J. Fixed Point Theory Appl. (2018) 20:23
https://doi.org/10.1007/s11784-018-0494-9
Published online February 8, 2018

Journal of Fixed Point Theory and Applications

Fixed point theorems and L^*-operators

W. Kulpa, A. Szymanski and M. Turzański

Abstract. Within the framework of spaces admitting special L^*-operators (such as continuous or L^*_n-operators) we prove fixed point theorems (of Brouwer or Schauder type) and discuss some related issues (e.g. the existence of symmetric equilibria).

Mathematics Subject Classification. Primary 46A19, 54H25; Secondary 46A03, 47H10.

Keywords. L^*-operator, simplex, sonvexity.

1. Introduction

In what follows, $\mathbf{R}$ denotes the space of real numbers, $\mathbf{R}^n$ is the n-dimensional Euclidean topological vector space, and $\mathbf{e}^i$ is the ith basic unit vector in $\mathbf{R}^n$, $n = 1, 2, \ldots$

For any $\varnothing \neq A \subseteq N = \{1, 2, \ldots, n\}$, let

$$\Delta^A = \left\{ (x_1, x_2, \ldots, x_n) : \sum_{i=1}^{n} x_i = 1,\ x_i \geq 0 \text{ for all } i, \text{ and } x_i = 0 \text{ for all } i \notin A \right\}.$$

Thus Δ^N is the *canonical* $(n-1)$-*dimensional simplex* in $\mathbf{R}^n$, and Δ^A is a $(|A|-1)$-*dimensional face* of that simplex.

The KKM Theorem refers to the following celebrated theorem due to Knaster et al. [5]:

Theorem 1. *If F_i, $i \in N$, are closed subsets of Δ^N such that $\Delta^A \subseteq \bigcup\{F_i : \varnothing \neq i \in A\}$ for each $\varnothing \neq A \subseteq N$, then $\bigcap\{F_i : i \in N\} \neq \varnothing$.*

The contrapositive version of the KKM theorem (i.e., in the form of the implication $\sim q \rightarrow \sim p$, with $\sim p$ denoting the negation of p), where the closed sets F_i have been replaced by their complements U_i, takes the following form:

(KKM). If $U_1, \ldots, U_n$ are open subsets of Δ^N that cover Δ^N, then there exists $A \subseteq N$ such that $\Delta^A \cap \bigcap_{i \in A} U_i \neq \emptyset$.

The (KKM) version of the original KKM-theorem can be restated in two more equivalent forms.

(KK) If $\{V_i : i = 1, 2, \ldots, n\}$ is an open cover of a topological space X and $f : \Delta^N \to X$ is a continuous function, then there exists $B \subseteq N$ such that $f\left(\Delta^B\right) \cap \bigcap \{V_i : i \in B\} \neq \emptyset$.

(K) If A is a finite subset of a topological vector space X and $\{V_x : x \in A\}$ is an open cover of X, then there exists $B \subseteq A$ such that $coB \cap \bigcap\{V_x : x \in B\} \neq \emptyset$, where coB stands for the convex hull of B.

Clearly (KKM) implies (KK) by taking $U_i = f^{-1}(V_i)$. To see that (KK) implies (K), let $A = \{x_1, x_2, \ldots, x_n\}$ and let $f : \Delta^N \to X$ be given by $f\left(\sum_{i=1}^{n} \lambda_i \mathbf{e}^i\right) = \sum_{i=1}^{n} \lambda_i x_i$. Then f is a continuous and since $f\left(\Delta^B\right) = co\{x_i : i \in B\}$, we are done. Finally, (K) implies (KKM) since KKM is a special version of (K) with $X = \mathbf{R}^n$, $A = \left\{\mathbf{e}^i : i = 1, 2, \ldots, n\right\}$, and $V_i = U_i \cup \left(\mathbf{R}^n - \Delta^A\right)$.

In [6], Kulpa derived statement (K) from Brouwer's Fixed Point Theorem and referred to it as The Theorem on Indexed Families. Statement (K) has served as a prototype for L^*-operators [9], which we define next.

For any non-empty set E, let $\langle E \rangle$ and $\exp(E)$ denote the collection of all non-empty finite subsets of E and the collection of all subsets of E, respectively.

An L^*-*operator* on a topological space X is any function $\Lambda : \langle X \rangle \to \exp(X)$ satisfying the following condition:

$(*)$ If $A \in \langle X \rangle$ and $\{U_x : x \in A\}$ is an open cover of X, then there exists $B \subseteq A$ such that $\Lambda(B) \cap \bigcap \{U_x : x \in B\} \neq \emptyset$.

Statement (K) just asserts that the convex hull operator on any topological vector space is an L^*-operator on that space. However, unlike the *co* operator, the existence of an L^*-operator on a topological space does not entail imposing any additional algebraic structure on that space. It is a topological property.

A topological space endowed with an L^*-operator is going to be referred to as an L^*-*space*. Those spaces constitute a common generalization of Kulpa's simplicial structures [6] and Park's–Ben Mechaiekh et al. L-stuctures[1] [1].

In the setting of L^*-spaces, an important feature attributed exclusively to simplices can be utilized without any involvement of simplices. In this way, proving various types of fixed point theorems (of Tychonoff or Schauder type), along with a version of Nash's equilibrium theorem, and generalization of the Maynard-Smith theorem has become achievable within L^*-spaces (see [7–10]). Since Park's partial KKM spaces are closely related to L^*-spaces, many results obtained by S. Park in his development of the KKM theory carry out to L^*-spaces (see [18–20, 22–29]).

---

[1] An L-*structure* on a topological space X is given by a function $\Gamma : \langle X \rangle \to \exp(X)$ such that for every $A \in \langle X \rangle$ there exists a continuous function $f^A : \Delta(A) \to \Gamma(A)$ so that $f^A(\Delta(B)) \subseteq \Gamma(B)$ for all $B \subseteq A$.

Simplicial structures and L-structures coincide if Γ is a T_1 operator (cf. [30]).

In [12], Kulpa presented versions of some classical theorems valid within L^*-spaces. The present paper provides further account of such possible generalizations. We prove two classical fixed-point theorems, that of Brouwer, Theorem 2, and of Schauder, Theorem 3, in the L^*-spaces realm. We generalize the theorem on signatures from [12] and derive from it the existence of symmetric equilibria (Theorems 4 and 5). We also introduce and discuss L_n^*-operators (the convex hull operator on $\mathbf{R}^n$ is an instance of such an operator). We prove a new fixed point theorem and a version of the classical Helly's theorem for spaces that admit L_n^*-operators.

2. L^*-operators and fixed point theorems

L^*-operators can be defined on arbitrary topological space: for a given space X, set $\Lambda(A)$ to be any dense subset of X. Notice also that if Λ is an L^*-operator on the space X and $\Upsilon : \langle X\rangle \to \exp(X)$ verifies that $\Lambda(A) \cap \Upsilon(A)$ is a dense subset of $\Lambda(A)$ for each $A \in \langle X\rangle$, then Υ is also an L^*-operator on X.

Those examples, notwithstanding being trivial and not interesting, witness at least that L^*-operators do not have to be monotone operators nor that A has to be a subset of $\Lambda(A)$. They also indicate that the property $(*)$ alone is not sufficient to get any kind of noteworthy properties of topological spaces that admit L^*-operators. More adequate L^*-operators to work with and topological spaces admitting them will be provided in the next sections.

For the following exposition of L^*-operators we will introduce some technicalities first:

For a relation $S \subseteq X \times Y$, we write xSy to mean that $(x, y) \in S$. We set $S_x = \{y \in Y : xSy\}$ and $S^y = \{x \in X : xSy\}$. Notice that $y \in S_x$ iff $x \in S^y$. Now, consider the following:

$(**)$ If $S \subseteq X \times X$, $A \in \langle X\rangle$, and $\{S_x : x \in A\}$ is an open cover of X, then there exist $B \subseteq A$ and $y \in \Lambda(B)$ such that $B \subseteq S^y$.

Proposition 1. *The conditions* $(*)$ *and* $(**)$ *are equivalent.*

Proof. For the sake of completeness we provide a simple proof.

$(\Rightarrow)$ Observe that if $y \in \Lambda(B) \cap \bigcap\{S_x : x \in B\}$, then $y \in \Lambda(B)$ and $B \subseteq S^y$.

$(\Leftarrow)$ Let $S \subseteq X \times X$ be any relation such that $S_x = U_x$ for each $x \in A \in \langle X\rangle$. Let $y \in \Lambda(B)$ and $B \subseteq S^y$ for some $B \subseteq A$. Then $y \in S_x = U_x$ for each $x \in B$. Thus $y \in \Lambda(B) \cap \bigcap\{U_x : x \in B\}$. □

Example 1. Let $\mathcal{P}$ be an open family in a space X and let $g : X \to X$ be continuous. We set: xSy if $\exists_{U\in\mathcal{P}}\ x, g(y) \in U$. Then $S_x = g^{-1}(star(x, \mathcal{P}))$ and $S^y = star(g(y), \mathcal{P})$ for each $x, y \in X$. Assume that X is endowed with an L^*-operator Λ. By $(**)$, if $\mathcal{P}$ is finite and $g(X)$ is covered by $\mathcal{P}$, then for any selection of points $x_U \in U \in \mathcal{P}$, there exist $B \subseteq \{x_U : U \in \mathcal{P}\}$ and $y \in \Lambda(B)$ such that $B \subseteq star(g(y), \mathcal{P})$.

Example 2. Let N be a neighborhood assignment on X, i.e., N is a function on X such that $N(x)$ is an open neighborhood of x for each $x \in X$,. Suppose that $g : X \to X$ is continuous. We set: xSy if $g(y) \in N(x)$. Then $S_x = g^{-1}(N(x))$; Assuming additionally that N is symmetric (i.e., $y \in N(x)$ iff $x \in N(y)$) we get that $S^y = N(g(y))$ for each $x, y \in X$.

Remark 1. Neighborhoods assignments that are symmetric can be defined, e.g., by means of any real valued continuous symmetric function $\mu(x, y)$: Fix any real number ε and set $N(x) = \{y : \mu(x, y) < \varepsilon\}$.

Let Λ be an L^*-operator on the space X. A set $C \subseteq X$ is called *convex* if $\Lambda(B) \subseteq C$ for each $B \in \langle C\rangle$; the family of all such convex sets is going to be denoted by $\mathcal{CON}(X, \Lambda)$.

Example 3. Let $f : X \times X \to \mathbf{R}$ be continuous. We set: xSy if $f(x, y) > f(y, y)$. For each $x \in X$, the function $g_x : X \to \mathbf{R}$ defined by $g_x(y) = f(x, y) - f(y, y)$ is continuous. Then $S_x = g_x^{-1}((0, \infty))$ and so S_x is open for each $x \in X$. By the similar argument, $S^y = \{x \in X : f(x, y) > f(y, y)\}$ is open (and convex if f is assumed to be quasi-concave) for each $y \in X$.

Proposition 2. *Let N be a symmetric neighborhood assignment on a space X endowed with an L^*-operator such that $N(x)$ is convex for each $x \in X$. If $g : X \to X$ is continuous and $g(X)$ can be covered by finitely many $N(x)$ -s, then there exists $a \in X$ such that $g(a) \in N(a)$.*

Proof. Consider the relation xSy if $g(y) \in N(x)$. Let Λ be the L^*-operator on X. In the light of Example 2 and utilizing $(**)$, there exist a finite set B and $a \in \Lambda(B)$ such that $B \subseteq S^a$. Since $S^a = N(g(a))$ is convex, $a \in \Lambda(B) \subseteq N(g(a))$. Since N is symmetric, $g(a) \in N(a)$. □

Theorem 2. (Brouwer Fixed Point Theorem) *Let X be space with an L^*-operator Λ satisfying the following condition:*
$\oplus$ For each finite open cover $\mathcal{P}$ of X there exists a symmetric and convex neighborhood assignment N on X so that $\{N(x) : x \in X\}$ is a refinement of $\mathcal{P}$.
Then each continuous function $g : X \to X$ so that $g(X)$ is contained in a compact subset of X has a fixed point.

Proof. Suppose otherwise and let $g : X \to X$ be continuous, $g(X)$ is contained in a compact subset of X and yet g has no fixed points. Then each point $x \in X$ has an open neighborhood U_x such that $U_x \cap g^{-1}(U_x) = \varnothing$. Pick finitely many among them that cover the closure of $g(X)$. Those sets together with the complement of the closure of $g(X)$ constitute a finite open cover $\mathcal{P}$ of X. By $\oplus$, there exists a symmetric and convex neighborhood assignment N on X so that $\{N(x) : x \in X\}$ is a refinement of $\mathcal{P}$. Since $g(X)$ can be covered by finitely many $N(x) - s$, by Proposition 2, there exists $a \in X$ such that $g(a) \in N(a)$. Since $N(a)$ intersects $g(X)$, $N(a)$ must be contained in one of U_x's. But then $g(a) \in g(U_x) \cap U_x$; a contradiction. □

Let $\langle V\rangle^{\leq n}$ denote the collection of all subsets of V of size at most n. If $\Phi : \langle X\rangle \to \exp(X)$ and n is a natural number, then Φ is said to be *n-continuous at a point $p \in X$* if each open neighborhood U of p contains a neighborhood V of p verifying that $\Phi(A) \subseteq U$ whenever $A \in [V]^{\leq n}$; Φ is *n-continuous on a set* Y if Λ is n-continuous at each point of Y.

We say that Φ is *continuous at a point* $p \in X$ if each open neighborhood U of p contains a neighborhood V of p verifying that $\Phi(A) \subseteq U$ whenever $A \in \langle V \rangle$. We say that Φ is *continuous on a set* Y if Λ is continuous at each point of Y.

The following two propositions establish continuity properties of the convex hull operator:

Proposition 3. (cf. e.g. [10]) *The convex hull operator on any topological vector space is n-continuous for all* $n = 1, 2, \ldots$

Proof. Let X be a topological vector space and let U be an open neighborhood of 0. Take first an open set W such that $0 \in W + W \subseteq U$, and then, take an open and balanced set V with $0 \in V \subseteq W$. It shows that the convex hull operator in X is 2-continuous, and, by induction, that it is n-continuous for all $n = 1, 2, \ldots$ □

Proposition 4. *The convex hull operator on a topological vector space is continuous if and only if the space is locally convex.*

Proof. Only the necessity of the statements requires some argument. So let X be a topological vector space with the continuous convex hull operator and let U be an open neighborhood of $\mathbf{0}$. If V is an open neighborhood of $\mathbf{0}$ verifying that $co(A) \subseteq U$ whenever $A \in \langle V \rangle$, then $co\,(V) \subseteq U$. Hence the interior of $co\,(V)$ is as required for $\mathbf{0}$ and by translation, for any point of X. □

Let us point out that since $X = L_p\,([0,1])$, the space of all Lebesgue integrable real functions on the interval $[0,1]$ with $\int_0^1 |f\,(x)|^p\,dx < \infty$ and $0 < p < 1$, is not locally convex, the convex hull operator is not continuous at $\mathbf{0}$ (hence it is continuous at no point) but it is n-continuous for all $n = 1, 2, \ldots$ (cf. [10]).

Theorem 3. (Brouwer–Schauder Fixed Point Theorem) *Let X be a space with an L^*-operator Λ and let $g : X \to X$ be continuous so that $g\,(X)$ is contained in a compact subset C of X. If Λ is continuous at each point of C, then g has a fixed point.*

Proof. Suppose, contrary to our claim, that $g(x) \neq x$ for each $x \in X$. For each $x \in X$, pick an open neighborhood W_x of x such that $W_x \cap g\,(W_x) = \emptyset$. Without loss of generality, we may assume that $g\,(X)$ is a dense subset of C. Next, for each $x \in C$, pick an open neighborhood V_x of x so that $V_x \subseteq W_x$ and $\Lambda(A) \subseteq W_x$ provided $A \in \langle V_x \rangle$. There exists an open finite covering $\mathcal{U}$ of the compact set C that constitutes a star-refinement of the family $\{V_x : x \in C\}$ on the compact set C (cf. Engelking [4, p. 377]), i.e., for each $y \in C$ there exists $x \in C$ such that $star\,(y,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : y \in U\} \subseteq V_x$. Choose points $x_U \in U \cap g\,(X)$ in each $U \in \mathcal{U}$. As was noticed in Example 1, there exist $B \subseteq \{x_U : U \in \mathcal{U}\}$ and $y \in \Lambda\,(B)$ such that $B \subseteq star\,(g\,(y)\,,\mathcal{U})$. Since

$star\,(g\,(y)\,,\mathcal{U}) \subseteq V_x$, $B \in \langle V_x \rangle$ and so $y \in \Lambda\,(B) \subseteq W_x$. Hence both y and $g\,(y)$ belong to W_x; a contradiction. □

In the theorem, above, the continuity assumption for Λ is essential. In [6], a simplicial compact space without the fixed point property is constructed; however, the space is locally convex at each point but one.

Corollary 1. *Let Λ be an L^*-operator on a compact Hausdorff space X and let $\mathcal{M}$ be a family of continuous function from $X \times X$ into $[0, \infty)$ verifying the following properties:*

(1) *If $\varrho \in \mathcal{M}$, $p \in X$, and $r > 0$, then the pseudoball $B_\varrho\,(p, r) = \{x \in X : \varrho(x, p) < r\}$ is convex;*
(2) *$\varrho\,(x, x) = 0$ for each $x \in X$ and $\varrho \in \mathcal{M}$;*
(3) *For each two distinct points $x, y \in X$ there is $\varrho \in \mathcal{M}$ with $\varrho\,(x, y) > 0$. Then any continuous function $h : X \to X$ has a fixed point.*

Proof. By compactness, the family

$$\left\{\bigcap \{B_\varrho\,(p,r) : \varrho \in F\} : p \in X \text{ and } r > 0 \text{ and } F \in Fin\,(\mathcal{M})\right\}$$

is an open base in X that consists of convex sets. Hence Λ is continuous and the corollary follows from Theorem 3. □

In Corollary 1, if $\mathcal{M} = \{\|\|\}$, we get the classical Schauder fixed point theorem.

3. Signatures

The main result of this section is a theorem called here the *Theorem on Signatures* (Theorem 4). The notion of signature along with the forthcoming Theorem on Signatures were first introduced and discussed in [10], but only in the context of simplicial spaces. In [12], Kulpa proved it in the context of L^*-spaces and in the case $h = id$. We derive from the Theorem on Signatures the existence of symmetric equilibrium points on compact Hausdorff L^*-spaces (see Theorem 4), which in turn, enables getting a proof of the Separating Hyperplane Theorem that is surprisingly simple and elegant (see Remark 2(b)).

Corollary 1 is sort of a prototype for the signature theorem. In our terminology, a signature is going to be a family of functions (like $\mathcal{M}$, above) that satisfies the first two (adequately modified) conditions. Unlike the original families, signatures may consist of functions defined on the product of two arbitrary topological spaces. This imposes the necessity of modifying property (2). Towards this goal, we introduce the property of *contingency at* 0, which is defined as follows:

Let $\mathcal{M}$ be a family of non-negative real functions defined on the product of two sets $X \times Y$. We say that the family $\mathcal{M}$ is (Y-)*contingent at* 0 if for each finite subfamily F of $\mathcal{M}$, for each $\varepsilon > 0$, and for each $y \in Y$ there exists $a \in X$ such that $\mu\,(a, y) < \varepsilon$ for each $\mu \in \mathcal{M}$.

Clearly, if $\mathcal{M}$ verifies condition (2) of Theorem 1, then $\mathcal{M}$ is $(X$-*) contingent at* 0. In case X is a compact Hausdorff space and each function $\mu \in \mathcal{M}$ is continuous, to be $(Y$-*)contingent at* 0 for $\mathcal{M}$ is equivalent to

($\bigstar$) For each $y \in Y$ there exists $a \in X$ such that $\mu(a, y) = 0$ for each $\mu \in \mathcal{M}$.

In the equivalence ($\bigstar$), compactness of X is necessary (see Example 4).

Example 4. Let $\mu : X \times Y \to [0, \infty)$ be such that for each $\varepsilon > 0$ and for each $y \in Y$ the pseudoball $A(y; \varepsilon) = \{x \in X : \mu(x, y) < \varepsilon\}$ is non-empty. Then $\mathcal{M} = \{\mu\}$ is $(Y$-*)contingent at* 0.

Example 5. Let $X_\alpha, \alpha \in A$, be a compact Hausdorff space and let $X = \prod_{\alpha \in A} X_\alpha$ be the product of the spaces X_α. Suppose that for each $\alpha \in A$, $f_\alpha : X \to \mathbf{R}$ is a continuous real function on X. We define $\mu_\alpha : X \times X \to [0, \infty)$, $\alpha \in A$, by setting $\mu_\alpha(x, y) = -f_\alpha(N_\alpha(x, y)) + \sup_{w \in X} f_\alpha(N_\alpha(w, y))$,where $N_\alpha : X \times X \to X$ denotes the α^{th} Nash projection: $N_\alpha(x, y) = z = (z_\xi)$, where $z_\xi = \begin{cases} y_\xi \text{ if } \xi \neq \alpha \\ x_\alpha \text{ if } \xi = \alpha \end{cases}$.
The family $\mathcal{M} = \{\mu_\alpha : \alpha \in A\}$ consists of non-negative continuous real functions on the space $X \times X$. We shall show that $\mathcal{M}$ is contingent at 0 by verifying ($\bigstar$). Pick $y \in X$. For each $\alpha \in A$, let $a^\alpha = \left(a_\xi^\alpha\right)$ be such that $\mu_\alpha(a^\alpha, y) = 0$. If $a = (a_\alpha^\alpha)$, then since $N_\alpha(a, y) = N_\alpha(a^\alpha, y)$, $\mu_\alpha(a, y) = 0$ for each $\alpha \in A$.

Let Λ *be an* L^*-operator on a space X. A collection $\mathcal{M}$ of continuous functions $\mu : X \times Y \to [0, \infty)$ is said to be a *signature* on $X \times Y$ if:

(i) For each $\varepsilon > 0$ and for each $y \in Y$ the pseudoball $A(y; \varepsilon) = \{x \in X : \mu(x, y) < \varepsilon\}$ is convex. [2]

(ii) $\mathcal{M}$ is $(Y$-*)contingent at* 0.

Theorem 4. (Theorem on Signatures) *Let* Λ *be an* L^**-operator on a compact Hausdorff space* X*. Let* $\mathcal{M}$ *be a signature on* $X \times Y$*. Then for each continuous function* $h : X \to Y$ *there exists* $a \in X$ *such that* $\mu(a, h(a)) = 0$ *for all* $\mu \in \mathcal{M}$*.*

Proof. Fix $\varepsilon > 0$ and a finite subset F of $\mathcal{M}$. For $x \in X$ and $y \in Y$ we set $A(y) = \{x \in X : \mu(x, y) < \varepsilon \text{ for each } \mu \in F\}$ and $B(x) = \{y \in Y : \mu(x, y) < \varepsilon \text{ for each } \mu \in F\}$. Observe that $x \in A(y)$ iff $y \in B(x)$, and therefore, since $A(y)$ is non-empty for each $y \in Y$, $\{B(x) : x \in X\}$ is an open cover of Y. We claim that there exists $a \in X$ such that $\mu(a, h(a)) < \varepsilon$ for each $\mu \in F$.

Proof of the claim. Let $A \in \langle X \rangle$ be such that $\{B(x) : x \in A\}$ is a finite cover of the compact space $h(X)$. There exist $B \subseteq A$ and a point $w \in X$ such that $w \in \Lambda(B) \cap \bigcap \left\{h^{-1}(B(x)) : x \in B\right\}$. Since $h(w) \in$

[2] In terms of more familiar notions, condition (i) can be restated as follows. *Let* Λ *be an* L^*-operator on X. Recall that a real function $f : X \to \mathbf{R}$ is said to be *quasi-convex* (resp. *quasi-concave*) if the set $\{x \in X : f(x) < r\}$ (resp., if the set $\{x \in X : f(x) > r\}$) is convex for each $r \in \mathbf{R}$. Thus (i) asserts that $\mu(., y)$ is *quasi-convex*.

$\bigcap\{h^{-1}(B(x)) : x \in B\}$, $B \subseteq A(h(w))$. Since $A(h(w))$ is convex, $w \in \Lambda(B) \subseteq A(h(w))$; the claim is proved.

For $\mu \in \mathcal{M}$ and $\varepsilon > 0$, let $K(\mu,\varepsilon) = \{x \in X : \mu(x,h(x)) \leq \varepsilon\}$. Every set $K(\mu,\varepsilon)$ is closed in X and, by the claim, the family $\{K(\mu,\varepsilon) : \mu \in \mathcal{M}$ and $\varepsilon > 0\}$ is centered. By compactness of X, $\bigcap\{K(\mu,\varepsilon) : \mu \in \mathcal{M}$ and $\varepsilon > 0\} \neq \emptyset$. Thus any $a \in \bigcap\{K(\mu,\varepsilon) : \mu \in \mathcal{M}$ and $\varepsilon > 0\}$ verifies that $\mu(a,h(a)) = 0$ for each $\mu \in \mathcal{M}$. □

Here is an alternative proof of Corollary 1 as a corollary to the above theorem:

Proof. Theorem on Signatures yields a point $a \in X$ such that $\mu(a,h(a)) = 0$ for all $\mu \in \mathcal{M}$. Since $\mathcal{M}$ is separating, $a = h(a)$. □

A point $a \in X$ is said to be a *symmetric equilibrium point for a function* $f : X \times X \to \mathbf{R}$ if $f(x,a) \leq f(a,a)$ for each $x \in X$. It is a Nash equilibrium point in the symmetric game associated with f (i.e., a two-person game with the same strategy set X for the two players and the payoff function g for the second player as given by $g(x,y) = f(y,x)$). Symmetric equilibrium points are generalizations of *evolutionarily stable strategies, ESS*, which were introduced by Smith and Price in their ground breaking paper [15] (see also [14,16,17]).

Theorem 5. *Let Λ be an L^*-operator on a compact Hausdorff space X. If $f : X \times X \to \mathbf{R}$ is continuous and quasi-concave with respect to the first variable, then there exists a symmetric equilibrium point for f.*

Proof. Let $\mu(x,y) = -f(x,y) + \sup_{z\in X} f(z,y)$. One easily checks that $\mu : X \times X \to [0,\infty)$ *is continuous and quasi-convex with respect to the first variable and that for each y there exists x such that $\mu(x,y) = 0$. By applying the Theorem on Signatures to* $\mathcal{M} = \{\mu\}$ and $h = id_X$, we get a point $a \in X$ such that $\mu(a,a) = 0$, i.e., a point a such that $f(a,a) = \sup_{z\in X} f(z,a)$. □

Remark 2. (a) Here is another proof of Theorem 5 (appealing directly to $(**)$).
Set xSy if $f(x,y) > f(y,y)$. Clearly, for each $x \in X$, S_x is open; and, for each $y \in Y$, S^y is convex. Since no y can belong to S^y, by $(**)$, $S_x - s$ do not cover the space X. Hence, for some $a \in X$, $f(x,a) \leq f(a,a)$ for each $x \in X$.

(b) Here is an elegant and very simple proof of the Separating Hyperplane Theorem based on that theorem (see [12]). Let C and D be two compact convex disjoint subsets of an inner product space V. Then $X = D - C$ is convex and compact and $0 \notin X$. The Maynard-Smith theorem applied to $f(x,y) = -\langle x,y\rangle$ *on* X yields a symmetric equilibrium a for f. Thus $\langle x,a\rangle \geq \langle a,a\rangle = \|a\| > 0$ for each $x \in X$. It follows that there exists α such that $\langle c,a\rangle < \alpha < \langle d,a\rangle$ for each $c \in C$ and $d \in D$. Thus $\langle x,a\rangle = \alpha$ is an equation of a separating hyperplane.

4. Topological spaces admitting L_n^*-operators

An operator $\Lambda : \langle X\rangle \to \exp(X)$ on a topological space X is called L_n^*-*operator* if it satisfies the following condition $(*^+)$:

$(*^+)$ If $A \in \langle X\rangle$ and $\{U_x : x \in A\}$ is an open cover of X, then there exists $B \subseteq A$ such that $|B| \le n+1$ and $\Lambda(B) \cap \bigcap \{U_x : x \in B\} \neq \emptyset$.

The condition defining L_n^*-operators is the original condition $(*)$ with an additional requirement imposed on the size of the set B.

The class of L_n^*-operators and spaces that admit them possess some important properties. Among them, a fixed point property (Theorem 6) and Helly's property (Theorem 10).

Lemma 1. *Let (X, Λ) be an L^*-space and let $f : X \to X$ be a continuous function. If $\mathcal{P}$ is a finite open family in X and $f(X) \subseteq \bigcup \mathcal{P}$, then there exists $p \in X$ and $F \in \langle st(f(p), \mathcal{P})\rangle$ such that $p \in \Lambda(F)$.*

Proof. There exists an $A \in \langle f(X)\rangle$ and $\mathcal{Q} = \{U_a : a \in A\} \subseteq \mathcal{P}$ such that $f(X) \subseteq \bigcup \mathcal{Q}$ and $a \in U_a$ for each $a \in A$ (e.g., let $\mathcal{Q} \subseteq \mathcal{P}$ irreducibly covers $f(X)$). Since $\left\{f^{-1}(U_a) : a \in A\right\}$ is an open cover of X, there exists $F \subseteq A$ such that $\Lambda(F) \cap \bigcap \left\{f^{-1}(U_a) : a \in F\right\} \neq \emptyset$. If p is any point from that non-empty set, then $f(p) \in \bigcap \{U_a : x \in F\}$ and since $a \in U_a$ for each $a \in F$, $F \in \langle st(f(p), \mathcal{Q})\rangle \subseteq \langle st(f(p), \mathcal{P})\rangle$. □

Theorem 6. *If X is a compact T_2 space that admits $(n+1)$-continuous L_n^*-operator, then X has the fixed point property.*

Proof. Let $f : X \to X$ be a continuous function and suppose that $f(x) \neq x$ for each $x \in X$. Hence there exists a finite open cover $\mathcal{U}$ of X such that $U \cap f(U) = \emptyset$ for each $U \in \mathcal{U}$. By $(n+1)$-continuity of Λ, there exists a finite open cover $\mathcal{V}$ of X such that $\left\{\bigcup \Lambda\left([V]^{\le n+1}\right) : V \in \mathcal{V}\right\}$ is a refinement of $\mathcal{U}$. There exists a finite open family $\mathcal{P}$ that is a star-refinement of the family $\mathcal{V}$ (cf. Engelking [4, p. 377]). Applying Lemma 1 to that $\mathcal{P}$ we get a point $p \in X$ and an $F \in [st(f(p), \mathcal{P})]^{\le n+1}$such that $p \in \Lambda(F)$. Pick $V \in \mathcal{V}$ that contains $st(f(p), \mathcal{P})$ and $U \in \mathcal{U}$ that contains $\bigcup \Lambda\left([V]^{\le n+1}\right)$. Then both p and $f(p)$ belong to U which is impossible. □

A natural question that arises is what kind of L^*-operators could be L_n^*-operators for some n. For that purpose, let's recall the concept of an n-ary operation.

Following van del Vel [31], a function $\Lambda : \langle X\rangle \to \exp(X)$ is said to be *n- ary*, $n = 1, 2, \ldots$, if $\Lambda(A) = \bigcup \left\{\Lambda(F) : F \in [A]^{\le n+1}\right\}$ for each $A \in \langle X\rangle$.

Theorem 7. *If $\Lambda : \langle X\rangle \to \exp(X)$ is an L^*-operator that is also n-ary, then Λ is an L_n^*-operator.*

Proof. Let $A \in \langle X\rangle$ and $\{U_x : x \in A\}$ is an open cover of X. There exists $B \subseteq A$ such that $\Lambda(B) \cap \bigcap \{U_x : x \in B\} \neq \emptyset$. Since Λ is n-ary, $\Lambda(C) \cap$

$\bigcap\{U_x : x \in B\} \neq \emptyset$ for some subset C of B of size at most $n+1$. Since $\bigcap\{U_x : x \in B\} \subseteq \bigcap\{U_x : x \in C\}$, $\Lambda(C) \cap \bigcap\{U_x : x \in C\} \neq \emptyset$. □

Theorem 8. *The convex hull operator on* $\mathbf{R}^n$ *is an* L_n^**-operator.*

Proof. By the classical Caratheodory's theorem , *co* is n-ary. □

We shall state more properties enjoyed by L_n^*-operators.

An operator $\Lambda : \langle X\rangle \to \exp(X)$ on a set X is called *n-monotone* if it $\Lambda(A) \subseteq \Lambda(B)$ provided that $A \subseteq B$ and $|B| \leq n+1$.

Theorem 9. *Let* $\Lambda : \langle X\rangle \to \exp(X)$ *be an* L_n^**-operator and n-monotone operator on a normal space X.*
If A is a subset of X of size $n+2$, then $\bigcap\{cl\Lambda(A - \{x\}) : x \in A\} \neq \emptyset$.

Proof. Assume otherwise. By the Swelling Theorem (see [4], Theorem 7.1.4), there exist open sets V_x such that $cl\Lambda(A - \{x\}) \subseteq V_x$ and $\bigcap\{V_x : x \in A\} = \emptyset$. For each $a \in A$ set $U_a = \left(X - \bigcup\{cl\Lambda(A - \{x\}) : x \in A\}\right) \cup V_a$. Since $\bigcup\{cl\Lambda(A - \{x\}) : x \in A\} \subseteq \bigcup\{V_x : x \in A\}$, the family $\{U_x : x \in A\}$ constitutes an open cover of X. So there exists $B \subseteq A$ such that $|B| \leq n+1$ and $\Lambda(B) \cap \bigcap\{U_x : x \in B\} \neq \emptyset$. Notice that since Λ is n-monotone,

$\Lambda(B) \subseteq \bigcap\{\Lambda(A - \{x\}) : x \in A - B\} \subseteq \bigcap\{V_x : x \in A - B\}$. Consequently, $\Lambda(B) \cap \bigcap\{U_x : x \in B\} \subseteq \left(\bigcup\{cl\Lambda(A - \{x\}) : x \in A\}\right) \cap \bigcap\{V_x : x \in A - B\} \cap \bigcap\{U_x : x \in B\} \subseteq \bigcap\{V_x : x \in A - B\} \cap \bigcap\{V_x : x \in B\} = \varnothing$; a contradiction. □

Remark 3. (a) The convex hull operator on $\mathbf{R}^n$ is L_n^*-operator but it is not L_{n-1}^*-operator. Towards this goal, consider an n-dimensional simplex Δ in $\mathbf{R}^n$ with vertices $x_0, x_1, \ldots, x_n$. Let Δ_i denote the $(n-1)$-dimensional face of the simplex Δ of all the original vertices but x_i. Thus $\Delta_i = co(A - \{x_i\})$, where $A = \{x_0, x_1, \ldots, x_n\}$. Since a point $x = \sum_{i=0}^{n} \lambda_i x_i$ of Δ belongs to the face Δ_k if and only if $\lambda_k = 0$, $\bigcap\{\Delta_i : i = 0, 1, \ldots, n\} = \emptyset$. Clearly, the convex hull operator in any vector space is n-monotone for any n. Hence Theorem 9 shows that *co* cannot be L_{n-1}^*-operator.

(b) The convex hull operator on any infinitely dimensional topological vector space cannot be an L_n^*-operator for any n. For any such vector space contains simplices of all possible finite dimensions and thus arguments from part (a) apply.

Helly's Theorem has its counterpart within topological spaces admitting L_n^*-operators.

Theorem 10. (Helly's Theorem) *Let* $\Lambda : \langle X\rangle \to \exp(X)$ *be an* L_n^**-operator and n-monotone operator on a normal space X. If $\mathcal{C}$ is a family of convex subset of X such that any $n+1$ of them have a non-empty intersection, then the family* $\{clF : F \in \mathcal{C}\}$ *is centered.*

Proof. Suppose otherwise and let $\mathcal{F}$ be a finite subfamily of $\mathcal{C}$ such that $\bigcap\{clF : F \in \mathcal{F}\} = \varnothing$. Assume that $\mathcal{F}$ is of size $n+2$ (the smallest possible), say $\mathcal{F} = \{F_1, \dots, F_n, F_{n+1}, F_{n+2}\}$. By the Swelling Theorem (see [4], Theorem 7.1.4), there exist open sets V_i such that $cl\Lambda F_i \subseteq V_i$ and $\bigcap\{V_i : i = 1, 2, \dots, n+2\} = \emptyset$. For each $i = 1, 2, \dots, n+2$, pick $a_i \in \bigcap\{F_j : j \neq i\}$ and set

$U_{a_i} = \left(X - \bigcup\{clF_j : j = 1, 2, \dots, n+2\}\right) \cup V_i$. The open sets U_{a_i} cover the space X and since Λ is an L^*_n-operator, there exists $A \subset \{1, 2, \dots, n+2\}$, $|A| \leq n+1$, such that $\Lambda(\{a_i : i \in A\}) \cap \bigcap\{U_{a_i} : i \in A\} \neq \emptyset$. Since $a_i \in \bigcap\{F_j : j \neq i\}$, for each $i = 1, 2, \dots, n$, we have $\{a_i : i \in A\} \subseteq \bigcap\{F_j : j \notin A\}$. In consequence, $\Lambda(\{a_i : i \in A\}) \subseteq \bigcap\{F_j : j \notin A\} \subseteq \bigcap\{V_j : j \notin A\}$. Clearly, the set $\Lambda(\{a_i : i \in A\})$ is also contained in $\bigcup\{clF_j : j = 1, 2, \dots, n+2\}$. Hence $\varnothing \neq \Lambda(\{a_i : i \in A\}) \cap \bigcap\{U_{a_i} : i \in A\} \subseteq \left(\bigcup\{clF_j : j = 1, 2, \dots, n+2\}\right) \cap \bigcap\{V_j : j \notin A\} \cap \bigcap\{U_{a_i} : i \in A\} \subseteq \bigcap\{V_j : j \notin A\} \cap \bigcap\{V_j : j \in A\} = \varnothing$; a contradiction.

By induction, Helly's Theorem holds true. □

The statement: *Every compact convex subset of a metric linear topological space has the fixed point property* is known as the Schauder conjecture. Since its publication in The Scottish Book in 1935 (cf. [13], Problem 54), the conjecture has stimulated a great deal of research in efforts to settle it. Schauder's conjecture was solved affirmatively (with a complicated and controversial [3] proof) by Cauty [2] (see also [3] for some details and for some pertinent references). Therefore it is desirable to come up with much simpler arguments than Cauty's. It was the motivation of the first author when he introduced the concept of the *characteristic of a metric* in [11].

Let (X, ρ) be a metric space. The *characteristic of the metric (pseudometric)* ρ is defined as follows:

$$\chi(\rho) = \inf\{order(\mathcal{U}) \cdot mesh(\mathcal{U}) : \mathcal{U} \text{ is an open point-finite open cover of } X\},$$

where $mesh(\mathcal{U}) = \sup\{diam(V) : V \in \mathcal{U}\}$ and $ord(\mathcal{U}) = \sup\{|\{V : x \in V \in \mathcal{U}\}| : x \in X\}$. In the same paper[11], he posed the following problem:

Problem 1. *Let $Y \subseteq E$ be a convex compact subset of a metrizable linear space E. Is there a subnorm $\|\cdot\| : E \to [0, \infty)$ such that the metric $\rho(x, y) = \|x - y\|$ for $x, y \in Y$, is of characteristic equal to zero, $\chi(\rho) = 0$, and ρ is compatible with the topology of Y?*

[3] Cauty's original proof and its follow up one stirred some controversies (cf. [21]). Even though Cauty's proof is accepted to be correct by some, "*Many people think that Cauty's proofs are incorrect and that the Schauder conjecture is still open.*"—to quote a referee of this paper.

It is worth pointing out that an affirmative answer to that problem would allow us to present an alternative, much simpler, proof of the Schauder conjecture based on the method presented in Theorems 3 and 6.

Acknowledgements

We would like to thank an anonymous referee for her/his valuable comments.

References

[1] Ben-El-Mechaiekh, H., Chebbi, S., Florenzano, M., Llinares, J.V.: ıAbstract convexity and fixed points. J. Math. Anal. Appl. **222**, 138–150 (1998)

[2] Cauty, R.: Solution du probleme de point fixe de Schauder. Fundam. Math. **170**, 231–246 (2001)

[3] Dobrowolski, T.: Revisiting Cauty's proof of the Schauder conjecture. Abstract Appl. Anal. **7**, 407–433 (2003)

[4] Engelking, R.: General Topology. PWN, Warszawa (1977)

[5] Knaster, B., Kuratowski, K., Mazurkiewicz, S.: Ein Beweis des Fixpunktsatzes für n-dimensionale Simplexe. Fund. Math. **14**, 132–137 (1929)

[6] Kulpa, W.: Convexity and the Brouwer fixed point theorem. Topol. Proc. **22**, 211–235 (1997)

[7] Kulpa, W., Szymanski, A.: On Nash theorem. Acta Univ. Carol. **43**(2), 51–65 (2002)

[8] Kulpa, W., Szymanski, A.: Infimum principle. Proc. AMS **192**(1), 203–204 (2004)

[9] Kulpa, W., Szymanski, A.: Applications of general infimum principles to fixed-point theory and game theory. Set Valued Anal. **16**, 375–398 (2008)

[10] Kulpa, W., Szymanski, A.: A note on n-continuous L*-operators. Topol. Proc. **37**, 403–407 (2011)

[11] Kulpa, W.: On metric of characteristic zero. Colloquium Mathematicum **130**(1), 139–146 (2013)

[12] Kulpa, W.: L*-operators of convexity. Math. Appl. **43**, 145–156 (2015)

[13] Mauldin, R.D. (ed.): The Scottish Book: Mathematics from the Scottish Cafe. Birkhäuser, Boston (1981)

[14] Smith, J.M.: Evolution and the Theory of Games. Cambridge University Press, Cambridge (1982). 12th printing (2005)

[15] Smith, J.M., Price, G.: The logic of animal conflict. Nature **246**, 15–18 (1973)

[16] Nash, J.F.: Equilibrium points in n-person games. Proc. Natl. Acad. Sci. USA **36**, 48–49 (1950)

[17] Nash, J.F.: Non-cooperative games. Ann. Math. **54**, 286–295 (1951)

[18] Park, S.: Elements of the KKM theory for generalized convex spaces. Korean J. Comput. Appl. Math. **7**, 1–28 (2000)

[19] Park, S.: Elements of the KKM theory on abstract convex spaces. J. Korean Math. Soc. **45**(1), 1–27 (2008)

[20] Park, S.: Foundations of the KKM theory. J. Nonlinear Convex Anal. **9**, 1–20 (2008)

[21] Park, S.: Compact Browder maps and equilibria of abstract economies. J. Appl. Math. Comput. **26**, 555–564 (2008). MR2383687

[22] Park, S.: Equilibrium existence theorems in KKM spaces. Nonlinear Anal. **69**, 4352–4364 (2008)

[23] Park, S.: The KKM principle in abstract convex spaces: equivalent formulations and applications. Nonlinear Anal. **73**, 1028–1042 (2010)

[24] Park, S.: Generalizations of the Nash Equilibrium Theorem in the KKM Theory, Hindawi Publishing Corporation, Fixed Point Theory and Applications, Article ID 234706 (2010)

[25] Park, S.: A genesis of general KKM theorems for abstract convex spaces. J. Nonlinear Anal. Optim. **2**, 133–146 (2011)

[26] Park, S.: New generalizations of basic theorems in the KKM theory. Nonlinear Anal. **74**, 3000–3010 (2011)

[27] Park, S.: A genesis of general KKM theorems for abstract convex spaces. J. Nonlinear Anal. Optim. **2**(1), 121–132 (2011)

[28] Park, S.: Remarks on simplicial spaces and L^*- spaces of Kulpa and Szymanski. Commun. Appl. Nonlinear Anal. **19**(1), 59–69 (2012)

[29] Park, S., Kim, H.: Admissible classes of multifunctions on generalized convex spaces. Proc. Coll. Nat. Sci. SNU **18**, 1–21 (1993)

[30] Szymanski, A.: Convexities generated by multifunctions. Nonlinear Anal. Forum **21**(2), 65–76 (2016)

[31] van de Vel, M.L.J.: Theory of Convex Structures, p. 50. North-Holland Mathematical Library, Amsterdam (1993)

W. Kulpa, A. Szymanski and M. Turzański
Faculty of Mathematics and Natural Sciences,
College of Sciences
Cardinal Stefan Wyszyński University
ul. Dewaitis
01-815 Warsaw
Poland

A. Szymanski
Department of Mathematics
Slippery Rock University
Slippery Rock PA16057
USA
e-mail: andrzej.szymanski@sru.edu