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Erratum to: Existence of solutions of scalar field equations with fractional operator

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In the latter part of [2, Proposition 3.6 (i)], it is claimed that any weak solution of

$$(1-\Delta)^{\alpha} u = f(u) \quad \text{in } \mathbf{R}^N, \qquad u \in H^{\alpha}(\mathbf{R}^N)$$

decays faster than any polynomial. However, the proof there is valid only for positive solutions since the function $g(x) := (f(u(x)) - (1-\delta_0)u(x))_+$ may not be compactly supported.

We shall modify the proof and show the decay estimate of any weak solution of

$$(1-\Delta)^{\alpha} u = f(x,u) \quad \text{in } \mathbf{R}^N, \qquad u \in H^{\alpha}(\mathbf{R}^N). \tag{1}$$

Proposition 0.1. *Let $f \in C(\mathbf{R}^N \times \mathbf{R}, \mathbf{R})$ satisfy $|f(x,s)| \le C(|s| + |s|^{2^*_\alpha - 1})$ for each $(x,s) \in \mathbf{R}^N \times \mathbf{R}$ and*

$$\limsup_{s\to 0} \sup_{x\in\mathbf{R}^N} \frac{f(x,s)}{s} < 1. \tag{2}$$

Then for every weak solution u of (1) and every $k \in \mathbf{N}$, there exists a $C_{k,u}$ such that $|u(x)| \le C_{k,u}(1+|x|)^{-k}$ for all $x \in \mathbf{R}^N$.

We first introduce the extension problem (see [1]):

$$\begin{cases} t^{1-2\alpha}(-\Delta_x + 1)w - (t^{1-2\alpha} w_t)_t = 0 & \text{in } \mathbf{R}^{N+1}_+, \\ w = u & \text{on } \partial\mathbf{R}^{N+1}_+ \end{cases} \tag{3}$$

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where $\Delta_x = \sum_{i=1}^N \partial_{x_i}^2$. We also set

$$X^\alpha := \{w : \mathbf{R}_+^{N+1} \to \mathbf{R} \mid \|w\|_{X^\alpha} < \infty\},$$
$$\|w\|_{X^\alpha}^2 := \int_{\mathbf{R}_+^{N+1}} t^{1-2\alpha} \left(|\nabla w|^2 + w^2\right) \mathrm{d}X, \quad X := (x,t)$$

where $\nabla = (\nabla_x, \partial_t)$. Recall [2, Proposition 5.2]:

Proposition 0.2. (i) *There exists the trace operator* $\mathrm{Tr} : X^\alpha \to H^\alpha(\mathbf{R}^N)$.

(ii) *For each* $u \in H^\alpha(\mathbf{R}^N)$*, there is a unique solution* $Eu \in X^\alpha$ *of* (3)*. In addition, there is a constant* $\kappa_\alpha > 0$ *so that*

$$\kappa_\alpha \langle u, \mathrm{Tr}\, w\rangle_\alpha = \int_{\mathbf{R}_+^{N+1}} t^{1-2\alpha}(\nabla(Eu)\cdot\nabla w + (Eu)w)\,\mathrm{d}X$$

for every $u \in H^\alpha(\mathbf{R}^N)$ *and* $w \in X^\alpha$.

(iii) *For each* $u \in H^\alpha(\mathbf{R}^N)$ *and* $w \in X^\alpha$ *with* $\mathrm{Tr}\, w = u$, $\kappa_\alpha\|u\|_\alpha^2 = \|Eu\|_{X^\alpha}^2 \le \|w\|_{X^\alpha}^2$ *hold.*

Now we prove Proposition 0.1:

Proof of Proposition 0.1. Let u be a weak solution of (1). By (2), we may choose $s_0 > 0$ and $\delta_0 > 0$ such that

$$\frac{f(x,s)}{s} < 1 - 2\delta_0 \quad \text{for any } (x,s) \in \mathbf{R}^N \times ([-s_0, s_0] \setminus \{0\}).$$

From the former part of the proof of [2, Proposition 3.6 (i)], we see that $u \in C_{\mathrm{b}}^\beta(\mathbf{R}^N)$ for any $\beta \in (0, 2\alpha)$ and $u(x) \to 0$ as $|x| \to \infty$. Hence, we may choose an $R_0 > 0$ such that

$$|x| \ge R_0,\ u(x) \ne 0 \quad \Rightarrow \quad \frac{f(x,u(x))}{u(x)} \le 1 - \delta_0. \tag{4}$$

Denote by $\chi_{R_0}(x)$ the characteristic function of $B_{R_0}(0)$. Notice that $\chi_{R_0}(x)|f(x,u(x))| \in L^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ is compactly supported. Let v be a unique solution of

$$(1-\Delta)^\alpha v - (1-\delta_0)v = \chi_{R_0}(x)|f(x,u(x))| \quad \text{in } \mathbf{R}^N, \quad v \in H^\alpha(\mathbf{R}^N).$$

Then [2, Proposition 5.1] asserts that for any $k \in \mathbf{N}$ there is a $c_k > 0$ so that

$$0 < v(x) \le c_k(1+|x|)^{-k} \quad \text{for all } x \in \mathbf{R}^N.$$

Therefore, it suffices to prove $-v(x) \le u(x) \le v(x)$ for all $x \in \mathbf{R}^N$.

To this end, remark that u satisfies

$$(1-\Delta)^\alpha u = \chi_{R_0}(x) f(x,u(x)) + (1-\chi_{R_0}(x)) f(x,u(x)) \quad \text{in } \mathbf{R}^N.$$

Setting $U(x) := v(x) - u(x)$, one finds that

$$\begin{aligned}(1-\Delta)^\alpha U = {} & \chi_{R_0}(x)\{|f(x,u(x))| - f(x,u(x))\} \\ & + \{-(1-\chi_{R_0}(x)) f(x,u(x)) + (1-\delta_0)v(x)\}.\end{aligned}$$

Now consider EU and $(EU)_-$ where $w_-(X) := \max\{0, -w(X)\}$. It is easily seen that $(EU)_- \in X^\alpha$ and $\mathrm{Tr}\,(EU)_- = U_-$. Applying Proposition 0.2, we get

$$\begin{aligned}\kappa_\alpha \langle U, U_- \rangle_\alpha \\ &= \int_{\mathbf{R}^N} t^{1-2\alpha}(\nabla EU \cdot \nabla (EU)_- + (EU)(EU)_-)\,\mathrm{d}X = -\|(EU)_-\|^2_{X^\alpha}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\kappa_\alpha \langle U, U_- \rangle_\alpha &= \kappa_\alpha \int_{\mathbf{R}^N} \left[\chi_{R_0}\left\{|f(x,u)| - f(x,u)\right\}\right. \\ &\qquad \left. + \left\{-(1-\chi_{R_0})f(x,u) + (1-\delta_0)v\right\}\right] U_- \mathrm{d}x \\ &\geq \kappa_\alpha \int_{\mathbf{R}^N} (1-\chi_{R_0})\left[-f(x,u) + (1-\delta_0)v\right] U_- \mathrm{d}x. \qquad (5)\end{aligned}$$

If $U_-(x) > 0$ and $|x| \geq R_0$, then from $u(x) > v(x) > 0$ and (4) it follows that

$$\begin{aligned}&(1-\chi_{R_0}(x))\left[-f(x,u(x)) + (1-\delta_0)v(x)\right] \\ &\quad \geq (1-\chi_{R_0}(x))\left[-(1-\delta_0)u(x) + (1-\delta_0)v(x)\right] \\ &\quad = (1-\delta_0)(1-\chi_{R_0}(x))U(x).\end{aligned}$$

Hence,

$$\kappa_\alpha \int_{\mathbf{R}^N} (1-\chi_{R_0})\left[-f(x,u) + (1-\delta_0)v\right] U_- \mathrm{d}x \geq -(1-\delta_0)\kappa_\alpha \|U_-\|^2_{L^2}. \quad (6)$$

Next, by the Plancherel theorem and Proposition 0.2, one sees

$$\kappa_\alpha \|U_-\|^2_{L^2} = \kappa_\alpha \int_{\mathbf{R}^N} |\widehat{U_-}|^2 \mathrm{d}\xi \leq \kappa_\alpha \|U_-\|^2_\alpha \leq \|(EU)_-\|^2_{X^\alpha}.$$

Combining this with (4)–(6), we finally obtain

$$-\|(EU)_-\|^2_{X^\alpha} \geq -(1-\delta_0)\|(EU)_-\|^2_{X^\alpha},$$

which implies $(EU)_- \equiv 0$, hence, $U_- \equiv 0$ and $u(x) \leq v(x)$.

For the opposite inequality $-v(x) \leq u(x)$, we proceed similarly. Set $V(x) := v(x) + u(x)$. Then we have

$$\begin{aligned}-\|(EV)_-\|^2_{X^\alpha} &= \kappa_\alpha \langle V, V_- \rangle_\alpha \\ &\geq \kappa_\alpha \int_{\mathbf{R}^N} (1-\chi_{R_0}(x))\left[f(x,u(x)) + (1-\delta_0)v(x)\right] V_-(x)\mathrm{d}x.\end{aligned}$$

If $V_-(x) > 0$ and $|x| \geq R_0$, then $u(x) < -v(x) < 0$. Thus, by (4), one gets

$$\begin{aligned}&(1-\chi_{R_0}(x))\left[f(x,u(x)) + (1-\delta_0)v(x)\right] \\ &\quad \geq (1-\chi_{R_0}(x))\left[(1-\delta_0)u(x) + (1-\delta_0)v(x)\right] \\ &\quad = (1-\delta_0)(1-\chi_{R_0}(x))V(x).\end{aligned}$$

The rest of the argument is same as the above and we obtain $V_- \equiv 0$. Thus, $-v(x) \leq u(x)$ holds and we complete the proof. $\square$

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