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Family of mappings with an equicontractive-type condition

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Abstract. In a real Banach space X and a complete metric space M, we consider a compact mapping C defined on a closed and bounded subset A of X with values in M and the operator $T\colon A \times C(A) \to X$. Using a new type of equicontractive condition for a certain family of mappings and β-condensing operators defined by the Hausdorff measure of noncompactness we prove that the operator $x \mapsto T(x, C(x))$ has a fixed point. The obtained results are applied to the initial value problem.

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1. Introduction and preliminaries

The investigations concerning compact operators together with contractive mappings have their origin in the famous Krasnosel'skiĭ's result [7]. This known theorem states that if M is a nonempty closed convex and bounded subset of the given Banach space X and there are given two mappings: a contraction $A\colon M \to X$ and a compact operator $B\colon M \to X$ satisfying $A(M) + B(M) \subset M$ then $A + B$ has a fixed point. In the literature, one can find many contributions, where the authors extend this idea. In [3], Burton replaced the Banach contractive condition with the more general so-called large contraction. In [4], the authors merged the concepts due to Krasnosel'skiĭ with the Schaefer's result [11]. In addition, in [9], Reich considered condensing mappings with bounded ranges and applied them to obtain the Schaefer's alternative and a Krasnosel'skiĭ type fixed point theorem. Using Krasnosel'skiĭ-Schaefer type method, Vetro and Wardowski [12] have recently proved an existence theorem producing a periodic solution of a nonlinear integral equation. Przeradzki in his work [8], using a concept of Hausdorff measure of noncompactness, relaxed a strong condition:

$$A(M) + B(M) \subset M,$$

by considering a weaker one

$$(A + B)(M) \subset M,$$

where a contractive type operator is a generalized contraction. In addition, in [14], there was investigated a wide class of (φ, F)-contractions. The author proved that a certain subclass of these mappings is β-condensing. Applying the Sadovskii's result, the fixed point result for the sum of compact mapping with (φ, F)-contraction was obtained. On the other hand in [6], Karakostas gave an extension of Krasnosel'skiĭ's theorem by involving both operators (contractive and compact) in the resulting one given in an implicit form. In this way, the author was interested in finding a solution of the equation given by the formula:

$$x = T(x, C(x)), \tag{1}$$

where $C\colon A \to Y$, $T\colon A \times C(A) \to X$, A is a subset of a real Banach space X and Y is a metric space.

In the present paper, we prove two theorems which improve the results in [6]. One of the derived tools will be applied to some nonlinear problem which, according to the author's knowledge, cannot be solved using the mentioned existing theorems. Before we formulate our results, we recall and establish the needed definitions and notations. Consider a real Banach space X with norm $\|\cdot\|$ and a complete metric space M with metric d. Let $A \subset X$ and consider the mappings $C\colon A \to M$ and $T\colon A \times C(A) \to X$. The mapping $C\colon A \to M$ is said to be *compact* if it is continuous and maps bounded sets into relatively compact subsets of M, i.e. if $K \subset A$ is bounded then $\overline{K}$ is compact. For the given T we can consider the families of mappings $\mathcal{M}_T$ and $\mathcal{N}_T$ of the form:

$$\mathcal{M}_T := \{A \ni x \mapsto T(x, y)\colon y \in C(A)\},$$
$$\mathcal{N}_T := \{C(A) \ni y \mapsto T(x, y)\colon x \in A\}.$$

We say that the family $\mathcal{N}_T$ is *uniformly equicontinuous* if for every $\varepsilon > 0$ there exists $\delta > 0$, such that for all $x \in A$ and $y_1, y_2 \in C(A)$,

$$d(y_1, y_2) < \delta \text{ implies } \|T(x, y_1) - T(x, y_2)\| \leq \varepsilon.$$

Definition 1.1. We say that the family $\mathcal{M}_T$ is equicontractive singularly if there exists $c > 0$, such that

$$c \leq \frac{1}{\|T(x, p) - T(y, p)\|} - \frac{1}{\|x - y\|} \tag{2}$$

for every $x, y \in A$, $p \in C(A)$ satisfying $T(x, p) \neq T(y, p)$.

The following example shows the nature of the introduced type of equicontractive family.

Example 1.1. Consider $X = \mathbb{R}$ with the standard norm, $M = [0, 1]$ with the standard metric, $A = [0, 1]$:

$$C(x) = x,\ x \in A \ \text{ and } \ T(x, y) = \frac{x}{1 + x} + y,\ x, y \in A.$$

Then, the family $\mathcal{M}_T$ is equicontractive singularly for $c = 1$ and is not equicontractive in the usual sense. Indeed, fix $y \in [0,1]$. For every $x_1, x_2 \in [0,1]$, $x_1 \neq x_2$, we have

$$\begin{aligned}\frac{1}{|T(x_1,y)-T(x_2,y)|} - \frac{1}{|x_1-x_2|} &= \frac{(1+x_1)(1+x_2)}{|x_1-x_2|} - \frac{1}{|x_1-x_2|}\\ &= \frac{x_1+x_1x_2+x_2}{|x_1-x_2|}\\ &\geq \frac{|x_1-x_2|+x_1x_2}{|x_1-x_2|}\\ &\geq 1.\end{aligned}$$

On the other hand, observe that

$$\frac{|T(x_1,y)-T(x_2,y)|}{|x_1-x_2|} = \frac{1}{(1+x_1)(1+x_2)} \to 1,\ \ x_1, x_2 \to 0.$$

Therefore, one cannot find $k \in (0,1)$, such that

$$|T(x_1,y)-T(x_2,y)| \leq k|x_1-x_2| \text{ for every } x_1, x_2, y \in [0,1].$$

In one of our result, we will apply a measure of noncompactness which determines how much the given bounded set is not compact. For our purposes we will use the Hausdorff measure of noncompactness, i.e. for any bounded subset $B \subset X$, there is assigned a nonnegative number $\beta(B)$ by the formula:

$$\beta(B) := \inf\left\{r > 0\colon B \subset \bigcup_{i=1}^{N} B(x_i, r),\ x_i \in X,\ i = 1, \ldots, N\right\},$$

where $B(x_i, r)$ denotes the closed ball centred at x_i with radius r. Some of the basic properties of the Hausdorff measure of noncompactness are the following:

(a) $\beta(B) = 0$ if and only if C is relatively compact,
(b) $\beta(B) = \beta(\overline{B})$,
(c) if $D \subset X$ is bounded then $B \subset D$ implies $\beta(B) \leq \beta(D)$ and $\beta(B+D) \leq \beta(B) + \beta(D)$.

More information about the measure of noncompactness and its properties can be found e.g. in [1]. The fundamental fixed point results where measure of noncompactness is applied are due to Darbo [5] and Sadovskii [10].

Theorem 1.1. *If M is a nonempty bounded closed convex subset of a Banach space X, $T\colon M \to M$ is a continuous mapping, such that there exists $k \in [0,1)$, such that for any set $C \subset M$:*

$$\beta(T(C)) \leq k\beta(C).$$

Then, T has a fixed point.

Theorem 1.2. *If M is a nonempty bounded closed convex subset of a Banach space X, $T\colon M \to M$ is a continuous mapping, such that for any set $C \subset M$ with positive measure of noncompactness:*

$$\beta(T(C)) < \beta(C).$$

Then T has a fixed point.

The mappings satisfying the contraction condition in Darbo's and Sadovskii's result are called *k-set contraction* and *β-condensing* respectively.

2. Results

In the first result, we prove the existence of solution of the Eq. (1) by combining Krasnosel'skiĭ's theorem and the classical Schauder's fixed point result.

Theorem 2.1. *Let $A \subset X$ be closed and convex, $C\colon A \to M$ continuous and sending A into relatively compact subset of M and let the mapping $T\colon A \times \overline{C(A)} \to A$ be continuous such that the family $\mathcal{M}_T$ is equicontractive singularly. Then the Eq. (1) has a solution in A.*

Proof. Let $y \in \overline{C(A)}$ be arbitrary. By Theorem 2.1 in [13], one can find a unique $x \in A$, such that

$$T(x, y) = x. \tag{3}$$

Denote by S a mapping that for a given $y \in \overline{C(A)}$ assigns $x \in A$ such that (3) holds. We will show that the mapping $S\colon \overline{C(A)} \to A$ is continuous. Let (y_n) be a sequence of elements in $\overline{C(A)}$ convergent to $y_0 \in \overline{C(A)}$. Observe that if $T(y_n, S(y_n)) = T(y_m, S(y_m))$ for some $m, n \in \mathbb{N}$ then obviously we get $S(y_n) = S(y_m)$. We can therefore assume that $T(y_n, S(y_n)) \neq T(y_m, S(y_m))$ for every $m, n \in \mathbb{N}$ and apply (2). We have

$$\begin{aligned}
\|S(y_n) - S(y_0)\| &= \|T(S(y_n), y_n) - T(S(y_0), y_0)\| \\
&\leq \|T(S(y_n), y_n) - T(S(y_0), y_n)\| \\
&\quad + \|T(S(y_0), y_n) - T(S(y_0), y_0)\| \\
&\leq \frac{\|S(y_n) - S(y_0)\|}{1 + c\|S(y_n) - S(y_0)\|} + \|T(S(y_0), y_n) - T(S(y_0), y_0)\|.
\end{aligned}$$

In consequence, we obtain

$$\frac{c\|S(y_n) - S(x_0)\|^2}{1 + c\|S(y_n) - S(x_0)\|} \leq \|T(S(y_0), y_n) - T(S(y_0), y_0)\|.$$

From the continuity of T and monotonicity of the function

$$(-\infty, 0] \ni t \mapsto \frac{ct^2}{1 + ct},$$

we get $\|S(y_n) - S(x_0)\| \to 0$, and thus S is continuous. Next, using Schauder's fixed point theorem for the mapping $S \circ C$ defined on a compact set

$$K := \overline{\mathrm{conv}(S(C(A))} \subset A,$$

we get the existence of $\overline{x} \in K$ satisfying $S(C(\overline{x})) = \overline{x}$ (for details, see [6]). Finally, we receive

$$\begin{aligned}
T(\overline{x}, C(\overline{x})) &= T(S(C(\overline{x})), C(\overline{x})) \\
&= S(C(\overline{x})) \\
&= \overline{x}.
\end{aligned}$$

□

In the second result, we will show that the Eq. (1) admits a solution by applying the Hausdorff measure of noncompactness and showing that the operator $T(\cdot, C(\cdot))$ is β-condensing.

Theorem 2.2. *Let $A \subset X$ be closed bounded and convex, $C\colon A \to M$ compact and let the mapping $T\colon A \times C(A) \to X$ be such that the family $\mathcal{M}_T$ is equicontractive singularly and the family $\mathcal{N}_T$ is uniformly equicontinuous. If $T(x, C(x)) \in A$ for every $x \in A$ then Eq. (1) has a solution in A.*

Proof. Consider the operator

$$E(x) := T(x, C(x)),\ x \in A.$$

We will show that E is β-condensing. Consider $B \subset A$ with a positive Hausdorff measure of noncompactness and take $\varepsilon > 0$ arbitrarily. From the uniform equicontinuity of $\mathcal{N}_T$ there exists $\delta > 0$ such that for all $x \in A$, $y_1, y_2 \in C(A)$:

$$d(y_1, y_2) \le \delta \ \text{ implies } \|T(x, y_1) - T(x, y_2)\| \le \varepsilon. \tag{4}$$

Let

$$R := \beta(B) + \varepsilon,$$

and take a finite R-net of B, that is

$$B \subset \bigcup_{i=1}^{k} B(x_i, R),\ x_1, \dots, x_k \in A.$$

From the compactness of C we have $\beta(C(B)) = 0$, and hence, one can find $y_1, \dots, y_l \in C(B)$, such that

$$C(B) \subset \bigcup_{j=1}^{l} B\left(y_j, \delta\right).$$

Therefore, we get

$$\begin{aligned} B &\subset \left(\bigcup_{i=1}^{k} B(x_i, R)\right) \cap \left(\bigcup_{j=1}^{n} C^{-1}\left(B\left(y_j, \delta\right)\right)\right) \\ &= \bigcup_{i=1}^{k}\bigcup_{j=1}^{l} B(x_i, R) \cap C^{-1}\left(B\left(y_j, \delta\right)\right). \end{aligned} \tag{5}$$

Set

$$R' := \frac{\beta(B)}{1 + c\beta(B)}.$$

We will show that open balls

$$B(z_{ij}, R'), \quad \text{where} \ \ z_{ij} := T(x_i, y_j),\ 1 \le i \le k,\ 1 \le j \le l,$$

cover $E(B)$. Take $w \in E(B)$. and let $u \in B$ be such that

$$w = E(u) = T(u, C(u)).$$

From (5) there exist $i \in \{1, \dots, k\}$, $j \in \{1, \dots, l\}$ such that

$$u \in B(x_i, R) \cap C^{-1}\left(B\left(y_j, \delta\right)\right).$$

First observe that

$$\|u - x_i\| < R \ \text{ and } \ d(Cu, y_j) < \delta.$$

If $T(u, C(u)) = T(x_i, C(u))$ then, by (4), the following holds

$$\begin{aligned}\|w - z_{ij}\| &= \|T(u, C(u)) - T(x_i, y_j)\| \\ &= \|T(x_i, C(u)) - T(x_i, y_j)\| \\ &\le \varepsilon.\end{aligned}$$

ε is arbitrary and hence $w = z_{ij}$ and obviously $w \in B(z_{ij}, R')$. If $T(u, C(u)) \neq T(x_i, C(u))$, then due to the fact that $\mathcal{M}_T$ is equicontractive singularly and from (4), we have

$$\begin{aligned}\|w - z_{ij}\| &= \|T(u, C(u)) - T(x_i, y_j)\| \\ &\le \|T(u, C(u)) - T(x_i, C(u))\| + \|T(x_i, C(u)) - T(x_i, y_j)\| \\ &\le \frac{\|u - x_i\|}{1 + c\|x - x_i\|} + \varepsilon \\ &\le \frac{R}{1 + cR} + \varepsilon \\ &\le \frac{\beta(B) + \varepsilon}{1 + c\beta(B) + c\varepsilon} + \varepsilon.\end{aligned}$$

Since ε was taken arbitrarily, we finally obtain

$$\|w - z_{ij}\| \le \frac{\beta(B)}{1 + c\beta(B)},$$

and thus $w \in B(z_{ij}, R')$. In consequence

$$E(B) \subset \bigcup_{i=1}^{k} \bigcup_{j=1}^{l} B(z_{ij}, R').$$

From the definition of Hausdorff measure of noncompactness we have

$$\beta(E(B)) \le \frac{\beta(B)}{1 + c\beta(B)} < \beta(B).$$

Sadovskii's fixed point result ends the proof. □

3. Initial value problem

We illustrate some of our results by considering the following initial value problem:

$$\begin{cases} x'(t) = h(t) + f(t, x(t)) \int_0^t D(t-s)x(s)ds, \ t \in I := [0, 1], \\ x(0) = 0, \end{cases} \tag{6}$$

where $h\colon I \to \mathbb{R}$, $D\colon I \to \mathbb{R}$ are continuous functions, $D \neq 0$, $f\colon I \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists $M > 0$, such that

$$|f(t, x)| \le M \ \text{ for every } \ t \in I, x \in \mathbb{R}.$$

Consider the space $X := C(I, \mathbb{R})$ with supremum norm. The following theorem holds.

Theorem 3.1. *Suppose that* $\|D\| \leq 1/M$, $\|h\| \leq 1 - M\|D\|$ *and there exists* $L \geq \|D\|$, *such that*

$$|f(t,x) - f(t,y)| \leq \frac{|x-y|}{L + |x-y|} \quad \text{for every} \ \ t \in I, \ \ x, y \in \mathbb{R}. \tag{7}$$

Then, the problem (6) has a solution in $B(0,1)$.

Proof. First, define the operator $C\colon X \to X$ by the formula:

$$(Cx)(t) := \int_0^t D(t-s)x(s)ds, \ \ x \in X.$$

Observe that C is compact. Taking any $x \in B(0,1)$, we have

$$\begin{aligned} |(Cx)(t)| &\leq \|D\|\|x\|t \\ &\leq \|D\|, \end{aligned}$$

so the operator C maps $B(0,1)$ into $B(0,\|D\|)$. Consider also the operator $T\colon B(0,1) \times B(0,\|D\|) \to X$ of the form:

$$T(x,y)(t) := \int_0^t h(s)ds + \int_0^t f(s,x(s))y(s)ds, \quad t \in I.$$

Take $x_1, x_2 \in B(0,1)$ and $y \in C(B(0,1))$. We obtain

$$\begin{aligned} |T(x_1,y)(t) - T(x_2,y)(t)| &\leq \int_0^t |y(s)||f(s,x_1(s)) - f(s,x_2(s))|ds \\ &\leq \|D\| \frac{|x_1(s) - x_2(s)|}{L + |x_1(s) - x_2(s)|} \\ &\leq \frac{\|x_1 - x_2\|}{1 + c\|x_1 - x_2\|}, \end{aligned}$$

where

$$c := 1/\|D\|.$$

Hence, the family $\mathcal{M}_T$ is equicontractive singularly.

In addition, observe that the operator

$$E(x) := T(x, C(x)), \quad x \in B(0,1),$$

is continuous and for every $x \in B(0,1)$ we see

$$\begin{aligned} |T(x,C(x))| &\leq \|h\| + M\|D\| \\ &\leq (1 - M\|D\|) + M\|D\| \\ &= 1. \end{aligned}$$

Thus, for every $x \in B(0,1)$, $E(x) \in B(0,1)$. Finally, note that the family $\mathcal{N}_T$ is uniformly equicontinuous. Theorem 2.2 ends the proof. $\square$

Remark 3.1. Observe that based on the methods used in the proof of Theorem 3.1 we cannot show that T is equicontractive in a common sense (see e.g. [6]). Indeed, taking any $x_1, x_2 \in B(0,1)$ and $y \in C(B(0,1))$, we get

$$\begin{aligned}
&\sup_{x_1,x_2\in B(0,1),x_1\neq x_2} \frac{\|T(x_1,y)-T(x_2,y)\|}{\|x_1-x_2\|}\\
&= \sup_{x_1,x_2\in B(0,1),x_1\neq x_2} \frac{\sup_{t\in I}|T(x_1,y)(t)-T(x_2,y)(t)|}{\|x_1-x_2\|}\\
&\leq \sup_{x_1,x_2\in B(0,1),x_1\neq x_2} \frac{1}{1+\frac{1}{\|D\|}\|x_1-x_2\|} = 1.
\end{aligned}$$

Remark 3.2. The form of the right side of inequality (7) plays an essential role in the proof of Theorem 3.1, where the equicontractive singularity is applied. The inequalities of such type appeared in the literature, e.g., in the joint work of Banaś and Dhage [2], where the so-called k-set contractions and Darbo type fixed point results were utilized. In the present paper, the posed claims have been proved with the help of condensing operators and Sadovskii's result, which as it is known, comparing to Darbo's result, require weaker assumptions.

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