J. Fixed Point Theory Appl. (2018) 20:20
https://doi.org/10.1007/s11784-018-0500-2
Published online February 8, 2018

Journal of Fixed Point Theory and Applications

Various extensions of Kannan's fixed point theorem

Jarosław Górnicki

Abstract. The aim of the paper is to give some extensions of Kannan's fixed point theorem. In particular, we give some criteria of the usual functional type for the convergence of iterations generated by a Kannan-type mapping to a fixed point of the mapping.

Mathematics Subject Classification. Primary 47H10, Secondary 54H25.

Keywords. Fixed point, complete metric space, Kannan mapping, approximating sequence, iterations, asymptotic regularity, control function.

1. Completeness

Let (X, d) be a metric space. A mapping $T : X \to X$ is called a *Kannan mapping* if there exists $K < \frac{1}{2}$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leqslant K \cdot \{d(x, Tx) + d(y, Ty)\}. \tag{1}$$

In 1968, Kannan [9] proved the following fixed point theorem, see [6].

Theorem 1.1. (Kannan) *Let (X, d) be a complete metric space and let $T : X \to X$ be a Kannan mapping. Then T has a unique fixed point $v \in X$ and for any $x \in X$ the sequence of iterates $\{T^n x\}$ converges to v.*

Using Kannan's mappings Subrahmanyam [16] proved the following characterization of complete metric spaces:

Theorem 1.2. *A metric space (X, d) is complete if and only if every Kannan mapping $T : X \to X$ has a fixed point.*

Here is a simple use of this result.

Example 1.3. Let $\mathbb{Q}$ be endowed with the Euclidean metric and $T : \mathbb{Q} \to \mathbb{Q}$ be a mapping defined by:

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$$\text{for } x \in \mathbb{Q} \text{ and } x < \sqrt{2} \text{ let } Tx \in \left\{ y \in \mathbb{Q} : y < \sqrt{2} \text{ and } \sqrt{2} - y < \frac{1}{4}(\sqrt{2} - x) \right\},$$

$$\text{for } x \in \mathbb{Q} \text{ and } x > \sqrt{2} \text{ let } Tx \in \left\{ y \in \mathbb{Q} : y > \sqrt{2} \text{ and } y - \sqrt{2} < \frac{1}{4}(x - \sqrt{2}) \right\}.$$

Since, $|x - Tx| > \frac{3}{4}|\sqrt{2} - x| > 0$ for $x \in \mathbb{Q}$, T is fixed point free.

Case 1. $a, b \in \mathbb{Q}$ and $a < \sqrt{2} < b$. Then, $|Ta - Tb| < \frac{1}{4}(b - a)$ and $|a - Ta| + |b - Tb| > \frac{3}{4}(b - a)$. Thus

$$|Ta - Tb| \leqslant \frac{1}{3} \cdot \{|a - Ta| + |b - Tb|\}.$$

Case 2. $a, b \in \mathbb{Q}$ and $a \leqslant b < \sqrt{2}$. Then, $|Ta - Tb| < \frac{1}{4}(\sqrt{2} - a)$ and $|a - Ta| + |b - Tb| > \frac{3}{4}(\sqrt{2} - a) + \frac{3}{4}(\sqrt{2} - b)$, so

$$|Ta - Tb| \leqslant \frac{1}{3} \cdot \{|a - Ta| + |b - Tb|\}.$$

Similarly, when $\sqrt{2} < b \leqslant a$.
Thus, $T : \mathbb{Q} \to \mathbb{Q}$ is Kannan mapping without a fixed point, so by Theorem 1.2, $(\mathbb{Q}, |\cdot|)$ is not complete metric space.

A Kannan-type mapping $T : X \to X$ such that

$$d(Tx, Ty) \leqslant \frac{1}{2} \cdot \{d(x, Tx) + d(y, Ty)\} \quad \text{for all} \quad x, y \in X,$$

in complete metric space (X, d) may not have a fixed point. It can be seen from the following example.

Example 1.4. Let $X = \mathbb{R}$ with metric $d_{0-1}(x, y) = \begin{cases} 0 \text{ if } x = y, \\ 1 \text{ if } x \neq y. \end{cases}$ Let $T : \mathbb{R} \to \mathbb{R}$ be a mapping defined by $Tx = x + 1$ for $x \in \mathbb{R}$. Then

$$d_{0-1}(Tx, Ty) \leqslant \frac{1}{2} \cdot \{d_{0-1}(x, Tx) + d_{0-1}(y, Ty)\} \quad \text{for all} \quad x, y \in \mathbb{R}$$

and T is fixed point free.

Analogically, even continuous Kannan-type mapping $T : X \to X$ such that

$$d(Tx, Ty) < \frac{1}{2} \cdot \{d(x, Tx) + d(y, Ty)\} \quad \text{for all} \quad x, y \in X \quad \text{with} \quad x \neq y,$$

in complete but noncompact metric space (X, d) may not have a fixed point. It can be seen from the following example (this is the answer to Question 2.4 from [6]).

Example 1.5. (G. Minak, personal communication, 2017). Let $X = \{1 + \frac{1}{n} : n = 1, 2, \ldots\}$ and define a metric $d(x, y) = \begin{cases} 0 \text{ if } x = y, \\ x + y \text{ if } x \neq y. \end{cases}$ Then, (X, d) is complete and noncompact. A mapping $T : X \to X$ define by $T(1 + \frac{1}{n}) =$

$1 + \frac{1}{n+1}$ is continuous and has no fixed point. Moreover, for $x = 1 + \frac{1}{n}$, $y = 1 + \frac{1}{m}$, we have

$$\begin{aligned}2 \cdot d(Tx, Ty) &= 2\Big(2 + \frac{1}{n+1} + \frac{1}{m+1}\Big)\\ &< 2 + \frac{1}{n} + \frac{1}{n+1} + 2 + \frac{1}{m} + \frac{1}{m+1} = d(x, Tx) + d(y, Ty),\end{aligned}$$

because $\frac{2}{k+1} < \frac{1}{k} + \frac{1}{k+1}$ for $k = 1, 2, \ldots$.

To ensure the existence of a fixed point for mappings of this type there are needed additional assumptions, see for example, Bogin [2], De Blasi [4], Górnicki [6]. These conditions are not discussed in this paper.

2. Approximating sequence

Let C be a nonempty subset of metric space (X, d) and $T : C \to C$ a mapping. Then, a sequence $\{x_n\}$ is said to be an *approximating fixed point sequence* of T if $d(x_n, Tx_n) \to 0$ as $n \to \infty$.

Brouwer [3] argues that only approximating fixed point sequences have a meaning for the intuitionist.

Theorem 2.1. *Let (X, d) be a metric space and let T map the closed subset $M \subset X$ into a compact subset $C \subset X$. Let T be a mapping such that there exists $K < 1$ satisfying (1). Then, T has a unique fixed point if and only if there exists an approximating fixed point sequence of T.*

Proof. Let $\{x_n\} \subset M$ be an approximating fixed point sequence of T. Since Tx_n in C, we may assume without loss of generality that $Tx_n \to y \in C$ as $n \to \infty$. By assumption, we also have $x_n \to y \in M$. Then

$$\begin{aligned}d(Ty, y) &\leqslant d(Ty, Tx_n) + d(Tx_n, y)\\ &\leqslant K \cdot \{d(y, Ty) + d(x_n, Tx_n)\} + d(Tx_n, y),\end{aligned}$$

and hence

$$d(Ty, y) \leqslant \frac{K}{1-K} d(x_n, Tx_n) + \frac{1}{1-K} d(Tx_n, y) \to 0$$

as $n \to \infty$, it follows $Ty = y$. Of course such the fixed point is exactly one. □

Obviously, the result holds for mapping $T : M \to C$ such that there exists $K < 1$ satisfying

$$d(Tx, Ty) \leqslant K \cdot \{d(x, Tx) + d(y, Ty) + d(x, y)\} \text{ for all } x, y \in M.$$

Remark 2.2. A mapping $T : X \to X$ such that

$$d(Tx, Ty) < d(x, Tx) + d(y, Ty) \text{ for all } x, y \in X \text{ and } x \neq y,$$

and there exists an approximating fixed point sequence of T, may not have a fixed point, see [6, Example 3.2].

3. Iterations

Very often, together with investigating a mapping $T : X \to X$, there is a need to consider the iterates $T^2 = T \circ T$, $T^3 = T^2 \circ T = T \circ T^2 = T \circ T \circ T, \ldots$. We use the notation $T^0 = I$, where I is the identity mapping on X. We always have $T^n \circ T^m = T^{n+m}$ for natural numbers $n, m = 1, 2, \ldots$.

If T is a Kannan mapping on a complete metric space (X, d) with constant K, then T^n, $n \geqslant 2$, satisfy the following condition

$$d(T^n x, T^n y) \leqslant K \cdot \left(\frac{K}{1-K}\right)^{n-1} \cdot \{d(x, Tx) + d(y, Ty)\} \text{ for all } x, y \in X,$$

and the unique fixed point of T is also the unique fixed point of T^n.

Consider now the situation in which $T : X \to X$ is not necessarily a Kannan mapping, but T^N is a Kannan mapping for some $N \geqslant 2$.

Example 3.1. [6] Let $X = [0, 1]$ be with usual metric and $T : [0, 1] \to [0, 1]$ be a mapping defined by $Tx = \frac{x}{3}$ for $0 \leqslant x < 1$ and $T1 = \frac{1}{6}$. T does not satisfy Kannan's condition because $|T0 - T\frac{1}{3}| = \frac{1}{2}\{|0 - T0| + |\frac{1}{3} - T\frac{1}{3}|\}$, and T is not continuous at $x = 1$. The mapping T^2 is defined by $T^2 x = \frac{x}{9}$ for $0 \leqslant x < 1$ and $T^2 1 = \frac{1}{18}$. Then, $d(T^2x, T^2y) \leqslant \frac{1}{9}(|x| + |y|)$ and $d(x, T^2x) + d(y, T^2y) \geqslant \frac{8}{9}(|x| + |y|)$. Thus

$$d(T^2x, T^2y) \leqslant \frac{1}{4} \cdot \{d(x, T^2x) + d(y, T^2y)\} \text{ for } x, y \in [0, 1],$$

so T^2 is Kannan mapping.

Therefore, we have a trivial lemma.

Lemma 3.2. *Let X be a nonempty set and $\mathcal{F}$ be a family of mappings*

$$\mathcal{F} = \{F : X \to X : \ F \text{ has a unique fixed point in } X\}.$$

If $T : X \to X$ is a mapping such that for some integer $N \geqslant 2$, $T^N \in \mathcal{F}$ then, T has a unique fixed point.

Hence, we have the following corollaries.

Corollary 3.3. (Kannan [9]) *Suppose (X, d) is a complete metric space and suppose $T : X \to X$ is a mapping such that for some positive integer $N \geqslant 2$, T^N is a mapping such that there exists $K < \frac{1}{2}$ satisfying for all $x, y \in X$,*

$$d(T^N x, T^N y) \leqslant K \cdot \{d(x, T^N x) + d(y, T^N y)\}.$$

Then, T has a unique fixed point.

We say $T : X \to X$ is *asymptotically regular* at x if $\lim_{n\to\infty} d(T^{n+1}x, T^n x) = 0$. If T is asymptotically regular at every $x \in X$, we simply say T is asymptotically regular.

Corollary 3.4. (Górnicki [6]) *Suppose (X, d) is a complete metric space and suppose $T : X \to X$ is a mapping such that for some positive integer $N \geqslant 2$, T^N is asymptotically regular and such that there exists $K < 1$ satisfying for all $x, y \in X$,*

$$d(T^N x, T^N y) \leqslant K \cdot \{d(x, y) + d(x, T^N x) + d(y, T^N y)\}.$$

Then, T has a unique fixed point.

Corollary 3.5. (De Blasi [4]) *Suppose $(H, \|\cdot\|)$ is a Hilbert space, $C \subset H$ is a nonempty weakly closed and suppose $T : C \to C$ is a mapping such that for some positive integer $N \geqslant 2$, $T^N : C \to C$ is continuous, asymptotically regular and satisfies for all $x, y \in C$,*

$$\|T^N x - T^N y\| \leqslant \|x - T^N x\| + \|y - T^N y\|.$$

Then, T has a unique fixed point.

Now, we prove the following

Lemma 3.6. *Let (X, d) be a metric space, $N \geqslant 2$ a positive integer and $K < \frac{1}{2}$. Let $T : X \to X$ be a mapping such that for all $x, y \in X$ we have*

$$d(T^N x, T^N y) \leqslant K \cdot \{d(x, Tx) + d(y, Ty)\}. \tag{2}$$

If there is an $x \in X$ such that $T^N x = x$, then x is a unique fixed point of T.

Proof. Let $x \in X$ and $T^N x = x$. Then, by (2),

$$d(x, Tx) = d(T^N x, T^{N+1} x) \leqslant K \cdot \{d(x, Tx) + d(Tx, T^2 x)\},$$

so

$$d(x, Tx) \leqslant \frac{K}{1-K} d(Tx, T^2 x).$$

Next,

$$d(Tx, T^2 x) = d(T^{N+1} x, T^{N+2} x) \leqslant K \cdot \{d(Tx, T^2 x) + d(T^2 x, T^3 x)\},$$

so

$$d(Tx, T^2 x) \leqslant \frac{K}{1-K} d(T^2 x, T^3 x),$$

etc. Similarly

$$d(T^{N-2} x, T^{N-1} x) \leqslant \frac{K}{1-K} d(T^{N-1} x, T^N x),$$

and finally

$$\begin{aligned} d(T^{N-1} x, T^N x) &= d(T^{N+(N-1)} x, T^{N+N} x) \\ &\leqslant K \cdot \{d(T^{N-1} x, T^N x) + d(T^N x, T^{N+1} x)\}, \end{aligned}$$

so

$$d(T^{N-1} x, T^N x) \leqslant \frac{K}{1-K} d(x, Tx).$$

But then

$$d(x, Tx) \leqslant \left(\frac{K}{1-K}\right)^N \cdot d(x, Tx).$$

Since $K < \frac{1}{2}$, $x = Tx$.

Assume $x, y \in X$ satisfy $Tx = x$ and $Ty = y$. Then $d(x, y) = d(T^N x, T^N y) \leqslant K \cdot \{d(x, Tx) + d(y, Ty)\} = 0$, so $x = y$. □

In this situation, it is obvious question: Does there exist a fixed point of T if T satisfies (2)? More generally conjecture is inspired by *Generalized Banach Contraction Conjecture* (see: [7,8,10,15]). Is that true?

Conjecture 3.7. *Let (X,d) be a complete metric space, $K<\frac{1}{2}$, and $T:X\to X$. Let J be a set of positive integers. Assume that for each pair $x,y\in X$,*

$$\inf\{d(T^ix,T^iy):i\in J\}\leqslant K\cdot\{d(x,Tx)+d(y,Ty)\}.$$

Then T has a fixed point.

Kannan's theorem is simply the case $J=\{1\}$.

Now, we give an extension of Lemma 3.6.

Lemma 3.8. *Let (X,d) be a metric space, J a set of positive integers, and $K<\frac{1}{2}$. Let $T:X\to X$ be a mapping such that for all $x,y\in X$ we have*

$$\inf\{d(T^ix,T^iy):i\in J\}\leqslant K\cdot\{d(x,Tx)+d(y,Ty)\}.$$

If there is an $x\in X$ such that $T^Nx=x$, then x is a unique fixed point of T.

Proof. (see [7, Lemma 1]). Note that for each integer $i\in\{0,1,\dots,N-1\}$ there is an integer $j_i\in J$ such that

$$d\left(T^{j_i}T^ix,T^{j_i}T^{i+1}x\right)\leqslant K\cdot\left\{d\left(T^ix,T^{i+1}x\right)+d(T^{i+1}x,T^{i+2}x)\right\}.$$

Since $T^Nx=x$, we can find a sequence $\{a_i:i=1,2,\dots\}\subset\{0,1,2,\dots,N-1\}$ such that

$$d(T^{a_i}x,T^{a_i+1}x)\leqslant K\cdot\{d(T^{a_{i-1}}x,T^{a_{i-1}+1}x)+d(T^{a_i}x,T^{a_i+1}x)\},$$

ie.

$$d(T^{a_i}x,T^{a_i+1}x)\leqslant\frac{K}{1-K}\cdot d(T^{a_{i-1}}x,T^{a_{i-1}+1}x)$$

as follows; define $a_0=0$, and for $i\geqslant 1$, apply $T^{j_{a_{i-1}}}$ to the pair $T^{a_{i-1}}x$ and $T^{a_{i-1}+1}x$. a_i is then defined as the remainder obtained when dividing $a_{i-1}+j_{a_{i-1}}$ by N. Since the a_i are contained in the finite set $\{0,1,2,\dots,N-1\}$, there are integers i and n such that $a_{i+n}=a_i$. But then,

$$d(T^{a_i}x,T^{a_i+1}x)=d(T^{a_{i+n}}x,T^{a_{i+n}+1}x)\leqslant\left(\frac{K}{1-K}\right)^n\cdot d(T^{a_i}x,T^{a_i+1}x).$$

Since $K<\frac{1}{2}$, $T^{a_i+1}x=T^{a_i}x$, so $T^{a_i}x$ is a fixed point of T. Note that $N-a_i>0$ and $T^{a_i}x$ is also a fixed point of T^{N-a_i}, which means that $T^{a_i}x=T^Nx=x$. So x is a fixed point of T.

Assume that $x,y\in X$ satisfy $Tx=x$ and $Ty=y$. Then there exists $j\in J$ such that $d(T^jx,T^jy)\leqslant K\cdot\{d(x,Tx)+d(y,Ty)\}$. Since x and y are fixed points of T, this implies that $d(x,y)\leqslant K\cdot\{d(x,Tx)+d(y,Ty)\}=0$. Hence, $x=y$. □

4. Localization

It may be the case that $T:X\to X$ is not Kannan's mapping on the whole space X, but rather Kannan's mapping on some neighbourhood of a given point. In this case, we have the following result.

Theorem 4.1. *Let (X, d) be a complete metric space and let $B(z, r) = \{x \in X : d(z, x) \leqslant r\}$, where $z \in X$ and $r > 0$. Let $T : B(z, r) \to X$ be a mapping such that there exists $K < \frac{1}{2}$ satisfying*

$$d(Tx, Ty) \leqslant K \cdot \{d(x, Tx) + d(y, Ty)\} \text{ for all } x, y \in B(z, r).$$

Further, assume that

$$d(z, Tz) \leqslant \Big(1 - \frac{3K}{1+K}\Big) r.$$

Then, T has a unique fixed point in $B(z, r)$.

Proof. Let $x \in B(z, r)$, then

$$\begin{aligned} d(z, Tx) &\leqslant d(z, Tz) + d(Tz, Tx) \\ &\leqslant d(z, Tz) + K \cdot \{d(z, Tz) + d(x, Tx)\} \\ &\leqslant (1 + K)d(z, Tz) + K \cdot \{d(x, z) + d(z, Tx)\}, \end{aligned}$$

so

$$(1 - K)d(z, Tx) \leqslant (1 + K)d(z, Tz) + Kd(x, z),$$

and

$$\begin{aligned} d(z, Tz) &\leqslant \frac{1+K}{1-K} d(z, Tz) + \frac{K}{1-K} d(x, z) \\ &\leqslant \frac{1+K}{1-K}\Big(1 - \frac{3K}{1+K}\Big) r + \frac{K}{1-K} r = r, \end{aligned}$$

and hence $T : B(z, r) \to B(z, r)$. Since $B(z, r)$ is a complete metric space, using Theorem 1.1, T has a unique fixed point $v \in B(z, r)$. □

5. Control function

We now consider some (important) generalization of Kannan theorem in which the constant $K < \frac{1}{2}$ is replaced by some real-valued control function. A presented idea is due to Geraghty [5].

Let $\mathcal{S}$ denote the class of functions which satisfy the simple condition

$$\mathcal{S} = \Big\{ f : (0, \infty) \to [0, \frac{1}{2}) : f(t_n) \to \frac{1}{2} \Rightarrow t_n \to 0 \text{ as } n \to \infty \Big\}.$$

We do not assume that f is continuous in any sense.

Theorem 5.1. *Let (X, d) be a complete metric space, let $T : X \to X$, and suppose there exists $f \in \mathcal{S}$ such that for each $x, y \in X$ with $x \neq y$,*

$$d(Tx, Ty) \leqslant f(d(x, y)) \cdot \{d(x, Tx) + d(y, Ty)\}. \tag{3}$$

Then, T has a unique fixed point $v \in X$ and for any $x \in X$ the sequence of iterates $\{T^n x\}$ converges to v.

Proof. Fix $x_0 \in X$ and let $x_n = Tx_{n-1}$, $n = 1, 2, \ldots$. Assume that there exists $p \in \mathbb{N}$ such that $T^p x_0 = T^{p+1} x_0$. Since $T^p x_0 = T(T^p x_0)$, so $T^p x_0$ is the fixed point of T. Therefore, suppose that $T^n x_0 \neq T^{n+1} x_0$ for all $n \geqslant 0$.
Step 1. $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$.

Since T satisfies (3), we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(T^{n+1}x_0, T^n x_0) \\ &\leqslant \frac{1}{2} \cdot \left\{ d(T^n x_0, T^{n+1}x_0) + d(T^{n-1}x_0, T^n x_0) \right\} \\ &= \frac{1}{2} \cdot \{ d(x_{n+1}, x_n) + d(x_n, x_{n-1}) \}, \end{aligned}$$

so

$$d(x_{n+1}, x_n) \leqslant d(x_n, x_{n-1}).$$

The sequence $\{d(x_{n+1}, x_n)\}$ is monotone decreasing and bounded below, so $\lim_{n\to\infty} d(x_{n+1}, x_n) = \gamma \geqslant 0$. Assume $\gamma \neq 0$. Then by (3),

$$d(x_{n+2}, x_{n+1}) \leqslant f(d(x_{n+1}, x_n)) \cdot \{ d(x_{n+1}, x_{n+2}) + d(x_n, x_{n+1}) \},$$

$$\frac{d(x_{n+2}, x_{n+1})}{d(x_{n+1}, x_{n+2}) + d(x_n, x_{n+1})} \leqslant f(d(x_{n+1}, x_n)), \quad n = 1, 2, \ldots.$$

Letting $n \to \infty$, we see that $\frac{1}{2} \leqslant \lim_{n\to\infty} f(d(x_{n+1}, x_n))$, and since $f \in \mathcal{S}$ this in turn implies $\gamma = 0$. This contradiction establishes step 1.
Step 2. $\{x_n\}$ is a Cauchy sequence.

Suppose $m > n$. By condition (3) and step 1, we get for $m > n$,

$$\begin{aligned} d(x_{n+1}, x_{m+1}) &\leqslant f(d(x_n, x_m)) \cdot \{ d(x_n, x_{n+1}) + d(x_m, x_{m+1}) \} \\ &\leqslant \frac{1}{2} \cdot \{ d(x_n, x_{n+1}) + d(x_m, x_{m+1}) \} \to 0, \end{aligned}$$

as $n \to \infty$, so $\{x_n\}$ is a Cauchy sequence.

Since X is complete and since $\{T^n x_0\}$ is a Cauchy sequence, $\lim_{n\to\infty} T^n x_0 = v \in X$. Then

$$\begin{aligned} d(Tv, v) &\leqslant d(Tv, Tx_n) + d(Tx_n, v) \\ &\leqslant f(d(v, x_n)) \cdot \{ d(v, Tv) + d(x_n, Tx_n) \} + d(x_{n+1}, v) \end{aligned}$$

and

$$d(Tv, v) \leqslant \frac{f(d(v, x_n))}{1 - f(d(v, x_n))} \cdot d(x_n, x_{n+1}) + \frac{1}{1 - f(d(v, x_n))} \cdot d(x_{n+1}, v) \to 0$$

as $n \to \infty$. Hence, $Tv = v$. It is obvious that v is unique. □

Let $\mathcal{U}$ denote the class of functions which satisfy the condition

$$\mathcal{U} = \left\{ f : (0, \infty) \to [0, \frac{1}{3}) : f(t_n) \to \frac{1}{3} \Rightarrow t_n \to 0 \text{ as } n \to \infty \right\}.$$

We do not assume that f is continuous in any sense.

Theorem 5.2. *Let (X, d) be a complete metric space, let $T : X \to X$, and suppose there exists $f \in \mathcal{U}$ such that for each $x, y \in X$ with $x \neq y$,*

$$d(Tx, Ty) \leqslant f(d(x, y)) \cdot \{ d(x, Tx) + d(y, Ty) + d(x, y) \}. \tag{4}$$

Then T has a unique fixed point $v \in X$ and for any $x \in X$ the sequence of iterates $\{T^n x\}$ converges to v.

Proof. Fix $x_0 \in X$ and let $x_n = Tx_{n-1}$, $n = 1, 2, \ldots$. Assume that there exists $p \in \mathbb{N}$ such that $T^p x_0 = T^{p+1} x_0$. Since $T^p x_0 = T(T^p x_0)$, so $T^p x_0$ is the fixed point of T. Therefore suppose that $T^n x_0 \neq T^{n+1} x_0$ for all $n \geqslant 0$.
STEP 1. $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$.

Since T satisfies (4), we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d\left(T^{n+1}x_0, T^n x_0\right) \\ &\leqslant \frac{1}{3} \cdot \left\{ d\left(T^n x_0, T^{n+1} x_0\right) + d\left(T^{n-1} x_0, T^n x_0\right) + d\left(T^n x_0, T^{n-1} x_0\right) \right\} \\ &= \frac{1}{3} \cdot \left\{ d(x_{n+1}, x_n) + 2d(x_n, x_{n-1}) \right\}, \end{aligned}$$

so

$$d(x_{n+1}, x_n) \leqslant d(x_n, x_{n-1}).$$

The sequence $\{d(x_{n+1}, x_n)\}$ is monotone decreasing and bounded below, so $\lim_{n\to\infty} d(x_{n+1}, x_n) = \gamma \geqslant 0$. Assume $\gamma \neq 0$. Then by (4),

$$d(x_{n+2}, x_{n+1}) \leqslant f(d(x_{n+1}, x_n)) \cdot \{d(x_{n+1}, x_{n+2}) + d(x_n, x_{n+1}) + d(x_{n+1}, x_n)\},$$
$$\frac{d(x_{n+2}, x_{n+1})}{d(x_{n+1}, x_{n+2}) + 2d(x_n, x_{n+1})} \leqslant f(d(x_{n+1}, x_n)), \quad n = 1, 2, \ldots.$$

Letting $n \to \infty$, we see that $\frac{1}{3} \leqslant \lim_{n\to\infty} f(d(x_{n+1}, x_n))$, and since $f \in \mathcal{U}$ this in turn implies $\gamma = 0$. This contradiction establishes step 1.
STEP 2. $\{x_n\}$ is a Cauchy sequence.

Suppose $m > n$. By condition (4) and step 1, we get for $m > n$,

$$\begin{aligned} d(x_{n+1}, x_{m+1}) &\leqslant f(d(x_n, x_m)) \cdot \{d(x_n, x_{n+1}) + d(x_m, x_{m+1}) + d(x_n, x_m)\} \\ &\leqslant \frac{1}{3} \cdot \{d(x_n, x_{n+1}) + d(x_m, x_{m+1}) + d(x_n, x_{n+1}) \\ &\quad + d(x_{n+1}, x_{m+1}) + d(x_{m+1}, x_m)\}, \end{aligned}$$

and

$$d(x_{n+1}, x_{m+1}) \leqslant \{d(x_{n+1}, x_n) + d(x_{m+1}, x_m)\} \to 0$$

as $n \to \infty$, so $\{x_n\}$ is a Cauchy sequence.

Since X is complete and since $\{T^n x_0\}$ is a Cauchy sequence, $\lim_{n\to\infty} T^n x_0 = v \in X$. Then

$$\begin{aligned} d(v, Tv) &\leqslant d(v, T^{n+1}x_0) + d(T^{n+1}x_0, Tv) \\ &\leqslant d\left(v, T^{n+1}x_0\right) + f\left(d(T^n x_0, v)\right) \\ &\quad \cdot \left\{ d\left(T^n x_0, T^{n+1} x_0\right) + d(v, Tv) + d\left(T^n x_0, v\right) \right\}, \end{aligned}$$

and

$$\begin{aligned} [1 - f(d(T^n x_0, v))] \cdot d(Tv, v) &\leqslant d(v, T^{n+1}x_0) + f(d(T^n x_0, v)) \\ &\quad \cdot \left\{ d\left(T^n x_0, T^{n+1} x_0\right) + d\left(T^n x_0, v\right) \right\}, \end{aligned}$$

and

$$d(Tv,v) \leqslant \frac{1}{1-f(d(T^n x_0,v))} \cdot d(v,T^{n+1}x_0) + \frac{f\left(d(T^n x_0,v)\right)}{1-f\left(d(T^n x_0,v)\right)} \cdot \left\{d\left(T^n x_0, T^{n+1}x_0\right) + d\left(T^n x_0, v\right)\right\} \to 0$$

as $n \to \infty$. Hence, $Tv = v$. Suppose u is another fixed point of T. Then

$$d(u,v) = d(Tu,Tv) \leqslant \frac{1}{3} \cdot \{d(u,Tu) + d(v,Tv) + d(u,v)\},$$

and

$$\frac{2}{3} d(u,v) \leqslant \frac{1}{3} \cdot \{d(u,Tu) + d(v,Tv)\} = 0,$$

so $d(u,v) = 0$. Hence, T has a unique fixed point $v \in X$, so for each $x \in X$ the sequence of iterates $\{T^n x\}$ converges to v. □

This theorem "lies between" Banach's theorem and Kannan's theorem. This is illustrated by the following example.

Example 5.3. Let $X = [0,1]$ be endowed with the Euclidean metric. Consider $Tx = \frac{x}{3}$ for $0 \leqslant x < 1$ and $T1 = \frac{1}{6}$. T does not satisfy Banach's theorem and T does not satisfy Kannan's condition. But T satisfies for all $x, y \in X$ the following condition

$$d(Tx,Ty) \leqslant f(d(x,y)) \cdot \{d(x,Tx) + d(y,Ty) + d(x,y)\},$$

where $f(t) = -\frac{t}{12} + \frac{1}{3}$ for $0 < t \leqslant 1$, $f(0) = \frac{1}{4}$, and $v = 0$ is the unique fixed point of T.

Other generalizations Kannan's fixed point theorem are discussed in [1,12–14].

6. Asymptotic regularity and control function

Let $\mathcal{V}$ denote the class of functions which satisfy the condition

$$\mathcal{V} = \{f : (0,\infty) \to [0,1) : f(t_n) \to 1 \Rightarrow t_n \to 0 \text{ as } n \to \infty\}.$$

We do not assume that f is continuous in any sense.

Remark 6.1. The class of Rakotch functions [11],

$$\{\alpha : (0,\infty) \to [0,1) : \ \alpha(t) \text{ is a decreasing function of } t\}$$

is in the class $\mathcal{V}$.

Theorem 6.2. *Let (X,d) be a complete metric space and let $T : X \to X$ be an asymptotically regular and continuous mapping. Suppose there exists $f \in \mathcal{V}$ such that for each $x, y \in X$ with $x \neq y$,*

$$d(Tx,Ty) \leqslant f(d(x,y)) \cdot \{d(x,Tx) + d(y,Ty) + d(x,y)\}. \tag{5}$$

Then, T has a unique fixed point $v \in X$ and for any $x \in X$ the sequence of iterates $\{T^n x\}$ converges to v.

Proof. Let $x \in X$ and define a sequence $\{x_n = T^n x\}$, $n = 0, 1, 2, \ldots$. Assume that there exists n_0 such that $T^{n_0}x = T^{n_0+1}x$, then $T^{n_0}x$ is a fixed point of T. Suppose $T^n x \neq T^{n+1}x$ for all $n \geqslant 0$.

Step 1. $\{x_n\}$ is a Cauchy sequence.

Assume $\limsup_{m,n\to\infty} d(x_n, x_m) > 0$. By triangle inequality and (5),

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{m+1}) + d(x_{m+1}, x_m) \\ &\leqslant d(x_n, x_{n+1}) + f(d(x_n, x_m)) \\ &\quad \cdot \{d(x_n, x_{n+1}) + d(x_m, x_{m+1}) + d(x_n, x_m)\} + d(x_{m+1}, x_m), \end{aligned}$$

so

$$\frac{d(x_n, x_m)}{d(x_n, x_{n+1}) + d(x_m, x_{m+1})} \leqslant \frac{1 + f(d(x_n, x_m))}{1 - f(d(x_n, x_m))}.$$

Under the assumption $\limsup_{m,n\to\infty} d(x_n, x_m) > 0$, the asymptotic regularity now implies

$$\limsup_{n,m\to\infty} \frac{1 + f(d(x_n, x_m))}{1 - f(d(x_n, x_m))} = +\infty,$$

from which

$$\limsup_{n,m\to\infty} f(d(x_n, x_m)) = 1.$$

But since $f \in \mathcal{V}$ this implies $\limsup_{m,n\to\infty} d(x_n, x_m) = 0$ which is a contradiction.

Step 2. Existence and uniqueness of fixed points.

Since X is complete and since $\{x_n = T^n x\}$ is a Cauchy sequence, $\lim_{n\to\infty} T^n x = v \in X$. Since T is continuous, $Tv = v$.

If $Tv = v$, $Tu = u$ and $d(u, v) > 0$, then by (5), we have

$$\begin{aligned} d(u, v) = d(Tu, Tv) &\leqslant f((u, v)) \cdot \{d(u, Tu) + d(v, Tv) + d(u, v)\} \\ &\leqslant f(d(u, v)) \cdot d(u, v), \end{aligned}$$

so

$$1 \leqslant f(d(u, v)),$$

which is a contradiction. Hence, T has a unique fixed point and for each $x \in X$ the sequence of iterates $\{T^n x\}$ converges to v. □

Corollary 6.3. *Let (X, d) be a complete metric space and let $T : X \to X$ be an asymptotically regular and continuous mapping. Suppose there exists $f \in \mathcal{V}$ such that for each $x, y \in X$,*

$$d(Tx, Ty) \leqslant f(d(x, y)) \cdot \{d(x, Tx) + d(y, Ty))\}.$$

Then T has a unique fixed point $v \in X$ and for any $x \in X$ the sequence of iterates $\{T^n x\}$ converges to v.

Acknowledgements

The author is grateful to the referee for some useful comments and suggestions.

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Jarosław Górnicki
Department of Mathematics and Applied Physics
Rzeszów University of Technology
P.O. Box 8535-959 Rzeszow
Poland
e-mail: `gornicki@prz.edu.pl`