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# Fixed points of some nonlinear operators in spaces of multifunctions and the Ulam stability

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**Abstract.** We prove a fixed point theorem for nonlinear operators, acting on some function spaces (of set-valued maps), which satisfy suitable inclusions. We also show some applications of it in the Ulam type stability.

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## 1. Introduction

The question *when we can replace an approximate solution to an equation by an exact solution to it (or conversely) and what error we thus commit* seems to be very natural. Some convenient tools to study such issues are provided by the theory of Ulam's (often also called the Hyers–Ulam) type stability. For some updated information and further references concerning the Ulam stability, we refer to [1,4,5]. Let us only mention that the investigation of that problem started with a question raised by Ulam in 1940 and an answer to it given by Hyers in [3].

It has been noticed in numerous papers that there are strict connections between some fixed point theorems and the results concerning the Ulam stability of various (differential, difference, functional, and integral) equations; for a suitable survey we refer to [2]. In this paper we continue those investigations by proving a fixed point result for a class of nonlinear operators acting on some spaces of set-valued mappings and showing several of its consequences.

Through this paper, we assume that $K$ is a nonempty set and $(Y, d)$ is a complete metric space. We denote by $n(Y)$ the family of all nonempty subsets of $Y$, by $bd(Y)$ the family of all nonempty and bounded subsets of

$Y$, and by $bcl(Y)$ the family of all closed sets from $bd(Y)$. Moreover, $h$ is the Hausdorff distance induced by the metric in $Y$ and given by

$$h(A,B) := \max\Big\{\sup_{x\in A}\inf_{y\in B} \mathrm{d}(x,y), \sup_{y\in B}\inf_{x\in A} \mathrm{d}(x,y)\Big\}, \quad A,B \in n(Y).$$

It is well known that $h$ is a metric if restricted to $bcl(Y)$.

The number (possibly also $\infty$)

$$\delta(A) = \sup\{\mathrm{d}(x,y) :\ x,y \in A\}$$

is said to be the diameter of $A \in n(Y)$. For $F\colon K \to n(Y)$ and $g\colon K \to Y$, we denote by $\mathrm{cl}\, F$ and $\widehat{g}$ the multifunctions defined by

$$(\mathrm{cl}\, F)(x) = \mathrm{cl}\, F(x), \quad \widehat{g}(x) := \{g(x)\}, \quad x \in K.$$

We write $a^0(x) = x$ for $x \in K$ and $a^{n+1} = a^n \circ a$ for $a\colon K \to K$, $n \in \mathbb{N}_0$ ($\mathbb{N}_0$ stands for the set of nonnegative integers).

We present a theorem, concerning fixed points of some operators acting on set-valued functions, and several of its consequences. To do this, we need to introduce some notations. Namely, given functions $a, b \in \mathbb{R}^K$ (as usually, $B^A$ denotes the family of all functions mapping a set $A \neq \emptyset$ into a set $B \neq \emptyset$) and $F, G \in n(Y)^K$, we write $a \le b$ provided

$$a(x) \le b(x), \quad x \in K,$$

and $F \subset G$ provided

$$F(x) \subset G(x), \quad x \in K;$$

moreover, we define $F \cup G \in n(Y)^K$ by $(F \cup G)(x) := F(x) \cup G(x)$ for $x \in K$.

We say that $\Lambda\colon {\mathbb{R}_+}^K \to {\mathbb{R}_+}^K$ (where $\mathbb{R}_+ := [0, +\infty)$) is non-decreasing if

$$\Lambda a \le \Lambda b, \quad a, b \in {\mathbb{R}_+}^K,\ a \le b.$$

We always assume the Tichonoff topology (of pointwise convergence) in $bcl(Y)^K$, with the Hausdorff metric in $bcl(Y)$.

We write

$$\Big(\lim_{n\to\infty} H_n\Big)(x) := \lim_{n\to\infty} H_n(x), \quad x \in K,$$

for each sequence $(H_n)_{n\in\mathbb{N}}$ in $bcl(Y)^K$ that is convergent in $bcl(Y)^K$. Next, an operator $\alpha\colon n(Y)^K \to n(Y)^K$ is i.p. (inclusion preserving) if

$$\alpha F \subset \alpha G, \quad F, G \in n(Y)^K,\ F \subset G;$$

$\alpha$ is l.p. (limit preserving) if

$$\alpha\Big(\lim_{n\to\infty} \mathrm{cl}\, H_n\Big) \subset \lim_{n\to\infty} \mathrm{cl}\,(\alpha H_n) \tag{1}$$

for each sequence $(H_n)_{n\in\mathbb{N}}$ in $bd(Y)^K$, such that the sequences $(\mathrm{cl}\, H_n)_{n\in\mathbb{N}}$ and $(\mathrm{cl}\,(\alpha H_n))_{n\in\mathbb{N}}$ are convergent in $bcl(Y)^K$.

We also need the following hypothesis for operators $\alpha\colon bd(Y)^K \to bd(Y)^K$.

$(H)$ $\alpha\widehat{f}$ is single valued for each $f \in Y^K$ and

$$\lim_{n\to\infty} \mathrm{cl}\,(\alpha H_n) \subset \mathrm{cl}\,\alpha\Big(\lim_{n\to\infty} \mathrm{cl}\, H_n\Big)$$

for each sequence $(H_n)_{n\in\mathbb{N}} \subset bd(Y)^K$, such that the sequences $(\mathrm{cl}\, H_n)_{n\in\mathbb{N}}$ and $(\mathrm{cl}\,(\alpha H_n))_{n\in\mathbb{N}}$ are convergent in $bcl(Y)^K$.

Clearly, $(H)$ is somewhat complementary to (1).

Finally, $\widetilde{\delta}\colon bd(Y)^K \to {\mathbb{R}_+}^K$ is given by the formula

$$\widetilde{\delta}F(x) = \delta(F(x)), \quad F \in bd(Y)^K, x \in K,$$

and, for every $t \in \mathbb{R}_+$ and $a \in \mathbb{R}_+^K$, we define the mapping $ta \in \mathbb{R}_+^K$ by $(ta)(x) := ta(x)$ for $x \in K$.

## 2. Main results

In the sequel $\alpha\colon bd(Y)^K \to bd(Y)^K$, $\mathcal{G}\colon bd(Y)^K \to bd(Y)^K$ and $\Lambda\colon \mathbb{R}_+^K \to \mathbb{R}_+^K$ are given. We consider functions $F \in bd(Y)^K$ that satisfy the equation:

$$\alpha F = F$$

$\mathcal{G}$–approximately, i.e., such that

$$\alpha F \cup F \subset \mathcal{G}F. \tag{2}$$

We use the following contraction condition on $\alpha$:

$$\widetilde{\delta}(\alpha H) \le \Lambda(\widetilde{\delta}H), \quad H \in bd(Y)^K. \tag{3}$$

Now, we are in a position to present the main result of this paper.

**Theorem 1.** *Assume that $\Lambda$ is non-decreasing, $\alpha$ is i.p. and satisfies (3), $F \in bd(Y)^K$, $\mathcal{G}\colon bd(Y)^K \to bd(Y)^K$, (2) holds, and*

$$\kappa(x) = \sum_{n=0}^{\infty} \Lambda^n(\widetilde{\delta}(\mathcal{G}F))(x) < \infty, \quad x \in K. \tag{4}$$

*Suppose that $\alpha$ is l.p. or $(H)$ is valid. Then, there exists a function $f\colon K \to Y$, such that $\widehat{f}$ is a fixed point of the operator $\alpha$ (i.e., $\alpha\widehat{f} = \widehat{f}$) and*

$$h\big(\widehat{f}(x), F(x)\big) \le \kappa(x), \quad x \in K.$$

*Moreover, if $G \in bd(Y)^K$ satisfies the conditions*

$$G \subset \alpha G,$$

$$h\big(G(x), F(x)\big) \le \mu(x), \quad x \in K,$$

*with some $\mu\colon K \to \mathbb{R}_+$ such that*

$$\liminf_{n\to\infty} \Lambda^n(\kappa + 2\mu)(x) = 0, \quad x \in K, \tag{5}$$

*then $G = \widehat{f}$.*

*Proof.* Fix $x \in K$. Since $\alpha$ is i.p., by (2), we get

$$\alpha^{n+1}F(x) \subset \alpha^n(\mathcal{G}F)(x), \quad \alpha^n F(x) \subset \alpha^n(\mathcal{G}F)(x)$$

for every $n \in \mathbb{N}_0$ (nonnegative integers). Hence

$$\begin{aligned} h(\alpha^{n+1}F(x), \alpha^n F(x)) &\leq \widetilde{\delta}(\alpha^n(\mathcal{G}F))(x) \\ &\leq \Lambda^n(\widetilde{\delta}(\mathcal{G}F))(x), \quad n \in \mathbb{N}_0. \end{aligned}$$

Therefore, for $k \in \mathbb{N}$, $n \in \mathbb{N}_0$, we have

$$\begin{aligned} h(\alpha^{n+k}F(x), \alpha^n F(x)) &\leq \sum_{i=0}^{k-1} h(\alpha^{n+i+1}F(x), \alpha^{n+i}F(x)) \\ &\leq \sum_{i=0}^{k-1} \Lambda^{n+i}(\widetilde{\delta}(\mathcal{G}F))(x) = \sum_{i=n}^{n+k-1} \Lambda^i(\widetilde{\delta}(\mathcal{G}F))(x). \end{aligned} \tag{6}$$

Furthermore, by (4), we get

$$\lim_{n\to\infty} \sum_{i=n}^{n+k-1} \Lambda^i(\widetilde{\delta}(\mathcal{G}F))(x) = 0, \quad k \in \mathbb{N}.$$

Moreover,

$$\widetilde{\delta}(\operatorname{cl} \alpha^n F(x)) = \widetilde{\delta}(\alpha^n F(x)), \tag{7}$$

whence $(\operatorname{cl} \alpha^n F(x))_{n\in\mathbb{N}_0}$ is a Cauchy sequence of closed and bounded sets and, as the space $(bcl(Y), h)$ is complete, there exists the limit

$$\rho(x) := \lim_{n\to\infty} \operatorname{cl} \alpha^n F(x) \in bcl(Y).$$

Furthermore, by (3) and (7), we have

$$\widetilde{\delta}(\operatorname{cl} \alpha^n F)(x) \leq \Lambda^n(\widetilde{\delta}F)(x)$$

and $(\Lambda^n(\widetilde{\delta}F)(x))_{n\in\mathbb{N}_0}$ is convergent to 0 as $n \to \infty$. Therefore, the set $\rho(x)$ has exactly one element for each $x \in K$ and we denote that element by $f(x)$.

If $\alpha$ is l.p., it is clear that

$$\alpha\widehat{f}(x) = \alpha\Big(\lim_{n\to\infty} \operatorname{cl} \alpha^n F\Big)(x) \subset \lim_{n\to\infty} \operatorname{cl} \alpha^{n+1}F(x) = \{f(x)\}.$$

Thus, $\alpha\widehat{f} = \widehat{f}$.

If $(H)$ holds, then

$$\begin{aligned} \{f(x)\} &= \lim_{n\to\infty} \operatorname{cl} \alpha^{n+1}F(x) \\ &\subset \operatorname{cl} \alpha\Big(\lim_{n\to\infty} \operatorname{cl} \alpha^n F\Big)(x) = \operatorname{cl} \alpha\widehat{f}(x) = \alpha\widehat{f}(x), \end{aligned}$$

whence again, $\alpha\widehat{f} = \widehat{f}$. Next, by (6), we have

$$h\big(\operatorname{cl} \alpha^n F(x), F(x)\big) = h\big(\alpha^n F(x), F(x)\big) \leq \sum_{i=0}^{n-1} \Lambda^i\big(\widetilde{\delta}(\mathcal{G}F)\big)(x)$$

for $n \in \mathbb{N}$, and consequently, with $n \to \infty$, we obtain $h(\widehat{f}(x), F(x)) \leq \kappa(x)$.

It remains to show the statement on the uniqueness of $\widehat{f}$. Therefore, fix $G \in bd(Y)^K$ and $\mu \in \mathbb{R}_+^K$, such that (5) holds, $G \subset \alpha G$, and $h(G(x), F(x)) \leq \mu(x)$ for $x \in K$. Define the multifunction $\mathcal{B}_F\colon K \to n(Y)$ by

$$\mathcal{B}_F(x) := \{y \in Y :\ d(y, F(x)) \leq \mu(x)\}, \quad x \in K.$$

Then, it is easily seen that $F, G \subset \mathcal{B}_F$, and consequently

$$\alpha^n F, \alpha^n G \subset \alpha^n \mathcal{B}_F, \quad n \in \mathbb{N}.$$

Next, for each $n \in \mathbb{N}$, we have $G \subset \alpha^n G$, whence

$$\begin{aligned} h(\widehat{f}(x), G(x)) &\leq h(\widehat{f}(x), \alpha^n G(x)) \\ &\leq h(\widehat{f}(x), \alpha^n F(x)) + h(\alpha^n F(x), \alpha^n G(x)) \\ &\leq h(\widehat{f}(x), \operatorname{cl} \alpha^n F(x)) + \widetilde{\delta}(\alpha^n \mathcal{B}_F)(x) \\ &\leq h(\widehat{f}(x), \operatorname{cl} \alpha^n F(x)) + \Lambda^n(\widetilde{\delta \mathcal{B}_F})(x), \quad x \in K. \end{aligned}$$

Note that for every $x \in K$, $y, z \in \mathcal{B}_F(x)$ and $w_1, w_2 \in F(x)$, we have

$$\begin{aligned} d(y, z) &\leq d(y, w_1) + d(w_1, w_2) + d(w_2, z) \\ &\leq d(y, w_1) + \delta(F(x)) + d(w_2, z). \end{aligned}$$

This means that $\delta(\mathcal{B}_F(x)) \leq \kappa(x) + 2\mu(x)$ for each $x \in K$. Therefore, we get

$$h(\widehat{f}(x), G(x)) \leq h(\widehat{f}(x), \operatorname{cl} \alpha^n F(x)) + \Lambda^n(\kappa + 2\mu)(x), \quad x \in K.$$

This completes the proof in view of (5). □

## 3. Some consequences

The next simple theorems show some direct applications of Theorem 1; they correspond to the results on stability of functional equations (for the set-valued mappings) in [6–10].

**Theorem 2.** *Let $F, G\colon K \to bd(Y)$, $\Psi\colon Y \to Y$, $\xi\colon K \to K$, $\lambda \in \mathbb{R}_+$,*

$$\kappa(x) := \sum_{n=0}^{\infty} \lambda^n \delta(F(\xi^n(x)) \cup G(\xi^n(x))) < \infty, \quad x \in K, \tag{8}$$

$$d(\Psi(x), \Psi(y)) \leq \lambda d(x, y), \quad x, y \in Y, \tag{9}$$

$$\Psi(F(\xi(x))) \subset F(x) \cup G(x), \quad x \in K. \tag{10}$$

*Then, there exists a unique function $f\colon K \to Y$, such that $\Psi \circ f \circ \xi = f$ and*

$$h(\widehat{f}(x), F(x)) \leq \kappa(x), \quad x \in K.$$

*Proof.* Define $\alpha\colon bd(Y)^K \to bd(Y)^K$ by

$$\alpha H(x) := \Psi(H(\xi(x))), \quad H \in bd(Y)^K.$$

Then, it is easily seen that it is i.p. Next, let $(H_n)_{n\in\mathbb{N}}$ be a sequence in $bd(Y)^K$, such that there exist $H_L := \lim_{n\to\infty} \operatorname{cl} H_n \in bcl(Y)^K$ and $\lim_{n\to\infty} \operatorname{cl}(\alpha H_n) \in bcl(Y)^K$. Clearly, on account of (9),

$$\begin{aligned}h(\operatorname{cl}(\alpha H_L(x)), \operatorname{cl}(\alpha H_n(x))) &= h(\alpha H_L(x), \alpha H_n(x))\\ &= h(\Psi(H_L(\xi(x))), \Psi(H_n(\xi(x))))\\ &\le \lambda h(H_L(\xi(x)), H_n(\xi(x)))\\ &= \lambda h(H_L(\xi(x)), \operatorname{cl} H_n(\xi(x)))\end{aligned}$$

for every $x \in K$ and $n \in \mathbb{N}$, which implies that

$$\operatorname{cl}(\alpha H_L(x)) = \lim_{n\to\infty} \operatorname{cl}(\alpha H_n(x)).$$

Consequently $\alpha$ is l.p. Let $\Lambda\colon \mathbb{R}_+^K \to \mathbb{R}_+^K$ be given by

$$\Lambda a(x) := \lambda a(\xi(x)), \quad a \in \mathbb{R}_+^K,\, x \in K.$$

Then, it is non-decreasing and (3) holds. Define $\mathcal{G}\colon bd(Y)^K \to bd(Y)^K$ by

$$\mathcal{G}H(x) := H(x) \cup G(x), \quad x \in K,\, H \in bd(Y)^K.$$

Then, in view of (10), (2) is valid, too. Hence, according to Theorem 1, there exists a function $f\colon K \to Y$, such that $\widehat{f}$ is a fixed point of the operator $\alpha$ (i.e., $\Psi \circ f \circ \xi = f$) and

$$h\big(\widehat{f}(x), F(x)\big) \le \kappa(x), \quad x \in K.$$

Moreover, by (8)

$$\lim_{n\to\infty} \lambda^n \kappa(\xi^n(x)) = 0, \quad x \in K,$$

thus (5) holds with $\mu = \kappa$, and consequently, such $f$ must be unique. □

**Theorem 3.** *Assume that $(Y, \cdot)$ is a group with the neutral element $e$ and $d$ is invariant (i.e., $d(xz, yz) = d(x, y) = d(zx, zy)$ for $x, y, z \in Y$). Let $F, G\colon K \to bd(Y)$, $e \in G(x)$ for $x \in K$, $\Psi\colon Y \to Y$, $\xi\colon K \to K$, $\lambda \in \mathbb{R}_+$, (9) holds,*

$$\gamma(x) := \sum_{n=0}^{\infty} \lambda^n \delta(G(\xi^n(x))) < \infty, \quad x \in K, \tag{11}$$

$$\nu(x) := \sum_{n=0}^{\infty} \lambda^n \delta(F(\xi^n(x))) < \infty, \quad x \in K, \tag{12}$$

$$\Psi(F(\xi(x))) \subset F(x)G(x), \quad x \in K, \tag{13}$$

*where $AB := \{ab : \ a \in A,\, b \in B\}$ for nonempty $A, B \subset Y$. Then, there exists a unique function $f\colon K \to Y$, such that $\Psi \circ f \circ \xi = f$ and*

$$h(\widehat{f}(x), F(x)) \le \nu(x) + \gamma(x), \quad x \in K. \tag{14}$$

*Proof.* It is sufficient to argue analogously as in the proof of Theorem 2 with function $\mathcal{G}\colon bd(Y)^K \to bd(Y)^K$ given by

$$\mathcal{G}H(x) := H(x)G(x), \quad x \in K,\, H \in bd(Y)^K.$$

Then, in view of (13), (2) is valid and, according to Theorem 1, there exists a function $f\colon K \to Y$, such that $\widehat{f}$ is a fixed point of $\alpha$ and

$$h\big(\widehat{f}(x), F(x)\big) \le \kappa(x) := \sum_{n=0}^{\infty} \Lambda^n(\tilde{\delta}(\mathcal{G}F))(x), \quad x \in K.$$

Since

$$\Lambda^n(\tilde{\delta}(\mathcal{G}F))(x) \le \lambda^n \delta\big(F(\xi^n(x)) + G(\xi^n(x))\big), \quad x \in K,\, n \in \mathbb{N},$$

and

$$\delta(F(x)G(x)) \le \delta(F(x)) + \delta(G(x)), \quad x \in K,$$

we get (14). Furthermore, since $\kappa(x) \le \mu(x) := \nu(x) + \gamma(x)$ for $x \in K$, (11) and (12) imply that

$$\lim_{n\to\infty} 2\lambda^n(\mu(\xi^n(x)) + \kappa(\xi^n(x))) = 0, \quad x \in K.$$

Therefore, (5) is valid whence $f$ is unique in view of Theorem 1. □

Clearly, in the particular case where $\lambda \in (0,1)$ and

$$M := \sup_{x\in K} \delta(F(x)) < \infty,$$

estimation (14) can be replaced by the following one:

$$h(\widehat{f}(x), F(x)) \le \frac{M}{1-\lambda} + \gamma(x), \quad x \in K.$$



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