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On some nonlinear operators in ΛBV-spaces

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Abstract. In this paper we investigate autonomous as well as nonautonomous superposition operators acting between spaces of functions of bounded Λ-variation. A particular emphasis is put on acting conditions as well as on continuity problems for such operators. In particular, we give necessary and sufficient conditions for nonautonomous superposition operators to map a space of functions of bounded Λ-variation into itself. Moreover, we prove the continuity of certain autonomous superposition operators acting between various spaces of functions of bounded Λ-variation. We also examine the existence of ΛBV-solutions as well as the topological structure of such solution sets to classical nonlinear integral equations.

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1. Introduction

One of the most interesting generalizations of the classical variation in the sense of Jordan seems to be the notion of Λ-variation. It was introduced by Waterman in 1972 [18], in connection with his investigation of Fourier series. More precisely, it appears that Fourier series of functions of bounded harmonic variation which is a particular Λ-variation converge pointwise to the arithmetic mean of the left- and the right-hand side limits and converge uniformly to the functions under consideration on closed intervals of their continuity (see [18] for more details).

It appears that functions of bounded Λ-variation possess some properties which are entitled to functions of bounded variation in the sense of Jordan. For example, functions of bounded Λ-variation are bounded and sets of points of their discontinuities are at most denumerable (see [14], Theorem

3 and Theorem 4). Moreover, for such functions a Helly-type theorem holds (see [11]). Let us add also that the space of functions of bounded Λ-variation from a functional point of view was investigated, for example, in the paper [16]. It has appeared that the so-called Shao–Sablin index plays an essential role in that approach.

The autonomous superposition operators acting in the spaces of functions of bounded Λ-variation have been investigated for example in the paper [15]. The authors proved in it an analogue of the well-known Josephy theorem, which describes the behavior of such operators in the space of functions of bounded variation in the sense of Jordan. On the other hand, the sufficient conditions for nonautonomous superposition operators to act in the spaces of functions of bounded Λ-variation were given in the paper [2].

The existence and the existence and uniqueness of global and local ΛBV-solutions, that is, solutions being functions of bounded Λ-variation, to nonlinear Hammerstein as well as to nonlinear Volterra–Hammerstein integral equations were investigated, for example, in the paper [4]. Let us also add that linear and semilinear differential equations in the spaces of continuous functions of bounded Λ-variation were investigated in the paper [2].

In this paper we would like to achieve a few goals. First, in Sect. 3 we focus on nonautonomous superposition operators acting in the spaces of functions of bounded Λ-variation. The main result in this section is Theorem 7 which gives necessary and sufficient conditions for nonautonomous superposition operators to map a space of functions of bounded Λ-variation into itself. The results included in this section are mainly inspired by the paper [3]. However, let us emphasize the fact that nonautonomous superposition operators in the spaces of functions of bounded Λ-variation behave in a slightly different way than in spaces of functions of bounded variation in the sense of Jordan (see Example 1).

In Sect. 4 we deal with the problem of continuity of autonomous superposition operators acting between spaces of functions of bounded Λ-variation. It appears that in some cases generators of such operators reduce to constant mappings (see Theorem 10). Let us recall that a necessary and sufficient condition for an autonomous superposition operator to map a space of functions of bounded Λ-variation into itself is the requirement of its generator to be a locally Lipschitz function (see [15] or [1], Theorem 5.14). We show that in case of generators being continuously differentiable functions the corresponding superposition operator is continuous. However, if one considers generators that do not have to be locally Lipschitz functions, then it appears that for a given domain it is possible to indicate a space of functions of bounded Λ-variation that the superposition operator acts to (see Theorem 11). According to Theorem 12 it appears that such operators are continuous without additional assumptions. Let us add that the problem of continuity of the superposition operators still remains open (even in the autonomous case) because we still cannot say anything about the continuity of operators with a general Lipschitz generator and acting from a space of functions of bounded Λ-variation into itself.

In Sect. 5 we focus on ΛBV-solutions to some classical nonlinear integral equations. We prove therein an existence result to the nonlinear Hammerstein integral equation, extending Theorem 1 from the paper [4]. Moreover, we prove an Aronszajn-type theorem concerning the topological structure of continuous ΛBV-solution sets to the nonlinear Volterra–Hammerstein integral equations.

Finally, in Sect. 6 we examine embeddings of compact subsets of a given space of functions of bounded Λ-variation into another space of this type. It appears, roughly speaking, that sets of functions which are continuous in Λ-variation possess some nice properties, as far as compactness is concerned (see Theorems 18, 19).

2. Preliminaries

In this section we collect some basic definitions and facts, which will be needed in the sequel.

Notation. *Throughout the paper, we will denote the unit interval* $[0, 1]$ *by* I. *By* S_n *we denote the symmetric group of the set* $\{1, 2, \dots, n\}$.

Definition 1. Let us consider a nondecreasing sequence of positive real numbers $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$. We call such sequence a *Waterman sequence* if

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = +\infty.$$

Definition 2. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a Waterman sequence, $J = [a, b] \subset I$ and let $x: I \to \mathbb{R}$. We say that x is of bounded Λ-variation over J if there exists a positive constant M such that for any finite sequence of nonoverlapping[1] subintervals $\{[a_1, b_1], [a_2, b_2], \dots, [a_n, b_n]\}$ of J, the following inequality holds

$$\sum_{i=1}^{n} \frac{|x(b_i) - x(a_i)|}{\lambda_i} \leq M.$$

The supremum of the above sums taken over the family of all the finite collections of nonoverlapping subintervals of J is called the Λ-variation of x over J and it is denoted by $\mathrm{var}_\Lambda(x; J)$. If $J = I$, then we simply write $\mathrm{var}_\Lambda(x)$.

Remark 1. In the above definition the finite collections of intervals and the finite sums may be replaced by countable collections and series, respectively (see [19], Theorem 1, p. 34).

The vector space of all the functions defined on the interval I and of bounded Λ-variation, endowed with the norm $\|x\|_{\Lambda BV} := |x(0)| + \mathrm{var}_\Lambda(x)$ forms a Banach space $\Lambda BV(I)$ (see [19], Section 3).

Below the norm in the normed space E will be denoted by $\|\cdot\|_E$. The exception is the supremum norm in the space $B(I)$ of all bounded functions

[1]Two closed intervals I_1 and I_2 are said to be *nonoverlapping* if $I_1 \cap I_2$ consists of at most one point.

on the interval I (and its subspaces) which will be denoted by $\|\cdot\|_\infty$. To shorten slightly the notation, we will also write $\|\cdot\|_{\Lambda BV}$ instead of $\|\cdot\|_{\Lambda BV(I)}$. The open ball, in the space E, centered at x_0 with radius $r > 0$ will be denoted as $B_E(x_0, r)$, while the closed ball as $\overline{B}_E(x_0, r)$.

Let $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ be a *partition of the interval* J. We will denote such partition by $\pi = \{t_0, t_1, \ldots, t_{n-1}, t_n\}$.
Now we are going to rephrase the definition of Λ-variation (in a slightly different way than it was done before by Prus-Wiśniowski—see [16]). This, very simple, observation will play an essential role in our considerations.

Proposition 1. *Let* $x : J \to \mathbb{R}$ *and let* Λ *be a Waterman sequence. The following conditions are equivalent:*

(1) there exists a positive constant M *such that for any finite partition* $\pi = \{c_0, c_1, \ldots, c_n\}$ *of* J *and any permutation* $\sigma \in S_n$*, we have*

$$\sum_{i=1}^{n} \frac{|x(c_i) - x(c_{i-1})|}{\lambda_{\sigma(i)}} \leq M;$$

(2) x *is of bounded* Λ*-variation.*

Proof. Let x be as in (1). Let us choose an arbitrary finite sequence $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$ of nonoverlapping subintervals of J. We complete it to be also a finite partition $\pi = \{c_0, c_1, \ldots, c_m\}$ of J. We have

$$\sum_{i=1}^{n} \frac{|x(b_i) - x(a_i)|}{\lambda_i} \leq \sup_{\sigma \in S_m} \sum_{i=1}^{m} \frac{|x(c_i) - x(c_{i-1})|}{\lambda_{\sigma(i)}} \leq \sup_{\sigma \in S_m} M = M.$$

The verification of the inverse direction is trivial. □

In the next lemma, to simplify the notation a little bit, instead of taking any interval $J \subset I$ we will focus on the entire interval I.

Lemma 1. *There exists a positive constant* $\tilde{c}_\Lambda$ *such that for any function* $x \in \Lambda BV(I)$ *we have*

$$\|x\|_\infty \leq \tilde{c}_\Lambda \|x\|_{\Lambda BV}.$$

Proof. Taking any $t \in I$, we get

$$|x(t)| \leq |x(0)| + |x(t) - x(0)| = |x(0)| + \lambda_1 \frac{|x(t) - x(0)|}{\lambda_1} \leq |x(0)| + \lambda_1 \mathrm{var}_\Lambda(x) \leq \max\{1, \lambda_1\} \|x\|_{\Lambda BV}.$$

Finally, for completeness, let us recall the definition of nonlinear superposition operators.

Definition 3. Let J be an arbitrary interval.

(a) Let $f\colon J \times \mathbb{R} \to \mathbb{R}$. The operator

$$F(g)(u) := f(u, g(u)),$$

where $g\colon J \to \mathbb{R}$ is an arbitrary function, is called the nonautonomous superposition operator generated by the function f.

(b) Let $f\colon \mathbb{R} \to \mathbb{R}$. The operator

$$F(g)(u){:=}f(g(u)),$$

where $g\colon J \to \mathbb{R}$ is an arbitrary function, is called the autonomous superposition operator generated by the function f.

3. Nonautonomous superposition operators in the space of functions of bounded Λ-variation.

In this section, for the sake of simplicity, instead of an arbitrary interval J, we will consider the unit interval I. In the paper [3] Bugajewska *et al.* have stated the necessary and sufficient conditions which guarantee that the nonautonomous superposition operator generated by a function $f\colon I\times\mathbb{R} \to \mathbb{R}$ maps the space $BV(I)$ of all functions of bounded variation in the sense of Jordan into itself and is locally bounded.

Theorem 1. ([3, Theorem 3.8]) *Suppose that $f\colon I\times\mathbb{R} \to \mathbb{R}$ is a given function. The following conditions are equivalent:*

(i) the nonautonomous superposition operator F, generated by f, maps the space $BV(I)$ into itself and is locally bounded;

(ii) for every $r > 0$ there exists a constant $M_r > 0$ such that for every $k \in \mathbb{N}$, every finite partition $0 = t_0 < \cdots < t_k = 1$ of the interval I and every finite sequence $u_0, u_1, \ldots, u_k \in [-r, r]$ with $\sum_{i=1}^{k}|u_i - u_{i-1}| \le r$, the following inequalities hold

$$\sum_{i=1}^{k}\bigl|f(t_i,u_i)-f(t_{i-1},u_i)\bigr| \le M_r \quad \textit{and} \quad \sum_{i=1}^{k}\bigl|f(t_{i-1},u_i)-f(t_{i-1},u_{i-1})\bigr| \le M_r.$$

Now we are going to present some results that show that the similar conditions may be stated for the more general case of nonlinear superposition operators between spaces $\Lambda BV(I)$ and $\Gamma BV(I)$ (to avoid confusion, let us explain that Λ and Γ in the above symbols mean that those spaces may be constructed with the help of two various Waterman sequences). But before we proceed to that more general case we are going to begin with the generators f satisfying the local Lipschitz condition with respect to the second variable.

First, let us explicitly write the definition which will be used in the sequel.

Definition 4. We say that a function $f\colon I \times \mathbb{R} \to \mathbb{R}$ is locally bounded if the image $f(I \times [-M, M])$ is bounded for any $M > 0$. Similarly we say that the superposition operator $F : \Lambda BV(I) \to \Gamma BV(I)$ is locally bounded if the image of each ball $F(B_{\Lambda BV}(0, M))$ is bounded in $\Gamma BV(I)$.

Theorem 2. *Let $f\colon I \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:*

(i) f satisfies the local Lipschitz condition on $\mathbb{R}$ uniformly in $t \in I$;

(ii) for any $r > 0$ there exists $M_r > 0$ such that for any positive number n, any partition π of the interval I, any $u_0, u_1, \ldots, u_n \in \mathbb{R}$ and any permutation $\sigma \in S_n$, the following implication holds

$$\sup_{\sigma\in S_n}\sum_{i=1}^{n}\frac{|u_i-u_{i-1}|}{\lambda_{\sigma(i)}}\le r \Longrightarrow \sup_{\sigma\in S_n}\sum_{i=1}^{n}\frac{|f(t_i,u_{i-1})-f(t_{i-1},u_{i-1})|}{\lambda_{\sigma(i)}}\le M_r.$$

Then the nonautonomous superposition operator F, generated by f, maps the space $\Lambda BV(I)$ into itself and is locally bounded.

Proof. Let $x\in\Lambda BV(I)$ be such that $\|x\|_{\Lambda BV}\le r$. Let L_r be the uniform Lipschitz constant corresponding to the function $u\mapsto f(t,u)$ restricted to the interval $[-\tilde{r},\tilde{r}]$, where $\tilde{r}=r\tilde{c}_\Lambda$. For any partition $\pi=\{t_0,t_1,\ldots,t_n\}$ we have

$$\begin{aligned}&\sum_{i=1}^{n}\frac{|f(t_i,x(t_i))-f(t_{i-1},x(t_{i-1}))|}{\lambda_{\sigma(i)}}\\&\quad\le\sum_{i=1}^{n}\frac{|f(t_i,x(t_i))-f(t_i,x(t_{i-1}))|}{\lambda_{\sigma(i)}}+\sum_{i=1}^{n}\frac{|f(t_i,x(t_{i-1}))-f(t_{i-1},x(t_{i-1}))|}{\lambda_{\sigma(i)}}\\&\quad\le L_r\sum_{i=1}^{n}\frac{|x(t_i)-x(t_{i-1})|}{\lambda_{\sigma(i)}}+M_r\le L_r\cdot r+M_r,\end{aligned}$$

which proves that F maps the space $\Lambda BV(I)$ into itself. Moreover, we have:

$$\begin{aligned}\|F(x)\|_{\Lambda BV}&=|F(x(0))|+\mathrm{var}_\Lambda(F(x))\le|f(0,x(0))|+L_r\cdot\mathrm{var}_\Lambda(x)+M_r\\&\le|f(0,x(0))-f(0,0)|+|f(0,0)|+L_r\mathrm{var}_\Lambda(x)+M_r\\&\le L_r\,\|x\|_{\Lambda BV}+M_r+|f(0,0)|\le L_r\cdot r+M_r+|f(0,0)|,\end{aligned}$$

which means that F is locally bounded.

It appears that in the case of functions satisfying the local Lipschitz condition in view of the first variable, the condition (ii) of Theorem 2 plays an important role, if one considers the space $BV(I)$. More precisely, the following result holds.

Theorem 3. ([3], Proposition 3.3) *Let us assume that $f\colon I\times\mathbb{R}\to\mathbb{R}$ satisfies the condition (i) of Theorem 2. If the nonautonomous superposition operator F, generated by f, maps the space $BV(I)$ into itself and is locally bounded, then f satisfies (ii).*

It is quite surprising that in the above result one cannot replace the space $BV(I)$ by an arbitrary space $\Lambda BV(I)$. To establish that, let us consider the following.

Example 1. Let us consider the function $f\colon I\times\mathbb{R}\to\mathbb{R}$, defined by the formula

$$f(t,x)=\begin{cases}0, & \text{if } t<1;\\ x, & \text{if } t=1.\end{cases}$$

The nonautonomous superposition operator, generated by the function f, maps any space $\Lambda BV(I)$ into any space $\Gamma BV(I)$ and it is locally bounded. Let $\lambda_n=n$ and $a_n=\frac{1}{\sqrt{n}}$, for $n\in\mathbb{N}$. Obviously $\sum_{n=1}^{\infty}a_n=+\infty$ and there exists $r>0$ such that $\sum_{n=1}^{\infty}\frac{a_n}{\lambda_n}\le r$. Therefore, one can take an arbitrarily large interval $[0, \sum_{n=1}^{N} a_n]$ and its partition consisting of points $u_i = \sum_{n=1}^{i} a_n$ ($i = 1, \ldots, N$) and $u_0 = 0$. For that partition, the predecessor of the implication which appears in the condition (ii) of Theorem 2 is satisfied. On the other hand, we have

$$\sum_{i=1}^{N} \frac{|f(t_i, u_{i-1}) - f(t_{i-1}, u_{i-1})|}{\lambda_{\sigma(i)}} = \frac{|f(1, u_{N-1})|}{\lambda_{\sigma(N)}},$$

so for some permutation $\sigma \in S_N$ the above sum is equal to $\frac{|u_{N-1}|}{\lambda_1}$, which means that it can be arbitrarily large.

Remark 2. Let us notice that the local Lipschitz condition imposed on a function $f\colon \mathbb{R} \to \mathbb{R}$ is a necessary and sufficient condition for the autonomous superposition operator to act from $\Lambda BV(I)$ to $\Lambda BV(I)$ (cf. [15]), so Theorem 2 presented above is the direct generalization of that result to the nonautonomous case.

The following three results describe some properties of nonautonomous superposition operators acting between spaces $\Lambda BV(I)$ and $\Gamma BV(I)$.

Theorem 4. *Assume that $f\colon I \times \mathbb{R} \to \mathbb{R}$ generates a nonautonomous superposition operator F which maps a space $\Lambda BV(I)$ into $\Gamma BV(I)$. If f is not locally bounded, then neither is F.*

Proof. Since f is not locally bounded, there exists $r > 0$ and sequences $(t_n)_{n\in\mathbb{N}}$, $t_n \in I$ for $n \in \mathbb{N}$ and $(u_n)_{n\in\mathbb{N}}$, $u_n \in [-r, r]$ for $n \in \mathbb{N}$, such that

$$\lim_{n\to+\infty} |f(t_n, u_n)| = +\infty.$$

For every $n \in \mathbb{N}$ we define

$$x_n(t) = \begin{cases} u_n, & \text{if} \quad t = t_n, \\ 0, & \text{otherwise.} \end{cases}$$

We have $\|x_n\|_{\Lambda BV} = |x_n(0)| + \mathrm{var}_\Lambda(x_n) \leq r + \frac{1}{\lambda_1} r + \frac{1}{\lambda_2} r$ for each n. By the assumptions, we infer that $F(x_n) \in \Gamma BV(I)$ and for fixed $s \neq t_n$ we have

$$\begin{aligned} |f(t_n, u_n)| - |f(s, 0)| &\leq |f(t_n, u_n) - f(s, 0)| \\ &= \gamma_1 \frac{|f(t_n, x_n(t_n)) - f(s, x_n(s))|}{\gamma_1} \leq \gamma_1 \mathrm{var}_\Gamma(F(x_n)), \end{aligned}$$

where s is an arbitrary point of the interval I. Hence, $\lim\limits_{n\to\infty} \mathrm{var}_\Gamma(F(x_n)) = +\infty$.

Theorem 5. *If the nonautonomous superposition operator F, generated by f, maps a space $\Lambda BV(I)$ into a space $\Gamma BV(I)$, then for every $r > 0$ the set $T_r = \{t \in I : \sup\limits_{u\in[-r,r]} |f(t, u)| = +\infty\}$ is finite.*

Proof. Let us assume that $F\colon \Lambda BV(I) \to \Gamma BV(I)$ and there exists such $r > 0$ that the set T_r is at least denumerable (countable and infinite). There exists a sequence $((t_n, u_n))_{n\in\mathbb{N}}$, $(t_n, u_n) \in I \times [-r, r]$ for $n \in \mathbb{N}$ such that:

- $|f(t_{n+1}, u_{n+1})| \geq |f(t_n, u_n)| + 1$;
- t_n, u_n are monotone;
- $t_n \to t_0$, $u_n \to u_0$ as $n \to +\infty$;
- $t_m \neq t_n$ if $m \neq n$.

Let us define the function

$$x(t) = \begin{cases} u_n, & \text{if} \quad t = t_n, \\ u_0, & \text{if} \quad t = t_0, \\ \text{linear}, & \text{if} \quad t \in (\min\{t_n, t_{n+1}\}, \max\{t_n, t_{n+1}\}), \\ x(\sup_{n\in\mathbb{N}} t_n), & \text{if} \quad t \in (\sup_{n\in\mathbb{N}} t_n, 1], \\ x(\inf_{n\in\mathbb{N}} t_n), & \text{if} \quad t \in [0, \inf_{n\in\mathbb{N}} t_n). \end{cases}$$

This function is monotone, so it is of bounded variation in the sense of Jordan and, therefore, $x \in \Lambda BV(I)$. For any n we have

$$\begin{aligned} \operatorname{var}_\Gamma(F(x)) &\geq \sum_{i=1}^{n} \frac{|F(x)(t_{i+1}) - F(x)(t_i)|}{\gamma_i} \\ &\geq \sum_{i=1}^{n} \frac{|f(t_{i+1}, x(t_{i+1})) - f(t_i, x(t_i))|}{\gamma_i} \geq \sum_{i=1}^{n} \frac{1}{\gamma_i}, \end{aligned}$$

which tends to infinity as $n \to +\infty$. This gives a contradiction.

Theorem 6. *Let F be a nonautonomous superposition operator, generated by $f\colon I \times \mathbb{R} \to \mathbb{R}$, which maps a space $\Lambda BV(I)$ into $\Gamma BV(I)$. Then for every $u \in \mathbb{R}$ the function $t \mapsto f(t, u)$ is of bounded Γ-variation. Furthermore, in general, nothing can be said about the function $u \mapsto f(t, u)$, where $t \in I$ is fixed.*

Proof. For every $u \in \mathbb{R}$ let us set $x_u = u$. In view of the assumption, $F(x_u) \in \Gamma BV(I)$, that is, the function $t \mapsto f(t, u)$ is of bounded Γ-variation.

To show the second claim, let us consider the function $f\colon I \times \mathbb{R} \to \mathbb{R}$ given by the formula $f(t, u) = h(t)g(u)$, where $g\colon \mathbb{R} \to \mathbb{R}$ is an arbitrary function and $h\colon I \to \mathbb{R}$ is defined as follows:

$$h(t) = \begin{cases} 0, & \text{if} \quad t \in (0, 1], \\ 1, & \text{if} \quad t = 0. \end{cases}$$

Then the nonautonomous superposition operator F, generated by the function f, is defined as follows:

$$F(x)(t) = \begin{cases} 0, & \text{if} \quad t \in (0, 1], \\ g(x(0)), & \text{if} \quad t = 0, \end{cases}$$

where $x \in \Lambda BV(I)$. Therefore, F maps the space $\Lambda BV(I)$ into $\Gamma BV(I)$ and nothing can be said about the function $u \mapsto f(0, u) = g(u)$.

Now, let us introduce the following.

Definition 5. Let $u = \{u_0, u_1, \ldots, u_k\}$ be a finite sequence of real numbers. The number

$$\operatorname{var}_\Lambda(u) = \sup_{\tilde{u}} \left\{ \sup_{\sigma\in S_n} \sum_{i=1}^{n} \frac{|u_{k_i} - u_{k_{i-1}}|}{\lambda_{\sigma(i)}} \right\},$$

where $\tilde{u} = \{u_{k_0}, u_{k_1}, \ldots, u_{k_n}\}$ is any subsequence of u, is called the Λ-*variation* of the sequence u.

We are ready to prove the main result of this section.

Theorem 7. *Let $f\colon I \times \mathbb{R} \to \mathbb{R}$. The following conditions are equivalent:*

(i) the nonautonomous superposition operator F, generated by f, maps the space $\Lambda BV(I)$ into $\Gamma BV(I)$ and is locally bounded,
(ii) for every $r > 0$ there exists $M_r > 0$ such that for every $k \in \mathbb{N}$, every partition $\pi = \{t_0, t_1, \ldots, t_k\}$ of the interval I and every sequence $u = \{u_0, u_1, \ldots, u_k\}$ of elements of the interval $[-r\tilde{c}_\Lambda, r\tilde{c}_\Lambda]$ such that $\mathrm{var}_\Lambda(u) \le r$, the following inequalities hold

$$\sup_{\sigma\in S_k} \sum_{i=1}^{k} \frac{|f(t_i, u_i) - f(t_{i-1}, u_i)|}{\gamma_{\sigma(i)}} \le M_r \quad \textit{and}$$
$$\sup_{\sigma\in S_k} \sum_{i=1}^{k} \frac{|f(t_{i-1}, u_i) - f(t_{i-1}, u_{i-1})|}{\gamma_{\sigma(i)}} \le M_r.$$

Proof. (ii) $\Rightarrow$ (i) Let $x \in \Lambda BV(I)$ be such that $\|x\|_{\Lambda BV} \le r$. Let M_r and π be as in (ii). We have

$$\begin{aligned}
&\sum_{i=1}^{k} \frac{|f(t_i, x(t_i)) - f(t_{i-1}, x(t_{i-1}))|}{\gamma_{\sigma(i)}} \\
&\quad\le \sum_{i=1}^{k} \frac{|f(t_i, u_i) - f(t_{i-1}, u_i)|}{\gamma_{\sigma(i)}} + \sum_{i=1}^{k} \frac{|f(t_{i-1}, u_i) - f(t_{i-1}, u_{i-1})|}{\gamma_{\sigma(i)}} \\
&\quad\le 2M_r,
\end{aligned}$$

where $u_i = x(t_i)$ and $\sigma \in S_k$. Furthermore, we have

$$\begin{aligned}
\|F(x)\|_{\Gamma BV} &= |F(x(0))| + \mathrm{var}_\Gamma(F(x)) = |f(0, x(0))| + \mathrm{var}_\Gamma(F(x)) \\
&\le |f(0, x(0)) - f(0,0)| + |f(0,0)| + \mathrm{var}_\Gamma(F(x)) \\
&= \gamma_1 \frac{|f(0, x(0)) - f(0,0)|}{\gamma_1} + |f(0,0)| + \mathrm{var}_\Gamma(F(x)) \\
&\le \gamma_1 \cdot M_r + |f(0,0)| + 2M_r = (\gamma_1 + 2)M_r + |f(0,0)|,
\end{aligned}$$

which means that F is locally bounded.

(i) $\Rightarrow$ (ii) Assume that F satisfies the condition (i) and there exists $r > 0$ such that for every $n \in \mathbb{N}$ there exists a partition $\pi_n = \{t_0^{(n)}, t_1^{(n)}, \ldots, t_{k_n}^{(n)}\}$ of the interval I and a sequence $u_0^{(n)}, u_1^{(n)}, \ldots, u_{k_n}^{(n)}$ of elements of the interval $[-r\tilde{c}_\Lambda, r\tilde{c}_\Lambda]$ such that $\mathrm{var}_\Lambda(u) \le r$ and

$$\sup_{\sigma\in S_{k_n}}\sum_{i=1}^{k_n}\frac{|f(t_i^{(n)},u_i^{(n)})-f(t_{i-1}^{(n)},u_i^{(n)})|}{\gamma_{\sigma(i)}}>n \tag{1}$$

or

$$\sup_{\sigma\in S_{k_n}}\sum_{i=1}^{k_n}\frac{|f(t_{i-1}^{(n)},u_i^{(n)})-f(t_{i-1}^{(n)},u_{i-1}^{(n)})|}{\gamma_{\sigma(i)}}>n. \tag{2}$$

For each $n\in\mathbb{N}$ we define the function $\xi^n\colon [0,1]\to\mathbb{R}$ by the formula

$$\xi^n(t)=\begin{cases}u_i^{(n)}, & \text{if}\quad t=t_i^{(n)},\\ \text{linear}, & \text{if}\quad t\in(t_{i-1}^{(n)},t_i^{(n)}).\end{cases}$$

Obviously, we have

$$\mathrm{var}_\Lambda(\xi^{(n)})=\mathrm{var}_\Lambda(u^{(n)})\le r.$$

Since F is locally bounded, there exists R (corresponding to r) such that $\sup_{n\in N}\|F(\xi^n)\|_{\Gamma BV}\le\ R$. For any $n\in\mathbb{N}$ we have

$$\begin{aligned}&\sum_{i=1}^{k_n}\frac{|f(t_i^{(n)},u_i^{(n)})-f(t_{i-1}^{(n)},u_i^{(n)})|}{\gamma_{\sigma(i)}}\\ &\quad\le\sum_{i=1}^{k_n}\frac{|f(t_i^{(n)},u_i^{(n)})-f(t_{i-1}^{(n)},u_{i-1}^{(n)})|}{\gamma_{\sigma(i)}}+\sum_{i=1}^{k_n}\frac{|f(t_{i-1}^{(n)},u_{i-1}^{(n)})-f(t_{i-1}^{(n)},u_i^{(n)})|}{\gamma_{\sigma(i)}}\\ &\quad\le\sup_n\mathrm{var}_\Gamma(F(\xi^n))+\sum_{i=1}^{k_n}\frac{|f(t_{i-1}^{(n)},u_{i-1}^{(n)})-f(t_{i-1}^{(n)},u_i^{(n)})|}{\gamma_{\sigma(i)}}.\end{aligned}$$

Taking the supremum over $\sigma\in S_{k_n}$ we get

$$\begin{aligned}\sup_{\sigma\in S_{k_n}}\sum_{i=1}^{k_n}\frac{|f(t_i^{(n)},u_i^{(n)})-f(t_{i-1}^{(n)},u_i^{(n)})|}{\gamma_{\sigma(i)}}&\le\sup_n\mathrm{var}_\Gamma(F(\xi^n))\\ &+\sup_{\sigma\in S_{k_n}}\sum_{i=1}^{k_n}\frac{|f(t_{i-1}^{(n)},u_{i-1}^{(n)})-f(t_{i-1}^{(n)},u_i^{(n)})|}{\gamma_{\sigma(i)}}.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}&\sum_{i=1}^{k_n}\frac{|f(t_{i-1}^{(n)},u_i^{(n)})-f(t_{i-1}^{(n)},u_{i-1}^{(n)})|}{\gamma_{\sigma(i)}}\\ &\quad\le\sum_{i=1}^{k_n}\frac{|f(t_{i-1}^{(n)},u_i^{(n)})-f(t_i^{(n)},u_i^{(n)})|}{\gamma_{\sigma(i)}}+\sum_{i=1}^{k_n}\frac{|f(t_i^{(n)},u_i^{(n)})-f(t_{i-1}^{(n)},u_{i-1}^{(n)})|}{\gamma_{\sigma(i)}}\\ &\quad\le\sup_n\mathrm{var}_\Gamma(F(\xi^n))+\sum_{i=1}^{k_n}\frac{|f(t_{i-1}^{(n)},u_i^{(n)})-f(t_i^{(n)},u_i^{(n)})|}{\gamma_{\sigma(i)}}.\end{aligned}$$

Taking the supremum over $\sigma \in S_{k_n}$ we get

$$\sup_{\sigma\in S_{k_n}} \sum_{i=1}^{k_n} \frac{|f(t_{i-1}^{(n)}, u_i^{(n)}) - f(t_{i-1}^{(n)}, u_{i-1}^{(n)})|}{\gamma_{\sigma(i)}} \leq \sup_n \mathrm{var}_\Gamma(F(\xi^n)) + \sup_{\sigma\in S_{k_n}} \sum_{i=1}^{k_n} \frac{|f(t_{i-1}^{(n)}, u_i^{(n)}) - f(t_i^{(n)}, u_i^{(n)})|}{\gamma_{\sigma(i)}}.$$

Hence, $\sup_{\sigma\in S_{k_n}} \sum_{i=1}^{k_n} \frac{|f(t_{i-1}^{(n)},u_i^{(n)})-f(t_{i-1}^{(n)},u_{i-1}^{(n)})|}{\gamma_{\sigma(i)}}$ tends to infinity (as $n \to \infty$) if and only if $\sup_{\sigma\in S_{k_n}} \sum_{i=1}^{k_n} \frac{|f(t_{i-1}^{(n)},u_i^{(n)})-f(t_i^{(n)},u_i^{(n)})|}{\gamma_{\sigma(i)}}$ tends to infinity. By this observation, we get

$$\lim_{n\to+\infty} \sup_{\sigma\in S_{k_n}} \sum_{i=1}^{k_n} \frac{|f(t_{i-1}^{(n)}, u_i^{(n)}) - f(t_{i-1}^{(n)}, u_{i-1}^{(n)})|}{\gamma_{\sigma(i)}} = +\infty$$

Let us fix $n \in \mathbb{N}$ and let us consider the finer partition π with a sequence of additional points $\tau_i^{(n)} \in (t_{i-1}^{(n)}, t_i^{(n)})$, $i = 1, 2, \ldots, k_n$. Let us define a sequence (s_i^n) in the following manner:

$$s_0^n{:=}t_0^{(n)}, \quad s_1^n{:=}\tau_1^{(n)}, \quad s_2^n{:=}t_1^{(n)}, \quad s_3^n{:=}\tau_2^{(n)}, \ldots, s_{2k_n-1}^n{:=}\tau_{k_n}^{(n)}, \quad s_{2k_n}^n{:=}t_{k_n}^{(n)}.$$

Let us define two functions:

$$x^n(t) = \begin{cases} u_i^{(n)}, & \text{if} \quad t \in [s_{2i-2}^n, s_{2i-1}^n] \quad \text{for some} \quad i \in \{1, \ldots, k_n\}, \\ \frac{s_{2i}^n - t}{s_{2i}^n - s_{2i-1}^n} u_i^{(n)} + \frac{t - s_{2i-1}^n}{s_{2i}^n - s_{2i-1}^n} u_{i+1}^{(n)} & \text{if} \quad t \in [s_{2i-1}^n, s_{2i}^n] \quad \text{for some} \quad i \in \{1, \ldots, k_n - 1\}, \\ u_{k_n}^{(n)}, & \text{if} \quad t \geq s_{2k_n-1}^n, \end{cases}$$

and

$$y^n(t) = \begin{cases} \frac{s_{2i-1}^n - t}{s_{2i-1}^n - s_{2i-2}^n} u_{i-1}^{(n)} + \frac{t - s_{2i-2}^n}{s_{2i-1}^n - s_{2i-2}^n} u_i^{(n)} & \text{if} \quad t \in [s_{2i-2}^n, s_{2i-1}^n] \quad \text{for some} \quad i \in \{1, \ldots, k_n\}, \\ u_i^{(n)}, & \text{if} \quad t \in [s_{2i-1}^n, s_{2i}^n] \quad \text{for some} \quad i \in \{1, \ldots, k_n\}. \end{cases}$$

Actually, by defining the above functions we were going to build two functions that are continuous, piecewise linear and constant in the left (the case of $x^{(n)}$) or in the right (the case of $y^{(n)}$) part of each interval $[t_{i-1}^{(n)}, t_i^{(n)}]$, and achieving values $u_i^{(n)}$ in points $t_i^{(n)}$ of the partition π.

We have $x^n(0) = u_1^{(n)}$, $y^n(0) = u_0^{(n)}$ and

$$\mathrm{var}_\Lambda(x^n) \leq \mathrm{var}_\Lambda(\xi^{(n)}) \leq r,$$

and

$$\mathrm{var}_\Lambda(y^n) \leq \mathrm{var}_\Lambda(\xi^{(n)}) \leq r.$$

Hence, $\|x^n\|_{\Lambda BV} \leq 2r$ and $\|y^n\|_{\Lambda BV} \leq 2r$ for each $n \in \mathbb{N}$. By the local boundedness of F we get $\|F(z^n)\|_{\Gamma BV} \leq R$ for $z^n \in \{x^n, y^n\}$. On the other hand, we have

$$\begin{aligned}
&\sum_{i=1}^{k_n} \frac{|f(t_{i-1}^{(n)}, u_i^{(n)}) - f(t_{i-1}^{(n)}, u_{i-1}^{(n)})|}{\gamma_{\sigma(i)}} \\
&\quad \leq \sum_{i=1}^{k_n} \frac{|f(t_{i-1}^{(n)}, u_i^{(n)}) - f(\tau_i^{(n)}, u_i^{(n)})|}{\gamma_{\sigma(i)}} + \sum_{i=1}^{k_n} \frac{|f(\tau_i^{(n)}, u_i^{(n)}) - f(t_{i-1}^{(n)}, u_{i-1}^{(n)})|}{\gamma_{\sigma(i)}} \\
&\quad = \sum_{i=1}^{k_n} \frac{|f(t_{i-1}^{(n)}, x^n(t_{i-1}^{(n)})) - f(\tau_i^{(n)}, x^n(\tau_i^{(n)}))|}{\gamma_{\sigma(i)}} \\
&\qquad + \sum_{i=1}^{k_n} \frac{|f(\tau_i^{(n)}, y^n(\tau_i^{(n)})) - f(t_{i-1}^{(n)}, y^n(t_{i-1}^{(n)}))|}{\gamma_{\sigma(i)}} \\
&\quad \leq \operatorname{var}_\Gamma(F(x^n)) + \operatorname{var}_\Gamma(F(y^n)) \leq 2R,
\end{aligned}$$

which contradicts (2).

4. Continuity of the autonomous superposition operator

Now we are going to concentrate on the continuity of an autonomous superposition operator mapping the space $\Lambda BV(I)$ into itself.

First, we will show that the space $\Lambda BV(I)$ is a Banach algebra under a certain norm, equivalent to the norm $\|\cdot\|_{\Lambda BV}$.

Let us consider an arbitrary Waterman sequence $(\lambda_n)_{n\in\mathbb{N}}$. If that sequence is bounded by M, then for any λ_i, $i \in \mathbb{N}$, we have

$$\frac{1}{M} \leq \frac{1}{\lambda_i} \leq \frac{1}{\lambda_1}$$

and, therefore, the norm $\|\cdot\|_{\Lambda BV}$ is equivalent to the norm $\|\cdot\|_{BV}$, what means that $\Lambda BV(I) = BV(I)$ and this space is a Banach algebra under certain equivalent norm.

If the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is *proper*, that is, unbounded, then by Lemma 1 there exists such constant $\tilde{c}_\Lambda \geq 1$ that for any function $x \in \Lambda BV(I)$ we have

$$\|x\|_\infty \leq \tilde{c}_\Lambda \|x\|_{\Lambda BV}.$$

Hence, for any two functions $x, y \in \Lambda BV(I)$, any partition $\pi = \{c_0, c_1, c_2, \ldots, c_n\}$ of I and any permutation $\sigma \in S_n$ we have

$$\begin{aligned}
&|x(0)y(0)| + \sum_{k=1}^{n} \frac{|x(c_i)y(c_i) - x(c_{i-1})y(c_{i-1})|}{\lambda_{\sigma(i)}} \\
&\quad \leq \|y\|_\infty \left(|x(0)| + \sum_{i=1}^{n} \frac{|x(c_i) - x(c_{i-1})|}{\lambda_{\sigma(i)}} \right)
\end{aligned}$$

$$+ \|x\|_\infty \left(|y(0)| + \sum_{i=1}^{n} \frac{|y(c_i) - y(c_{i-1})|}{\lambda_{\sigma(i)}} \right)$$
$$\leq \|y\|_\infty \|x\|_{\Lambda BV} + \|x\|_\infty \|y\|_{\Lambda BV}$$
$$\leq 2\tilde{c}_\Lambda \|x\|_{\Lambda BV} \|y\|_{\Lambda BV} .$$

Therefore, we get

$$\|xy\|_{\Lambda BV} \leq 2\tilde{c}_\Lambda \|x\|_{\Lambda BV} \|y\|_{\Lambda BV} .$$

Let us multiply both sides of the above inequality by $2\tilde{c}_\Lambda$ and let us denote $\|x\|'_{\Lambda BV} = 2\tilde{c}_\Lambda \|x\|_{\Lambda BV}$. Then we get

$$\|xy\|'_{\Lambda BV} \leq \|x\|'_{\Lambda BV} \|y\|'_{\Lambda BV}$$

and, therefore, $\Lambda BV(I)$ is a Banach algebra under a norm, $\|\cdot\|'_{\Lambda BV}$ which is equivalent to the norm $\|\cdot\|_{\Lambda BV}$.

We begin our considerations concerning the problem of continuity of autonomous superposition operators with the case of analytic generators of such operators.

Theorem 8. *Let $f \colon \mathbb{R} \to \mathbb{R}$ be a sum of a power series centered at 0 with the radius of convergence $\rho = +\infty$, that is, there exist real numbers $a_0, a_1, \ldots$ such that*

$$f(u) = \sum_{i=0}^{\infty} a_i u^i \quad \textit{for} \quad u \in \mathbb{R}.$$

Then the autonomous superposition operator F, generated by f, which maps the Banach space $\Lambda BV(I)$ into itself is continuous.

Proof. Let us note that the operator F is well defined, since the function f satisfies a local Lipschitz condition (see Remark 2).

Write

$$f_n(u) = \sum_{i=1}^{n} a_i u^i, \qquad g_n(u) = f_n'(u) = \sum_{i=1}^{n} i a_i u^{i-1},$$

where $u \in \mathbb{R}$ and $u^0 = 1$. Furthermore, let $x^0(t) \equiv 1$ and

$$F_n(x) = \sum_{i=1}^{n} a_i x^i \quad \text{for every } n \in \mathbb{N} \text{and } x \in \Lambda BV(I).$$

Since $\Lambda BV(I)$ is a Banach algebra under a certain norm equivalent to $\|\cdot\|_{\Lambda BV}$, the mapping $F_n \colon \Lambda BV(I) \to \Lambda BV(I)$ is continuous. Therefore, to show the continuity of F it suffices to show that the sequence of mappings $(F_n)_{n\in\mathbb{N}}$ converges to F uniformly on bounded sets. Since the function $u \mapsto (f_n - f)(u)$ satisfies the Lipschitz condition on every interval $[-a, a]$ with the constant $L_n(a) = \sup_{u\in[-a,a]} |f'(u) - g_n(u)|$, we obtain

$$\mathrm{var}_\Lambda(F_n(x) - F(x)) \leq L_n(b)\mathrm{var}_\Lambda(x),$$

where $b = \tilde{c}_\Lambda \|x\|_{\Lambda BV}$, hence

$$\begin{aligned}\|F_n(x) - F(x)\|_{\Lambda BV} &= |f_n(x(0)) - f(x(0))| + \mathrm{var}_\Lambda(F_n(x) - F(x)) \\ &\leq |f_n(x(0)) - f(x(0))| + L_n(a\tilde{c}_\Lambda)\mathrm{var}_\Lambda(x),\end{aligned}$$

which shows that $F_n(x) \to F(x)$ as $n \to +\infty$ uniformly for every $x \in \overline{B}_{\Lambda BV}(0,a)$, where a is an arbitrary, fixed real number.

Now, we are going to consider generators of C^1-class. For that we will use the well-known concept of Bernstein polynomials. For the definition and properties (especially those required in the proof below) of Bernstein polynomials we would like to refer the reader to [5] and references therein.

Theorem 9. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Then the autonomous superposition operator $F\colon \Lambda BV(I) \to \Lambda BV(I)$, generated by the function f, is continuous.*

Proof. First observe that the operator F is well defined due to the continuity of f' and the mean-value theorem.

To show its continuity we will approximate F by an almost uniformly convergent sequence of continuous mappings on the space $\Lambda BV(I)$.

For a given $a > 0$ let ϕ_a denote the restriction of f to the interval $[-a,a]$. Moreover, let $F_n\colon \overline{B}_{\Lambda BV}(0,r) \to \Lambda BV(I)$ be the autonomous superposition operator generated by the nth Bernstein polynomial $B_n^a(\phi_a)$ of the function ϕ_a, where $a = \tilde{c}_\Lambda r$. Since $\Lambda BV(I)$ is a Banach algebra, the operators F_n are continuous.

Now we are going to show that the sequence $(F_n)_{n\in\mathbb{N}}$ converges uniformly to F on $\overline{B}_{\Lambda BV}(0,r)$. Note that the function $u \mapsto [f - B_n^a(\phi_a)](u)$ satisfies the Lipschitz condition on the interval $[-a,a]$ with the constant $L_n(a) = \sup_{u\in[-a,a]}|f'(u) - \frac{\mathrm{d}}{\mathrm{d}u}B_n^a(\phi_a)(u)|$, and thus we have

$$\mathrm{var}_\Lambda(F(x) - F_n(x)) \leq L_n(a)\mathrm{var}_\Lambda(x) \quad \text{for } x \in \overline{B}_{\Lambda BV}(0,r).$$

Therefore, by [5] Proposition 1 and Proposition 2, for $x \in \overline{B}_{\Lambda BV}(0,r)$, we get

$$\begin{aligned}\|F(x) - F_n(x)\|_{\Lambda BV} &\leq |f(x(0)) - B_n^a(\phi_a)(x(0))| + L_n(a)\mathrm{var}_\Lambda(x) \\ &\leq |\phi_a(x(0)) - B_n^a(\phi_a)(x(0))| + rL_n(a) \to 0,\end{aligned}$$

as $n \to +\infty$, which ends the proof.

We should note here that we cannot expect a similar result to hold for the codomain being the proper subspace of the domain. To show it, first we introduce some useful notation. Given two Waterman sequences $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$ and $\Gamma = (\gamma_n)_{n\in\mathbb{N}}$ we will write $\Gamma < \Lambda$, whenever

$$\lim_{n\to+\infty} \frac{\gamma_n}{\lambda_n} = 0.$$

It is quite easy to show that if $\Gamma < \Lambda$, then

$$\lim_{n\to+\infty}\frac{\sum_j^n \frac{1}{\lambda_j}}{\sum_j^n \frac{1}{\gamma_j}}=0,$$

what implies that $\Gamma BV(I) \subsetneq \Lambda BV(I)$ (see [13], Theorem 3).

We may also observe that since there exists such $M>0$ that $\frac{1}{\lambda_n}\le M\frac{1}{\gamma_n}$ we have

$$\|x\|_{\Lambda BV}\le M\,\|x\|_{\Gamma BV},$$

for all $x\in\Gamma BV(I)$, which makes the inclusion $\Gamma BV(I) \subsetneq \Lambda BV(I)$ a continuous map.

Theorem 10. *If $\Gamma<\Lambda$ is a Waterman sequence and $F:\Lambda BV(I)\to\Gamma BV(I)$ is an autonomous superposition operator, generated by $f:\mathbb{R}\to\mathbb{R}$, then f is a constant function.*

Proof. We may assume that $\gamma_n\ge 1$ for every $n\in\mathbb{N}$. We will prove that f is differentiable at every point of its domain and that its derivative is equal to zero.

Assume, contrary to our claim, that there exists $x_0\in\mathbb{R}$, $\varepsilon>0$ and a sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers such that $x_n\to 0$ and

$$\left|\frac{f(x_0+x_n)-f(x_0)}{x_n}\right|\ge\varepsilon. \tag{3}$$

We may assume that $(x_n)_{n\in\mathbb{N}}$ is monotone and, moreover, that all terms are of the same sign. To abbreviate a notation, let us assume that they are positive. Since $(x_n)_{n\in\mathbb{N}}$ tends to zero, there exists an infinite set $P\subseteq\mathbb{N}$ such that for every $p\in P$ there is m such that $p\le x_m^{-1}<p+1$.

Now, we are going to define some sequences $(N_k)_{k\in\mathbb{N}}, (m_k)_{k\in\mathbb{N}}, (p_k)_{k\in\mathbb{N}}$ of positive integers. Put $N_1=1$. Choose a positive integer $N_2>N_1$ such that $\lambda_n\ge 2\gamma_n$ for every $n\ge N_2$ and

$$p_1\le\sum_{n=N_1}^{N_2-1}\frac{1}{\gamma_n}<p_1+1 \tag{4}$$

for some positive integer $p_1\in P, p_1\ge 2$. Such N_2 must exist since $\sum_{n=N_1}^{\infty}\frac{1}{\gamma_n}=+\infty$ and $\gamma_n\ge 1$ for every positive integer n. Let m_1 be such a positive integer that $p_1\le x_{m_1}^{-1}<p_1+1$.

Assume that $k>1$ and that N_k, m_{k-1}, p_{k-1} are already defined. Choose a positive integer $N_{k+1}>N_k$ such that $\lambda_n\ge 2^k\gamma_n$ for every $n\ge N_{k+1}$ and

$$p_k\le\sum_{N_k}^{N_{k+1}-1}\frac{1}{\gamma_n}<p_k+1 \tag{5}$$

for some positive integer $p_k\in P$ such that $p_k>\max\{p_{k-1},2^k\}$. As above, such N_{k+1} must exist since $\sum_{n=N_k}^{\infty}\frac{1}{\gamma_n}=+\infty$ and $\gamma_n\ge 1$ for every positive integer n. Let m_k be such a positive integer that $p_k\le x_{m_k}^{-1}<p_k+1$, that is

$$\frac{1}{p_k+1}<x_{m_k}\le\frac{1}{p_k}. \tag{6}$$

Let us notice that

$$\frac{1}{p_k} < \frac{1}{2^k} \tag{7}$$

for every positive integer k. Put $y_n = x_{m_k}$ for n such that $N_k \le n < N_{k+1}$. By (5) and (6), we get

$$\sum_{n=1}^{\infty} \frac{y_n}{\gamma_n} = \sum_{k=1}^{\infty} x_{m_k} \sum_{n=N_k}^{N_{k+1}-1} \frac{1}{\gamma_n} > \sum_{k=1}^{\infty} \frac{1}{p_k+1} \cdot p_k = +\infty. \tag{8}$$

Using additionally (7) and the fact that $\lambda_n \ge 2^{k-1}\gamma_n$ for every $n \ge N_k$, we get

$$\sum_{n=1}^{\infty} \frac{y_n}{\lambda_n} \le \sum_{k=1}^{\infty} \frac{x_{m_k}}{2^{k-1}} \sum_{n=N_k}^{N_{k+1}-1} \frac{1}{\gamma_n} < \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} \frac{p_k+1}{p_k} < +\infty. \tag{9}$$

Now we are ready to define a function $g : [0,1] \to \mathbb{R}$ such that $g \in \Lambda BV(I)$ and $F(g) \notin \Gamma BV(I)$ what will contradict our assumption. Let

$$g(t) = \begin{cases} y_n + x_0 & \text{if } t = \frac{1}{2n} \text{for} n \in \mathbb{N}, \\ x_0 & \text{if } t \neq \frac{1}{2n}. \end{cases}$$

Let $t_n = \frac{1}{n}$ for $n \in \mathbb{N}$. The function g is of bounded Λ-variation if and only if $\sum_{j=1}^{\infty} \frac{|g(t_{j+1})-g(t_j)|}{\lambda_j} < \infty$; however, by (9), we get

$$\sum_{n=1}^{\infty} \frac{|g(t_{n+1}) - g(t_n)|}{\lambda_n} \le \sum_{n=1}^{\infty} \frac{|g(t_{2n}) - g(t_{2n-1})| + |g(t_{2n+1}) - g(t_{2n})|}{\lambda_n} < +\infty$$

since $|g(t_{2n}) - g(t_{2n-1})| = |g(t_{2n+1}) - g(t_{2n})| = y_n$. Furthermore, by (3) and (8), we have

$$\sum_{n=1}^{\infty} \frac{|F(g)(t_{2n}) - F(g)(t_{2n-1})|}{\gamma_n} = \sum_{n=1}^{\infty} \frac{|f(g(t_{2n})) - f(g(t_{2n-1}))|}{\gamma_n} =$$

$$= \sum_{n=1}^{\infty} \frac{|f(x_0+y_n) - f(x_0)|}{\gamma_n} \ge \varepsilon \sum_{n=1}^{\infty} \frac{y_n}{\gamma_n} = \infty.$$

It means that $F(g)$ is not of bounded Γ-variation, which gives a contradiction.

We will end this section considering the general case of continuous functions $f : \mathbb{R} \to \mathbb{R}$. Since f is not necessarily Lipschitz continuous, then the superposition operator generated by f and defined in $\Lambda BV(I)$ does not necessarily act in $\Lambda BV(I)$. However, we can show that it acts in some space $\Gamma BV(I)$. For that, we will need the following three lemmas.

Let $\Gamma_k = (\gamma_n^k)_{n\in\mathbb{N}}$ for $k \in \mathbb{N}$.

Lemma 2. *Assume that Γ_k is a Waterman sequence for every $k \in \mathbb{N}$ and that*

$$\gamma_n^{k+1} \ge \gamma_n^k \tag{10}$$

for $n, k \in \mathbb{N}$. Then there exists a Waterman sequence Γ such that $\Gamma_k BV(I) \subseteq \Gamma BV(I)$ for all $k \in \mathbb{N}$.

Proof. There exists a sequence $(N_k)_{k\in\mathbb{N}}$ of positive integers such that $N_1 = 1$ and

$$\sum_{n=N_k}^{N_{k+1}-1} \frac{1}{\gamma_n^k} \geq 1.$$

Let us define $\gamma_n = \gamma_n^k$ for $N_k \leq n < N_{k+1}$. The sequence Γ is monotone by (10) and the monotonicity of each Γ_k. Furthermore, $\sum_{n=1}^{\infty} \frac{1}{\gamma_n} = +\infty$, by the definition of the sequence $(N_k)_{k\in\mathbb{N}}$ and therefore Γ is a Waterman sequence.

Inequality (10) implies that for every $k \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $\gamma_n^k \leq \gamma_n$ for $n \geq N$ and hence $\Gamma_k BV(I) \subseteq \Gamma BV(I)$ for every positive integer k, which ends the proof.

Lemma 3. *If $(w_n)_{n\in\mathbb{N}}$ is a monotone sequence of positive real numbers tending to zero and Γ is a Waterman sequence, then there exists a Waterman sequence $\Gamma' = (\gamma'_n)_{n\in\mathbb{N}}$ such that $\gamma'_n \geq \gamma_n$ for every positive integer n and $\sum_{n=1}^{\infty} \frac{w_n}{\gamma'_n} < +\infty$.*

Proof. Let $(N_k)_{k\in\mathbb{N}}$ be such a strongly increasing sequence of positive integers that

(i) $N_1 = 1$;
(ii) $\forall_{k>1} \forall_{n\geq N_k} w_n \leq \frac{1}{2^k}$;
(iii) $\forall_{k\geq 1} \sum_{n=N_{k+1}}^{N_{k+2}-1} \frac{1}{\gamma_n} \geq \sum_{n=N_k}^{N_{k+1}-1} \frac{1}{\gamma_n} \geq 1$.

Let us define

$$\gamma'_n = \left(\sum_{j=N_k}^{N_{k+1}-1} \frac{1}{\gamma_j} \right) \gamma_n$$

for $N_k \leq n < N_{k+1}$ and $k \in \mathbb{N}$. It is readily seen from (iii) that Γ' is monotone and $\gamma'_n \geq \gamma_n$ for every positive integer n. It is a Waterman sequence, since

$$\sum_{n=1}^{\infty} \frac{1}{\gamma'_n} = \sum_{k=1}^{\infty} \sum_{n=N_k}^{N_{k+1}-1} \frac{1}{\gamma'_n} = \sum_{k=1}^{\infty} \sum_{n=N_k}^{N_{k+1}-1} \left(\sum_{j=N_k}^{N_{k+1}-1} \frac{1}{\gamma_j} \right)^{-1} \frac{1}{\gamma_n} = +\infty.$$

Finally, we get

$$\sum_{n=2}^{\infty} \frac{w_n}{\gamma'_n} = \sum_{k=2}^{\infty} \sum_{n=N_k}^{N_{k+1}-1} \frac{w_n}{\gamma'_n} \leq \sum_{k=2}^{\infty} \frac{1}{2^k} \sum_{n=N_k}^{N_{k+1}-1} \left(\sum_{N_k}^{N_{k+1}-1} \frac{1}{\gamma_j} \right)^{-1} \frac{1}{\gamma_n} = \sum_{k=2}^{\infty} \frac{1}{2^k} < +\infty.$$

Lemma 4. *If Λ is a Waterman sequence, then there exists a Waterman sequence $\Lambda' = ({\lambda'}_n)_{n\in\mathbb{N}}$ such that $\Lambda < \Lambda'$. Moreover, the sequence $(\frac{\lambda_n}{\lambda'_n})_{n\in\mathbb{N}}$ is monotone.*

Proof. Let $(N_k)_{k\in\mathbb{N}}$ be such a sequence that $N_1 = 1$ and $\sum_{n=N_k}^{N_{k+1}-1} \frac{1}{\lambda_n} \geq 2^k$ for $k \in \mathbb{N}$. Let us define $\lambda'_n = 2^k \lambda_n$ for $N_k \leq n < N_{k+1}$ and $k \in \mathbb{N}$. Obviously, Λ' is monotone, $\frac{\lambda_n}{\lambda'_n} \to 0$, as $n \to \infty$ and

$$\sum_{n=1}^{\infty} \frac{1}{\lambda'_n} = \sum_{k=1}^{\infty} \sum_{n=N_k}^{N_{k+1}-1} \frac{1}{\lambda'_n} = \sum_{k=1}^{\infty} \frac{1}{2^k} \sum_{n=N_k}^{N_{k+1}-1} \frac{1}{\lambda_n} = +\infty$$

Hence, Λ' is a Waterman sequence having all the desired properties.

Theorem 11. *If $f : \mathbb{R} \to \mathbb{R}$ is continuous, F is the autonomous superposition operator, generated by f and Λ is a Waterman sequence, then there exists a Waterman sequence Γ such that $F(\Lambda BV(I)) \subseteq \Gamma BV(I)$.*

Proof. First, let $\Lambda' = (\lambda'_n)_{n\in\mathbb{N}}$ be the Waterman sequence given in Lemma 4. For every positive integer k let ω_k denote a modulus of continuity of the function $f|_{[-k,k]}$. We have $\omega_k(\delta) \to 0$, as $\delta \to 0$, because obviously a continuous function on compact interval is uniformly continuous.

We shall define a sequence $(\Gamma_k)_{k\in\mathbb{N}}$ of Waterman sequences such that $\gamma_n^k \le \gamma_n^{k+1}$ for all positive integers n, k. Let Γ_1 be a Waterman sequence such that

(i) $\gamma_n^1 \ge \lambda'_n$;

(ii) $\sum_{n=1}^{\infty} \frac{\omega_1(\frac{\lambda_n}{\lambda'_n})}{\gamma_n^1} < +\infty$.

Assume that Γ_k is already defined. Let Γ_{k+1} be a Waterman sequence such that

(i) $\gamma_n^{k+1} \ge \gamma_n^k$;

(ii) $\sum_{n=1}^{\infty} \frac{\omega_{k+1}(\frac{\lambda_n}{\lambda'_n})}{\gamma_n^{k+1}} < +\infty$.

The above construction is possible by Lemma 3. Let Γ be such Waterman sequence that $\Gamma_k BV(I) \subseteq \Gamma BV(I)$ for all $k \in \mathbb{N}$. Lemma 2 implies the existence of such a sequence.

Now it suffices to prove that if $x : I \to \mathbb{R}$ is of bounded Λ-variation, then $F(x)$ is of bounded Γ-variation. Let $x \in \Lambda BV(I)$. Obviously, there exists a positive integer k such that $|x(t)| \le k$ for $t \in [0, 1]$. Let $(I_n)_{n\in\mathbb{N}} \subseteq [0, 1]$ be a sequence of nonoverlapping intervals. Let us define $A = \{n \in \mathbb{N} : |x(I_n)| \le \frac{\lambda_n}{\lambda'_n}\}$. Let us notice that $\sum_{n\notin A} \frac{1}{\lambda'_n} \le \sum_{n\notin A} \frac{|x(I_n)|}{\lambda_n} < +\infty$, since $x \in \Lambda BV(I)$. Using this fact, properties (i), (ii), and the monotonicity of ω_k for every $k \in \mathbb{N}$, we get

$$\sum_{n\in\mathbb{N}} \frac{|(f\circ x)(I_n)|}{\gamma_n^k} \le \sum_{n\in\mathbb{N}} \frac{\omega_k(|x(I_n)|)}{\gamma_n^k} = \sum_{n\in A} \frac{\omega_k(|x(I_n)|)}{\gamma_n^k} + \sum_{n\notin A} \frac{\omega_k(|x(I_n)|)}{\gamma_n^k}$$

$$\le \sum_{n\in A} \frac{\omega_k(\frac{\lambda_n}{\lambda'_n})}{\gamma_n^k} + \sum_{n\notin A} \frac{\omega_k(2k)}{\gamma_n^k} \le \sum_{n\in A} \frac{\omega_k(\frac{\lambda_n}{\lambda'_n})}{\gamma_n^k} + \omega_k(2k) \sum_{n\notin A} \frac{1}{\lambda'_n} < +\infty.$$

Hence, $F(x)$ is of bounded Γ_k-variation and, therefore, it is also of bounded Γ-variation.

Now we are going to prove that the superposition operator considered above is actually continuous. We will use again Bernstein polynomials and will keep the notation used in Theorem 9. More precisely, we will need one more property of Bernstein polynomials which we will state below. But first let us remind the important definition.

Definition 6. (cf. [7],p. 40) The optimal modulus of continuity of a function $f\colon [-a,a] \to \mathbb{R}$ is a function $\omega_f\colon [0,2a] \to \mathbb{R}$ given by

$$\omega_f(\delta) = \sup\big\{|f(t)-f(s)| : t,s \in [-a,a] \text{ and } |t-s| \le \delta\big\}, \qquad \delta \ge 0.$$

Proposition 2. *(cf. [10]) If $\phi_a\colon [-a,a] \to \mathbb{R}$ is a continuous function with the optimal modulus of continuity ω_{ϕ_a}, then $\omega_{B_n^a(\phi_a)}(t) \le 4\omega_{\phi_a}(t)$ for $t \ge 0$.*

As an obvious consequence of the above proposition, we get

$$\omega_{\phi_a - B_n^a(\phi_a)}(t) \le 5\omega_{\phi_a}(t), \quad \text{for} \quad t \in [-a,a].$$

An easy observation (the proof will be omitted) is the following.

Proposition 3. $\omega_{\phi_a - B_n^a(\phi_a)}(2a) \to 0$ *as* $n \to +\infty$.

The following theorem refers to the superposition operator acting from the space $\Lambda BV(I)$ to certain space $\Gamma BV(I)$, which actually depends on the generator f (as it is constructed in Theorem 11).

Theorem 12. *If $f : \mathbb{R} \to \mathbb{R}$ is continuous, F is the autonomous superposition operator, generated by f and Λ is a Waterman sequence, then there exists such a Waterman sequence Γ that the autonomous superposition operator $F\colon \Lambda BV(I) \to \Gamma BV(I)$ is continuous.*

Proof. Let Γ and Γ_k ($k \in \mathbb{N}$) be the Waterman sequences constructed in the proof of Theorem 11. Hence, $F(\Lambda BV(I)) \subseteq \Gamma BV(I)$ and we may focus on the proof of the continuity of the superposition operator F.

Let us fix $r > 0$ and put $a = \tilde{c}_\Lambda r$. Let us define $F_n\colon \overline{B}_{\Lambda BV}(0,r) \to \Gamma BV(I)$ as the sequence of autonomous superposition operators, generated by Bernstein polynomials $B_n^a(\phi_a)$, where $\phi_a = f|_{[-a,a]}$. Let us fix $k > a$. Since $\Gamma_k BV(I)$ is a Banach algebra, we conclude that the superposition operator F_n is continuous, for every $n \in \mathbb{N}$. Now we are going to show that $F_n(x)$ converges to $F(x)$, uniformly in view of $x \in \overline{B}_{\Lambda BV}(0,r)$, in the norm of the space $\Gamma_k BV(I)$ (and hence in the norm of the space $\Gamma BV(I)$).

Let us take any $x \in \overline{B}_{\Lambda BV}(0,r)$ and estimate the norm $\|F(x) - F_n(x)\|_{\Gamma_k BV}$. Let us take any collection $(I_m)_{m\in\mathbb{N}}$ of nonoverlapping intervals, contained in I. Let Λ' be the Waterman sequence given as in the Lemma 4; without loss of generality we may assume that $\frac{\lambda_m}{\lambda'_m} \le 2a$ for every $m \in \mathbb{N}$. Let $A = \{m \in \mathbb{N} : |x(I_m)| \le \frac{\lambda_m}{\lambda'_m}\}$. As above one can check that $\sum_{m\notin A} \frac{1}{\lambda'_m} \le \sum_{m\notin A} \frac{|x(I_m)|}{\lambda_m} \le \|x\|_{\Lambda BV} \le r$. Hence, we get

$$\begin{aligned}
\sum_{m\in\mathbb{N}} \frac{|((\phi_a - B_n^a(\phi_a)) \circ x)(I_m)|}{\gamma_m^k} &\le \sum_{m\in\mathbb{N}} \frac{\omega_{\phi_a - B_n^a(\phi_a)}(|x(I_m)|)}{\gamma_m^k} \\
&= \sum_{m\in A} \frac{\omega_{\phi_a - B_n^a(\phi_a)}(|x(I_m)|)}{\gamma_m^k} + \sum_{m\notin A} \frac{\omega_{\phi_a - B_n^a(\phi_a)}(|x(I_m)|)}{\gamma_m^k} \\
&\le \sum_{m\in\mathbb{N}} \frac{\omega_{\phi_a - B_n^a(\phi_a)}\big(\frac{\lambda_m}{\lambda'_m}\big)}{\gamma_m^k} + \omega_{\phi_a - B_n^a(\phi_a)}(2a) \sum_{m\notin A} \frac{1}{\lambda'_m}
\end{aligned}$$

$$\leq \sum_{m\in\mathbb{N}} \frac{\omega_{\phi_a - B_n^a(\phi_a)}\left(\frac{\lambda_m}{\lambda'_m}\right)}{\gamma_m^k} + r\omega_{\phi_a - B_n^a(\phi_a)}(2a).$$

Now, let us concentrate on the sum

$$\sum_{m\in\mathbb{N}} \frac{\omega_{\phi_a - B_n^a(\phi_a)}\left(\frac{\lambda_m}{\lambda'_m}\right)}{\gamma_m^k}.$$

Fix $\varepsilon > 0$. Since the series $\sum_{m\in\mathbb{N}} \frac{\omega_k\left(\frac{\lambda_m}{\lambda'_m}\right)}{\gamma_m^k}$ converges, we can find such $m_0 \in \mathbb{N}$ that

$$\sum_{m\geq m_0} \frac{5\omega_k\left(\frac{\lambda_m}{\lambda'_m}\right)}{\gamma_m^k} < \frac{\varepsilon}{3}$$

Then also

$$\sum_{m\geq m_0} \frac{\omega_{\phi_a - B_n^a(\phi_a)}\left(\frac{\lambda_m}{\lambda'_m}\right)}{\gamma_m^k} \leq \sum_{m\geq m_0} \frac{5\omega_k\left(\frac{\lambda_m}{\lambda'_m}\right)}{\gamma_m^k} < \frac{\varepsilon}{3}.$$

Now let us take such large $n \in \mathbb{N}$ that

$$\omega_{\phi_a - B_n^a(\phi_a)}(2a) \sum_{m\notin A} \frac{1}{\lambda'_m} < \frac{\varepsilon}{3},$$

and

$$\sum_{m\leq m_0} \frac{\omega_{\phi_a - B_n^a(\phi_a)}\left(\frac{\lambda_m}{\lambda'_m}\right)}{\gamma_m^k} \leq \omega_{\phi_a - B_n^a(\phi_a)}(2a) \sum_{m\leq m_0} \frac{1}{\gamma_m^k} < \frac{\varepsilon}{3}.$$

Now for n big enough we have

$$\sum_{m\in\mathbb{N}} \frac{|((\phi_a - B_n^a(\phi_a)) \circ x)(I_m)|}{\gamma_m^k}$$

$$\leq r\omega_{\phi_a - B_n^a(\phi_a)}(2a) + \sum_{m\geq m_0} \frac{\omega_{\phi_a - B_n^a(\phi_a)}\left(\frac{\lambda_m}{\lambda'_m}\right)}{\gamma_m^k} + \sum_{m\leq m_0} \frac{\omega_{\phi_a - B_n^a(\phi_a)}\left(\frac{\lambda_m}{\lambda'_m}\right)}{\gamma_m^k} < \varepsilon.$$

which proves that F_n converges to F, as $n \to \infty$, uniformly on the ball $\overline{B}_{\Lambda BV}(0, r)$. It implies the continuity of the operator F.

5. Applications

As applications of our results from previous sections we are going to give some theorems concerning the so-called ΛBV-solutions to some nonlinear integral equations.

5.1. Hammerstein integral equation

In this subsection we will be interested in the problem of the existence of solutions of bounded Λ-variation to the following nonlinear Hammerstein integral equation

$$x(t) := \lambda \int_0^1 k(t,s) f(x(s)) \mathrm{d}s, \qquad t \in I, \tag{11}$$

where $\lambda \in \mathbb{R}$. Let us make the following assumptions:

1 ° $f\colon \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function which is sublinear, that is, $\lim_{u\to\infty} |f(u)|/|u| = 0$;

2 ° the kernel $k\colon I \times I \to \mathbb{R}$ is such that:

(a) for every $t \in I$ the function $s \mapsto k(t,s)$ is Lebesgue measurable;

(b) the function $s \mapsto k(0,s)$ is Lebesgue integrable;

(c) there exists such a Waterman sequence Λ that $\mathrm{var}_\Lambda(k(\cdot,s)) \le m(s)$ for a. e. $s \in I$, where $m\colon I \to [0,+\infty]$ is a Lebesgue integrable function.

Remark 3. Let us note that if the function k satisfies assumptions 2 ° (a)–(c), then for every $t \in I$, the function $s \mapsto k(t,s)$ is Lebesgue integrable on I. Indeed

$$\begin{aligned} |k(t,s)| &\le |k(0,s)| + |k(0,s) - k(t,s)| \le |k(0,s)| + \lambda_1 \mathrm{var}_\Lambda(k(\cdot,s)) \\ &\le \lambda_1 m(s) + |k(0,s)| \text{ for a.e. } s \in I, \end{aligned}$$

which confirms our claim.

Proposition 4. *Let $k\colon I \times I \to \mathbb{R}$ satisfy 2 °(a) – (c). Then the operator K, defined by the formula*

$$Kx(t) = \int_0^1 k(t,s)x(s)\mathrm{d}s \quad t \in I, \quad x \in \Lambda BV(I), \tag{12}$$

maps the space $\Lambda BV(I)$ into $\Lambda BV(I)$ and is compact.

Proof. Let $x \in \Lambda BV(I)$. Then, by Remark 3, we have

$$\left| \int_0^1 k(t,s)x(s)\mathrm{d}s \right| \le \int_0^1 |k(t,s)x(s)|\mathrm{d}s \le \|x\|_\infty m_1 \le \tilde{c}_\Lambda m_1 \|x\|_{\Lambda BV},$$

where $m_1 = \int_0^1 (\lambda_1 m(s) + |k(0,s)|)\mathrm{d}s$. Hence, $Kx(t)$ exists and is finite for every $t \in I$. Moreover, for any partition $\pi = \{t_0, t_1, \ldots, t_n\}$ of the interval I and any permutation $\sigma \in S_n$ we have

$$\begin{aligned} \sum_{i=1}^n \frac{|Kx(t_i) - Kx(t_{i-1})|}{\lambda_{\sigma(i)}} &\le \int_0^1 \sum_{i=1}^n \frac{|(k(t_i,s) - k(t_{i-1},s))x(s)|}{\lambda_{\sigma(i)}} \mathrm{d}s \\ &\le \int_0^1 \mathrm{var}_\Lambda(k(\cdot,s))|x(s)|\mathrm{d}s \\ &\le \|x\|_\infty \int_0^1 m(s)\mathrm{d}s \le \tilde{c}_\Lambda m_2 \|x\|_{\Lambda BV}, \end{aligned}$$

where $m_2 = \int_0^1 m(s)\mathrm{d}s$. Since

$$\|Kx\|_{\Lambda BV} \leq \int_0^1 |k(0,s)x(s)|\mathrm{d}s + \tilde{c}_\Lambda m_2 \|x\|_{\Lambda BV} \\ \leq \tilde{c}_\Lambda \left(\int_0^1 |k(0,s)|\mathrm{d}s + m_2 \right) \|x\|_{\Lambda BV} = c \|x\|_{\Lambda BV},$$

where

$$c = \tilde{c}_\Lambda \left(\int_0^1 |k(0,s)|\mathrm{d}s + m_2 \right),$$

we conclude that K is well defined and continuous.

Now we will show that K is compact. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\overline{B}_{\Lambda BV}(0,1)$. In view of Helly's extraction theorem (cf. [11], Theorem 3.2), there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$, pointwise convergent to some $x \in \overline{B}_{\Lambda BV}(0,1)$. Let us set $y_k = x_{n_k} - x$ for $k \in \mathbb{N}$. We will show that $Ky_k \to 0$ with respect do the ΛBV-norm. Given $\varepsilon > 0$ let k_0 be such that

$$\int_0^1 |k(0,s)y_k(s)|\mathrm{d}s \leq \frac{\varepsilon}{2} \quad \text{and} \quad \int_0^1 m(s)|y_k(s)|\mathrm{d}s \leq \frac{\varepsilon}{2} \quad \text{for } k \geq k_0.$$

Let π and σ be as above. Then

$$\sum_{i=1}^n \frac{|Ky(t_i) - Ky(t_{i-1})|}{\lambda_{\sigma(i)}} \leq \int_0^1 \sum_{i=1}^n \frac{|k(t_i,s) - k(t_{i-1},s)|}{\lambda_{\sigma(i)}} |y_k(s)|\mathrm{d}s \\ \leq \int_0^1 m(s)|y_k(s)|\mathrm{d}s \leq \frac{\varepsilon}{2},$$

and therefore $\operatorname{var}_\Lambda(Ky_k) \leq \frac{\varepsilon}{2}$ for $k \geq k_0$. Finally, we obtain

$$\|Ky_k\|_{\Lambda BV} \leq \int_0^1 |k(0,s)y_k(s)|\mathrm{d}s + \int_0^1 m(s)|y_k(s)|\mathrm{d}s \leq \varepsilon$$

which ends the proof.

Since we have the completely continuous linear map we can apply the Schauder fixed point theorem to get the following existence result.

Theorem 13. *Let $k\colon I\times I \to \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$ satisfy 2° and 1°, respectively. Then for every $\lambda \in \mathbb{R}$ there exists a ΛBV-solution to equation (11).*

Proof. If $\lambda = 0$ the claim is obvious. Without loss of generality we can assume that $\lambda = 1$. Let us consider the operator $G = K \circ F\colon \Lambda BV(I) \to \Lambda BV(I)$, where the integral operator $K\colon \Lambda BV(I) \to \Lambda BV(I)$ is defined in (12) and $F\colon \Lambda BV \to \Lambda BV$ is the autonomous superposition operator generated by the function f. Due to Proposition 4, the operator K is continuous and due to Theorem 9, the operator F is also continuous, so the G is continuous, too. Moreover, it is completely continuous, because F maps bounded sets into bounded ones as a consequence of f being a locally Lipschitz function.

Therefore, it is enough to find a closed ball $\overline{B}_{\Lambda BV}(0,a) \subset \Lambda BV(I)$ which is invariant under the completely continuous map G. First, let us observe that for any ball $\overline{B}_{\Lambda BV}(0,a)$ and $x \in \overline{B}_{\Lambda BV}(0,a)$ there is $\|x\|_\infty \le \tilde{c}_\Lambda a$ and

$$\|F(x)\|_\infty \le \sup_{u\in[-\tilde{c}_\Lambda a,\tilde{c}_\Lambda a]} |f(u)|.$$

By assumption 1 ° (a) there exists $R > 0$ such that

$$\sup_{u\in[-\tilde{c}_\Lambda R,\tilde{c}_\Lambda R]} |f(u)| \left(\int_0^1 |k(0,s)|\mathrm{d}s + \int_0^1 m(s)\mathrm{d}s \right) \le R.$$

Otherwise, there would exist a sequence $(u_n)_{n\in\mathbb{N}}$ of real numbers for which we would have

$$|f(u_n)| \cdot \left(\int_0^1 |k(0,s)|\mathrm{d}s + \int_0^1 m(s)\mathrm{d}s \right) > n \quad \text{and} \quad |u_n| \le \tilde{c}_\Lambda n.$$

The sequence $(u_n)_{n\in\mathbb{N}}$ is not bounded, so there must exist an appropriate subsequence $(w_n)_{n\in\mathbb{N}}$, such that $|w_n| \to +\infty$ and

$$\frac{|f(w_n)|}{|w_n|} \ge \left(\int_0^1 |k(0,s)|\mathrm{d}s + \int_0^1 m(s)\mathrm{d}s \right)^{-1} \tilde{c}_\Lambda^{-1},$$

which contradicts 1 ° (a). Therefore,

$$\begin{aligned} \|G(x)\|_{\Lambda BV} = |G(x)(0)| + \mathrm{var}_\Lambda(G(x)) \le \int_0^1 |k(0,s)||f(x(s))|\mathrm{d}s \\ + \int_0^1 m(s)|f(x(s))|\mathrm{d}s \\ \le \sup_{u\in[-\tilde{c}_\Lambda R,\tilde{c}_\Lambda R]} |f(u)| \left(\int_0^1 |k(0,s)|\mathrm{d}s + \int_0^1 m(s)\mathrm{d}s \right) \le R \end{aligned}$$

for $x \in \overline{B}_{\Lambda BV}(0,R)$, which implies that the ball $\overline{B}_{\Lambda BV}(0,R)$ is invariant under the mapping G. By the Schauder fixed point theorem, this implies that there exists a fixed point of G, which completes the proof.

Let us focus on the continuity of the operator K which appears in Proposition 4. In the paper [6] the authors gave the sufficient and necessary conditions for the integral operator $K\colon BV(I) \to BV(I)$ of the form (12) to be continuous (see [6], Theorem 4). It appears that similar conditions may be stated for such integral operators $K\colon BV(I) \to \Lambda BV(I)$.

Notation. *For a given function $f \in BV(I)$, by v_f we will denote the function $t \to \mathrm{var}(f,[0,t]), t \in I$, where $\mathrm{var}(f,[0,t])$ denotes the variation in the sense Jordan of the function f over the interval $[0,t]$. Moreover, to avoid misunderstandings in the next theorem as well as in its proof, the Lebesgue integral will be denoted by "(L) $\int$" while the Riemann–Stieltjes integral is denoted by "(RS) $\int$".*

Theorem 14. *The linear integral operator K, defined by (12), maps continuously the space $BV(I)$ into $\Lambda BV(I)$ if and only if the following conditions are satisfied:*

(i) *for every $t \in I$ the function $s \mapsto k(t,s)$ is Lebesgue integrable on I,*

(ii) *there exists a constant* $M > 0$ *such that* $\sup_{\xi \in I} \mathrm{var}_\Lambda\left((L)\int_0^\xi k(\cdot, s)\mathrm{d}s\right) \le M$.

Proof. Suppose that the conditions (i) and (ii) hold. We will show that the operator K maps $BV(I)$ into $\Lambda BV(I)$ and is continuous.

Let us note that by [6], Remark 5, the function Kx is well defined for every $x \in BV(I)$. Suppose now that $I_1, \ldots, I_n$ is an arbitrary finite family of nonoverlapping subintervals of I of the form $I_i = [a_i, b_i]$. Then, by [6], Proposition 2 and the condition (ii), for every $x \in BV(I)$, we get

$$\begin{aligned}\sum_{i=1}^n \frac{\left|Kx(I_i)\right|}{\lambda_i} &= \sum_{i=1}^n \frac{1}{\lambda_i}\left|(L)\int_0^1 [k(b_i, s) - k(a_i, s)]x(s)\mathrm{d}s\right| \\ &\le |x(1)| \cdot \sum_{i=1}^n \frac{1}{\lambda_i}\left|(L)\int_0^1 [k(b_i, s) - k(a_i, s)]\mathrm{d}s\right| \\ &\quad + (RS)\int_0^1 \sum_{i=1}^n \frac{1}{\lambda_i}\left|(L)\int_0^\xi [k(b_i, s) - k(a_i, s)]\mathrm{d}s\right| dv_x(\xi) \\ &\le M\left\|x\right\|_\infty + Mv_x(1) \le M\left\|x\right\|_{BV},\end{aligned}$$

which implies that $Kx \in \Lambda BV(I)$ and $\mathrm{var}_\Lambda(Kx) \le 2M\left\|x\right\|_{BV}$. Therefore, $\left\|Kx\right\|_{\Lambda BV} \le \left(\left\|k(0, \cdot)\right\|_{L^1} + 2M\right) \cdot \left\|x\right\|_{BV}$ for $x \in BV(I)$. This shows that the linear operator K is continuous.

Now, we shall show that the conditions (i) and (ii) are necessary. The function $x \equiv 1$ is clearly of bounded Jordan variation, and hence the function

$$t \mapsto (L)\int_0^1 k(t, s)\mathrm{d}s$$

is of bounded Λ-variation. Thus, for every $t \in I$, the function $s \mapsto k(t, s)$ is Lebesgue integrable on I. It remains to show the condition (ii). Since the linear operator K is continuous, there exists a positive number $M > 0$ such that $\left\|Kx\right\|_{\Lambda BV} \le M\left\|x\right\|_{BV}$ for every $x \in BV(I)$. Hence,

$$\mathrm{var}_\Lambda(K\chi_{[0,\xi]}) \le \left\|K\chi_{[0,\xi]}\right\|_{\Lambda BV} \le M\left\|\chi_{[0,\xi]}\right\|_{BV} \le 2M.$$

Therefore,

$$\sup_{\xi \in I} \mathrm{var}_\Lambda\left((L)\int_0^\xi k(t, s)\mathrm{d}s\right) = \sup_{\xi \in I} \mathrm{var}_\Lambda(K\chi_{[0,\xi]}) \le 2M,$$

which ends the proof. □

5.2. Volterra–Hammerstein integral equation.

In this subsection we are going to focus not only on the existence of ΛBV-solutions to the nonlinear Volterra–Hammerstein integral equation but we would like to examine also the topological structure of such solution sets. More precisely, let us consider the following nonlinear Volterra–Hammerstein integral equation

$$x(t) = g(t) + \int_0^t k(t, s)f(s, x(s))ds, \quad t \in I, \tag{13}$$

where

3° $g \in C(I) \cap \Lambda BV(I)$;
4° $f\colon I \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions, that is:
(a) for every $u \in \mathbb{R}$ the function $t \mapsto f(t,u)$ is Lebesgue measurable;
(b) for a.e. $t \in I$ the function $u \mapsto f(t,u)$ is continuous;
(c) $|f(t,u)| \le m_1(t)$ for $(t,u) \in I \times \mathbb{R}$ with $m_1 \in L^p(I)$, where $p \in (1,+\infty]$;
5° the kernel $k\colon \Delta \to \mathbb{R}$, where $\Delta := \{(t,s) \in I \times I : 0 \le s \le t \le 1\}$, is such that:
(a) for every $t \in I$ the function $s \mapsto k(t,s)$ is Lebesgue measurable on $[0,t]$;
(b) $|k(s,s)| + \operatorname{var}_\Lambda(k(\cdot,s),[s,1]) \le m_2(s)$ for a.e $s \in I$ with $m_2 \in L^q(I)$, where $q^{-1} + p^{-1} = 1$;
(c) for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_0^1 |k(\tau,s) - k(t,s)| m_1(s) ds \le \varepsilon,$$

for all $(\tau,t) \in \Delta$ such that $0 \le \tau - t \le \delta$.

Remark 4. Let us note that if the kernel k satisfies the assumptions 5° (a) and (b), then for every $t \in I$ the function $s \mapsto k(t,s)$ belongs to $L^q[0,t]$. Indeed, given any $t \in I$, we have

$$\begin{aligned}|k(t,s)| &\le \frac{\lambda_1 |k(s,s) - k(t,s)|}{\lambda_1} + |k(s,s)| \le \lambda_1 \operatorname{var}_\Lambda(k(\cdot,s),[s,1]) + |k(s,s)| \\ &\le \max(\lambda_1,1) m_2(s),\end{aligned}$$

for almost every $s \in [0,t]$, which confirms our claim.

First, let us focus on the continuity of a certain linear Volterra integral operator.

Lemma 5. *Let* $p \in (1,+\infty]$. *If the kernel* k *satisfies assumptions* 5° (a) *and* (b), *then the linear Volterra integral operator* K *defined by*

$$Kx(t) = \int_0^t k(t,s)x(s)ds, \quad t \in I, \tag{14}$$

maps the space $L^p(I)$ *into* $\Lambda BV(I)$ *and is continuous.*

Proof. Let $x \in L^p(I)$. First let us observe that the integral

$$\int_0^t k(t,s)x(s)ds$$

exists and is finite for every $t \in I$ (cf. Remark 4), hence the operator K is well defined. Given an arbitrary partition $0 = t_1 \le t_2 \le \ldots \le t_n = 1$ and $\sigma \in S_n$ we have

$$\sum_{i=1}^{n}\frac{|Kx(t_i)-Kx(t_{i-1})|}{\lambda_{\sigma(i)}}=\sum_{i=1}^{n}\frac{|\int_0^{t_i}k(t_i,s)x(s)ds-\int_0^{t_{i-1}}k(t_{i-1},s)x(s)ds|}{\lambda_{\sigma(i)}}$$
$$\leq\int_0^1\sum_{i=1}^{n}\frac{|\tilde{k}(t_i,s)-\tilde{k}(t_{i-1},s)|}{\lambda_{\sigma(i)}}|x(s)|ds,$$

where

$$\tilde{k}(t,s)=\begin{cases}k(t,s), & \text{if} \quad (t,s)\in\Delta,\\ 0, & \text{if} \quad (t,s)\notin\Delta.\end{cases} \tag{15}$$

Since

$$\sum_{i=1}^{n}\frac{|\tilde{k}(t_i,s)-\tilde{k}(t_{i-1},s)|}{\lambda_{\sigma(i)}}\leq\frac{|k(s,s)|}{\lambda_1}+\operatorname{var}_\Lambda(k(\cdot,s),[s,1])$$
$$\leq\max(\frac{1}{\lambda_1},1)m_2(s) \quad \text{for a.e. } s\in I,$$

we infer that

$$\sum_{i=1}^{n}\frac{|Kx(t_i)-Kx(t_{i-1})|}{\lambda_{\sigma(i)}}\leq\max(\frac{1}{\lambda_1},1)\int_0^1 m_2(s)|x(s)|ds,$$

which proves that $\|Kx\|_{\Lambda BV}\leq\max(\frac{1}{\lambda_1},1)\,\|m_2\|_{L^q}\cdot\|x\|_{L^p}$, which means that K maps the space $L^p(I)$ into $\Lambda BV(I)$ and is continuous. □

We will also need the following.

Lemma 6. *Let $p\in(1,+\infty]$. Suppose the assumptions 5° (a) and (b) hold. If a bounded sequence $(x_n)_{n\in\mathbb{N}}$, $x_n\in L^p(I)$ for $n\in\mathbb{N}$ converges almost everywhere (or in measure) to a function $x\in L^p(I)$, then the sequence $(Kx_n)_{n\in\mathbb{N}}$, where K is given by (14), converges to Kx with respect to the ΛBV-norm.*

Proof. First let us note that Kx, $Kx_n\in\Lambda BV(I)$, by Lemma 5. Let us take an arbitrary partition $0=t_0\leq t_1\leq\ldots\leq t_n=1$ of the interval I and a permutation $\sigma\in S_n$. Then for the function $\tilde{k}\colon I\times I\to\mathbb{R}$, defined by (15), we have

$$\sum_{i=1}^{n}\frac{|Kx(t_i)-Kx_n(t_i)-Kx(t_{i-1})+Kx_n(t_{i-1}))|}{\lambda_{\sigma(i)}}$$
$$\leq\int_0^1\sum_{i=1}^{n}\frac{|\tilde{k}(t_i,s)-\tilde{k}(t_{i-1},s)|}{\lambda_{\sigma(i)}}|x_n(s)-x(s)|ds.$$

Hence,

$$\operatorname{var}_\Lambda(Kx_n-Kx)\leq\max(\frac{1}{\lambda_1},1)\int_0^1 m_2(s)|x_n(s)-x(s)|ds.$$

Therefore, in view of the assumptions and Vitali's Convergence Theorem we get $\|Kx_n-Kx\|_{\Lambda BV}\to 0$, as $n\to+\infty$. □

Now we are going to present the Aronszajn-type result for ΛBV-solutions to the Volterra–Hammerstein equation in $\Lambda BV(I)$. We are going to apply the certain Vidossich-type result. In what follows, assume that

$\mathcal{K}$ is a bounded and convex subset of a normed space, and E is a Banach space. Denote by $C(\mathcal{K}, E)$ the space of all bounded and continuous functions $x\colon \mathcal{K} \to E$ endowed with the supremum norm.

Theorem 15. *([17],Theorem 2) Let $F\colon C(\mathcal{K}, E) \to C(\mathcal{K}, E)$ be a continuous mapping satisfying the following conditions:*

(i) *the set $F(C(\mathcal{K}, E))$ is equiuniformly continuous;*
(ii) *there exists $t_0 \in \mathcal{K}$ and $x_0 \in E$ such that $F(x)(t_0) = x_0$ for every $x \in C(\mathcal{K}, E)$;*
(iii) *for every $\varepsilon > 0$ and every $x, y \in C(\mathcal{K}, E)$ the following implication holds*

$$x|_{\mathcal{K}_\varepsilon} = y|_{\mathcal{K}_\varepsilon} \Rightarrow F(x)|_{\mathcal{K}_\varepsilon} = F(y)|_{\mathcal{K}_\varepsilon},$$

where $\mathcal{K}_\varepsilon = \{t \in \mathcal{K} : \|t - t_0\| \le \varepsilon\}$;
(iv) every sequence $(x_n)_{n\in\mathbb{N}}$ in $C(\mathcal{K}, E)$ such that $\lim_{n\to+\infty}(x_n - F(x_n)) = 0$ has a limit point.

Then the set of fixed points of the mapping F is a compact R_δ, that is, it is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts.

Theorem 16. *If the assumptions 3° - 5° hold, then the set T of all continuous solutions of bounded Λ-variation to the nonlinear Volterra–Hammerstein integral equation (13) is a compact R_δ in the Banach space $C(I) \cap \Lambda BV(I)$, endowed with the ΛBV-norm.*

Proof. The proof falls into two parts. First, we shall show that the mapping

$$F(x)(t) = g(t) + \int_0^t k(t, s)f(s, x(s))ds, \quad t \in I,$$

defined for $x \in C(I)$ satisfies the assumptions of Theorem 15. Let $x \in C(I)$ and $\varepsilon > 0$. In view of the assumptions, there exists $\delta > 0$ such that:

- $|g(t) - g(\tau)| \le \frac{\varepsilon}{3}$ for $t, \tau \in I, |t - \tau| < \delta$;
- $\int_0^t |k(\tau, s) - k(t, s)|m_1(s)ds \le \frac{\varepsilon}{3}$ for $(\tau, t) \in \Delta, |t - \tau| \le \delta$;
- $\max\{1, \lambda_1\} \int_A m_1(s)m_2(s)ds \le \frac{\varepsilon}{3}$ for any Lebesgue measurable set $A \subset I$ such that $\mu(A) \le \delta$.

Therefore, for $t, \tau \in I$ such that $0 \le \tau - t \le \delta$, we have

$$\begin{aligned}
|F(x)(t) - F(x)(\tau)| &\le |g(t) - g(\tau)| + \left|\int_0^t k(t, s)f(s, x(s))ds - \int_0^\tau k(\tau, s)f(s, x(s))ds\right| \\
&\le |g(t) - g(\tau)| + \int_0^t |k(t, s) - k(\tau, s)||f(s, x(s))|ds + \int_t^\tau |k(\tau, s)||f(s, x(s))|ds \\
&\le |g(t) - g(\tau)| + \int_0^t |k(t, s) - k(\tau, s)|m_1(s)ds + \max\{1, \lambda_1\} \int_t^\tau m_1(s)m_2(s)ds \\
&\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

This shows that $F(x) \in C(I)$. Furthermore, let us observe that the number δ is independent of x, which implies, that the set $F(C(I))$ is equiuniformly continuous.

The continuity of the mapping F is a consequence of Lemma 6 and the fact that if a sequence $(x_n)_{n\in\mathbb{N}}$ is uniformly convergent to $x \in C(I)$, then the sequence $(f(\cdot, x_n(\cdot)))_{n\in\mathbb{N}}$, bounded in $L^p(I)$, converges almost everywhere to the function $t \mapsto f(t, x(t)), t \in I$.

The assumptions (ii) and (iii) of Theorem 15 are obviously satisfied for $t_0 = 0$ and $x_0 = g(0)$.

Hence, it suffices to prove that the mapping F satisfies the Palais–Smale condition (iv). Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $C(I)$ such that $\lim_{n\to+\infty}(x_n - F(x_n)) = 0$ with respect to the supremum norm. In view of the Assumption 5.2 and Lemma 5 we get

$$\operatorname{var}_\Lambda(F(x)) \leq \|g\|_{\Lambda BV} + \max(\frac{1}{\lambda_1}, 1)\, \|m_1\|_{L^p}\, \|m_2\|_{L^q}\,, \quad \text{for } x \in C(I). \tag{16}$$

Therefore, by Helly's Extraction Theorem, there exists a subsequence $(F(x_{n_k}))_{k\in\mathbb{N}}$ of $(F(x_n))_{n\in\mathbb{N}}$ pointwise convergent to a function $y \in \Lambda BV(I)$. Thus, $(x_{n_k})_{k\in\mathbb{N}}$ is also pointwise convergent to y. Hence, for almost all $t \in I$, we have $f(t, x_{n_k}(t)) \to f(t, y(t))$ as $k \to +\infty$, and the sequence $(f(\cdot, x_{n_k}(\cdot)))_{k\in\mathbb{N}}$ is bounded in $L^p(I)$. This, by Lemma 6, implies that $(F(x_{n_k}))_{k\in\mathbb{N}}$ converges to $F(y) = y$ with respect to the ΛBV-norm, and so, since the supremum norm is weaker than the ΛBV-norm, the sequence $(x_n)_{n\in\mathbb{N}}$ has a limit point in $C(I)$.

All the assumptions of Theorem 15 are satisfied and, therefore, the set S of all continuous solutions of the equation (13) is a compact R_δ in $C(I)$. To end the proof, it suffices to show that S endowed with the metric d_∞ induced by the supremum norm is homeomorphic to the set T of all continuous solutions of (13) of bounded Λ-variation, endowed with the metric $d_{\Lambda BV}$ induced by the ΛBV-norm. Note that $S = T$ as sets, and since the ΛBV-norm is stronger than the supremum norm, we get that the identity map $id\colon S \to T$ is continuous. Now, we shall show that $id\colon T \to S$ is also continuous. Let us take a sequence $(x_n)_{n\in\mathbb{N}}$ in S convergent to $x_0 \in S$. Reasoning as above, we infer that the sequence $(F(x_n)_{n\in\mathbb{N}})$ converges to $F(x_0)$ with respect to the metric $d_{\Lambda BV}$ induced by the norm $\|\cdot\|_{\Lambda BV}$. But $F(x_n) = x_n$ for all $n \in \mathbb{N} \cup \{0\}$, and hence $\|x_n - x_0\|_{\Lambda BV} \to 0$ as $n \to +\infty$. This shows that the identity map constitutes a homeomorphism between the metric spaces T and S and, in consequence, proves that the set of all continuous solutions of (13) of bounded Λ-variation is a compact R_δ set in $C(I) \cap \Lambda BV(I)$ with respect to the ΛBV-norm.

6. Compactness

Let us start with some notation. Let $\omega\colon [0, +\infty) \to [0, +\infty)$ be a continuous strictly increasing function such that $\omega(0) = 0$. By $H^\omega(I)$ we will denote the vector space of real-valued functions defined on I which satisfy the following condition:

$$|x(t) - x(s)| \leq k\omega(|t-s|) for t, s \in I, \text{where } k \geq 0.$$

In the paper [6] the authors gave a sufficient condition for a subset of $\Lambda BV(I)$ space to be relatively compact. More precisely, let $\Lambda BV^{\omega}(I)$ denote the Banach space $H^{\omega}(I) \cap \Lambda BV(I)$, endowed with the norm $|x|_{\Lambda} = \max\{\|x\|_{\omega}, \|x\|_{\Lambda}\}$, where

$$\|x\|_{\omega} = \inf\{k \geq 0 : |x(t) - x(s)| \leq k\omega(|t-s|) \text{ for } t, s \in I\}.$$

Proposition 5. (cf. [6], Proposition 5) *Let $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$ and $\Gamma = (\gamma_n)_{n\in\mathbb{N}}$ be two Waterman sequences such that $\Lambda < \Gamma$. Then the space $\Lambda BV^{\omega}(I)$ can be compactly embedded into the space $\Gamma BV(I)$.*

It is interesting to know if it is possible, roughly speaking, to reverse Proposition 5, that is, if it is possible to prove that for each compact subset K of $\Lambda BV(I)$ there exists such Waterman sequence $\Gamma = (\gamma_n)_{n\in\mathbb{N}}$, that $\Gamma < \Lambda$, $K \subset \Gamma BV(I)$ and K is bounded in the norm $\|\cdot\|_{\Gamma BV}$. Below we are going to show that in general the answer is negative, but for the wide range of Waterman sequences the answer is positive. First, let us recall two definitions. For given Waterman sequence Λ, let us denote by $\Lambda_{(m)}$ the Waterman sequence constructed by deleting the first $m-1$ terms of Λ.

Definition 7. (cf. [16]) A function $x \in \Lambda BV(I)$ is said to be continuous in Λ-variation, if $\lim_{m\to\infty} \operatorname{var}_{\Lambda_{(m)}}(x) = 0$.

Definition 8. (cf. [16]) If Λ is a Waterman sequence, then:

$$S_{\Lambda} = \limsup_{n\to\infty} \frac{\sum_{i=1}^{2n} \frac{1}{\lambda_i}}{\sum_{i=1}^{n} \frac{1}{\lambda_i}}$$

is called the Shao–Sablin index of Λ.

The set of functions which are continuous in Λ-variation will be denoted by $\Lambda BV_c(I)$ while its subset consisting of all the functions which are also continuous will be denoted by $C\Lambda BV_c(I)$. Let us recall that this concept was introduced by Waterman in [20] to provide a sufficient condition for the so-called (C, β)-summability of the Fourier series of a function. Waterman conjectured in [21] that not every function of bounded Λ-variation is continuous in Λ-variation. An example of such function was given in the paper [8]. Moreover, Prus-Wiśniowski in [16] described the relation between the Shao–Sablin index of a Waterman sequence Λ and the existence of functions that are not continuous in Λ-variation.

Theorem 17. (cf. [16]) *For every proper Waterman sequence Λ, the following statements are equivalent:*

(i) *the space $C\Lambda BV(I)$ is separable,*
(ii) $C\Lambda BV_c(I) = C\Lambda BV(I)$,
(iii) $\Lambda BV_c(I) = \Lambda BV(I)$,
(iv) $S_{\Lambda} < 2$.

Surprisingly, the concept of a function continuous in Λ-variation is closely related to the compactness in the space $\Lambda BV(I)$. We are going to begin with the following.

Lemma 7. *For every proper Waterman sequence Λ and $K \subseteq \Lambda BV(I)$, the following conditions are equivalent:*

(i) *there exists such $\Gamma < \Lambda$ that $K \subseteq \Gamma BV(I)$ and K is bounded in ΓBV-norm,*

(ii) *K is bounded and the following condition holds:*

$$\forall_{\varepsilon>0}\exists_{N\in\mathbb{N}}\forall_{n\geq N}\forall_{x\in K}\mathrm{var}_{\Lambda_{(n)}}(x) < \varepsilon. \tag{17}$$

Proof. Let us start with the implication $(ii) \Rightarrow (i)$ There exists such $M > 0$ that $\|x\|_{\Lambda BV} < 2M$ for any $x \in K$. By (17), there exists a strongly increasing sequence $(N_k)_{k\in\mathbb{N}}$ of positive integers such that $N_1 = 1$ and $\mathrm{var}_{\Lambda_{(N_k)}}(x) < \frac{M}{2^k}$ for any $k \in \mathbb{N}$ and $x \in K$. Let us define the sequence Γ in the following way:

$$\frac{1}{\gamma_1} = \frac{1}{\lambda_1},$$

$$\frac{1}{\gamma_{k+1}} = \min\left\{\frac{1}{\gamma_k}, m\cdot\frac{1}{\lambda_{k+1}}\right\},$$

where m is a positive integer such that $N_m \leq k < N_{m+1}$.

The sequence $\Gamma < \Lambda$ is monotone directly by its definition, and $\frac{1}{\gamma_k} \geq \frac{1}{\lambda_k}$ and, therefore, it is a Waterman sequence. Let $(I_n)_{n\in\mathbb{N}}$ be any sequence of nonoverlapping intervals. Then, for any $x \in K$ we have

$$\sum_{n=1}^{\infty}\frac{|x(I_n)|}{\gamma_n} = \sum_{k=1}^{\infty}\sum_{n=N_k}^{N_{k+1}-1}\frac{|x(I_n)|}{\gamma_n} \leq \sum_{k=1}^{\infty}\sum_{n=N_k}^{N_{k+1}-1}\frac{k\cdot|x(I_n)|}{\lambda_n}$$
$$\leq \sum_{k=1}^{\infty}k\cdot\mathrm{var}_{\Lambda_{N_k}}(x) < \sum_{k=1}^{\infty}k\cdot\frac{M}{2^k} < +\infty.$$

This implies that x is of bounded Γ-variation and, moreover, that

$$\|x\|_{\Gamma BV} \leq |x(0)| + \sum_{k=1}^{\infty}k\cdot\frac{M}{2^k} \leq 2M + \sum_{k=1}^{\infty}k\cdot\frac{M}{2^k} < +\infty$$

for any $x \in K$, that is, that K is bounded in ΓBV-norm which ends the first part of the proof.

Now let us proceed to the proof of the implication $(i) \Rightarrow (ii)$ and let us assume that there exists $\Gamma < \Lambda$ such that $K \subseteq \Gamma BV(I)$ is bounded in ΓBV-norm and that (17) does not hold. Then there exists such $\delta > 0$ and a sequence $(N_n)_{n\in\mathbb{N}}$, $N_n \to +\infty$ and functions $x_n \in K$ such that $\mathrm{var}_{\Lambda_{(N_n)}}(x_n) > \delta$. Take any $M > 0$. There exists $N \in \mathbb{N}$ such that

$$\frac{1}{\gamma_n} \geq M\frac{1}{\lambda_n},$$

for $n \geq N$. We may also assume that $N_n \geq N$. Take $x_n \in K$ and a sequence of nonoverlapping intervals $(I_k)_{k\in\mathbb{N}}$, $(I_k) \subset I$ for every $k \in \mathbb{N}$, such that $\sum_{k=N_n}^{\infty}\frac{|x_n(I_k)|}{\lambda_k} \geq \delta$. Then, we get

$$\|x_n\|_{\Gamma BV} \geq \sum_{k=1}^{\infty}\frac{|x(I_k)|}{\gamma_k} \geq \sum_{k=N_n}^{\infty}\frac{|x(I_k)|}{\gamma_k} \geq \sum_{k=N_n}^{\infty}M\cdot\frac{|x(I_k)|}{\lambda_k} \geq M\delta$$

and, therefore, K is not bounded in ΓBV-norm, which gives a contradiction. □

Corollary 1. *If $x \in \Lambda BV$, then the following conditions are equivalent*

(i) *there exists a Waterman sequence $\Gamma < \Lambda$ such that $x \in \Gamma BV(I)$,*
(ii) *x is continuous in Λ-variation.*

Remark 5. As we mentioned above, if $S_\Lambda = 2$, then there exists such a continuous function $x_0 \in \Lambda BV(I)$ that is not continuous in Λ-variation and thus it does not belong to any $\Gamma BV(I) \subsetneq \Lambda BV(I)$. The set $\{x_0\}$ is compact but, at the same time, there does not exist a space $\Gamma BV(I)$ where $\Gamma < \Lambda$ and $x_0 \in \Gamma BV(I)$. This shows that if $S_\Lambda = 2$, then not every compact subset of $\Lambda BV(I)$ is a bounded subset of some proper subspace $\Gamma BV(I) \subsetneq \Lambda BV(I)$. As one can check, there exist proper Waterman sequences having the Shao–Sablin index equal to 2. As an example of such a sequence one can consider the sequence $(\ln(n+1))_{n\in\mathbb{N}}$.

Theorem 18. *If $K \subset \Lambda BV(I)$ is compact and $K \subseteq \Lambda BV_c(I)$, then there exists a Waterman sequence $\Gamma < \Lambda$ such that $K \subseteq \Gamma BV(I)$ and K is bounded in $\Gamma BV(I)$.*

Proof. Fix $\varepsilon > 0$. Since K is compact, there exist $x_1, x_2, \ldots, x_k \in K$ such that

$$K \subseteq \bigcup_{n=1}^{k} B_{\Lambda BV}\left(x_n, \frac{\varepsilon}{2}\right).$$

Since $x_1, \ldots, x_k \in \Lambda BV_c(I)$, there exists $N \in \mathbb{N}$ such that $\operatorname{var}_{\Lambda_{(n)}}(x_m) < \frac{\varepsilon}{2}$ for every $n \geq N$ and $m = 1, 2, \ldots k$. Take any $x \in K$. There is such m that $x \in B(x_m, \frac{\varepsilon}{2})$. Then for $n \geq N$ we have

$$\begin{aligned}\operatorname{var}_{\Lambda_{(n)}}(x) &= \operatorname{var}_{\Lambda_{(n)}}(x - x_m + x_m) \leq \operatorname{var}_{\Lambda_{(n)}}(x - x_m) + \operatorname{var}_{\Lambda_{(n)}}(x_m) \\ &\leq \operatorname{var}_{\Lambda_{(n)}}(x - x_m) + \operatorname{var}_{\Lambda_{(n)}}(x_m) < \varepsilon.\end{aligned}$$

We have just proved (17) so by Lemma 7 there exists such $\Gamma < \Lambda$ that $K \subset \Gamma BV(I)$ and K is bounded in $\|\cdot\|_{\Gamma BV}$. □

Corollary 2. *If $S_\Lambda < 2$ and $K \subseteq \Lambda BV(I)$ is compact, then there exists a Waterman sequence $\Gamma < \Lambda$ such that $K \subseteq \Gamma BV(I)$.*

Proof. By Theorem 17, if $S_\Lambda < 2$, then $\Lambda BV(I) = \Lambda BV_c(I)$. □

Now we are going to consider σ-compact subsets of the space $\Lambda BV(I)$.

Denote $\Gamma_k = (\gamma_n^k)_{n\in\mathbb{N}}$ for $k \in \mathbb{N}$, $\Gamma = (\gamma_n)_{n\in\mathbb{N}}$ and $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$. We will start with the slight modification of the Lemma 2.

Lemma 8. *If Λ is a Waterman sequence, $\Gamma_k < \Lambda$ are Waterman sequences for every $k \in \mathbb{N}$ and*

$$\lambda_n \geq \gamma_n^{k+1} \geq \gamma_n^k \tag{18}$$

for $n, k \in \mathbb{N}$, then there exists a Waterman sequence $\Gamma < \Lambda$ such that $\Gamma_k BV(I) \subseteq \Gamma BV(I)$ for every $k \in \mathbb{N}$.

Proof. There exists a strongly increasing sequence $(N_k)_{k\in\mathbb{N}}$ of positive integers such that $N_1 = 1$ and

$$\lambda_n \geq k \cdot \gamma_n^k \tag{19}$$

for every $n \geq N_k$, $k \in \mathbb{N}$.

Put $\gamma_n = \gamma_n^k$ for $N_k \leq n < N_{k+1}$ and $k \in \mathbb{N}$. Then

(i) $\Gamma = (\gamma_n)_{n\in\mathbb{N}}$ is monotone, by (18) and the monotonicity of Γ_k for $k \in \mathbb{N}$,
(ii) $\sum_{n=1}^{\infty} \frac{1}{\gamma_n} \geq \sum_{n=1}^{\infty} \frac{1}{\lambda_n} = +\infty$, by (18),

which implies that Γ is a Waterman sequence. Moreover, (19) implies that for an arbitrary $k \in \mathbb{N}$ and $n \in \mathbb{N}$ such that $N_k \leq n < N_{k+1}$:

$$\frac{\lambda_n}{\gamma_n} = \frac{\lambda_n}{\gamma_n^k} \geq k,$$

so $\Gamma < \Lambda$. Finally, by the definition of Γ and (18), we get $\gamma_n \geq \gamma_n^k$ for every $n \geq N_k$, $k \in \mathbb{N}$, so $\Gamma_k BV(I) \subseteq \Gamma BV(I)$ for all $k \in \mathbb{N}$.

Lemma 9. *If $\Gamma_1, \Gamma_2 < \Lambda$ are Waterman sequences, then there exists a Waterman sequence $\Gamma < \Lambda$ such that $\gamma_n^i \leq \gamma_n$ for every $n \in \mathbb{N}$ $(i = 1, 2)$.*

Proof. We may assume that $\gamma_n^1, \gamma_n^2 \leq \lambda_n$ for every $n \in \mathbb{N}$, since $\Gamma_1 < \Lambda$, $\Gamma_2 < \Lambda$. Let $\gamma_n = \max\{\gamma_n^1, \gamma_n^2\}$ for $n \in \mathbb{N}$. Then Γ is monotone and $\sum_{n=1}^{\infty} \frac{1}{\gamma_n} \geq \sum_{n=1}^{\infty} \frac{1}{\lambda_n} = +\infty$, so Γ is a Waterman sequence. Obviously, $\Gamma < \Lambda$. □

Theorem 19. *If K is σ-compact and $K \subseteq \Lambda BV_c(I)$, then there exists $\Gamma < \Lambda$ such that $K \subseteq \Gamma BV(I)$.*

Proof. Let $K = \bigcup_{n\in\mathbb{N}} K_n$, where K_n is compact for every positive integer n. By Theorem 18, for every $n \in \mathbb{N}$ there exists a Waterman sequence $\Lambda_n < \Lambda$ such that $K_n \subseteq \Lambda_n BV(I)$. We shall define a sequence $(\Gamma_k)_k$ of Waterman sequences, fulfilling the assumptions of the Lemma 8 and such that $\Lambda_k BV(I) \subseteq \Gamma_k BV(I)$ for $k \in \mathbb{N}$. Let $\Gamma_1 = \Lambda_1$. Assume that Γ_k is a Waterman sequence such that $\Gamma_k < \Lambda$. Applying Lemma 9 to the sequences $\Lambda_{k+1}, \Gamma_k < \Lambda$ we obtain a Waterman sequence Γ_{k+1}, such that $\gamma_n^k \leq \gamma_n^{k+1}$ and $\lambda_n^{k+1} \leq \gamma_n^{k+1}$ for every $n \in \mathbb{N}$. Moreover, $\Lambda_{k+1} BV(I) \subseteq \Gamma_{k+1} BV(I)$ for every $k \in \mathbb{N} \cup \{0\}$. This completes the construction.

Finally, applying Lemma 8 to the sequence $(\Gamma_k)_{k\in\mathbb{N}}$ we get the desired Waterman sequence Γ. □

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