Around averaged mappings

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Abstract. This paper is intended for a general mathematical audience. The examples show how the study of existence of fixed points of averaged mappings $T_\lambda = (1 - \lambda)I + \lambda T$, where $0 < \lambda < 1$ and I is the identity operator, can help in the study of existence of fixed points of mappings T .

Mathematics Subject Classification. Primary 47H10, Secondary 54H25.

Keywords. Asymptotic regularity, average mapping, contraction, fixed point, nonexpansive mapping.

1. Known results

Let X be a Banach space with norm $\|\cdot\|$, and C be a nonempty closed convex subset of X . Let $T : C \rightarrow C$ be a map. We denote the set of fixed points of T by $\text{Fix}(T)$.

Definition 1.1. [1] Let T be a self-mapping on a nonempty convex subset C of a Banach space. By an *averaged mapping* we mean one of the form $T_\lambda = (1 - \lambda)I + \lambda T : C \rightarrow C$, where $0 < \lambda < 1$, I is the identity operator.

Generally,

- (1) T and T_λ , both have the same fixed point set ($\text{Fix}(T_\lambda) = \text{Fix}(T)$).

Indeed, if $p \in \text{Fix}(T)$, then $T_\lambda p = (1 - \lambda)p + \lambda T p = p$. If $p \in \text{Fix}(T_\lambda)$, then $p = T_\lambda p = (1 - \lambda)p + \lambda T p$, so $T p = p$.

- (2) An averaged mapping of the averaged mapping is an averaged mapping.

Let $\lambda_1, \lambda_2 \in (0, 1)$. If $T_{\lambda_1} = (1 - \lambda_1)I + \lambda_1 T$, then $(T_{\lambda_1})_{\lambda_2} = (1 - \lambda_2)I + \lambda_2 T_{\lambda_1} = (1 - \lambda_2)I + \lambda_2 [(1 - \lambda_1)I + \lambda_1 T] = (1 - \lambda_1 \lambda_2)I + \lambda_1 \lambda_2 T$.

- (3) In the class of nonexpansive mappings $F : C \rightarrow C$ (satisfying $\|Fx - Fy\| \leq \|x - y\|$ for all $x, y \in C$), if $S = (1 - \lambda)T + \lambda V$ for some $\lambda \in (0, 1)$ and if T is averaged and V is nonexpansive, then S is averaged.

Indeed, if T is averaged, we have a number $\alpha \in (0, 1)$ and a non-expansive mapping $U : C \rightarrow C$ such that $T = (1 - \alpha)I + \alpha U$. Put $\gamma = \lambda + \alpha(1 - \lambda) \in (0, 1)$ and $W = \frac{(1-\lambda)\alpha}{\gamma}U + \frac{\lambda}{\gamma}V$. Since W is a convex combination of two nonexpansive mappings, W remains nonexpansive. Now, we can express S as

$$\begin{aligned} S &= (1 - \lambda)T + \lambda V = (1 - \lambda)((1 - \alpha)I + \alpha U) + \lambda V = \\ &= (1 - \lambda)(1 - \alpha)I + (1 - \lambda)\alpha U + \lambda V = (1 - \gamma)I + \gamma W. \end{aligned}$$

So S is averaged.

- (4) In the class of nonexpansive mappings $F : C \rightarrow C$, if S and T are both averaged, then the composite ST is averaged.

If S is averaged, we have a number $\alpha \in (0, 1)$ and a nonexpansive mapping $U : C \rightarrow C$ such that $S = (1 - \alpha)I + \alpha U$. Then $ST = (1 - \alpha)T + \alpha UT$. Since UT is again nonexpansive, by (2), ST is averaged.

- (5) Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and C given a closed convex subset of H . Projection $P_C : H \rightarrow C$ is the unique point in C such that $\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}$. In particular, P_C is nonexpansive and can be written as the mean average of a nonexpansive mapping and the identity: $P_C = \frac{1}{2}(I + S)$.

Mapping $T : H \rightarrow H$ such that $\langle x - y, Tx - Ty \rangle \geq \|Tx - Ty\|^2$ for all $x, y \in H$ is *firmly nonexpansive* (a projection is firmly nonexpansive). It is easily seen that T is firmly nonexpansive if and only if T can be expressed as $\frac{1}{2}(I + V)$, where V is nonexpansive, see [14].

Thus, a firmly nonexpansive mapping (in particular, a projection) is an averaged map with $\lambda = \frac{1}{2}$, see (2).

- (5) Let H be a real Hilbert space. If $S, T : H \rightarrow H$ are both averaged (in the class of nonexpansive mappings) and if S and T have a common fixed point, then $\text{Fix}(S) \cap \text{Fix}(T) = \text{Fix}(ST)$, details in [24].

In Banach spaces when T is nonexpansive mapping, T_λ may have much more felicitous asymptotic behavior than the original mapping.

Krasnoselskij [19] was the first to notice this regularizing effect (for $\lambda = \frac{1}{2}$). He proved

Theorem 1.2. *Let C be a nonempty compact convex subset of a uniformly convex Banach space X and $T : C \rightarrow C$ a nonexpansive mapping. Then the iterates $\{T_{\frac{1}{2}}^n\}$ (i.e. the sequence $x_{n+1} = \frac{1}{2}(x_n + Tx_n)$, $x_1 \in C$ be an arbitrary, $n = 1, 2, 3, \dots$) converges strongly to a fixed point of T .*

Since the existence of a fixed point is guaranteed by Schauder's theorem, the novelty of this result lays in this asymptotic statement.

Shortly thereafter, Schaefer [23] proved that $\{T_\lambda^n\}$ converges strongly for all $0 < \lambda < 1$, and in 1966, Edelstein [13] noted that the uniform convexity of X can be relaxed to strict convexity.

In 1966, Browder and Petryshyn [9] separated out the notion of asymptotic regularity: T is said to be *asymptotically regular* on C if for each $x \in C$, $T^{n+1}x - T^n x \rightarrow 0$ as $n \rightarrow \infty$. Obviously, every Banach contraction or Kannan mapping is an asymptotically regular mapping having exactly one fixed point [7], [15]. For more information, see [1], [10], [12], [21], [22].

In 1976, Ishikawa [17] proved the truly astonishing result with no restrictions on the geometry of the Banach space!

Theorem 1.3. *If C is nonempty bounded closed convex subset of a Banach space X and $T : C \rightarrow C$ is nonexpansive, then the mapping T_λ is asymptotically regular for each $0 < \lambda < 1$.*

Kirk [18] has extended Ishikawa's method to metric spaces with convexity structure. For the rate of asymptotic regularity, see Bruck [11] and reference therein.

2. Examples

The search for fixed points for mappings $T : C \rightarrow C$ that do not fit into the "known patterns" is difficult.

Instead of the mapping T , we can, for example, look for the fixed points of mapping T^n , $n = 2, 3, \dots$, n th iterate of T . Sometimes it is easier, see the following example.

Example 2.1. Let $\|\cdot\|$ be the usual norm on $\mathbb{R}$. Define $Tx = 0$ if $x \leq 1$ and $Tx = 1$ if $x > 1$. Then T is discontinuous, T does not satisfy the Kannan type condition, i.e. there is no constant $K \in [0, \infty)$ such that for all $x, y \in X$ satisfying $\|Tx - Ty\| \leq K \cdot \{\|x - Tx\| + \|y - Ty\|\}$, take $x = 0$ and $y = 1 + \varepsilon$, $\varepsilon > 0$.

On the other hand, $T^2x \equiv 0$, so T^2 is a contraction and $p = 0$ is the unique fixed point of T .

When the examination of successive iterations fails, we can look for fixed points of the averaged mapping instead of the T mapping. As we already know, the averaged mappings may have more favorable properties. In particular, averaged mapping T_λ (for $\lambda \in (0, 1)$) has a fixed point if and only if mapping T has it, see (1).

Example 2.2. Let $X = \mathbb{C}$ with the norm $\|z\| = |z|$ for complex numbers, $C = \{z \in \mathbb{C} : |z| \leq 1\}$, and $T : C \rightarrow C$ be a mapping defined by $Tz = iz$ (T is an anticlockwise rotation of $\frac{\pi}{4}$ about the origin of coordinates). Then T is nonexpansive ($|T((0, 0)) - T((1, 0))| = 1 = |(0, 0) - (1, 0)|$) with the origin as the only fixed point and T is not asymptotically regular. Indeed, let $z \in \mathbb{C} \setminus \{0\}$. Then,

$$|T^{n+1}z - T^n z| = |i^{n+1}z - i^n z| = |i^n||1 - i||z| = \sqrt{2}|z| \not\rightarrow 0, \text{ as } n \rightarrow \infty.$$

However, $T_{\frac{1}{2}} = \frac{1}{2}(z + iz)$, $z \in \mathbb{C}$, is a contraction, for $z_1, z_2 \in C$ we have,

$$|T_{\frac{1}{2}}z_1 - T_{\frac{1}{2}}z_2| = \left|\frac{1}{2}(1+i)z_1 - \frac{1}{2}(1+i)z_2\right| = \left|\frac{1+i}{2}\right||z_1 - z_2| = \frac{\sqrt{2}}{2}|z_1 - z_2|.$$

Thus, $\text{Fix}(T_{\frac{1}{2}}) = \{0\} = \text{Fix}(T)$ and $T_{\frac{1}{2}}^n z \rightarrow 0$ for all $z \in C$ (so $T_{\frac{1}{2}}$ is asymptotically regular).

Example 2.3. Let $X = \mathbb{R}$ with the usual norm $\|x\| = |x|$. Define a mapping $T : \mathbb{R} \rightarrow \mathbb{R}$ by $Tx = -3x + 2$. For $x = 0$ and $y = 1$, we have $|T0 - T1| = 3 > 1 = |0 - 1|$, so T is not a nonexpansive mapping. But then $T_{\frac{1}{2}}x = \frac{1}{2}(x + Tx) = 1 - x$, $x \in \mathbb{R}$, is isometry and $T_{\frac{1}{2}}$ does not satisfy Kannan's condition ($|T_{\frac{1}{2}}0 - T_{\frac{1}{2}}1| = 1$ and $\{|0 - T_{\frac{1}{2}}0| + |1 - T_{\frac{1}{2}}1|\} = 2$). But averaged mapping $(T_{\frac{1}{2}})_{\frac{1}{2}} = T_{\frac{1}{4}}x = (1 - \frac{1}{4})x + \frac{1}{4}Tx = \frac{1}{2}$ for all $x \in \mathbb{R}$, and $T_{\frac{1}{4}}$ is a contraction (see [4, Example 1 (2)]). Thus, $\text{Fix}(T_{\frac{1}{4}}) = \text{Fix}(T)$. Unfortunately, this procedure is not the rule. We will show this in the next example.

Example 2.4. Let $X = [1, \infty)$ with the usual norm $\|x\| = |x|$. Define a mapping $T : X \rightarrow X$ by $Tx = 2x - 1$. For $x = 1$ and $y = 2$, we have $|T1 - T2| = 2 > 1 = |1 - 2|$, so T is not a nonexpansive mapping. Similarly for mappings $T_{\lambda} = (1 - \lambda)I + \lambda T$, $0 < \lambda < 1$, we have $|T_{\lambda}1 - T_{\lambda}2| = 1 + \lambda > 1 = |1 - 2|$.

Example 2.5. Let $X = l_{\infty}$ be a normed space of all bounded sequences $x = (x_1, x_2, \dots)$ of real numbers with the norm $\|x\| = \sup_i |x_i|$. Let $C = \{x \in l_{\infty} : \|x\| \leq 1\}$. Define $T : C \rightarrow C$ by $Tx = (0, x_1^2, x_2^2, x_3^2, \dots)$ for $x = (x_1, x_2, x_3, \dots)$ in C . Obviously, $\text{Fix}(T) = \{0\}$, where $0 = (0, 0, 0, \dots)$.

If $a = (\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \dots)$ and $b = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots)$, then $\|a - b\| = \|(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \dots)\| = \frac{1}{4}$, and

$$\|Ta - Tb\| = \left\| \left(0, \frac{5}{16}, \frac{5}{16}, \dots \right) \right\| = \frac{5}{16} > \frac{1}{4} = \|a - b\|.$$

Let $T_{\lambda}x = (1 - \lambda)x + \lambda Tx$, $0 < \lambda < 1$. Then

$$\begin{aligned} \|T_{\lambda}a - T_{\lambda}b\| &= \left\| \left((1 - \lambda)\left(\frac{3}{4} - \frac{1}{2}\right), (1 - \lambda)\left(\frac{3}{4} - \frac{1}{2}\right) + \lambda\left(\left(\frac{3}{4}\right)^2 - \left(\frac{1}{2}\right)^2\right), \right. \right. \\ &\quad \left. \left. (1 - \lambda)\left(\frac{3}{4} - \frac{1}{2}\right) + \lambda\left(\left(\frac{3}{4}\right)^2 - \left(\frac{1}{2}\right)^2\right), \dots \right) \right\| = \\ &= \left\| \left((1 - \lambda)\frac{1}{4}, \frac{1}{4} + \lambda \cdot \frac{1}{16}, \frac{1}{4} + \lambda \cdot \frac{1}{16}, \dots \right) \right\| = \frac{1}{4} + \lambda \cdot \frac{1}{16} > \frac{1}{4} = \|a - b\|. \end{aligned}$$

Hence, T and averaged mappings T_{λ} for all $0 < \lambda < 1$ are not nonexpansive.

We will illustrate the procedure described above with another examples.

Example 2.6. [15, Example 2.12] Let $X = [0, \frac{\pi}{2}]$ with the usual norm $\|x\| = |x|$ and $T : X \rightarrow X$ be given by $Tx = \cos x$. Then T is not a contraction. Suppose there exists $M < 1$ such that $|\frac{\cos x - \cos y}{x - y}| \leq M$ for all $x \neq y$. Letting $y \rightarrow x$, we get $|\sin x| \leq M$ for all $x, y \in [0, \frac{\pi}{2}]$, which is false. Of course, T as a continuous mapping, by Schauder's Theorem has a fixed point. This can be obtained by weaker means. Consider the mapping $T_{\frac{1}{2}}x = \frac{1}{2}(x + \cos x)$, $x \in X$. Let $x, y \in X$ and $x < y$. By Mean Value Theorem,

$$|T_{\frac{1}{2}}x - T_{\frac{1}{2}}y| = \left| \frac{1}{2} - \frac{1}{2} \sin \xi \right| \cdot |x - y| \leq \frac{1}{2} |x - y|,$$

for some $x < \xi < y$, so $T_{\frac{1}{2}}$ is a contraction and $\text{Fix}(T) = \text{Fix}(T_{\frac{1}{2}}) = \{p\}$, $p \approx 0.739$.

Example 2.7. Let $X = [1, \infty)$ with the usual norm $\|x\| = |x|$ and $T : X \rightarrow X$ be given by $Tx = \frac{1}{x^2}$. Since $|T1 - T(1 + \frac{1}{10})| = \frac{21}{121} \approx 0.17 > 0.1 = |1 - (1 + \frac{1}{10})|$, T is not a nonexpansive mapping. Consider the mapping $T_{\frac{1}{2}}x = \frac{1}{2}(x + \frac{1}{x^2})$, $x \geq 1$. By Mean Value Theorem,

$$|T_{\frac{1}{2}}x - T_{\frac{1}{2}}y| = |\frac{1}{2} - \frac{1}{\xi^3}| \cdot |x - y| \leq \frac{1}{2}|x - y|, \quad x, y \geq 1$$

for some $x < \xi < y$, so $T_{\frac{1}{2}}$ is a contraction and $\text{Fix}(T) = \text{Fix}(T_{\frac{1}{2}}) = \{1\}$.

Remark 2.8. Let F be a self-mapping on a nonempty convex subset C of a Banach space X . By a *backwards mapping* for F we mean the mapping $T_F : D \rightarrow X$ ($D \subset X$) such that $F = (1 - \lambda)I + \lambda T_F$ for some $0 < \lambda < 1$, i.e., $T_F = (1 - \frac{1}{\lambda})I + \frac{1}{\lambda}F$.

Generally,

- (6) F and T_F (=the backwards mapping for F) both have the same fixed point set, $\text{Fix}(F) = \text{Fix}(T_F)$.

Indeed, if $p \in \text{Fix}(F)$, then $T_F p = (1 - \frac{1}{\lambda})p + \frac{1}{\lambda}Fp = p$. If $p \in \text{Fix}(T_F)$, then $T_F p = (1 - \frac{1}{\lambda})p + \frac{1}{\lambda}Fp = p$, so $Fp = p$.

- (7) A backwards mapping of the backwards mapping is a backwards mapping.

Let $\lambda_1, \lambda_2 \in (0, 1)$ and $F : C \rightarrow C$. If $T_F = (1 - \frac{1}{\lambda_1})I + \frac{1}{\lambda_1}F$, then $G_{T_F} = (1 - \frac{1}{\lambda_2})I + \frac{1}{\lambda_2}T_F = (1 - \frac{1}{\lambda_2})I + \frac{1}{\lambda_2}[(1 - \frac{1}{\lambda_1})I + \frac{1}{\lambda_1}F] = (1 - \frac{1}{\lambda_1 \lambda_2})I + \frac{1}{\lambda_1 \lambda_2}F$.

3. New results

The following two problems are significant in the study of fixed points:

- (1) What conditions on the structure of the ambient space X and / or on the properties of T must be added to assure that mapping T has at least one fixed point?
- (2) How one can locate and approximate such a fixed point?

The literature on the subject is very extensive and multi-threaded. In recently published papers by Berinde [2], [3], Berinde and Păcurar [5], [6] drew attention to the so-called enriched nonexpansive (contractive, Kannan, Chatterjea) mappings. In the next theorems, we will give some extension of the Berinde and Păcurar result for enriched contractions [4] and some earlier results from paper [16].

Let $b \in [0, \infty)$ and let $\mathcal{S}$ denote the class of those functions $\alpha : [0, \infty) \rightarrow [0, b+1)$ which satisfy the simple condition: $\alpha(t_n) \rightarrow b+1 \Rightarrow t_n \rightarrow 0$. (We do not assume that α is continuous in any sense.) For example, $\alpha_1(t) = (b+1)e^{-t}$, $\alpha_2(t) = \frac{b+1}{t+1}$, $t > 0$.

Theorem 3.1. Let $(X, \|\cdot\|)$ be a Banach space. Suppose $T : X \rightarrow X$ is a mapping for which exists $0 \leq b, K < \infty$, $\alpha \in \mathcal{S}$, such that for each $x, y \in X$, $\|b(x - y) + Tx - Ty\| \leq \alpha(\|x - y\|) \cdot \|x - y\| + K \cdot \{\|x - Tx\| + \|y - Ty\|\}$ (8) and $T_{\frac{1}{b+1}}$ is an asymptotically regular mapping. If T is continuous, then T has a unique fixed point $z \in X$ and for each $x_0 \in X$ a sequence given by $x_{n+1} = T_{\frac{1}{b+1}}x_n$, $n = 0, 1, \dots$, converges to z as $n \rightarrow \infty$.

Proof. Let us consider two cases.

Case 1. When $b = 0$, then we can apply [16, Theorem 2.1].

Case 2. Let $b > 0$ and let us denote $\lambda = \frac{1}{b+1} \in (0, 1)$. By (8), we get

$$\left\| \left(\frac{1-\lambda}{\lambda} \right) (x-y) + Tx - Ty \right\| \leq \alpha(\|x-y\|) \cdot \|x-y\| + K \cdot \{\|x-Tx\| + \|y-Ty\|\},$$

which can be written in an equivalent form as

$$\begin{aligned} \|[(1-\lambda)x + \lambda Tx] - [(1-\lambda)y + \lambda Ty]\| &\leq \lambda \cdot \alpha(\|x-y\|) \cdot \|x-y\| \\ &\quad + K \cdot \{\|x - [(1-\lambda)x + \lambda Tx]\| \\ &\quad + \|y - [(1-\lambda)y + \lambda Ty]\|\}, \end{aligned}$$

and

$$\|T_\lambda x - T_\lambda y\| \leq \lambda \cdot \alpha(\|x-y\|) \cdot \|x-y\| + K \cdot \{\|x - T_\lambda x\| + \|y - T_\lambda y\|\}. \quad (9)$$

Fix $x_0 \in X$ and let $x_{n+1} = T_\lambda x_n$, $n = 0, 1, 2, \dots$. If $x_{n+1} = x_n$ for some $n \in \mathbb{N} \cup \{0\}$, then $(1-\lambda)x_n + \lambda Tx_n = x_n$, so $Tx_n = x_n$ is the fixed point of T . Suppose $x_{n+1} \neq x_n$ for all $n \geq 0$. Assume that $\{x_n\}$ is not a Cauchy sequence. Then $\limsup_{n,m \rightarrow \infty} \|x_n - x_m\| > 0$. In view

of triangle inequality and (9), we get

$$\begin{aligned} \|x_n - x_m\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_{m+1}\| + \|x_{m+1} - x_m\| \\ &\leq \lambda \cdot \alpha(\|x_n - x_m\|) \|x_n - x_m\| + (K+1) \{\|x_n - x_{n+1}\| + \|x_m - x_{m+1}\|\}. \end{aligned}$$

Under the assumption $\limsup_{n,m \rightarrow \infty} \|x_n - x_m\| > 0$, asymptotic regularity of $\{x_n\}$, the inequality

$$\frac{\|x_n - x_m\|}{\|x_n - x_{n+1}\| + \|x_m - x_{m+1}\|} \leq \frac{K+1}{1 - \lambda \cdot \alpha(\|x_n - x_m\|)}$$

now implies

$$\limsup_{n,m \rightarrow \infty} \frac{K+1}{1 - \lambda \cdot \alpha(\|x_n - x_m\|)} = +\infty,$$

from which

$$\limsup_{n,m \rightarrow \infty} \alpha(\|x_n - x_m\|) = \frac{1}{\lambda} = b+1.$$

But since $\alpha \in \mathcal{S}$ this implies $\limsup_{n,m \rightarrow \infty} \|x_n - x_m\| = 0$, which is a contradiction. Thus, $\{x_n\}$ is a Cauchy sequence. Because X is a Banach space $\lim_{n \rightarrow \infty} x_n = z \in X$.

Using the continuity of T_λ , we immediately obtain $z = T_\lambda z$, so by (1), $Tz = z$.

Assume that $u \neq z$ is another fixed point of T_λ . Then by (9),

$$0 < \|z - u\| \leq \lambda \cdot \alpha(\|z - u\|) \|z - u\| < \|z - u\|,$$

a contradiction. Hence, z is the unique fixed point of T .

It is clear that for each $x_0 \in X$, the sequence $\{x_{n+1} = T_\lambda x_n\}$ converges to z . $\square$

The next result is inspired by theorem of Boyd and Wong [8]. A mapping T satisfying $\|Tx - Ty\| \leq \varphi(\|x - y\|)$, $\varphi(t) < t$ for each $t > 0$, may not possess a fixed point unless some additional condition is assumed on φ . Boyd and Wong [8] assumed φ to be upper semi-continuous from the right.

Let $b \in [0, \infty)$ and let $\mathcal{U}$ denote the class of all mappings $\varphi : [0, \infty) \rightarrow [0, \infty)$ satisfying

- (a) $\varphi(t) < t(b+1)$ for all $t > 0$,
- (b) φ is upper semi-continuous, that is $t_n \rightarrow t \geq 0 \Rightarrow \limsup_{n \rightarrow \infty} \varphi(t_n) \leq \varphi(t)$.

Theorem 3.2. *Let $(X, \|\cdot\|)$ be a Banach space. Suppose $T : X \rightarrow X$ is a mapping for which exists $0 \leq b, K < \infty$, $\varphi \in \mathcal{U}$ such that for each $x, y \in X$,*

$$\|b(x - y) + Tx - Ty\| \leq \varphi(\|x - y\|) + K \cdot \{\|x - Tx\| + \|y - Ty\|\} \quad (10)$$

and $T_{\frac{1}{b+1}}$ is an asymptotically regular mapping. If T is continuous, then T has a unique fixed point $z \in X$ and for each $x_0 \in X$ a sequence given by $x_{n+1} = T_{\frac{1}{b+1}} x_n$, $n = 0, 1, 2, \dots$, converges to z as $n \rightarrow \infty$.

Proof. Let us consider two cases.

Case 1. When $b = 0$, then we can apply [16, Theorem 2.2].

Case 2. Let $b > 0$ and let us denote $\lambda = \frac{1}{b+1} \in (0, 1)$. Analogically as in the proof of theorem 2.6, we get

$$\|T_\lambda x - T_\lambda y\| \leq \lambda \cdot \varphi(\|x - y\|) + K \cdot \{\|x - T_\lambda x\| + \|y - T_\lambda y\|\}. \quad (11)$$

Let $x_0 \in X$ and let $x_{n+1} = T_\lambda x_n$, $n = 0, 1, 2, \dots$. If $x_{n+1} = x_n$ for some $n \in \mathbb{N} \cup \{0\}$, then $(1 - \lambda)x_n + \lambda T x_n = x_n$, so $T x_n = x_n$ is the fixed point of T . Suppose $x_{n+1} \neq x_n$ for all $n \geq 0$. Assume that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and integers $n_i, m_i \in \mathbb{N}$ such that $m_i > n_i \geq i$ and

$$\|x_{n_i} - x_{m_i}\| \geq \varepsilon \text{ for } i = 1, 2, \dots$$

Also, choosing m_i small as possible, it may be assumed that

$$\|x_{n_i} - x_{m_i-1}\| < \varepsilon.$$

Hence, for each $i \in \mathbb{N}$, we have

$$\begin{aligned} \varepsilon &\leq \|x_{n_i} - x_{m_i}\| \leq \|x_{n_i} - x_{m_i-1}\| + \|x_{m_i-1} - x_{m_i}\| < \\ &< \varepsilon + \|x_{m_i-1} - x_{m_i}\|, \end{aligned}$$

and it follows from asymptotic regularity that

$$\lim_{i \rightarrow \infty} \|x_{n_i} - x_{m_i}\| = \varepsilon.$$

Next observe, by (11), that

$$\begin{aligned}\|x_{n_i} - x_{m_i}\| &\leq \|x_{n_i} - x_{n_i+1}\| + \|x_{n_i+1} - x_{m_i+1}\| + \|x_{m_i+1} - x_{m_i}\| \leq \\ &\leq \lambda \cdot \varphi(\|x_{n_i} - x_{m_i}\|) + \\ &\quad + (K+1)\{\|x_{n_i} - x_{n_i+1}\| + \|x_{m_i} - x_{m_i+1}\|\}.\end{aligned}$$

Letting $i \rightarrow \infty$, using asymptotic regularity and upper semi-continuity of φ , we obtain

$$0 < \varepsilon = \lim_{i \rightarrow \infty} \|x_{n_i} - x_{m_i}\| \leq \limsup_{i \rightarrow \infty} \lambda \cdot \varphi(\|x_{n_i} - x_{m_i}\|) \leq \lambda \cdot \varphi(\varepsilon) < \varepsilon,$$

which is a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence. The conclusion of the proof follows as in the proof of Theorem 3.1. $\square$

Example 3.3. [16, Example 2.3] If $b = 0$ and $Tx = \frac{x}{x+1}$ if $x \geq 0$, then T does not satisfy the Banach theorem, take $x = 0$ and $y = \varepsilon$, where $\varepsilon > 0$ and $\varepsilon \downarrow 0$. On the other hand, T satisfy $|Tx - Ty| \leq \varphi(|x - y|)$ with $\varphi(t) = \frac{t}{t+1}$. Obviously, the mapping T is asymptotically regular and has a unique fixed point $p = 0$.

Now we note that T does not satisfy the Kannan type condition, i.e. there is no constant $K \in [0, \infty)$ such that for all $x, y \geq 0$ satisfying $|Tx - Ty| \leq K \cdot \{|x - Tx| + |y - Ty|\}$, take $x = 0$ and $y = \varepsilon$, where ε is sufficiently small. Therefore we cannot apply [15, Theorem 2.6] or [7, Theorem 2.1].

In the next result, we show that the assumption of the existence of constant $K \in [0, \infty)$ can be relaxed to weaker concept using certain functions, see [20].

Let $\mathcal{B}$ denote the class of functions $\beta : [0, \infty) \rightarrow [0, \infty)$ satisfying the following condition, $\limsup_{t \rightarrow 0} \beta(t) < \infty$.

Using the approach of Theorem 3.2, we prove the following result:

Theorem 3.4. *Let $(X, \|\cdot\|)$ be a Banach space. Suppose $T : X \rightarrow X$ is a mapping for which exists $0 \leq b < \infty$, $\varphi \in \mathcal{U}$, $\beta_1, \beta_2 \in \mathcal{B}$ such that for each $x, y \in X$,*

$$\begin{aligned}\|b(x - y) + Tx - Ty\| &\leq \\ &\leq \varphi(\|x - y\|) + \beta_1\left(\frac{1}{b+1}\|x - Tx\|\right) \cdot \|x - Tx\| \\ &\quad + \beta_2\left(\frac{1}{b+1}\|y - Ty\|\right) \cdot \|y - Ty\|\end{aligned}\tag{12}$$

and $T_{\frac{1}{b+1}}$ is an asymptotically regular mapping. If T is continuous, then T has a unique fixed point $z \in X$ and for each $x_0 \in X$ a sequence given by $x_{n+1} = T_{\frac{1}{b+1}}x_n$, $n = 0, 1, 2, \dots$, converges to z as $n \rightarrow \infty$.

Proof. Let us consider two cases.

Case 1. When $b = 0$, then we can apply [20, Theorem 3, for $\mathbf{a} = \mathbf{b} = 0$ and $\ell_1 = \ell_2 = 1$].

Case 2. Let $b > 0$ and let us denote $\lambda = \frac{1}{b+1} \in (0, 1)$. Analogically as in the proof of Theorem 3.1, we get

$$\begin{aligned} \|T_\lambda x - T_\lambda y\| &\leq \\ &\leq \lambda \cdot \varphi(\|x - y\|) + \beta_1(\|x - T_\lambda x\|) \cdot \|x - T_\lambda x\| \\ &\quad + \beta_2(\|y - T_\lambda y\|) \cdot \|y - T_\lambda y\|. \end{aligned} \quad (13)$$

Let $x_0 \in X$ and let $x_{n+1} = T_\lambda x_n$, $n = 0, 1, 2, \dots$. If $x_{n+1} = x_n$ for some $n \in \mathbb{N} \cup \{0\}$, then $(1 - \lambda)x_n + \lambda T x_n = x_n$, so $T x_n = x_n$ is the fixed point of T . Suppose $x_{n+1} \neq x_n$ for all $n \geq 0$. Assume that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and integers $n_i, m_i \in \mathbb{N}$ such that $m_i > n_i \geq i$ and

$$\|x_{n_i} - x_{m_i}\| \geq \varepsilon \text{ for } i = 1, 2, \dots$$

We may choose m_i small as possible such that

$$\|x_{n_i} - x_{m_i-1}\| < \varepsilon.$$

In view of triangle inequality, we have for each $i \in \mathbb{N}$ that

$$\varepsilon \leq \|x_{n_i} - x_{m_i}\| \leq \|x_{n_i} - x_{m_i-1}\| + \|x_{m_i-1} - x_{m_i}\| < \varepsilon + \|x_{m_i-1} - x_{m_i}\|.$$

It follows from asymptotic regularity that

$$\lim_{i \rightarrow \infty} \|x_{n_i} - x_{m_i}\| = \varepsilon.$$

Next observe from (13), we have for each i that

$$\begin{aligned} \|x_{n_i} - x_{m_i}\| &\leq \|x_{n_i} - x_{n_i+1}\| + \|x_{m_i} - x_{m_i+1}\| + \|x_{n_i+1} - x_{m_i+1}\| \leq \\ &\leq \|x_{n_i} - x_{n_i+1}\| + \|x_{m_i} - x_{m_i+1}\| + \\ &\quad + \lambda \cdot \varphi(\|x_{n_i} - x_{m_i}\|) + \\ &\quad + \beta_1(\|x_{n_i} - x_{n_i+1}\|) \cdot \|x_{n_i} - x_{n_i+1}\| + \\ &\quad + \beta_2(\|x_{m_i} - x_{m_i+1}\|) \cdot \|x_{m_i} - x_{m_i+1}\|. \end{aligned}$$

It follows from $\mathcal{B}$ that there exist $i_1 \geq i_0 \in \mathbb{N}$ and $L > 0$ such that

$$\beta_1(\|x_{n_i} - x_{n_i+1}\|) \leq L, \quad \text{and} \quad \beta_2(\|x_{m_i} - x_{m_i+1}\|) \leq L$$

for all $i \geq i_1$. Thus, by above inequality, we have for all $i \geq i_1$,

$$\begin{aligned} \|x_{n_i} - x_{m_i}\| &\leq \|x_{n_i} - x_{n_i+1}\| + \|x_{m_i} - x_{m_i+1}\| + \\ &\quad + \lambda \cdot \varphi(\|x_{n_i} - x_{m_i}\|) + \\ &\quad + L\{\|x_{n_i} - x_{n_i+1}\| + \|x_{m_i} - x_{m_i+1}\|\}. \end{aligned}$$

Making $i \rightarrow \infty$, using asymptotic regularity and upper semi-continuity of φ , we get

$$0 < \varepsilon = \lim_{i \rightarrow \infty} \|x_{n_i} - x_{m_i}\| \leq \limsup_{i \rightarrow \infty} \lambda \cdot \varphi(\|x_{n_i} - x_{m_i}\|) \leq \lambda \cdot \varphi(\varepsilon) < \varepsilon,$$

which is a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence. The conclusion of the proof follows as in the proof of Theorem 3.1. $\square$

Now notice that mappings satisfying the condition

$$|Tx - Ty| < |x - y| + K \cdot \{|x - Tx| + |y - Ty|\} \text{ for all } x, y \in X, x \neq y,$$

where $0 \leq K < \infty$, may not have a fixed point.

Example 3.5. [16, Example 2.5] Let $X = \mathbb{R}$ be with the usual norm and let $Tx = \ln(1 + e^x)$, $x \in \mathbb{R}$. Then T is asymptotically regular ($T^n x = \ln(n + e^x)$, $n = 1, 2, \dots$) and fixed point free ($Tx > x$ for all $x \in \mathbb{R}$). It is easy to see that T satisfies

$$|Tx - Ty| < |x - y| \text{ for } x \neq y.$$

Now we note, that T does not satisfy the Kannan type condition, i.e. there is no constant $K \in [0, \infty)$ such that for all $x, y \geq 0$ satisfying $|Tx - Ty| \leq K \cdot \{|x - Tx| + |y - Ty|\}$: for sufficiently large values $n \in \mathbb{N}$, we have

$$|T(2n) - T(n)| \sim n \text{ and } \max\{|n - T(n)|, |2n - T(2n)|\} \sim 0.$$

Added in proof. Recently published paper Popescu, O., A new class of contractive mappings, Acta Math. Hungar. <https://doi.org/10.1007/s10474-021-01154-6>, contains additional information about enriched contractions, enriched Kannan mappings and enriched Chatterjea mappings.

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Accepted: July 20, 2021.