrak{L}_{\omega,x_i},[a,b]]},$$

where

$$\chi[\mathfrak{L}_{\omega,x_i},[a,b]] = \sum_{\lambda_{x_i} \in \Sigma(\mathfrak{L}_{\omega,x_i}) \cap [a,b]} \chi[\mathfrak{L}_{\omega,x_i}, \lambda_{x_i}].$$

Therefore, by (5.13)

$$\deg(f,\Omega,\varepsilon) = \varepsilon(p) \cdot \sum_{i=1}^{n} (-1)^{\chi[\mathfrak{L}_{\omega,x_i},[a,b]]},$$

which ends the proof of the theorem in the regular case. We have actually proven that in the regular case, any map satisfying the axioms (N), (A), and (N) must coincide with the degree of Fitzpatrick, Pejsachowicz and Rabier [13].

If the general case, when $(f,\Omega,\varepsilon) \notin \mathscr{R}$, for every $\eta > 0$, by a theorem of Quinn and Sard, [34], there exists a regular value x_0, such that $\|x_0\| < \eta$. Let H be the homotopy defined by

$$\begin{aligned} H : [0,1] \times \overline{\Omega} &\longrightarrow Y \\ (t,x) &\mapsto f(x) - tx_0. \end{aligned}$$

Then, $H \in \mathcal{C}^1([0,1] \times \overline{\Omega}, Y)$ and it is proper. Obviously $H \in \mathcal{C}^1([0,1] \times \overline{\Omega}, Y)$ and

$$D_x H(t,\cdot) = Df(\cdot) \in \Phi_0(X,Y).$$

First, we will prove that $H(t,\cdot)$ is proper for each $t \in [0,1]$. By the definition of H, for any compact subset, K, of Y

$$H(t,\cdot)^{-1}(K) = f^{-1}(K + tx_0)$$

is compact, because $K + tx_0$ is compact and f proper. Therefore, as the map H is uniformly continuous in t, as above, it follows from Theorem 3.9.2 of [42] that H is proper.

Now, we will show that $0 \notin H(t,\partial\Omega)$ for each $t \in [0,1]$. On the contrary, suppose that $0 \in H(t,\partial\Omega)$ for some $t \in [0,1]$. Then, there exists $x \in \partial\Omega$, such that $f(x) = tx_0$. In particular, $tx_0 \in f(\partial\Omega)$. Since f is proper, by Lemma 3.9.1 of [42], f is a closed map, and since $\partial\Omega$ is closed, $f(\partial\Omega)$ is closed. Since $0 \notin f(\partial\Omega)$ and $f(\partial\Omega)$ is closed, there exists $\eta > 0$, such that $B_\eta(0) \cap f(\partial\Omega) = \emptyset$. As we have already taken $\|x_0\| < \eta$, we also have that $tx_0 \in B_\eta(0)$ and, therefore, $tx_0 \notin f(\partial\Omega)$. This contradicts $tx_0 \in f(\partial\Omega)$.

Since $D_x H(\{t\} \times \Omega) = Df(\Omega)$ for each $t \in [0,1]$, $\mathcal{R}_{D_x H} = [0,1] \times \mathcal{R}_{Df}$ and if we define

$$\varepsilon^H : [0,1] \times \mathcal{R}_{Df} \longrightarrow \mathbb{Z}_2, \quad \varepsilon^H(t,x) = \varepsilon(x),$$

where ε is the orientation of Df, then H is a proper $\mathcal{C}^1$-Fredholm homotopy with orientation ε^H and $0 \notin H([0,1] \times \partial\Omega)$. Observe that in this case, the domain is the whole Ω and, therefore, the orientation ε might not be, in general, constant. Moreover, since $D_x H(t,\cdot) = Df(\cdot)$ and Df is orientable with orientation $\varepsilon(= \varepsilon_t^H)$ for each $t \in [0,1]$, necessarily $(H(t,\cdot), \Omega, \varepsilon_t^H) \in \mathscr{A}$ for each $t \in [0,1]$. Owing to the axiom (H) and taking into account that $\varepsilon_j^H = \varepsilon$ for each $j \in \{0,1\}$

$$\deg(f, \Omega, \varepsilon) = \deg(f - x_0, \Omega, \varepsilon).$$

Since x_0 is a regular value of f, we have that 0 is a regular value of $f - x_0$ and $(f - x_0, \Omega, \varepsilon) \in \mathscr{R}$. This reduces the case to the regular one discussed previously and proves that the map $\deg : \mathscr{A} \to \mathbb{Z}$ coincides with the one constructed by Fitzpatrick, Pejsachowicz and Rabier [13]. This concludes the proof. □

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