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The least number of 2-periodic points of a smooth self-map of S^2 of degree 2 equals 2

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Abstract. We show that there exists a smooth self-map of the sphere $f : S^2 \to S^2$ which has degree 2 and has only two 2-periodic points.

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Keywords. Fixed point, periodic point, Nielsen fixed point theory, Dold congruences, least number of periodic points.

1. Introduction

We fix a self-map $f : M \to M$ of a compact connected manifold and a natural number n. What is the least number of n-periodic points $\#\text{Fix}(h^n)$ where h runs through the homotopy class of f? If we moreover restrict to simply connected M and we allow all continuous maps homotopic to f, then the least number is always 1 or 0 [4]. However, if h runs only through the smooth homotopy class of f, then the least number may be much larger, which was noticed by Shub and Sullivan [10]. This gave rise to $D_n^m(f)$ an algebraic lower bound of the number of n-periodic points in the smooth homotopy class of f, see [4]. In dimension $m \geq 3$, the homotopy invariant $D_n^m(f)$ turned out to be the best lower bound, i.e., it can be realized by a smooth map homotopic to the given f (a Wecken-type theorem) [4] .

On the other hand, the sphere S^2 is the unique two-dimensional closed simply connected manifold. Surprisingly, the methods of reducing fixed and periodic points do not work in general on surfaces. The reason is that the Whitney trick of canceling intersection points does not hold in low dimensions, thus the Wecken theorem for periodic points works only from dimension 3 on. See [8,9]. This makes the problem of minimizing the number of periodic points open in dimension 2, in particular for self-maps of S^2.

In [6], we started to study this case.

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Theorem 1.1. (Theorem 2.6 in [6])

Let $f : S^2 \to S^2$ be a map of degree $d \in \mathbb{Z}$ and let n be a natural number. Then f is homotopic to a smooth map h so that $\mathrm{Fix}(h^n)$ *is a point (or empty set)* $\iff$ *one of conditions*

(1) $|d| \leq 1$
(2) $n = 1$
(3) $d = -2$ *and* $n = 2$

is satisfied. □

In this paper and in [6] by smooth we mean C^∞, since the maps in Lemma 2.2, which are given explicitly in the proof of Theorem 3.7 in [1], may by represented by C^∞ maps.

In Sect. 3 we give the algebraic necessary condition to homotope a self-map of S^2 to a map with at most two n-periodic points. The main result of the paper is Theorem 3.2 saying that the above condition is also sufficient for $\deg(f) = 2$ and $n = 2$.

This must be done directly, since the techniques of reducing isolated periodic orbits with opposite indices, used in the Nielsen fixed and periodic point theory, are not available in dimension 2.

2. Indices of iterations of a smooth map

In 1983, Dold [3] noticed that a sequence of fixed point indices $A_k = \mathrm{ind}(f^k; x_0)$, where f is a continuous self-map of a Euclidean space $\mathbb{R}^m$ and x_0 is an isolated fixed point for each k, must satisfy some congruences. Namely for each $n \in \mathbb{N}$

$$\sum_{k|n} \mu(n/k) \cdot \mathrm{ind}(f^k; x_0) \equiv 0 (\text{modulo } n)$$

where μ denotes the Möbius function.

It was shown [1] that each sequence of integers (A_k) satisfying Dold congruences can be realized as $A_k = \mathrm{ind}(f^k; x_0)$, for a continuous self-map of $\mathbb{R}^m$ for $m \geq 3$. In other words, Dold congruences are the only restriction for a sequence of integers to realize the fixed point index of a continuous map.

Surprisingly, it turned out that there are much more restrictions on sequences $A_k = \mathrm{ind}(f^k; x_0)$ when f is smooth [2,7,10]. In [5] it is proved that the necessary conditions given in [2] are, in dimension ≥ 3, also sufficient and the full description of all such sequences is given in [5]. We call them *smoothly realizable in dimension m*.

It is convenient to present the sequences of integers as the sum of the following elementary periodic sequences

Definition 2.1. For a given $l \in \mathbb{N}$ we define

$$\mathrm{reg}_l(n) = \begin{cases} l & \text{if } l \text{ divides } n, \\ 0 & \text{if } l \text{ does not divide } n. \end{cases}$$

It is easy to notice that each integer sequence (A_n) can be written down uniquely in the following form of a periodic expansion: $A_n = \sum_{l=1}^{\infty} a_l \mathrm{reg}_l(n)$, where $a_n = \frac{1}{n}\sum_{l|n} \mu(\frac{n}{l})\ A_l$ for suitable $a_l \in \mathbb{R}$. Moreover, all coefficients a_l are integers if and only if the sequence (A_n) satisfies Dold congruences.

The above observations, applied in dimension 2, resulted in the full description of all possible sequences smoothly realizable in dimension 2. In the next Lemma, we reformulate Theorem 3.7 in [1] using our notations

Lemma 2.2. [1] see also Lemma 1.1 in [6].

Let $U \subset \mathbb{R}^2$ be a neighborhood of $(0,0)$ and let $f: U \to \mathbb{R}^2$ be a smooth map such that $(0,0)$ is an isolated periodic point. Then, the periodic expansion of the local fixed point index of f takes one of the following forms:

(1) $\mathrm{ind}(f^k;(0,0)) = c \cdot \mathrm{reg}_1(k)$ for $c \in \mathbb{Z}$
(2) $\mathrm{ind}(f^k;(0,0)) = \mathrm{reg}_1(k) + c \cdot \mathrm{reg}_m(k)$ for $c \in \mathbb{Z}$, $m \in \mathbb{N}$, $c \neq 0$
(3) $\mathrm{ind}(f^k;(0,0)) = c \cdot \mathrm{reg}_2(k)$ for $c \in \mathbb{Z}$, $c \neq 0$
(4) $\mathrm{ind}(f^k;(0,0)) = -\mathrm{reg}_1(k) + c \cdot \mathrm{reg}_2(k)$ for $c \in \mathbb{Z}$, $c \neq 0$ □

Remark 2.3. The right-hand side of the above formulae may be also written as

(1) constant
(2) $\begin{cases} 1 + cm \text{ if } m \mid k \\ 1 \text{ otherwise} \end{cases}$
(3) $0, 2c, 0, 2c, \ldots$
(4) $-1, -1+2c, -1, -1+2c, \ldots$. □

3. Algebraic necessary condition

Let us fix a pair of numbers $(d, n) \in \mathbb{Z} \times \mathbb{N}$. Does there exist a smooth map $f: S^2 \to S^2$ satisfying $\deg(f) = d$ and $\#\mathrm{Fix}(f^n) \leq 2$? Theorem 1.1 allows to concentrate on $|d| \geq 2$ and $n \geq 2$. Suppose that there exists f a smooth map of degree d with at most two n-periodic points. We may assume that $\mathrm{Fix}(f^n) \subset \{N\mathrm{Pole}, \mathrm{SPole}\}$. Then

$$\begin{aligned} d^k + 1 = L(f^k) = \mathrm{ind}(f^k) &= \mathrm{ind}(f^k; S\mathrm{Pole}) + \mathrm{ind}(\mathrm{f^k}; \mathrm{NPole}) \\ &= C_1(k) + C_2(k) \end{aligned}$$

for some expressions C_1, C_2 of types $(1)--(4)$ for all $k|n$. In other words, the existence of expressions C_1, C_2 satisfying $d^k + 1 = C_1(k) + C_2(k)$ is a necessary algebraic condition to reduce $\mathrm{Fix}(f^n)$ to two points.

Lemma 3.1. *Let $f: S^2 \to S^2$ be a self-map of degree d satisfying $|d| \geq 2$ and let $n \in \mathbb{N}$, $n \neq 1$. Then there exist expressions C_1 , C_2 of types $(1)-(4)$ such that the equality $d^k + 1 = C_1(k) + C_2(k)$ is satisfied for all $k|n$ if and only if n is a prime number.*

Proof. $\Rightarrow$. We assume that $d^k + 1 = C_1(k) + C_2(k)$ for $k|n$ where C_1, C_2 are of type $(1)-(4)$. We will show that n is a prime. First we assume that one of C_1 , C_2 is of type (1). Then, the sum $C_1(k)+C_2(k)$ takes at most two values.

Since d^k+1 takes distinct values (for a fixed $|d|\geq 2$) and equality holds for all divisors of n, n must be a prime.

On the other hand if no of C_1, C_2 is of type (1) then $-2\leq C_1(1)+C_2(1)\leq 2$ hence $-3\leq d\leq 1$. This proves $\Rightarrow$ for $d\notin\{-3,-2,-1,0,+1\}$. It remains to consider the cases $d=-2$, $d=-3$ where no C_i is of type (1).

Let $d=-2$. Then $C_1(1)+C_2(1)=-1$, hence one of C_i must be of type (4) and the other of type (3). Then C_1+C_2 takes at most two values.

Let $d=-3$. Then $C_1(1)+C_2(1)=-2$, hence both C_1, C_2 must be of type (4), so their sum takes only two values.

$\Leftarrow$. If p is a prime number then the equality $d^p+1=d+1+(\frac{d^p-1}{p})p$, implies

$$d^k+1=d\cdot \mathrm{reg}_1(k)+\mathrm{reg}_1(k)+\left(\frac{d^p-d}{p}\mathrm{reg}_p(k)\right)$$

for $k=1$ and $k=p$. Moreover $\frac{d^p-d}{p}$ is an integer by small Fermat's theorem. □

In the rest of the paper, we will assume that $n=d=2$. We will show that then the above condition is also sufficient. We will show that

Theorem 3.2. *There exists a smooth map $f: S^2\to S^2$ of degree 2 which has only two 2-periodic points.*

This will be done as follows. First, we give a convenient formula of a map of degree 2. Then, we deform smoothly this map near the poles to realize their appropriate values of the fixed point index. Some extra 2-periodic points appear. The last step is to remove these points.

Remark 3.3. To get a smooth map we start with a map which is smooth near the poles and we use continuous deformations which are constant near the poles. Finally we deform the obtained map, with only two 2-periodic points, to a smooth map by a homotopy constant near the poles. If the last deformation is sufficiently small, the poles remain the unique 2-periodic points.

4. Notation

Let us introduce some notation. We will consider the sphere S^2 as the quotient set $[-\frac{\pi}{2},\frac{\pi}{2}]\times S^1/\sim$ where we identify the points $[\theta,\phi],[\theta',\phi']\in[-\frac{\pi}{2},\frac{\pi}{2}]\times S^2$ if $\theta=\theta'=\frac{\pi}{2}$ or $\theta=\theta'=-\frac{\pi}{2}$. Moreover $S^1=\mathbb{R}/(2\pi\cdot\mathbb{Z})$ and ϕ denotes a real number modulo 2π. We will refer to θ ,ϕ as latitude and longitude, respectively. We also denote by

$$S^2_+=\{[\theta,\phi]\in S^2; \frac{\pi}{6}\leq\theta\leq\frac{\pi}{2}\}\ ,\ \ S^2_0=\{[\theta,\phi]\in S^2; -\frac{\pi}{6}\leq\theta\leq\frac{\pi}{6}\}$$
$$S^2_-=\{[\theta,\phi]\in S^2; -\frac{\pi}{2}\leq\theta\leq-\frac{\pi}{6}\}$$

three sectors of the sphere. We will denote $N\mathrm{Pole}=[\frac{\pi}{2},*]$ and $S\mathrm{Pole}=[-\frac{\pi}{2},*]$. Please notice that symbols S^2_+ , S^2_- have a different meaning in [6].

We define a continuous map $f_0 : S^2 \to S^2$.

$$f_0[\theta,\phi] = \begin{cases} [3\theta-\pi,\phi+\pi] & \text{for } [\theta,\phi]\in S^2_+ \\ [-3\theta,0] & \text{for } [\theta,\phi]\in S^2_0 \\ [3\theta+\pi,\phi] & \text{for } [\theta,\phi]\in S^2_- \end{cases}$$

Remark 4.1. The degree of f_0 equals 2, since the restrictions of f_0 to S^2_- and S^2_+ are orientation-preserving diffeomorphisms .

Lemma 4.2. $\mathrm{Fix}(f_0^2)$ *consists of three fixed points* NPole*,* SPole *and* $[0,0]$ *and two 2-orbits* $\{[\frac{3\pi}{10};0];[-\frac{\pi}{10};\pi]\}$ *,* $\{[-\frac{3\pi}{10};0];[\frac{\pi}{10};0]\}$*.*

Proof. We notice that f_0 has exactly one fixed point in each sector: $N\text{Pole}\in \mathrm{S}^2_+$, $[0,0]\in S^2_0$, $S\text{Pole}\in \mathrm{S}^2_-$, since the coordinate θ is being expanded on each sector.

Now we look for 2-orbits. The above argument (θ is expanding) implies that there is no 2-orbit contained in a sector. Moreover the component ϕ excludes a 2-orbit with a point in S^2_+ and the other one in S^2_-. Now each 2-orbit must have one element in S^2_0 and the second in S^2_+ or in S^2_-. Let $[\theta,\phi]\in S^2_+$ be a 2-periodic point. Then $f_0[\theta,\phi]=[3\theta-\pi,\phi+\pi]\in S^2_0$ which implies $f_0^2[\theta,\phi]=f[3\theta-\pi,\phi+\pi]=[-9\theta+3\pi,0]$. Now $[-9\theta+3\pi,0]=[\theta,\phi]$ implies $-9\theta+3\pi=\theta$, $\phi=0$, hence $\theta=\frac{3\pi}{10}$, $\phi=0$. Since $f_0[\frac{3\pi}{10};0]=[-\frac{\pi}{10};\pi]$, we get the orbit $\{[\frac{3\pi}{10};0];[-\frac{\pi}{10};\pi]\}$.

Let $[\theta,\phi]\in S^2_-$ be a 2-periodic point. Then $f_0[\theta,\phi]=[3\theta+\pi,\phi]\in S^2_0$ which implies $f_0^2[\theta,\phi]=f[3\theta+\pi,\phi]=[-9\theta-3\pi,0]$. Now $[-9\theta-3\pi,0]=[\theta,\phi]$ implies $-9\theta-3\pi=\theta$, $\phi=0$, hence $\theta=-\frac{3\pi}{10}$, $\phi=0$. Since $f_0[-\frac{3\pi}{10};0]=[-\frac{9\pi}{10}+\pi;0]=[\frac{\pi}{10};0]$, we get the orbit $\{[-\frac{3\pi}{10};0];[\frac{\pi}{10};0]\}$. □

Thus

$$\mathrm{Fix}(f_0^2)=\{N\text{Pole},\text{SPole},[0,0];\text{a}=[\frac{3\pi}{10};0],\text{f(a)}=[-\frac{\pi}{10};\pi];$$
$$b=[-\frac{3\pi}{10};0],f(b)=[\frac{\pi}{10};0]\}$$

The aim of this paper is to reduce the above set to $\{N\text{Pole},\text{SPole}\}$.

In Sect. 5 we deform the map f_0 near SPole to a smooth map f_1 satisfying $\mathrm{ind}(f_1;S\text{Pole})=2$. Then a new fixed point b' appears and we remove simultaneously the points $b',b,f(b),[0,0]$ by a homotopy with the carrier in a neighborhood near the meridian $\phi=0$. We get a map f_2 with $\mathrm{Fix}(f_2^2)=\{N\text{Pole},\text{SPole};\text{a},\text{f(a)}\}$.

In Sect. 6 we consider again the original map f_0 (not f_2) and we deform it near NPole to get a map f_3 satisfying $\mathrm{ind}(f_3;N\text{Pole})=1$, $\mathrm{ind}(f_3;N\text{Pole})=3$. This will give a new 2-orbit $\{w_0,w_2\}$. In the next deformation the orbit $a,f(a)$ reduces with the new one. We get a map f_4 with $\mathrm{Fix}(f_4^2)=\{N\text{Pole},\text{SPole};\text{b}',\text{b},\text{f(b)},[0,0]\}$. Finally the maps f^2 and f^4 define a map $\tilde{f}$, also of degree 2 satisfying $\mathrm{Fix}(\tilde{f}^2)=\{N\text{Pole},\text{SPole}\}$.

5. Removing the orbit $\{b, f(b)\}$

In this section, we will remove the orbit $\{b, f(b)\}$ and the fixed point $[0,0]$ by a deformation with the carrier in a neighborhood of the arc $< b, f(b) >$. Here is the sketch of the deformation. We start by a smooth deformation of f_0 near SPole adding an additional fixed point b'. The orbit $\{b, f_0(b)\}$, the points $[0,0]$ and b' belong to the arc $< S\text{Pole}, [\pi/6, 0] >$. The map f_0 sends the ends of the last arc to SPole, the middle point of the arc goes to NPole and the f_0 is linear on each half of the arc. The restrictions of f_0 to neighbor arcs $< S\text{Pole}, [\pi/6, \phi_0] >$ look similar. We consider the region $V_1 = \{[\theta, \phi]; -\pi/2 \leq \theta \leq \pi/6, |\phi| \leq \epsilon\}$ for a small $\epsilon > 0$. We notice that f_0 sends V_1 to the region $V_2 = \{[\theta, \phi]; -\pi/2 \leq \theta \leq \pi/2, |\phi| \leq \epsilon\}$. We consider the restriction $f_{0|} : V_1 \to V_2$. Now if we denote $f_0[\theta, \phi] = [\theta', \phi']$ then $\phi' = \phi$ for $-\pi/2 \leq \theta \leq -\pi/6$ and $\phi' = 0$ for $-\pi/6 \leq \theta \leq \pi/6$. We deform the restriction of f_0 to the arc $< S\text{Pole}, [\pi/6, 0] >$, keeping the end points fixed, by squeezing the arc to a neighborhood of SPole so that there is no periodic point inside the arc. Then, we extend this deformation to V_1 by a homotopy which keeps the boundary $\text{bd}V_1$ fixed, the meridians are sent into themselves, or to the meridian 0. Finally, we compose the obtained deformation with a homeomorphism of V_1 which is constant on the boundary and for $\phi = 0$ and which makes $|\phi|$ smaller elsewhere. The final map is a self-map of degree 2 with no periodic points inside V_1. This gives the maps f_2 with $\text{Fix}(f_2^2) = \text{Fix}(f_2^2) \backslash \{b, f(b)\} = \{N\text{Pole}, \text{SPole}, \text{a}, \text{f}_2(\text{a})\}$.

Now we go to the details. We start with a deformation of f_0 near SPole. We introduce polar coordinates in $S^2 \backslash N$Pole by identifying

$$S^2 \backslash N\text{Pole} \ni [\theta, \phi] \to (\theta + \frac{\pi}{2}) \exp(\phi \cdot \text{i}) \in \text{K}(0; \pi) \subset \mathbb{C}$$

Since the map f_0 near the South Pole has the form $f_0[\theta, \phi] = [3\theta + \pi, \phi]$, in the new coordinates we get the map $\hat{f}(z) = 3z$ (for $|z| < \frac{\pi}{3}$). Lemma 7.1, Remark 7.3 and the above coordinates give a deformation of f_0 to a map which we will denote by f_1. Then $\text{ind}(f_1^k; S\text{Pole}) = 2$ and a new fixed point b' with $\text{ind}(f_1^k; b') = -1$ appears.

Now we are ready to remove the periodic points $b, f(b), [0,0]$ and b'. We will modify the map f_1 on $[-\pi/2, \pi/6] \times [-\epsilon, \epsilon]$ to a map f_2 so that

(1) $f_2 = f_1$ in $[-\pi/2; b''] \times [-\epsilon, \epsilon]$, for some $b'' \in (-\pi/2, b')$, and on the boundary of $[-\pi/2, \pi/6] \times [-\epsilon, \epsilon]$.
(2) $f_2([-\pi/2, \pi/6] \times [-\epsilon, \epsilon]) \subset [-\pi/2, \pi/2] \times [-\epsilon, \epsilon]$.
(3) $f_2^2(x) \neq x$ for $x \in (-\pi/2, \pi/6] \times [-\epsilon, \epsilon]$.

Then f_2 has no 2-periodic points in $(-\pi/2, \pi/6] \times [-\epsilon, \epsilon]$ which implies

$$\begin{aligned}\text{Fix}(f_2^2) &= \text{Fix}(f_1^2) \backslash \{ \text{ union of orbits in } \text{Fix}(f_1^2) \text{ which are disjointed from} \\ &\qquad \times (-\pi/2, \pi/6] \times [-\epsilon, \epsilon]\} \\ &= \{S\text{Pole}, \text{NPole}, \text{a}, \text{f}_1(\text{a})\}\end{aligned}$$

as required.

It remains to construct a map f_2 satisfying (1)-(3).

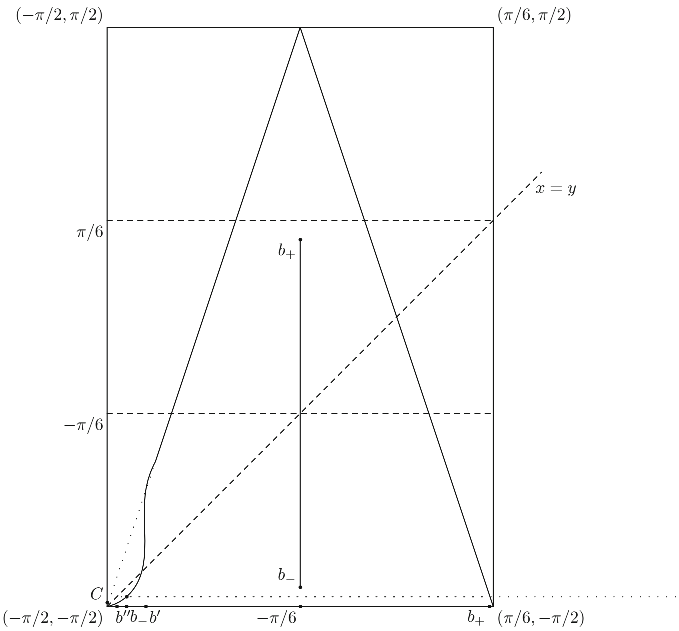

FIGURE 1. $f_2 : [-\pi/2; \pi/6] \times 0 \to [-\pi/2; \pi/2] \times 0$

We start by analyzing the restriction $f_{1|} : [-\pi/2, \pi/6] \times 0 \to [-\pi/2, \pi/2] \times 0$. The graph is given in Fig. 1, where the asymmetry in the left lower corner is resulted by the above deformation of f_0 to f_1. Let $(b', 0)$ be the unique fixed point in $(-\pi/2, 0) \times 0$. We fix a point $b_- \in (-\pi/2, b')$ and we denote $C = f_1(b_-)$. Then $-\pi/2 < C < b_-$ and we fix another point $b'' \in (C, b')$ (see the bottom of Fig. 1). Now the formula

$$\bar{f}_1(\theta) = \min(p_1 f_1(\theta, 0); C)$$

contracts interval $[-\pi/2, \pi/6]$ near SPole. Here p_1, p_2 denote the projections of $[-\pi/2; +\pi/2] \times [-\epsilon; +\epsilon]$.

Now we define the map $f_{2|} : [-\pi/2, \pi/6] \times [-\epsilon, +\epsilon] \to [-\pi/2, \pi/2] \times [-\epsilon, +\epsilon]$ by

$$f_{2|}(\theta, \phi) = ((1 - \eta(\theta, \phi)) p_1 f_1(\theta, \phi) + \eta(\theta, \phi) \cdot \bar{f}_1(\theta); R(\theta, \phi) \cdot (p_2 f_1(\theta, \phi)))$$

where

$\eta : [-\pi/2, \pi/6] \times [-\epsilon, +\epsilon] \to [0, 1]$ is a Urysohn function satisfying $\eta^{-1}(0) = \mathrm{bd}([-\pi/2, -\pi/6] \times [-\epsilon, +\epsilon]) \cup ([-\pi/2, b''] \times [-\epsilon, \epsilon])$, $\eta^{-1}(1) = [b'_-, b'_+] \times 0$ and

$R : [-\pi/2, \pi/6] \times [-\epsilon, +\epsilon] \to [-\epsilon, \epsilon]$ satisfies the following: if $R(\theta, \phi) = \phi'$ then

- $\mathrm{sign}(\phi') = \mathrm{sign}(\phi)$,
- $\phi' = \phi$ on the boundary and for $\theta \leq b''$
- $|\phi'| < |\phi|$ for ($0 < |\phi| < \epsilon$ and $\theta \geq b''$).

Roughly speaking: $[b_-; b_+] \times 0$ is squeezed near SPole and R makes $|\phi|$ smaller. Since $f_{2|}$ coincides with f_1 on the boundary, we may define the map

$$f_2(\theta,\phi) = \begin{cases} f_{2|}(\theta,\phi) & \text{for } (\theta,\phi) \in [-\pi/2, -\pi/6] \times [-\epsilon, +\epsilon] \\ f_1(\theta,\phi) & \text{for } (\theta,\phi) \notin [-\pi/2, -\pi/6] \times [-\epsilon, +\epsilon] \end{cases}$$

We check that f_2 satisfies $(1) - (3)$. (1),(2) follow straight from the definition. We check (3).

First we assume that $\phi \neq 0$ and we show $(\theta,\phi) \in (-\pi/2, \pi/6] \times [-\epsilon, +\epsilon]$ is not a periodic point. First we assume that $\theta \leq b''$. Then $f_2(\theta,\phi) = f_1(\theta,\phi)$. If we denote $f_2(\theta,\phi) = (\theta', \phi')$ then $|\phi'| < |\phi|$ because $\hat{f}$ has also this property near 0 (Remark 7.3). If $b'' < \theta$ then $R(\theta,\phi)$ makes $|\phi'| < |\phi|$. This proves that there is no periodic point for $\phi \neq 0$.

Now we consider a point $(\theta, 0)$.

If $-\pi/2 \leq \theta \leq b''$ then $p_1 f_|(\theta,0) < \theta$, since the similar inequality holds for the map $\hat{f}$.

If $b'' \leq \theta \leq b_+$ then $p_1 f_|(\theta,0) = \bar{f}_|(\theta) = \min(C, p_1 f_1(\theta)) \leq C < b'' \leq \theta$.

If $b_+ \leq \theta \leq \pi/6$ then $p_1 f_|(\theta,0) < b_+ \leq \theta$. □

6. Removing the orbit $a, f(a)$

In this section, we deform the map f_0 (not f_2) and we remove the other 2-orbit $\{a, f(a)\}$. The carrier of the deformation will be disjointed from the carrier of the previous deformation. At the end of the section, we will show that the two deformations give a map of degree 2 whose 2-periodic points are only NPole and SPole. This will end the proof of Theorem 3.2

We will say that a subset $A \subset S^2$ is S^2-convex if A does not contain antipodal points and for each $a, a' \in A$ the geodesic joining the points is contained in A.

We will deform the map f_0 only in the northern hemisphere, hence we introduce other polar coordinates

$$S^2 \backslash S\text{Pole} \ni [\theta,\phi] \to (\pi/2 - \theta) \cdot \exp(\phi \cdot \mathrm{i}) \in \mathrm{K}(0,\pi) \subset \mathbb{C}$$

Let $\hat{f}_2$ denote the induced map of $\mathbb{C}$. Now in a neighborhood of $0 \in \mathbb{C}$, $\hat{f}(z) = -3z$, hence Lemma 7.1 (for $n = 2$, $a = 3$) gives a small local deformation of $\hat{f}$. After the deformation $\mathrm{Fix}(\hat{f}^2) = \{0; w_0, w_2\}$ and $\mathrm{ind}(\hat{f}^2, 0) = 3$, $\mathrm{ind}(\hat{f}^k, w_k) = -1$ for $k = 0, 2$.

Let f_3 be the induced map of S^2. We will cancel simultaneously the orbits; $\{a = [\frac{3\pi}{10}; 0]; f_0(a) = [-\frac{\pi}{10}; \pi]\}$ and $\{w_0, w_2\}$ by a homotopy with the carrier in an arbitrarily prescribed neighborhood of the arc $< f_3(a); w_2 > \subset S^2$.

We consider the arc $< w_0, a >$, its images $f_3 < w_0, a > = < w_2, f_3(a) >$ and $f_3^2 < w_0, a > = f_3 < w_2, f_3(a) > = < w_0, S\text{Pole} >$. See Fig. 2. Since the above arcs contain no antipodal points, we can choose an S^2-convex neighborhood $V_3 \subset < w_0, S\text{Pole} >$ then an S^2-convex neighborhood $V_2 \supset < w_2, f_3(a) >$ satisfying $f_3(\mathrm{cl}(\mathrm{V}_2)) \subset \mathrm{V}_3$ and an S^2-convex neighborhood $V_1 \supset < w_0, a >$ satisfying $f_3(\mathrm{cl}(\mathrm{V}_1)) \subset \mathrm{V}_2$.

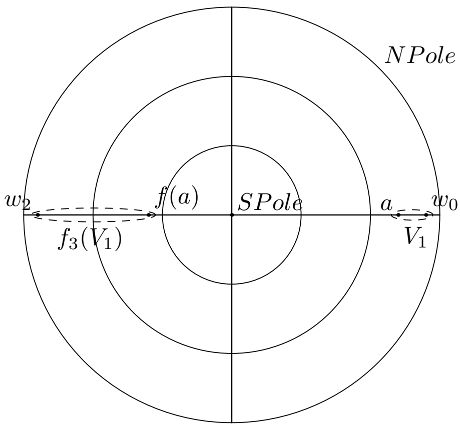

FIGURE 2. Orbit $\{a, f(a)\}$

Now $f_3^2(clV_1) \subset f_3(clV_2) \subset V_3$ and the last set is S^2-convex. On the other hand

$$\mathrm{ind}(f_3^2; \mathrm{cl}(\mathrm{V}_1)) = \mathrm{ind}(\mathrm{f}_3^2; \mathrm{w}_1) + \mathrm{ind}(\mathrm{f}_3^2; \mathrm{a}) = -1 + 1 = 0$$

By Hopf theorem, there is a homotopy $h_t : \mathrm{cl}(\mathrm{V}_1) \to \mathrm{cl}(\mathrm{V}_3)$ constant on the boundary, satisfying $h_0 = (f_3^2)_{|\mathrm{cl}(\mathrm{V}_1)}$, $h_1(x) \neq x$. Now Lemma 6.1 gives a homotopy so that $f_0 = f$, f_t is constant on $bd(V_1)$ and $f_1^2(x) \neq x$ for $x \in \mathrm{cl}(\mathrm{V}_1)$. To see the last, we check the assumptions of Lemma 6.1 for $X = S^2$, $A = V_1$, $f = f_3$, $h_t = h_t$.

1. $\mathrm{cl}(\mathrm{V}_1) \cap \mathrm{f}_3(\mathrm{cl}(\mathrm{V}_1)) = \emptyset$, since the elements of both sets have different longitudes
2. $f_{3|\mathrm{cl}(\mathrm{V}_1)}$ is a homeomorphism, since $\mathrm{cl}(\mathrm{V}_1) \subset \mathrm{int}(\mathrm{S}^2_+)$ and f_3 is a homeomorphism on $int(S^2_+)$.
3. Follows from the definitions of both functions h_t.
4. $f_3(\mathrm{cl}(\mathrm{V}_1)) \cap \mathrm{h}_1(\mathrm{cl}(\mathrm{V}_1)) = \emptyset$, since $h_1(\mathrm{cl}(\mathrm{V}_1)) \subset \mathrm{V}_3$ and the last is disjointed from $f_3(\mathrm{cl}(\mathrm{V}_1)$.

In the consequence Lemma 6.1 gives a homotopy with carrier in $f(clV_1)$ from $f : S^2 \to S^2$ to a map f_4 satisfying $\mathrm{Fix}(f_4^2) \subset \mathrm{Fix}(f_4^2)\backslash(\mathrm{cl}(\mathrm{A}) \cup \mathrm{f}(\mathrm{cl}(\mathrm{A}))$. This implies $\mathrm{Fix}(f_4^2) = \mathrm{Fix}(f^2)\backslash\{w_0, w_2; a, f(a)\}$.

Lemma 6.1. *Let $f : X \to X$ be a continuous map of a topological space, $A \subset X$ an open subset. We assume that*

(1) $\mathrm{cl}(\mathrm{A}) \cap \mathrm{f}(\mathrm{cl}(\mathrm{A})) = \emptyset$
(2) *the restriction* $f_| : \mathrm{cl}(\mathrm{A}) \to f(\mathrm{cl}(\mathrm{A}))$ *is a homeomorphism*
(3) $h_t : \mathrm{cl}(\mathrm{A}) \to \mathrm{X}$ *is a homotopy satisfying:* $h_0 = (f^2)_{|\mathrm{cl}(\mathrm{A})}$, h_t *is constant on the boundary,* $h_1(a) \neq a$ *for all* $a \in \mathrm{cl}(\mathrm{A})$.
(4) $f(\mathrm{cl}(\mathrm{A})) \cap \mathrm{h}_1(\mathrm{cl}(\mathrm{A})) = \emptyset$

Then the formula

$$f_t(x) = \begin{cases} h_t(f^{-1}_{|\mathrm{cl(A)}}(x)) & \text{for } \mathrm{x} \in \mathrm{f(cl(A))} \\ f(x) & \text{for } \mathrm{x} \notin \mathrm{f(A)} \end{cases}$$

gives a homotopy from $f_0 = f$ to a map $f_1 : X \to X$ satisfying

(1) *f_1 is homotopic to f by a homotopy constant outside $f(\mathrm{cl(A)})$*
(2) *$f_1^2(a) \neq a$ for all $a \in \mathrm{cl(A)} \cup \mathrm{f(cl(A))}$*
(3) *$\mathrm{Fix}(f_1^2) \subset \mathrm{Fix}(f^2) \backslash (\mathrm{cl(A)} \cup \mathrm{f(cl(A))})$.*

Proof. We notice that the map f_t is correctly defined, since for $x \in bd(f(\mathrm{cl}$ $(\mathrm{A})))$ the upper formula gives

$$h_t f^{-1}_{|\mathrm{cl(A)}}(x) = h_0 f^{-1}_{|\mathrm{cl(A)}}(x) = f^2 f^{-1}_{|\mathrm{cl(A)}}(x) = f(x)$$

Now (1) follows straight from the formula.

To get (2) we fix $a \in \mathrm{cl(A)}$. Then $f_1^2(a) = f_1 f(a) = h_1 f^{-1}_{|\mathrm{cl(A)}} f(a) = h_1(a) \neq a$.

To show (3) we first prove that $\mathrm{Fix}(f_1^2) \subset \mathrm{Fix}(f^2) \backslash \mathrm{cl(A)}$.

By (2) it is enough to show that $\mathrm{Fix}(f_1^2) \subset \mathrm{Fix}(f^2)$. Let $f_1^2(x) = x$. If moreover no of the points $x, f(x)$ belongs to $\mathrm{cl}(A)$ then $f^2(x) = f_1^2(x) = x$. Otherwise we may assume that $x \in \mathrm{cl}(A)$. But now (2) implies $f_1^2(x) \neq x$ which is a contradiction. □

Proof of Theorem (3.2). Since $(-\pi/2, \pi/6] \times [-\epsilon, \epsilon] \subset S^2$ (carrier of the homotopy from f_0 to f_2 in Sect. 5) is disjointed from $f_3(\mathrm{cl}V_1)$ (carrier of the homotopy from f_0 to f_4), the map

$$\tilde{f}(x) = \begin{cases} f_2(x) & \text{for } x \in (-\pi/2; \pi/6] \times [-\epsilon, \epsilon] \\ f_4(x) & \text{for } x \in f_3(\mathrm{cl}V_1) \\ f_0(x) & \text{otherwise} \end{cases}$$

is defined.

It remains to show that $\mathrm{Fix}(\tilde{f}^2) = \{N\mathrm{Pole}, \mathrm{SPole}\}$. $\supset$ is evident. To prove $\subset$ we consider an orbit in $\mathrm{Fix}(\tilde{f}^2)$ which contains no pole. If the orbit is disjointed from $(-\pi/2, \pi/6) \times (-\epsilon, \epsilon)$ then $x = \tilde{f}^2(x) = f_4^2(x)$. But $\mathrm{Fix}(f_4^2) = \{N\mathrm{Pole}, \mathrm{SPole}, \mathrm{b}, \mathrm{f_0(b)}\}$ implies that the orbit coincides with $\{b, f_0(b)\}$. But $\tilde{f}$ and f_2 coincide in $(-\pi/2, \pi/6] \times [-\epsilon, \epsilon]$ and f_2 has no periodic points there. □

7. Lemmas

We consider the complex plane as the union of sectors $\mathbb{C} = \bigcup_{k=0}^{2n-1} S_k$ where $S_k = \{z; \frac{k\pi}{n} \leq \arg(z) \leq \frac{(k+1)\pi}{n}\}$. See Fig. 3 for $n = 4$.

In this section we will show

Lemma 7.1. (Compare Lemma 6.2 in [6]) For given $\rho > 0$ and $a > 1$ there exists a smooth map $K_1 : \mathbb{C} \to \mathbb{C}$ so that $K_1(z) = az$ for $|z| \geq \rho$ and

(1) K_1 maps each sector $S_k = \{[r, \phi]; \frac{k\pi}{2n} \leq \phi \leq \frac{(k+1)\pi}{2n}\}$ into itself,

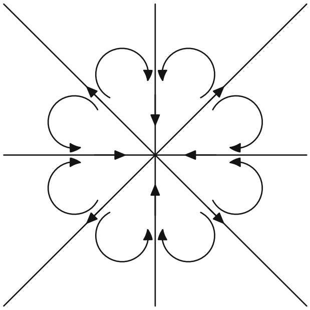

Figure 3. Vector field for $n = 4$

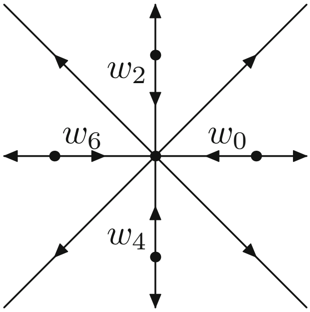

Figure 4. Zeroes of the vector field

(2) K_1 maps each half-line $L_k = \{[r,\phi] \in S^2; \frac{k\pi}{n} = \phi\}$ into itself,
(3) K_1 has exactly $n+1$ fixed points $w_0, w_2, \ldots, w_{2n-2}$ (lying on lines L_0 , L_2 ,..., L_{2n-2}, respectively) and $(0,0)$. See Fig. 4.
(4) K_1 has no periodic points different than $w_0, w_2, \ldots, w_{2n-2}$ and $(0,0)$
(5) $\mathrm{ind}(K_1^m; \omega_i) = -1$ for all $m \in \mathbb{N}$, $i = 0, 2, \ldots, 2n-2$.
(6) $\mathrm{ind}(K_1^m; 0) = 1 + n$

We are going to define map K_1. We consider vector field Φ given by Fig. 3.

Lemma 7.2. (Compare Lemma 6.1 in [6]) The time-1 map $\phi(v) := \Phi_1(v)$ satisfies.

(1) 0 is the unique fixed point
(2) $\phi(S_k) \subset S_k$ for each sector $S_k = \{z; \frac{k\pi}{n} \leq \arg(z) \leq \frac{(k+1)\pi}{n}\}$ for $k = 0, \ldots, 2n-1$.
(3) In particular ϕ maps each half-line $L_k = \{z; \arg(z) = \frac{k}{n}\}$ into itself.
(4) Points $0, z, \phi(z)$ belong to a line $\iff$ $z \in L_k = \{z; \arg(z) = \frac{k}{n}\}$ for a $k = 0, \ldots, 2n-1$. Moreover
$z \in (0, \phi(z)) \iff z \in L_k$ for k odd ;
$\phi(z) \in (0, z) \iff z \in L_k$ for k even .
(5) If $z \in intS_k$ then $\arg(\phi(z)) < \arg(z)$ $(\arg(\phi(z)) > \arg(z))$ for k- even (k- odd). □

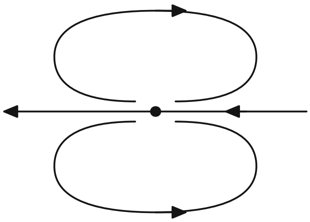

FIGURE 5. Vector field for $n = 1$

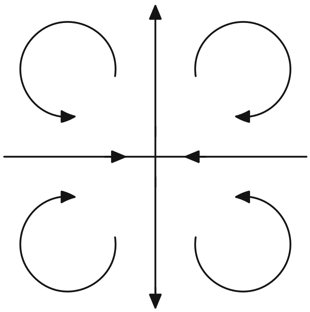

FIGURE 6. Vector field for $n = 2$

Let us fix two numbers $0 < \epsilon_1 < \epsilon_2$ and a smooth Urysohn function $\eta : [0, \infty) \to \mathbb{R}$ satisfying $\eta(t) = 1$ for $0 \le t < \epsilon_1$, $\eta(t) = 0$ for $t \ge \epsilon_2$ and moreover $t \le t'$ implies $\eta(t) \ge \eta(t')$.

We define a new vector field on $\mathbb{C}$ as the convex combination $K(z) = \eta(|z|) \cdot \Phi_1(z) + (1 - \eta(|z|)) \cdot az$, where $a > 1$.

We define K_1 as the time-1 map of the vector field K.

Proof of Lemma 7.1. The first two properties follow from Lemma 7.2, since the map Φ_1 preserves sectors S_k and half-lines L_k. To prove the third property we notice that the vectors $\phi(z)$, az are collinear $\iff$ $z \in L_k$ for a $k = 0, \ldots, 2n - 1$. Moreover they have the same direction for k odd and are opposite for k even. Now, for k odd, their convex combination never vanishes $K(z) \neq 0$. Similarly, for k even, K has exactly one zero for $\epsilon_1 \le |z| \le \epsilon_2$, $z \in L_k$, since η is nonincreasing. Now we prove (5). We notice that in each fixed point z_{2k} the map K_1 is expanding the line L_{2k} and is squeezing at the orthogonal direction (since so does Φ_1). Now fixed point index equals $(-1) \cdot (+1) = -1$. The same argument works for all iterations of K_1. To prove (6) we notice that the total index must be $+1$, hence $\mathrm{ind}(K_1^m; (0, 0)) = 1 - n(-1) = 1 + n$ for any $m \in \mathbb{N}$. □

Remark 7.3. For $n = 1$ Fig. 3 becomes Fig. 5. Now Fig. 5 and Lemma 7.1 give a map K_1 with a single additional fixed point with index -1.

Similarly for $n = 2$ we get the Fig. 6 and a map K_1 with two additional fixed points each of index -1.

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