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Symplectomorphisms and discrete braid invariants

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Dedicated to Professor Paul Rabinowitz.

Abstract. Area and orientation preserving diffeomorphisms of the standard 2-disc, referred to as symplectomorphisms of $\mathbb{D}^2$, allow decompositions in terms of *positive* twist diffeomorphisms. Using the latter decomposition, we utilize the Conley index theory of discrete braid classes as introduced in Ghrist et al. (Invent. Math. 152:369–432, 2003) and Ghrist et al. (C. R. Acad. Sci. Paris Sér. I Math. 331:861–865, 2000) to obtain a Morse type forcing theory of periodic points: a priori information about periodic points determines a mapping class which may force additional periodic points.

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Contents

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1. Prelude

Let $\mathbb{D}^2 \subset \mathbb{R}^2$ be the standard unit 2-disc with coordinates $z = (x, y) \in \mathbb{R}^2$, and let $\omega = \mathrm{d}x \wedge \mathrm{d}y$ be the standard area 2-form on $\mathbb{R}^2$. A diffeomorphism $F \colon \mathbb{D}^2 \to \mathbb{D}^2$ is said to be symplectic if $F^*\omega = \omega$—area and orientation preserving—and is referred to as a *symplectomorphism* of $\mathbb{D}^2$. Symplectomorphisms of the 2-disc form a group which is denoted by $\mathrm{Symp}(\mathbb{D}^2)$. A diffeomorphism F is *Hamiltonian* if it is given as the time-1 mapping of a Hamiltonian system

$$\dot{x} = \partial_y H(t, x, y);$$
$$\dot{y} = -\partial_x H(t, x, y), \tag{1}$$

where $H \in C^\infty(\mathbb{R} \times \mathbb{D}^2)$ is a the Hamiltonian function with the additional property that $H(t, \cdot)|_{\partial \mathbb{D}^2} = \text{const.}$ for all $t \in \mathbb{R}$. The set of Hamiltonians satisfying these requirements is denoted by $\mathcal{H}(\mathbb{D}^2)$ and the associated flow of (1) is denoted by $\psi_{t,H}$. The group $\mathrm{Ham}(\mathbb{D}^2)$ signifies the group of Hamiltonian diffeomorphisms of $\mathbb{D}^2$. Hamiltonian diffeomorphisms are symplectic by construction. For the 2-disc, these notions are equivalent, i.e., $\mathrm{Symp}(\mathbb{D}^2) = \mathrm{Ham}(\mathbb{D}^2)$ and we may therefore study Hamiltonian systems to prove properties about symplectomorphisms of $\mathbb{D}^2$, cf. [3], and Appendix B.

A subset $B \subset \mathbb{D}^2$ is a invariant set for F if $F(B) = B$. We are interested in finite invariant sets. Such invariant sets consist of periodic points, i.e., points $z \in \mathbb{D}^2$ such that $F^k(z) = z$ for some $k \geq 1$. Since $\partial \mathbb{D}^2$ is also invariant, periodic point are either in $\mathrm{int}\, \mathbb{D}^2$, or $\partial \mathbb{D}^2$.

The main result of this paper concerns a *forcing* problem. Given a finite invariant set $B \subset \mathrm{int}\, \mathbb{D}^2$ for $F \in \mathrm{Symp}(\mathbb{D}^2)$, do there exist additional periodic points? More generally, does there exist a finite invariant set $A \subset \mathrm{int}\, \mathbb{D}^2$, with $A \cap B = \varnothing$? This is much like a similar question for the discrete dynamics on an interval, where the famous Sharkovskii theorem establishes a forcing order among periodic points based on their period. In the 2-dimensional situation such an order is much harder to establish, cf. [3]. The main results are based on braiding properties of periodic points and are stated and proved in Sect. 7. The braid invariants introduced in this paper add additional information to existing invariants in the area-preserving case. For instance, Example 7.3 describes a braid class which forces additional invariant sets solely in the area-preserving case, hence extending the non-symplectic methods described in [4].

The theory in this paper can be further generalized to include symplectomorphisms of bounded subsets of $\mathbb{R}^2$ with smooth boundary, e.g., annuli, and symplectomorphisms of $\mathbb{R}^2$.

2. Mapping class groups

A priori knowledge of finite invariant sets B for F categorize mappings in the so-called mapping classes. Traditionally mapping class groups are defined for orientation preserving homeomorphisms, cf. [5] for an overview. Denote by $\mathrm{Homeo}^+(\mathbb{D}^2)$ the space of orientation preserving homeomorphisms and by $\mathrm{Homeo}_0^+(\mathbb{D}^2)$ the homeomorphisms that leave the boundary point wise invariant. Two homeomorphisms $F, G \in \mathrm{Homeo}^+(\mathbb{D}^2)$ are isotopic if there exists an isotopy ϕ_t, with $\phi_t \in \mathrm{Homeo}^+(\mathbb{D}^2)$ for all $t \in [0,1]$, such that $\phi_0 = F$ and $\phi_1 = G$. The equivalence classes in $\pi_0(\mathrm{Homeo}^+(\mathbb{D}^2)) = \mathrm{Homeo}^+(\mathbb{D}^2)/\sim$ are called *mapping classes* and form a group under composition. The latter is referred to as the *mapping class group* of the 2-disc and is denoted by $\mathrm{Mod}(\mathbb{D}^2)$. For homeomorphisms that leave the boundary point wise invariant the mapping class group is denoted by $\mathrm{Mod}_0(\mathbb{D}^2) = \pi_0(\mathrm{Homeo}_0^+(\mathbb{D}^2))$. In Appendix A we provide proofs of the relevant facts about mapping class groups.

Proposition 2.1. *Both mapping class groups* $\mathrm{Mod}(\mathbb{D}^2)$ *and* $\mathrm{Mod}_0(\mathbb{D}^2)$ *are trivial.*

The mapping class groups $\mathrm{Mod}(\mathbb{D}^2)$ and $\mathrm{Mod}_0(\mathbb{D}^2)$ may also be defined using diffeomorphisms, cf. Appendix A. In Proposition B.1, we show that $\pi_0(\mathrm{Symp}(\mathbb{D}^2)) = \mathrm{Mod}(\mathbb{D}^2)$ and in Proposition B.3 we show that $\mathrm{Ham}(\mathbb{D}^2) = \mathrm{Symp}(\mathbb{D}^2)$, which implies that every homeomorphism, or diffeomorphism is isotopic to a Hamiltonian symplectomorphism.

Proposition 2.2. $\pi_0(\mathrm{Symp}(\mathbb{D}^2)) = \pi_0(\mathrm{Ham}(\mathbb{D}^2)) = \mathrm{Mod}(\mathbb{D}^2) \cong 1.$

More refined information about mapping classes is obtained by considering finite invariant sets B. This leads to the notion of the *relative mapping classes*. Two homeomorphisms $F, G \in \mathrm{Homeo}^+(\mathbb{D}^2)$ are of the same mapping class *relative to B* if there exists an isotopy ϕ_t, with $\phi_t \in \mathrm{Homeo}^+(\mathbb{D}^2)$ and $\phi_t(B) = B$ for all $t \in [0,1]$, such that $\phi_0 = F$ and $\phi_1 = G$. The subgroup of such homeomorphisms is denoted by $\mathrm{Homeo}^+(\mathbb{D}^2 \operatorname{rel} B)$ and $\mathrm{Homeo}_0^+(\mathbb{D}^2 \operatorname{rel} B)$ in case $\partial\mathbb{D}^2$ is point wise invariant. The associated mapping class groups are denoted by $\mathrm{Mod}(\mathbb{D}^2 \operatorname{rel} B) = \pi_0(\mathrm{Homeo}^+(\mathbb{D}^2 \operatorname{rel} B))$ and $\mathrm{Mod}_0(\mathbb{D}^2 \operatorname{rel} B) = \pi_0(\mathrm{Homeo}_0^+(\mathbb{D}^2 \operatorname{rel} B))$, respectively.

Proposition 2.3. $\mathrm{Mod}(\mathbb{D}^2 \operatorname{rel} B) \cong \mathscr{B}_m/Z(\mathscr{B}_m)$ *and* $\mathrm{Mod}_0(\mathbb{D}^2 \operatorname{rel} B) \cong \mathscr{B}_m$, *where* $\mathscr{B}_m$ *is the Artin braid group, with* $m = \#B$ *and* $Z(\mathscr{B}_m)$ *is the center of the braid group.*

Let $\mathcal{C}_m\mathbb{D}^2$ be the *configuration space* of unordered configurations of m points in $\mathbb{D}^2$. *Geometric braids* on m strands on $\mathbb{D}^2$ are closed loops in $\mathcal{C}_m\mathbb{D}^2$ based at $B_0 = \{z_1, \ldots, z_m\}$, where the points z_i are defined as follows: $z_i = (x_i, 0)$, $x_0 = -1$, and $x_{i+1} = x_i + 2/(m+1)$. The *classical braid group* on $\mathbb{D}^2$ is the fundamental group $\pi_1(\mathcal{C}_m\mathbb{D}^2, B_0)$ and is denote by $\mathcal{B}_m\mathbb{D}^2$. The (algebraic) *Artin braid group* $\mathscr{B}_m$ is a free group spanned by the $m-1$ generators σ_i,

modulo the following relations:

$$\begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i, & |i-j| \geq 2, \quad i,j \in \{1,\ldots,m-1\} \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, & 1 \leq i \leq m-2. \end{cases} \quad (2)$$

Full twists are denoted algebraically by $\square = (\sigma_1 \ldots \sigma_{m-1})^m$ and generate the center of the braid group $\mathscr{B}_m$. Presentation of words consisting only of the σ_i's (not the inverses) and the relations in (2) form a monoid which is called the *positive braid monoid* $\mathscr{B}_m^+$.

There exists a canonical isomorphism $\boldsymbol{i}_m \colon \mathscr{B}_m \to \mathcal{B}_m\mathbb{D}^2$, cf. [6, Sect. 1.4]. For closed loops $\boldsymbol{\beta}(t)$ based at $B \in \mathcal{C}_m\mathbb{D}^2$ we have a canonical isomorphism $\boldsymbol{j}_B \colon \pi_1\big(\mathcal{C}_m\mathbb{D}^2, B\big) \to \pi_1\big(\mathcal{C}_m\mathbb{D}^2, B_0\big) = \mathcal{B}_m\mathbb{D}^2$. Let $p \colon [0,1] \to \mathcal{C}_m\mathbb{D}^2$ be a path connecting B_0 to B, then define $\boldsymbol{j}_B\big([\boldsymbol{\beta}]_B\big) := [(p \cdot \boldsymbol{\beta}) \cdot p^*]_{B_0} = [p \cdot (\boldsymbol{\beta} \cdot p^*)]_{B_0}$, where p^* is the inverse path connecting B to B_0. The definition of $\boldsymbol{j}_B$ is independent of the chosen path p. This yields the isomorphism $\iota_B = \boldsymbol{i}_m^{-1} \circ \boldsymbol{j}_B \colon \pi_1\big(\mathcal{C}_m\mathbb{D}^2, B\big) \to \mathscr{B}_m$.

The construction of the isomorphism $\mathrm{Mod}_0(\mathbb{D}^2 \operatorname{rel} B) \cong \mathcal{B}_m\mathbb{D}^2$ can be understood as follows, cf. [6,7]. For $F \in \mathrm{Homeo}_0^+(\mathbb{D}^2 \operatorname{rel} B)$ choose an isotopy $\phi_t \in \mathrm{Homeo}_0^+(\mathbb{D}^2)$, $t \in [0,1]$, such that $\phi_1 = F$. Such an isotopy exists since $\mathrm{Homeo}_0^+(\mathbb{D}^2)$ is contractible, cf. Proposition 2.1. For $G \in [F] \in \mathrm{Mod}_0(\mathbb{D}^2 \operatorname{rel} B)$, the composition and scaling of the isotopies defines isotopic braids based at $B \in \mathcal{C}_m\mathbb{D}^2$. The isomorphism $\jmath_B \colon \mathrm{Mod}_0(\mathbb{D}^2 \operatorname{rel} B) \to \mathscr{B}_m$ is given by $\jmath_B([F]) = \iota_B\big(d_*^{-1}([F])\big) = \iota_B([\boldsymbol{\beta}]_B)$, with $\boldsymbol{\beta}(t) = \phi_t(B)$ the geometric braid generated by ϕ_t. The isomorphism d_* is given in Appendix A.2 and $[\boldsymbol{\beta}]_B$ denotes the homotopy class in $\pi_1\big(\mathcal{C}_m\mathbb{D}^2, B\big)$. For $\mathrm{Mod}(\mathbb{D}^2 \operatorname{rel} B)$ we use the same notation for the isomorphism which is given by

$$\jmath_B \colon \mathrm{Mod}(\mathbb{D}^2 \operatorname{rel} B) \cong \mathscr{B}_m/Z(\mathscr{B}_m), \quad [F] \mapsto \jmath_B([F]) = \beta \bmod \square,$$

where $\beta = \iota_B\big([\boldsymbol{\beta}]_B\big)$. The above mapping class groups can also be defined using diffeomorphisms and symplectomorphisms.

Proposition 2.4. $\pi_0\big(\mathrm{Ham}(\mathbb{D}^2 \operatorname{rel} B)\big) = \mathrm{Mod}(\mathbb{D}^2 \operatorname{rel} B) \cong \mathscr{B}_m/Z(\mathscr{B}_m)$.

In Appendix B we show that $\pi_0\big(\mathrm{Symp}(\mathbb{D}^2 \operatorname{rel} B)\big) = \mathrm{Mod}(\mathbb{D}^2 \operatorname{rel} B)$ and that $\mathrm{Symp}(\mathbb{D}^2 \operatorname{rel} B) = \mathrm{Ham}(\mathbb{D}^2 \operatorname{rel} B)$, and therefore that every mapping class can be represented by Hamiltonian symplectomorphisms.

3. Braid classes

Considering free loops in a configuration space as opposed to based loops leads to classes of closed braids which are the key tool for studying periodic points.

3.1. Discretized braids
From [1] we recall the notion of positive piecewise linear braid diagrams and discretized braids.

Definition 3.1. The space of *discretized period d closed braids on m strands*, denoted $\mathscr{D}_m^d$, is the space of all pairs $(\boldsymbol{b}, \tau)$ where $\tau \in S_m$ is a permutation on m elements, and $\boldsymbol{b}$ is an unordered set of m strands, $\boldsymbol{b} = \{\boldsymbol{b}^\mu\}_{\mu=1}^m$, defined as follows:

(a) each strand $\boldsymbol{b}^\mu = (x_0^\mu, x_1^\mu, \dots, x_d^\mu) \in \mathbb{R}^{d+1}$ consists of $d+1$ *anchor points* x_j^μ;

(b) $x_d^\mu = x_0^{\tau(\mu)}$ for all $\mu = 1, \dots, m$;

(c) for any pair of distinct strands $\boldsymbol{b}^\mu$ and $\boldsymbol{b}^{\mu'}$ such that $x_j^\mu = x_j^{\mu'}$ for some j, the *transversality* condition $\left(x_{j-1}^\mu - x_{j-1}^{\mu'}\right)\left(x_{j+1}^\mu - x_{j+1}^{\mu'}\right) < 0$ holds.

Remark 3.2. Two discrete braids $(\boldsymbol{b}, \tau)$ and $(\boldsymbol{b}', \tau')$ are close if the strands $\boldsymbol{b}^{\zeta(\mu)}$ and $\boldsymbol{b}'^\mu$ are close in $\mathbb{R}^{md}$ for some permutation ζ such that $\tau' = \zeta\tau\zeta^{-1}$. We suppress the permutation τ from the notation. Presentations via the braid monoid $\mathscr{B}_m^+$ store the permutations.

Definition 3.3. (cf. [1]) The closure $\bar{\mathscr{D}}_m^d$ of the space $\mathscr{D}_m^d$ consists of pairs $(\boldsymbol{b}, \tau)$ for which (a)-(b) in Definition 3.1 are satisfied.

The path components of $\mathscr{D}_m^d$ are the *discretized braids classes* $[\boldsymbol{b}]$. Being in the same path connected component is an equivalence relation on $\mathscr{D}_m^d$, where the braid classes are the equivalence classes expressed by the notation $\boldsymbol{b}, \boldsymbol{b}' \in [\boldsymbol{b}]$, and $\boldsymbol{b} \sim \boldsymbol{b}'$. The associated permutations τ and τ' are conjugate. A path connecting $\boldsymbol{b}$ and $\boldsymbol{b}'$ is called a *positive isotopy* and the equivalence relation is referred to *positively isotopic*.

To a configuration $\boldsymbol{b} \in \mathscr{D}_m^d$ one can associate a *piecewise linear braid diagram* $\mathrm{B}(\boldsymbol{b})$. For each strand $\boldsymbol{b}^\mu \in \boldsymbol{b}$, consider the piecewise linear (PL) interpolation

$$\mathrm{B}^\mu(t) := x_{\lfloor d \cdot t \rfloor}^\mu + (d \cdot t - \lfloor d \cdot t \rfloor)\left(x_{\lceil d \cdot t \rceil}^\mu - x_{\lfloor d \cdot t \rfloor}^\mu\right), \tag{3}$$

for $t \in [0, 1]$. The braid diagram $\mathrm{B}(\boldsymbol{b})$ is then defined to be the superimposed graphs of all the functions $\mathrm{B}^\mu(t)$. A braid diagram $\mathrm{B}(\boldsymbol{b})$ is not only a good bookkeeping tool for keeping track of the strands in $\mathrm{B}(\boldsymbol{b})$, but also plays natural the role of a braid diagram projection with only positive intersections, cf. Sect. 4.

The set of t-coordinates of intersection points in $\mathrm{B}(\boldsymbol{b})$ is denoted by $\{t_i\}$, $i = 1, \dots, |\boldsymbol{b}|$, where $|\boldsymbol{b}|$ is the total number of intersections in $\mathrm{B}(\boldsymbol{b})$ counted with multiplicity. The latter is also referred to as the *word metric* and is an invariant for $\boldsymbol{b}$. A discrete braid $\boldsymbol{b}$ is *regular* if all points t_i and anchor points x_j^μ are distinct. The regular discrete braids in $[\boldsymbol{b}]$ form a dense subset and every discrete braid is positively isotopic to a regular discrete braid. To a regular discrete braid $\boldsymbol{b}$ one can assign a unique positive word $\beta = \beta(\boldsymbol{b})$ defined as follows:

$$\boldsymbol{b} \mapsto \beta(\boldsymbol{b}) = \sigma_{k_1} \dots \sigma_{k_\ell}, \tag{4}$$

where k_i and $k_i + 1$ are the positions that intersect at t_i, cf. [8, Def. 1.13]. On the positive braid monoid $\mathscr{B}_m^+$ two positive words β and β' are positively equal, notation $\beta \doteq \beta'$, if they represent the same element in $\mathscr{B}_m^+$ using the

relations in (2). On $\mathscr{B}_m^+$ we define an equivalence relation which acts as an analogue of conjugacy in the braid group, cf. [9, Sect. 2.2]. For a given word $\sigma_{i_1} \ldots \sigma_{i_n}$, define the relation

$$\sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_n} \equiv \sigma_{i_2} \ldots \sigma_{i_n} \sigma_{i_1}.$$

Definition 3.4. Two positive words $\beta, \beta' \in \mathscr{B}_m^+$ are *positively conjugate*, notation $\beta \doteq \beta'$, if there exists a sequence of words $\beta_0, \ldots, \beta_\ell \in \mathscr{B}_m^+$, with $\beta_0 = \beta$ and $\beta_\ell = \beta'$, such that for all k, either $\beta_k \doteq \beta_{k+1}$, or $\beta_k \equiv \beta_{k+1}$

   Positive conjugacy is an equivalence relation on $\mathscr{B}_m^+$ and the set of positive conjugacy classes $[\![\beta]\!]$ of the braid monoid $\mathscr{B}_m^+$ is denoted by $\mathscr{C}\mathscr{B}_m^+$.
   The above defined assignment $\boldsymbol{b} \mapsto \beta(\boldsymbol{b})$ can be extended to all discrete braids. A discrete braid $\boldsymbol{b}$ is positively isotopic to a regular braid $\boldsymbol{b}'$ and the mapping $\mathscr{D}_m^d \to \mathscr{C}\mathscr{B}_m^+$, given by $\boldsymbol{b} \mapsto [\![\beta(\boldsymbol{b})]\!]$, is well-defined by choosing $\beta(\boldsymbol{b})$ to be any representative in the positive conjugacy class $[\![\beta(\boldsymbol{b}')]\!]$. Observe that for fixed d the mapping $\mathscr{D}_m^d \to \mathscr{C}\mathscr{B}_m^+$ is not surjective.

Remark 3.5. The positive conjugacy relation defined in Definition 3.4 is symmetric by construction since it is defined on finite words. For instance, consider $\sigma_1 \sigma_2 \sigma_3 \equiv \sigma_2 \sigma_3 \sigma_1$. The question whether the $\sigma_2 \sigma_3 \sigma_1 \equiv \sigma_1 \sigma_2 \sigma_3$ is answered as follows: $\sigma_2 \sigma_3 \sigma_1 \equiv \sigma_3 \sigma_1 \sigma_2 \equiv \sigma_1 \sigma_2 \sigma_3$, which, by Definition 3.4, shows that $\sigma_2 \sigma_3 \sigma_1 \equiv \sigma_1 \sigma_2 \sigma_3$.

   The presentation of discrete braids via words in $\mathscr{B}_m^+$ yields the following alternative equivalence relation.

Definition 3.6. Two discretized braids $\boldsymbol{b}, \boldsymbol{b}' \in \mathscr{D}_m^d$ are *topologically equivalent* if $\beta(\boldsymbol{b}) \doteq \beta(\boldsymbol{b}')$ in $\mathscr{B}_m^+$, i.e., $\beta(\boldsymbol{b})$ and $\beta(\boldsymbol{b}')$ are positively conjugate. Notation: $\boldsymbol{b} \doteq \boldsymbol{b}'$.

   Summarizing, $\boldsymbol{b} \sim \boldsymbol{b}'$ implies $\boldsymbol{b} \doteq \boldsymbol{b}'$ and $\doteq$ defines a coarser equivalence relation on $\mathscr{D}_m^d$. The equivalence classes with respect to $\doteq$ are denote by $[\boldsymbol{b}]_{\doteq}$. The converse is not true in general, cf. [1, Fig. 8]. Following [1, Def. 17], a discretized braid class $[\boldsymbol{b}]$ is *free* if $[\boldsymbol{b}] = [\boldsymbol{b}]_{\doteq}$.

Proposition 3.7. [1, Prop. 27] *If $d > |\boldsymbol{b}|$, then $[\boldsymbol{b}]$ is a free braid class.*

   Taking d sufficiently large is a sufficient condition to ensure free braid classes, but this condition is not a necessary condition.

Example 3.8. Given the braids $\boldsymbol{b} \in \mathscr{D}_3^2$ with $\boldsymbol{b}^1 = (1, 4, 1)$, $\boldsymbol{b}^2 = (2, 2, 2)$ and $\boldsymbol{b}^3 = (3, 3, 3)$ and consider the braid class $[\boldsymbol{b}]$, see Fig. 1 (left and middle). Since $\boldsymbol{b}$ is regular, $\beta(\boldsymbol{b})$ is uniquely defined and $\beta(\boldsymbol{b}) = \sigma_1 \sigma_2^2 \sigma_1$. Also, define $\boldsymbol{b}' \in \mathscr{D}_3^2$ with $\boldsymbol{b}'^1 = (4, 1, 4)$, $\boldsymbol{b}'^2 = \boldsymbol{b}^2$ and $\boldsymbol{b}'^3 = \boldsymbol{b}^3$ and the braid class $[\boldsymbol{b}']$. Since $\boldsymbol{b}'$ is also regular we have the unique braid word $\beta(\boldsymbol{b}') = \sigma_2 \sigma_1^2 \sigma_2$. Observe that $\sigma_1 \sigma_2^2 \sigma_1 \doteq \sigma_2 \sigma_1^2 \sigma_2$, which implies that $\boldsymbol{b}$ and $\boldsymbol{b}'$ are topologically equivalent. However, $\boldsymbol{b}$ and $\boldsymbol{b}'$ are not positively isotopic in $\mathscr{D}_3^2$ and $[\boldsymbol{b}]$ and $[\boldsymbol{b}']$ are two different path components of $\mathscr{D}_3^2$. The positive conjugacy class of $\sigma_1 \sigma_2^2 \sigma_1$ is given by $[\![\sigma_1 \sigma_2^2 \sigma_1]\!] = \{\sigma_1 \sigma_2^2 \sigma_1, \sigma_2^2 \sigma_1^2, \sigma_2 \sigma_1^2 \sigma_2, \sigma_1^2 \sigma_2^2\}$. The words $\sigma_2^2 \sigma_1^2$ and $\sigma_1^2 \sigma_1^2$ are not represented in $\mathscr{D}_3^2$. If we consider $\boldsymbol{b}'' \in \mathscr{D}_3^3$ given by

$\boldsymbol{b}'' = \{(1,4,1,1),(2,2,2,2),(3,3,3,3)\}$, then the associated braid class $[\boldsymbol{b}'']$ is free, which confirms that the condition in Proposition 3.7 is not a necessary condition, see Fig. 1 (right).

Let $\beta = \sigma_{i_1}\dots\sigma_{i_d} \in \mathscr{B}_m^+$ be a positive braid word, then define

$$\mathcal{E}_q(\beta) := \boldsymbol{b} = \{\boldsymbol{b}^\mu\} \in \mathscr{D}_m^{d+q}, \quad \boldsymbol{b}^\mu = (x_j^\mu), \quad \mu = 1,\dots,m, \quad q \geq 0,$$

with $x_0^\mu = \mu$, $x_j^\mu = x_0^{\sigma_{i_1}\dots\sigma_{i_j}(\mu)}$, $j = 1,\dots,d$, and $x_{d+q}^\mu = \dots = x_d^\mu$. The expression $\sigma_{i_1}\dots\sigma_{i_j}(\mu)$, $\mu = 1,\dots,m$ describes the permutation of the set $\{1,\dots,m\}$, where $\sigma_{i_1}\dots\sigma_{i_j}$ is regarded as a concatenation of permutations given by the generators σ_i interpreted as a basic permutation of i and $i+1$. By Proposition 3.7, $[\mathcal{E}_q(\beta)]$ is free for all $q \geq 1$, and every $[\![\beta]\!] \in \mathscr{CB}_m^+$ defines a free discrete braid class $[\mathcal{E}_q(\beta)]$ in $\mathscr{D}_m^{d+q}$ for all $q \geq 1$.

3.2. Discrete 2-colored braid classes

On closed configuration spaces we define the following product:

$$\bar{\mathscr{D}}_n^d \times \bar{\mathscr{D}}_m^d \to \bar{\mathscr{D}}_{n+m}^d, \quad (\boldsymbol{a},\boldsymbol{b}) \mapsto \boldsymbol{a} \sqcup \boldsymbol{b},$$

where $\boldsymbol{a} \sqcup \boldsymbol{b}$ is the disjoint union of the strands in $\boldsymbol{a}$ and $\boldsymbol{b}$ regarded as an element in $\bar{\mathscr{D}}_{n+m}^d$. The definition yields a canonical permutation on the labels in $\boldsymbol{a} \sqcup \boldsymbol{b}$. Define the space of *2-colored discretized braids* as the space of ordered pairs

$$\mathscr{D}_{n,m}^d := \{\boldsymbol{a}\,\mathrm{rel}\,\boldsymbol{b} := (\boldsymbol{a},\boldsymbol{b}) \mid \boldsymbol{a} \sqcup \boldsymbol{b} \in \mathscr{D}_{n+m}^d\}. \tag{5}$$

The strand labels in $\boldsymbol{a}$ range from $\mu = 1,\dots,n$ and the strand labels in $\boldsymbol{b}$ range from $\mu = n+1,\dots,n+m$. The associated permutation $\tau_{\boldsymbol{a},\boldsymbol{b}} = \tau_{\boldsymbol{a}} \oplus \tau_{\boldsymbol{b}} \in S_{n+m}$, where $\tau_{\boldsymbol{a}} \in S_n$ and $\tau_{\boldsymbol{b}} \in S_m$, and $\tau_{\boldsymbol{a}}$ acts on the labels $\{1,\dots,n\}$ and $\tau_{\boldsymbol{b}}$ acts on the labels $\{n+1,\dots,n+m\}$. The strands $\boldsymbol{a} = \{x_j^\mu\}$, $\mu = 1,\dots,n$ are the *red*, or *free* strands and the strands $\boldsymbol{b} = \{x_j^\mu\}$, $\mu = n+1,\dots,n+m$ are the *black*, or *skeletal* strands. A path component $[\boldsymbol{a}\,\mathrm{rel}\,\boldsymbol{b}]$ in $\mathscr{D}_{n,m}^d$ is called a *2-colored discretized braid class*. The canonical projections are given by $\varpi\colon \mathscr{D}_{n,m}^d \to \mathscr{D}_m^d$ with $\boldsymbol{a}\,\mathrm{rel}\,\boldsymbol{b} \mapsto \boldsymbol{b}$ and by $\varpi^*\colon \mathscr{D}_{n,m}^d \to \mathscr{D}_n^d$ with $\boldsymbol{a}\,\mathrm{rel}\,\boldsymbol{b} \mapsto \boldsymbol{a}$. The mapping ϖ yields a fibration

$$[\boldsymbol{a}]\,\mathrm{rel}\,\boldsymbol{b} \to [\boldsymbol{a}\,\mathrm{rel}\,\boldsymbol{b}] \to [\boldsymbol{b}]. \tag{6}$$
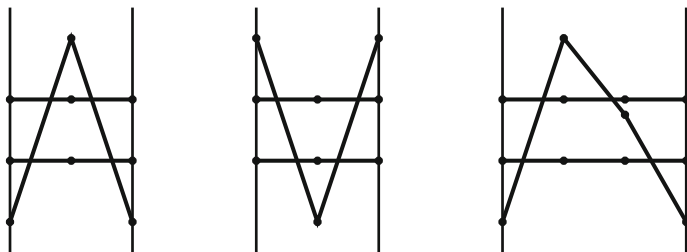
FIGURE 1. The *left* and *middle* diagrams show representatives $\boldsymbol{b},\boldsymbol{b}' \in \mathscr{D}_3^2$ in Example 3.8. The *right* diagram shows a representative the same topological braid class in $\mathscr{D}_3^3$ (free)

The pre-images $\varpi^{-1}(\boldsymbol{b}) = [\boldsymbol{a}]\operatorname{rel}\boldsymbol{b} \subset \mathscr{D}_n^d$, are called the *relative discretized braid class fibers*.

There exists a natural embedding $\mathscr{D}_{n,m}^d \hookrightarrow \mathscr{D}_{n+m}^d$, defined by $\boldsymbol{a}\operatorname{rel}\boldsymbol{b} \mapsto \boldsymbol{a} \sqcup \boldsymbol{b}$. Via the embedding, we define the notion of topological equivalence of two 2-colored discretized braids: $\boldsymbol{a}\operatorname{rel}\boldsymbol{b} \doteq \boldsymbol{a}'\operatorname{rel}\boldsymbol{b}'$ if $\boldsymbol{a} \sqcup \boldsymbol{b} \doteq \boldsymbol{a}' \sqcup \boldsymbol{b}'$. The associated equivalence classes are denoted by $[\boldsymbol{a}\operatorname{rel}\boldsymbol{b}]_{\doteq}$, which are not necessarily connected sets in $\mathscr{D}_{n,m}^d$. A 2-colored discretized braid class $[\boldsymbol{a}\operatorname{rel}\boldsymbol{b}]$ is free if $[\boldsymbol{a}\operatorname{rel}\boldsymbol{b}] = [\boldsymbol{a}\operatorname{rel}\boldsymbol{b}]_{\doteq}$. If $d > |\boldsymbol{a}\sqcup\boldsymbol{b}|$, then $[\boldsymbol{a}\operatorname{rel}\boldsymbol{b}]$ is free by Proposition 3.7.

The set of collapsed singular braids in $\bar{\mathscr{D}}_{n,m}^d$ is given by:

$$\Sigma^- := \left\{ \boldsymbol{a}\operatorname{rel}\boldsymbol{b} \in \bar{\mathscr{D}}_{n,m}^d \mid \boldsymbol{a}^\mu = \boldsymbol{a}^{\mu'}, \text{ or } \boldsymbol{a}^\mu = \boldsymbol{b}^{\mu'} \text{ for some } \mu \neq \mu', \text{ and } \boldsymbol{b} \in \mathscr{D}_m^d \right\}.$$

A 2-colored discretized braid class $[\boldsymbol{a}\operatorname{rel}\boldsymbol{b}]$ is *proper* if $\partial[\boldsymbol{a}\operatorname{rel}\boldsymbol{b}]_{\doteq} \cap \Sigma^- = \varnothing$. If a braid class $[\boldsymbol{a}\operatorname{rel}\boldsymbol{b}]$ is not proper it is called *improper*. In [9] properness is considered in a more general setting. The notion of properness in this paper coincides with weak properness in [9].

A 2-colored discretized braid class $[\boldsymbol{a}\operatorname{rel}\boldsymbol{b}]$ is called *bounded* if its fibers are bounded as sets in $\mathbb{R}^{nd}$. Note that $[\boldsymbol{a}\operatorname{rel}\boldsymbol{b}]$ is *not* a bounded set in $\mathbb{R}^{(n+m)d}$.

3.3. Algebraic presentations

Discretized braid classes are presented via the positive conjugacy classes of the positive braid monoid $\mathscr{B}_m^+$. For 2-colored discretized braids we seek a similar presentation.

To keep track of colors we define coloring on words in $\mathscr{B}_{n+m}^+$. Words in $\mathscr{B}_{n+m}^+$ define associated permutations τ and the permutations τ yield partitions of the set $\{1,\ldots,n+m\}$. Let $\gamma \in \mathscr{B}_{n+m}^+$ be a word for which the induced partition contains a union of equivalence classes $\textsc{a} \subset \{1,\ldots,n+m\}$ consisting of n elements. The set $\textsc{a}$ is the *red coloring* of length n and the remaining partitions are colored black, denoted by $\textsc{b}$. The pair $(\gamma, \textsc{a})$ is called a 2-colored positive braid word, see Fig. 2. For a given coloring $\textsc{a} \subset \{1,\ldots,n+m\}$ of length n the set of all words $(\gamma, \textsc{a})$ forms a monoid which is denoted by $\mathscr{B}_{n,m,\textsc{a}}^+$ and is referred as the *2-colored braid monoid* with coloring $\textsc{a}$.

Two pairs $(\gamma, \textsc{a})$ and $(\gamma', \textsc{a}')$ are positively conjugate if $\gamma \doteq \gamma'$ and $\textsc{a}' = \zeta^{-1}(\textsc{a})$, where ζ is a permutation conjugating the induced permutations τ_γ and $\tau_{\gamma'}$, i.e., $\tau_{\gamma'} = \zeta\tau_\gamma\zeta^{-1}$. If ξ is another permutation such that $\tau_{\gamma'} = \xi\tau_\gamma\xi^{-1}$, then $\zeta\tau_\gamma\zeta^{-1} = \xi\tau_\gamma\xi^{-1}$. This implies that $\tau_\gamma = \zeta^{-1}\xi\tau_\gamma\xi^{-1}\zeta$, and thus $\xi^{-1}\zeta(\textsc{a}) = \textsc{a}$, which is equivalent to $\zeta^{-1}(\textsc{a}) = \xi^{-1}(\textsc{a})$. This shows that
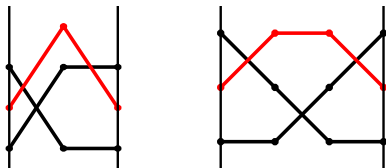

FIGURE 2. Representations of the relative braid class $[\boldsymbol{a}\operatorname{rel}\boldsymbol{b}]$

the conjugacy relation is well-defined. Positive conjugacy for 2-colored braid words is again denoted by $(\gamma, \text{A}) \doteq (\gamma', \text{A}')$ and a conjugacy class is denoted by $[\![\gamma, \text{A}]\!]$. The set of 2-colored positive conjugacy classes with red colorings of length n is denoted by $\mathscr{CB}^+_{n,m}$.

The words corresponding to the different colors (γ, A) can be derived from the information in (γ, A). Let $\text{A}_0 \subset \text{A}$ be a cycle of length $\ell \leq n$ and let $k \in \text{A}_0$. If $\gamma = \sigma_{i_1} \ldots \sigma_{i_d}$, then we define an ℓ-periodic sequence $\{k_j\}$, with

$$k_0 = k, \quad \text{and} \quad k_j = \sigma_{i_j}(k_{j-1}), \quad j = 1, \ldots, \ell d,$$

by considering the word γ^ℓ. Now use the following rule: if $k_j - k_{j-1} \neq 0$, remove $\sigma_{i_{j'}}$ from γ, for $j = 1, \ldots, \ell d$, where $j' = j \bmod d \in \{1, \ldots, d\}$. Moreover, σ_{i_j} is replaced by $\sigma_{i_j - 1}$, if $k_j = k_{j-1} < i_j$ and σ_{i_j} remains unchanged otherwise. We repeat this procedure for all cycles in A and we obtain the mapping $(\gamma, \text{A}) \mapsto \beta \in \mathscr{B}^+_m$ denoted by

$$\pi_\text{A} \colon \mathscr{B}^+_{n,m,\text{A}} \to \mathscr{B}^+_m.$$

By considering the complementary color B we construct a mapping $(\gamma, \text{B}) \mapsto \alpha \in \mathscr{B}^+_n$ using the same scheme.

With the notion of coloring braid words we can encode the information of a 2-colored discretized braid $\boldsymbol{a} \operatorname{rel} \boldsymbol{b}$ in a 2-colored word (γ, A). Given $\boldsymbol{a} \operatorname{rel} \boldsymbol{b} \in \mathscr{D}^d_{n,m}$ (regular), we define

$$\boldsymbol{a} \operatorname{rel} \boldsymbol{b} \mapsto \gamma(\boldsymbol{a} \operatorname{rel} \boldsymbol{b}) := \beta(\boldsymbol{a} \sqcup \boldsymbol{b}), \tag{7}$$

cf. (4) and the coloring $\text{A} = \text{C}^{-1}(\{1, \ldots, n\})$, where the permutation C is defined as follows. Order the coordinates $x_0^{\mu_1} < \cdots < x_0^{\mu_{n+m}}$ and define

$$\text{C}^{-1} = \begin{pmatrix} 1 & 2 & \cdots & n+m \\ \mu_1 & \mu_2 & \cdots & \mu_{n+m} \end{pmatrix}.$$

The permutations τ_γ and $\tau_{\boldsymbol{a},\boldsymbol{b}}$ are conjugated: $\tau_\gamma = \text{C}\tau_{\boldsymbol{a},\boldsymbol{b}}\text{C}^{-1}$. The mapping $\boldsymbol{a} \operatorname{rel} \boldsymbol{b} \mapsto (\gamma, \text{A})$ is well-behaved under positive conjugacy: $\boldsymbol{a} \operatorname{rel} \boldsymbol{b} \doteq \boldsymbol{a}' \operatorname{rel} \boldsymbol{b}'$ implies $(\gamma, \text{A}) \doteq (\gamma', \text{A}')$.

Conversely, every positive conjugacy class $[\![\gamma, \text{A}]\!]$ determines a 2-colored discretized braid class $[\boldsymbol{a} \operatorname{rel} \boldsymbol{b}]_{\doteq}$ via the mapping $\gamma \mapsto \mathcal{E}_q(\gamma) \in \mathscr{D}^{d+q}_{n+m}$, with d the number of generators in γ and $q \geq 0$, cf. Fig. 2 (right). The representation $\boldsymbol{a} \operatorname{rel} \boldsymbol{b} \in \mathscr{D}^{d+q}_{n,m}$ is obtained from the coloring A.

Definition 3.9. A positive conjugacy class $[\![\gamma, \text{A}]\!]$ is called proper if the associated discretized braid class $[\boldsymbol{a} \operatorname{rel} \boldsymbol{b}]_{\doteq}$ is proper, cf. Sect. 3.2.

Example 3.10. Consider $\boldsymbol{a} \operatorname{rel} \boldsymbol{b} \in \mathscr{D}^2_{1,2}$, with strands $\boldsymbol{a} = \{(2, 4, 2)\}$, $\boldsymbol{b} = \{(1, 3, 3), (3, 1, 1)\}$. Since the strands in $\boldsymbol{b}$ have labels $\mu = 2, 3$, the permutation is given by $\tau_{\boldsymbol{a},\boldsymbol{b}} = (23)$ and $\gamma = \beta(\boldsymbol{a} \sqcup \boldsymbol{b}) = \sigma_2 \sigma_1 \sigma_2$. The coloring permutation is given as follows: $x_0^2 < x_0^1 < x_0^3$ and therefore $\text{C}^{-1} = (12)$. The red coloring is given by $\text{A} = \text{C}(\{1\}) = \{2\}$. We verify that $\tau_\gamma = \text{C}\tau_{\boldsymbol{a},\boldsymbol{b}}\text{C}^{-1} = (2)(13) = (13)$, see Fig. 2 (left). The topological type of $\boldsymbol{a} \operatorname{rel} \boldsymbol{b}$ is given by the $[\![\sigma_2 \sigma_1 \sigma_2, \{2\}]\!]$ and $[\boldsymbol{a} \operatorname{rel} \boldsymbol{b}]_{\doteq}$ is a proper and free 2-colored discrete braid
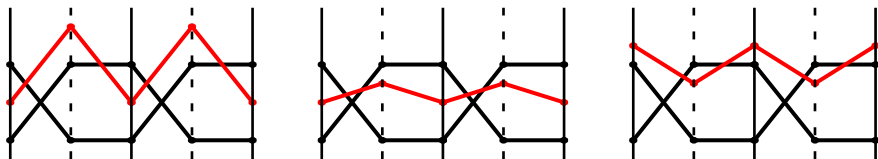
FIGURE 3. Positively conjugate representations

class, see Fig. 2 (right). To compute the skeletal braid word β we consider the following sequence:

$$k_0 = 2, \quad k_1 = \sigma_2(2) = 3, \quad k_2 = \sigma_1(3) = 3, \quad k_3 = \sigma_2(3) = 2.$$

This yields the differences $k_1 - k_0 = 1$, $k_2 - k_1 = 0$ and $k_3 - k_2 = -1$, and therefore both letters σ_2 are removed from γ, which gives $\gamma \mapsto \beta = \sigma_1$.

Figure 3 (left) shows shows a representation of (γ, A) and Fig. 3 (right) shows a positively conjugate representation (γ', A'). In the latter case $\text{A}' = \{3\}$, $\text{C} = (123)$ and $\text{A}' = \text{C}^{-1}(\{1\}) = (132)(1) = \{3\}$. It follows that $\tau_{\gamma'} = \text{C}\tau_{\boldsymbol{a},\boldsymbol{b}}\text{C}^{-1} = (123)(23)(132) = (12)$. Moreover, since $\gamma' = \zeta\gamma\zeta^{-1}$, with $\zeta = (23)$ it follows that $\tau_{\gamma'} = \zeta\tau_\gamma\zeta^{-1} = (23)(13(23) = (12)$.

4. Discrete braid invariants

We summarize the construction of a topological invariant for 2-colored relative braid classes as described in [1,2]. Let $\boldsymbol{a}\,\text{rel}\,\boldsymbol{b} \in \mathscr{D}_{n,m}^d$ represent a proper, bounded discretized 2-colored braid class $[\boldsymbol{a}\,\text{rel}\,\boldsymbol{b}]$. Then, the fiber $[\boldsymbol{a}]\,\text{rel}\,\boldsymbol{b}$ defines a bounded set in $\mathbb{R}^{nd}$.

A sequence $\mathcal{R} = \{\mathcal{R}_j\}$ of functions $\mathcal{R}_j \colon \mathbb{R}^3 \to \mathbb{R}$, which satisfy $\partial_1 \mathcal{R}_j > 0$ and $\partial_3 \mathcal{R}_j > 0$ is called a *parabolic recurrence relation*. From [1, Lem. 55–57] there exists a parabolic recurrence relation $\mathcal{R} = \{\mathcal{R}_j\}$ such that $\boldsymbol{b}$ is a zero for $\mathcal{R}$, i.e., $\mathcal{R}_j(x_{j-1}^{\mu_\nu}, x_j^{\mu_\nu}, x_{j+1}^{\mu_\nu}) = 0$ for all $j \in \mathbb{Z}$ and for all $\nu = 1, \ldots, m$. The recurrence relation $\mathcal{R}$ may regarded as vector field and is integrated via the equations

$$\frac{\mathrm{d}}{\mathrm{d}s}x_j^{\mu_\nu} = \mathcal{R}_j\left(x_{j-1}^{\mu_\nu}, x_j^{\mu_\nu}, x_{j+1}^{\mu_\nu}\right), \quad \nu = 1, \ldots, m. \tag{8}$$

Let N denoted the closure in $\mathbb{R}^{nd}$ of $[\boldsymbol{a}]\,\text{rel}\,\boldsymbol{b}$. By [1, Prop. 11, Thm. 15], the set N is an isolating neighborhood for the parabolic flow generated by Eq. (8). We define $\text{h}(\boldsymbol{a}\,\text{rel}\,\boldsymbol{b})$ as the homotopy Conley index of $\text{Inv}(N, \mathcal{R})$, cf. [1,10]. The Conley index is independent of the choice of parabolic recurrence relations $\mathcal{R}$ for which $\mathcal{R}(\boldsymbol{b}) = 0$, cf. [1, Thm. 15(a), (b)], as well as the choice of the fiber, i.e., $\boldsymbol{a}\,\text{rel}\,\boldsymbol{b} \sim \boldsymbol{a}'\,\text{rel}\,\boldsymbol{b}'$, then $\text{h}(\boldsymbol{a}\,\text{rel}\,\boldsymbol{b}) = \text{h}(\boldsymbol{a}'\,\text{rel}\,\boldsymbol{b}')$, cf. [1, Thm. 15(c)]. This makes $\text{h}(\boldsymbol{a}\,\text{rel}\,\boldsymbol{b})$ an invariant of the discrete 2-colored braid class $[\boldsymbol{a}\,\text{rel}\,\boldsymbol{b}]$.

There is an intrinsic way to define $\text{h}(\boldsymbol{a}\,\text{rel}\,\boldsymbol{b})$ without using parabolic recurrence relations. We define $N^- \subset \partial N$ to be the set of boundary points for which the word metric is locally maximal. The pair (N, N^-) is an index pair for any parabolic system $\mathcal{R}$ such that $\mathcal{R}(\boldsymbol{b}) = 0$, and thus by the independence
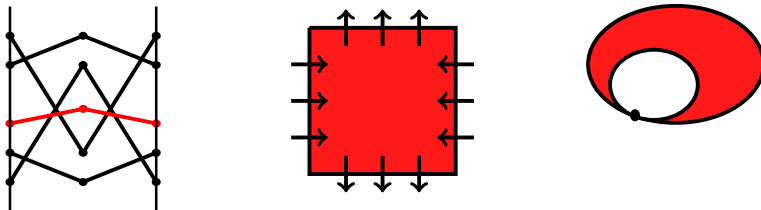
FIGURE 4. The Conley index for the braid in Example 7.2. The homotopy of the pointed space in given by $h(\boldsymbol{a} \operatorname{rel} \boldsymbol{b}) = \mathbb{S}^1$

of Conley index on $\mathcal{R}$, the pointed homotopy type of N/N^- gives the Conley index: $\operatorname{h}(\boldsymbol{a} \operatorname{rel} \boldsymbol{b}) = [N/N^-]$, see Fig. 4 and [1, Sect. 4.4] for more details on the construction.

The invariant $\operatorname{h}(\boldsymbol{a} \operatorname{rel} \boldsymbol{b})$ is not necessarily invariant with respect to the number of discretization points d. To have invariance also with respect to d, another invariant for discrete braid classes was introduced in [1]. Consider the equivalence class induced by the relation $\boldsymbol{a} \operatorname{rel} \boldsymbol{b} \doteq \boldsymbol{a}' \operatorname{rel} \boldsymbol{b}'$ on $\mathscr{D}_{n,m}^d$, which defines the class $[\boldsymbol{a} \operatorname{rel} \boldsymbol{b}]_{\doteq}$ of proper discrete 2-colored braids. Via the projection $\varpi \colon [\boldsymbol{a} \operatorname{rel} \boldsymbol{b}]_{\doteq} \to [\boldsymbol{b}]_{\doteq}$ we obtain fibers $\varpi^{-1}(\boldsymbol{b})$. Suppose $[\boldsymbol{a} \operatorname{rel} \boldsymbol{b}]_{\doteq}$ is a bounded class, i.e., all fibers $\varpi^{-1}(\boldsymbol{b})$ are bounded sets in $\mathbb{R}^{nd}$. Following [1, Def. 18] the closure N of a fiber $\varpi^{-1}(\boldsymbol{b})$ is an isolating neighborhood since $\boldsymbol{a} \operatorname{rel} \boldsymbol{b}$ is proper. Define $\operatorname{H}(\boldsymbol{a} \operatorname{rel} \boldsymbol{b})$ as the homotopy Conley index of N. If $[\boldsymbol{a}_k] \operatorname{rel} \boldsymbol{b}$ are the fibers belonging to the components $[\boldsymbol{a}_k \operatorname{rel} \boldsymbol{b}]$ of $[\boldsymbol{a} \operatorname{rel} \boldsymbol{b}]_{\doteq}$, then

$$\operatorname{H}(\boldsymbol{a} \operatorname{rel} \boldsymbol{b}) := \bigvee_k \operatorname{h}([\boldsymbol{a}_k] \operatorname{rel} \boldsymbol{b}). \tag{9}$$

Define the following extension mapping $\mathbb{E} : \mathscr{D}_m^d \to \mathscr{D}_m^{d+1}$, cf. [1], via concatenation with the trivial braid of period one:

$$(\mathbb{E}\boldsymbol{b})^\mu := \begin{cases} x_j^\mu & j = 0, \ldots, d; \\ x_d^\mu & j = d+1. \end{cases} \tag{10}$$

Properness remains unchanged under the extension mapping $\mathbb{E}$, however, boundedness may not be preserved. Define the skeletal augmentation:

$$\mathbb{A} \colon \mathscr{D}_m^d \to \mathscr{D}_{m+2}^d, \quad \boldsymbol{b} \mapsto \mathbb{A}\boldsymbol{b} = \boldsymbol{b}^* = \boldsymbol{b} \cup \boldsymbol{b}^- \cup \boldsymbol{b}^+,$$

where $\boldsymbol{b}^- = \{\min_\mu \{x_j^\mu\} - 1\}_j$ and $\boldsymbol{b}^+ = \{\max_\mu \{x_j^\mu\} + 1\}_j$. If $[\boldsymbol{a} \operatorname{rel} \boldsymbol{b}]_{\doteq}$ is bounded, then $\operatorname{h}([\boldsymbol{a}_k] \operatorname{rel} \boldsymbol{b}) = \operatorname{h}([\boldsymbol{a}_k] \operatorname{rel} \boldsymbol{b}^*)$ for all k, and therefore $\operatorname{H}(\boldsymbol{a} \operatorname{rel} \boldsymbol{b}) = \operatorname{H}(\boldsymbol{a} \operatorname{rel} \boldsymbol{b}^*)$. One can define second skeletal augmentation:

$$\mathbb{B} \colon \mathscr{D}_m^d \to \mathscr{D}_{m+2}^d, \quad \boldsymbol{b} \mapsto \mathbb{B}\boldsymbol{b} = \boldsymbol{b}^\# = \boldsymbol{b} \cup \boldsymbol{b}^s \cup \boldsymbol{b}^n,$$

where $\boldsymbol{b}^s = \{(-1)^j \min_\mu \{x_j^\mu\} - (-1)^j\}_j$ and $\boldsymbol{b}^n = \{(-1)^j \max_\mu \{x_j^\mu\} + (-1)^j\}_j$. As before, if $[\boldsymbol{a} \operatorname{rel} \boldsymbol{b}]_{\doteq}$ is bounded, then $\operatorname{h}([\boldsymbol{a}_k] \operatorname{rel} \boldsymbol{b}) = \operatorname{h}([\boldsymbol{a}_k] \operatorname{rel} \boldsymbol{b}^\#)$ for all k, and therefore $\operatorname{H}(\boldsymbol{a} \operatorname{rel} \boldsymbol{b}) = \operatorname{H}(\boldsymbol{a} \operatorname{rel} \boldsymbol{b}^\#)$.

Consider the proper, bounded 2-colored braid classes $[\boldsymbol{a}\operatorname{rel}\boldsymbol{b}]_{\pm}$ and $[\mathbb{E}\boldsymbol{a}\operatorname{rel}\mathbb{E}\boldsymbol{b}]_{\pm}$. The main result in [1, Thm. 20] is the stabilization theorem which states that

$$\mathrm{H}(\boldsymbol{a}\operatorname{rel}\boldsymbol{b}^*) = \mathrm{H}(\mathbb{E}\boldsymbol{a}\operatorname{rel}\mathbb{E}\boldsymbol{b}^*). \tag{11}$$

The independence of H on the skeleton $\boldsymbol{b}$ can be derived from the stabilization theorem. Since a 2-colored discretized braid class is free when d is sufficiently large, we have that $[\mathbb{E}^p\boldsymbol{a}\operatorname{rel}\mathbb{E}^p\boldsymbol{b}^*]$ is free for some $p > 0$ sufficiently large, and by stabilization $\mathrm{H}(\boldsymbol{a}\operatorname{rel}\boldsymbol{b}^*) = \mathrm{H}(\mathbb{E}^p\boldsymbol{a}\operatorname{rel}\mathbb{E}^p\boldsymbol{b}^*)$. Let $\boldsymbol{a}\operatorname{rel}\boldsymbol{b} \doteq \boldsymbol{a}'\operatorname{rel}\boldsymbol{b}'$, then $\mathbb{E}^p\boldsymbol{a}\operatorname{rel}\mathbb{E}^p\boldsymbol{b}^* \sim \mathbb{E}^p\boldsymbol{a}'\operatorname{rel}\mathbb{E}^p\boldsymbol{b}'^*$. By [1, Thm. 15(c)], a continuation can be constructed which proves that $\mathrm{H}(\mathbb{E}^p\boldsymbol{a}\operatorname{rel}\mathbb{E}^p\boldsymbol{b}^*) = \mathrm{H}(\mathbb{E}^p\boldsymbol{a}'\operatorname{rel}\mathbb{E}^p\boldsymbol{b}'^*)$. Consequently,

$$\mathrm{H}(\boldsymbol{a}\operatorname{rel}\boldsymbol{b}^*) = \mathrm{H}(\mathbb{E}^p\boldsymbol{a}\operatorname{rel}\mathbb{E}^p\boldsymbol{b}^*) = \mathrm{H}(\mathbb{E}^p\boldsymbol{a}'\operatorname{rel}\mathbb{E}^p\boldsymbol{b}'^*) = \mathrm{H}(\boldsymbol{a}'\operatorname{rel}\boldsymbol{b}'^*),$$

which shows that the index H only depends on the topological type $[\![\gamma, \mathrm{A}]\!]$, with $\gamma = \beta(\boldsymbol{a}\operatorname{rel}\boldsymbol{b})$.

Definition 4.1. Let $[\![\gamma, \mathrm{A}]\!]$ be proper, positive conjugacy class. Then, the *braid Conley index* is defined as

$$\mathrm{H}[\![\gamma, \mathrm{A}]\!] := \mathrm{H}(\boldsymbol{a}\operatorname{rel}\boldsymbol{b}^*). \tag{12}$$

The braid Conley index H may be computed using any representative $\boldsymbol{a}\operatorname{rel}\boldsymbol{b}^*$ for any sufficiently large d and any associated recurrence relation $\mathcal{R}$.

Finally, we mention that besides the extension $\mathbb{E}$, we also have a *half twist* extension operator $\mathbb{T}$:

$$(\mathbb{T}\boldsymbol{b})^\mu := \begin{cases} x_j^\mu & j = 0, \ldots, d \\ -x_d^\mu & j = d+1. \end{cases} \tag{13}$$

Every discretized braid can be dualized via the mapping $\{x_j^\mu\} \mapsto \{(-1)^j x_j^\mu\}$. On $\mathscr{D}_m^{2d}$ this yields a well-defined operator $\mathbb{D}\colon \mathscr{D}_m^{2d} \to \mathscr{D}_m^{2d}$ mapping proper, bounded discretized braid classes $[\boldsymbol{a}\operatorname{rel}\boldsymbol{b}]$ to proper, bounded discretized braid classes $[\mathbb{D}\boldsymbol{a}\operatorname{rel}\mathbb{D}\boldsymbol{b}]$. From [1, Cor. 31] we recall the following result. Let $\boldsymbol{a}\operatorname{rel}\boldsymbol{b} \in \mathscr{D}_{n,m}^{2d}$ be proper, then

$$\mathrm{H}\big(\mathbb{T}^2 \circ \mathbb{D}(\boldsymbol{a}\operatorname{rel}\boldsymbol{b}^*)\big) = \mathrm{H}\big(\mathbb{D}(\boldsymbol{a}\operatorname{rel}\boldsymbol{b}^*)\big) \wedge \mathbb{S}^{2n}, \tag{14}$$

where the wedge is the $2n$-suspension of the Conley index.

From the singular homology $H_*(\mathrm{H}(\boldsymbol{a}\operatorname{rel}\boldsymbol{b}^*))$ the Poincaré polynomial is denoted by $P_t(\boldsymbol{a}\operatorname{rel}\boldsymbol{b}^*)$, or $P_t[\![\gamma, \mathrm{A}]\!]$ in terms of the topological type. This yields an important invariant: $|P_t(\boldsymbol{a}\operatorname{rel}\boldsymbol{b}^*)| = |P_t[\![\gamma, \mathrm{A}]\!]|$, which is the number of monomial term in the Poincaré polynomial.

5. The variational formulation

For a given symplectomorphism $F \in \mathrm{Symp}(\mathbb{D}^2)$, the problem of finding periodic points can be reformulated in terms of parabolic recurrence relations.

5.1. Twist symplectomorphisms

Let $F(x, y) = \big(f(x, y), g(x, y)\big)$ be a symplectomorphism of $\mathbb{R}^2$, with f, g smooth functions on $\mathbb{R}^2$. Recall that $F \in \mathrm{Symp}(\mathbb{R}^2)$ is a *positive* twist symplectomorphism if

$$\frac{\partial f(x, y)}{\partial y} > 0.$$

For twist symplectomorphisms, there exists a variational principle for finding periodic points, cf. [11, 12]. Such a variational principle also applies to symplectomorphisms that are given as a composition:

$$F = F_d \circ \cdots \circ F_1,$$

with $F_j \in \mathrm{Symp}(\mathbb{R}^2)$ positive twist symplectomorphisms for all j. It is important to point out that F itself is *not* twist in general. An important question is whether every mapping $F \in \mathrm{Symp}(\mathbb{R}^2)$ can be written as a composition of (positive) twist symplectomorphisms, cf. [11]. Suppose $F \in \mathrm{Ham}(\mathbb{R}^2)$, and F allows a Hamiltonian isotopy $\psi_{t,H}$ with appropriate asymptotic conditions near infinity, such that $\psi_{t_i,H} \circ \psi_{t_{i-1},H}^{-1}$ is close to the identity mapping in the C^1-norm for sufficiently small time steps $t_i - t_{i-1}$. Then, define $G_i = \psi_{t_i,H} \circ \psi_{t_{i-1},H}^{-1}$, $i = 1, \ldots, k$, and $F = G_k \circ \cdots \circ G_1$. We remark that in this construction the individual mappings G_i are not twist necessarily. The following observation provides a decomposition consisting solely of positive twist symplectomorphisms. Consider the 90° degree clockwise rotation

$$\psi(x, y) = (y, -x), \quad \psi^4 = \mathrm{id},$$

which is positive twist symplectomorphism. This yields the decomposition:

$$F = (G_k \circ \psi) \circ \psi \circ \psi \circ \psi \circ \cdots \circ (G_1 \circ \psi) \circ \psi \circ \psi \circ \psi, \tag{15}$$

where $F_{4i} = G_i \circ \psi$ and $F_j = \psi$ for $j \neq 4i$ for some i and $d = 4k$. Since the mappings G_i are close to the identity, the compositions $G_i \circ \psi$ are positive twist symplectomorphisms. The above procedure intertwines symplectomorphisms with k full rotations. As we will see later on this results in positive braid representations of mapping classes. The choice of ψ is arbitrary since other rational rotations also yield twist symplectomorphisms.

For symplectomorphisms $F \in \mathrm{Symp}(\mathbb{D}^2)$, we establish a similar decomposition in terms of positive twist symplectomorphisms, with the additional property that the decomposition can be extended to symplectomorphisms of $\mathbb{R}^2$, which is necessary to apply the variational techniques in [1].

5.2. Interpolation

A symplectomorphism $F \in \mathrm{Symp}(\mathbb{R}^2)$ satisfies the *uniform twist condition* if the there exists a $\delta > 0$ such that

$$\delta^{-1} \geq \frac{\partial f(x, y)}{\partial y} \geq \delta > 0, \quad \forall (x, y) \in \mathbb{R}^2. \tag{16}$$

The subset of such symplectomorphism is denoted by $SV(\mathbb{R}^2)$, cf. [11]. A result by Moser implies that all symplectomorphisms of $\mathbb{R}^2$ with a uniform twist condition are Hamiltonian.

Proposition 5.1. (cf. [12]) *Let $F \in SV(\mathbb{R}^2)$. Then, there exists a Hamiltonian $H \in \mathcal{H}(\mathbb{R}^2)$ such that $0 < \delta \leq H_{yy} \leq \delta^{-1}$ and $\psi_{1,H} = F$, where $\psi_{t,H}$ is the associated Hamiltonian flow. All orbits of $\psi_{t,H}$ project to straight lines in (t,x)-plane, and $\psi_{t,H} \in SV(\mathbb{R}^2)$ for all $t \in (0,1]$.*

For completeness we give a self-contained proof of Proposition 5.1, which is the same as the proof in [12] modulo a few alterations.

Proof. Following [12] we consider action integral $\int_0^1 L(t, x(t), \dot{x}(t)) \mathrm{d}t$ for functions $x(t)$ with $x(0) = x_0$ and $x(1) = x_1$. We require that extremals are affine lines, i.e., $\ddot{x}(t) = 0$. For extremals the action is given by $S(x_0, x_1) = \int_0^1 L(t, x(t), \dot{x}(t)) \mathrm{d}t$ and we seek Lagrangians such that $S = \boldsymbol{h}$, where $\boldsymbol{h}$ is the generating function for F.

For Lagrangians this implies

$$\frac{\mathrm{d}}{\mathrm{d}t}(\partial_p L) - \partial_x L = (\partial_t + p\partial_x)\partial_p L - \partial_x L = 0, \tag{17}$$

where $p = \dot{x}$. Solving the first order partial differential equation yields $L = L_0(t, x, p) + p\partial_x m + \partial_t m$, with

$$L_0 := -\int_0^p (p - p')\partial^2_{x_0 x_1}\boldsymbol{h}(x - p't, x + p'(1-t))dp'. \tag{18}$$

and $m = m(t, x)$ to be specified later, cf. see [12] for details. The extremals $x(t)$ are also extremals for L_0. Let $S_0(x_0, x_1) = \int_0^1 L_0(t, x(t), \dot{x}(t)) \mathrm{d}t$, then

$$\int_0^1 p\partial_x m(t, x(t)) + \partial_t m(t, x(t)) \mathrm{d}t = m(1, x_1) - m(0, x_0) \tag{19}$$

and hence

$$S(x_0, x_1) = S_0(x_0, x_1) + m(1, x_1) - m(0, x_0). \tag{20}$$

Differentiating S yields

$$\partial_{x_0} S = -\partial_p L(0, x_0, x_1 - x_0), \quad \partial_{x_1} S = \partial_p L(0, x_1, x_1 - x_0)$$

and for the mixed derivate

$$\partial^2_{x_0 x_1} S_0(x_0, x_1) = -\partial^2_{pp} L(0, x_0, x_1 - x_0) = \partial^2_{x_0 x_1}\boldsymbol{h}(x_0, x_1). \tag{21}$$

Then, $S_0(x_0, x_1) - h(x_0, x_1) = u(x_0) + v(x_1)$ and the choice

$$m(t, x) := (1 - t)u(x) - tv(x)$$

implies $S = \boldsymbol{h}$. Differentiating the relation $y = -\partial_x\boldsymbol{h}(x, x_1)$ with respect to y and using the fact that $x_1 = f(x, y)$, yields

$$\begin{aligned} 1 &= -\partial^2_{yx}\boldsymbol{h}(x, x_1) = -\partial^2_{xx_1}\boldsymbol{h}(x, x_1)\partial_y f(x, x_1) \\ &= -\partial^2_{xx_1}\boldsymbol{h}(x, x_1)\partial_y f(x, x_1) \end{aligned}$$

and thus $\delta \leq \partial_y f \leq \delta^{-1}$ if and only if $-\delta^{-1} \leq \partial^2_{xx_1}\boldsymbol{h} \leq -\delta$. By relation (21) we have $\partial^2_{pp} L \in [\delta, \delta^{-1}]$.

The Hamiltonian is obtained via the Legendre transform

$$H(t, x, y) := yp - L(t, x, p), \tag{22}$$

where

$$y = \partial_p L(t, x, p), \qquad (23)$$

and we can solve for p, i.e., $p = \lambda(x, y)$. As before, differentiating (23) gives $1 = \partial_{pp}^2 L \cdot \partial_y \lambda$ and differentiating (22) gives $\partial_y H = \lambda$. Combining these two identities yields $\partial_{pp}^2 L \cdot \partial_{yy}^2 H = 1$, from which the desired property $\partial_{yy}^2 H \in [\delta, \delta^{-1}]$ follows.

From the above analysis we obtain the following expression for the isotopy $\psi_{t,H}$:

$$\psi_{t,H}(x, y) = \Big(x + \lambda(x, y)t, \partial_p L\big(t, x + \lambda(x, y)t, \lambda(x, y)\big) \Big). \qquad (24)$$

Let π_x denote the projection onto the x-coordinate. Then, $\partial_y \pi_x \psi_{t,H}(x, y) = \partial_y \lambda(x, y)t = \partial_{yy}^2 Ht$, which proves that $\psi_{t,H}$ is positive twist for all $t \in (0, 1]$. $\square$

Using Proposition 5.1, we obtain a decomposition of symplectomorphisms $F \in \mathrm{Symp}(\mathbb{R}^2)$ as given in (15) and which satisfy additional properties such that the discrete braid invariants in [1] are applicable.

Proposition 5.2. *Let $F \in \mathrm{Symp}(\mathbb{D}^2)$. Then, there exists an isotopy $\phi_t \subset \mathrm{Symp}(\mathbb{R}^2)$ for all $t \in [0, 1]$, an integer $d \in \mathbb{N}$ and a sequence $\{t_j\}_{j=0}^d \subset [0, 1]$ with $t_j = j/d$, such that*

(i) *$\phi_0 = \mathrm{id}$, $\phi_1|_{\mathbb{D}^2} = F$;*
(ii) *ϕ_t is smooth with respect to t on the intervals $[t_j, t_{j+1}]$ (piecewise smooth);*
(iii) *$\widehat{F}_j := \phi_{t_j} \circ \phi_{t_{j-1}}^{-1} \in SV(\mathbb{R}^2)$ for all $1 \leq j \leq d$, and $F_j := \widehat{F}_j|_{\mathbb{D}^2}$;*
(iv) *the projection of the graph of $\phi_t(x, y)$ onto (t, x)-plane is linear on the intervals $t \in (t_{j-1}, t_j)$ for all $1 \leq j \leq d$, and for all $(x, y) \in \mathbb{R}^2$;*
(v) *$\phi_t(\mathbb{D}^2) \subset [-1, 1] \times \mathbb{R}$ for all $t \in [0, 1]$;*
(vi) *the points $z_\pm = (\pm 2, 0)$ are fixed points of $\widehat{F}_j = \phi_{t_j} \circ \phi_{t_{j-1}}^{-1}$ for all $1 \leq j \leq d$;*
(vii) *the points $z'_\pm = (\pm 4, 0)$ are period-2 points of $\widehat{F}_j = \phi_{t_j} \circ \phi_{t_{j-1}}^{-1}$ for all $1 \leq j \leq d$, i.e., $\widehat{F}_j(z'_\pm) = z'_\mp = -z'_\pm$, for all j.*

The decomposition

$$\widehat{F} = \widehat{F}_d \circ \cdots \circ \widehat{F}_1, \qquad (25)$$

is a generalization of the decomposition given in (15).

The isotopy constructed in Proposition 5.2 is called a *chained Moser isotopy*. Before proving Proposition 5.2 we construct analogues of the rotation mapping used in (15).

Lemma 5.3. *For every integer $\ell \geq 3$, there exists a positive Hamiltonian twist diffeomorphism Ψ of the plane $\mathbb{R}^2$, such that:*

(i) *the restriction $\Psi|_{\mathbb{D}^2}$ is a rotation over angle $2\pi/\ell$ and $\Psi|_{\mathbb{D}^2}^\ell = \mathrm{id}$;*
(ii) *the points $z_\pm = (\pm 2, 0)$ are fixed points for Ψ;*
(iii) *the points $z'_\pm = (\pm 4, 0)$ are period-2 points for Ψ, i.e., $\Psi(z'_\pm) = z'_\mp$.*

Proof. A linear rotation mapping on $\mathbb{R}^2$ is a positive twist mapping for all rotation angles $\vartheta \in (0, \pi)$. The generating function for a rotation is given by

$$\boldsymbol{h}_\vartheta(x, x') = \tfrac{1}{2}\cot(\vartheta)x^2 - \csc(\vartheta)xx' + \tfrac{1}{2}\cot(\vartheta)x'^2. \tag{26}$$

To construct the mappings Ψ, we construct special generating functions. Let $\ell \geq 3$ be an integer and let $\vartheta_\ell = 2\pi/\ell \in (0, \pi)$. Consider generating functions of the form

$$\boldsymbol{h}_\Psi(x, x') = \xi_\ell(x) - \csc(\vartheta_\ell)xx' + \xi_\ell(x'), \tag{27}$$

which generate positive twist mappings for all $\ell \geq 3$. We choose ξ_ℓ as follows: $\xi_\ell(x) = \tfrac{1}{2}\cot(\vartheta_\ell)x^2$ for all $|x| \leq 1$, $\xi_\ell(x) = \tfrac{1}{2}\csc(\vartheta_\ell)x^2$ for all $3/2 \leq |x| \leq 5/2$, and $\xi_\ell(x) = -\tfrac{1}{2}\csc(\vartheta_\ell)x^2$ for all $7/2 \leq |x| \leq 9/2$. The mapping Ψ is defined by h_Ψ and $y = -\partial_1 \boldsymbol{h}_\Psi(x, x')$ and $y' = \partial_2 \boldsymbol{h}_\Psi(x, x')$.[1] For $|x|, |x'| \leq 1$, the generating function restricts to (26) which yields the rotation over ϑ_ℓ on $\mathbb{D}^2$ and establishes (i). For $3/2 \leq x, x' \leq 5/2$ we have

$$y = \csc(\vartheta_\ell)(x' - x), \quad \text{and} \quad y' = \csc(\vartheta_\ell)(x' - x),$$

which verifies that z_+ are fixed points and same holds for z_-, completing the verification of (ii). For $7/2 \leq x \leq 9/2$ and $-9/2 \leq x' \leq -7/2$, we have

$$y = \csc(\vartheta_\ell)(x' + x), \quad \text{and} \quad y' = -\csc(\vartheta_\ell)(x' + x),$$

then z'_+ is mapped to z'_- and similarly z'_- is mapped to z'_+, which completes (iii) and proof of the lemma. $\qquad\square$

To extend chained Moser isotopies yet another type of Hamiltonian twist diffeomorphism is needed.

Lemma 5.4. *For every integer $\ell \geq 3$, there exists a positive Hamiltonian twist symplectomorphism Υ of the plane $\mathbb{R}^2$, such that:*

(i) *the restriction $\Upsilon|_{\mathbb{D}^2}$ is a rotation over angle $2\pi/\ell$, i.e., $\Upsilon|_{\mathbb{D}^2}^\ell = \text{id}$;*

(ii) *the points $z_\pm = (\pm 2, 0)$ and $z'_\pm = (\pm 4, 0)$ are period-2 points for Υ, i.e., $\Upsilon(z'_\pm) = z'_\mp$.*

Proof. As before consider generating functions of the form

$$\boldsymbol{h}_\Upsilon(x, x') = \xi_\ell(x) - \csc(\vartheta_\ell)xx' + \xi_\ell(x'), \tag{28}$$

which generate positive twist mappings for all $\ell \geq 3$. We choose ξ_ℓ as follows: $\xi_\ell(x) = \tfrac{1}{2}\cot(\vartheta_\ell)x^2$ for all $|x| \leq 1$, and $\xi_\ell(x) = -\tfrac{1}{2}\csc(\vartheta_\ell)x^2$ for all $3/2 \leq |x| \leq 9/2$. The mapping Υ is defined by h_Υ. For $|x|, |x'| \leq 1$, the generating function restricts to (26) which yields the rotation over ϑ_ℓ on $\mathbb{D}^2$ and establishes (i). For $3/2 \leq x, \leq 9/2$ and $-9/2 \leq x' \leq -7/2$, we have

$$y = \csc(\vartheta_\ell)(x' + x), \quad \text{and} \quad y' = -\csc(\vartheta_\ell)(x' + x),$$

then z_+ and z'_+ are mapped to z_- and z'_-, respectively, and similarly, z_- and z'_- are mapped to z_+ and z'_+, respectively, which completes proof. $\qquad\square$

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[1] To simplify notation we express the derivatives of $\boldsymbol{h}$ with respect to its two coordinates by $\partial_1 \boldsymbol{h}$ and $\partial_2 \boldsymbol{h}$.

Proof of Proposition 5.2. Consider the subgroup $\mathrm{Symp}_c(\mathbb{R}^2)$ formed by compactly supported symplectomorphisms of the plane.[2] Recall that due to the uniform twist property the set $SV(\mathbb{R}^2)$ is open in the topology given by C^1-convergence on compact sets, cf. [11]. Let $\Psi \in \mathrm{Ham}(\mathbb{R}^2)$ be given by Lemma 5.3 for some $\ell \geq 3$. Then, there exists an open neighborhood $\mathscr{W} \subset \mathrm{Symp}_c(\mathbb{R}^2)$ of the identity, such that $\varphi \circ \Psi \in SV(\mathbb{R}^2)$ for all $\varphi \in \mathscr{W}$.

For $F \in \mathrm{Symp}(\mathbb{D}^2)$, Proposition 2.2 provides a Hamiltonian $H \in \mathcal{H}(\mathbb{D}^2)$ such that $F = \psi_{1,H}$. Let $H^\dagger$ be a smooth extension to $\mathbb{R} \times \mathbb{R}^2$ and $\mathscr{U}_\epsilon(\mathbb{D}^2) = \{z \in \mathbb{R}^2 \mid |z| < 1 + \epsilon\}$ and let $\alpha \colon \mathbb{R}^2 \to \mathbb{R}$ be a smooth bump function satisfying $\alpha|_{\mathbb{D}^2} = 1$, $\alpha = 0$ on $\mathbb{R}^2 \backslash \mathscr{U}_\epsilon(\mathbb{D}^2)$. Take $\epsilon \in (0, 1/2)$ and define $\widetilde{H} = \alpha H^\dagger$ with $\widetilde{H} \in \mathcal{H}(\mathbb{R}^2)$. The associated Hamiltonian isotopy is denoted by $\psi_{t,\widetilde{H}}$ and $\widetilde{F} = \psi_{1,\widetilde{H}} \in \mathrm{Ham}(\mathbb{R}^2)$. Moreover, $\psi_{t,\widetilde{H}}$ equals the identity on $\mathbb{R}^2 \backslash \mathscr{U}_\epsilon(\mathbb{D}^2)$, i.e., $\psi_{t,\widetilde{H}}$ is supported in $\mathscr{U}_\epsilon(\mathbb{D}^2)$, and $\widetilde{F}|_{\mathbb{D}^2} = F$.

Fix $\ell \geq 3$ and choose $k > 0$ sufficiently large such that the symplectomorphisms

$$G_i = \psi_{i/k,\widetilde{H}} \circ \psi^{-1}_{(i-1)/k,\widetilde{H}}, \quad i \in \{1, \ldots, k\} \tag{29}$$

are elements of $\mathscr{W}$. Each G_i restricted to $\mathbb{D}^2$ can be decomposed as follows:

$$G_i|_{\mathbb{D}^2} = \left(G_i|_{\mathbb{D}^2} \circ \Psi\right) \circ \underbrace{\Psi' \circ \cdots \circ \Psi'}_{\kappa(\ell-1)}, \quad \ell \geq 3, \ \kappa \in \mathbb{N}, \tag{30}$$

where Ψ and Ψ' are obtained from Lemma 5.3 by choosing rotation angles $2\pi/\ell$ and $2\pi/\kappa\ell$, respectively. Observe that $\Psi \circ \Psi'^{\kappa(\ell-1)}|_{\mathbb{D}^2} = \mathrm{id}$. From $\widetilde{F}$ we define the mapping $\widehat{F} \in \mathrm{Symp}(\mathbb{R}^2)$:

$$\widehat{F} = (G_k \circ \Psi) \circ \underbrace{\Psi' \circ \cdots \circ \Psi'}_{\kappa(\ell-1)} \circ \cdots \circ (G_1 \circ \Psi) \circ \underbrace{\Psi' \circ \cdots \circ \Psi'}_{\kappa(\ell-1)}. \tag{31}$$

By construction we have $\widehat{F}|_{\mathbb{D}^2} = F$. Let $\ell_\kappa = \kappa(\ell - 1) + 1$ and $d = \ell_\kappa k$ and put

$$\widehat{F}_j = \begin{cases} G_{j/\ell_\kappa} \circ \Psi & \text{for } j \in \{\ell_\kappa, 2\ell_\kappa, \ldots, d\} \\ \Psi' & \text{for } j \in \{1, \ldots, d\} \backslash \{\ell_\kappa, 2\ell_\kappa, \ldots, d\}. \end{cases} \tag{32}$$

with $\widehat{F}_j \in SV(\mathbb{R}^2)$ for $j \in \{1, \ldots, d\}$ and $F_j = \widehat{F}_j|_{\mathbb{D}^2}$. Using the latter we obtain a decomposition of F as given in (15), and with the additional property that the mappings F_j extend to twist symplectomorphisms of the $\mathbb{R}^2$, which proves (25).

Each symplectomorphism $\widehat{F}_j$ can be connected to identity by a Hamiltonian path. Let H^j be the Hamiltonian given by Proposition 5.1, which connects $\widehat{F}_j$ to the identity via the Moser isotopy ψ_{s,H^j}, $s \in [0, 1]$. Let $t_j = j/d$ for all $j \in \{0, \ldots, d\}$ and define

$$\phi_t = \psi_{s^j(t),H^j} \circ \widehat{F}_{j-1} \circ \cdots \circ \widehat{F}_0, \quad t \in [t_{j-1}, t_j], \ j \in \{1, \ldots, d\}, \tag{33}$$

---

[2] A symplectomorphism is compactly supported in $\mathbb{R}^2$ if it is the identity outside a compact subset of $\mathbb{R}^2$.

with $s^j(t) = d(t - t_j)$ and $\widehat{F}_0 = \mathrm{id}$. Observe that, by construction, $\phi_{t_j} \circ \phi_{t_{j-1}}^{-1} = \widehat{F}_j$, for all $j = 1, \ldots, d$ and (i)–(iv) is satisfied. Condition (v) follows from (iv) and from the fact that each $\widehat{F}_j$ leaves the disc $\mathbb{D}^2$ invariant.

All the symplectomorphisms in the decomposition are supported in the disc $\mathscr{U}_\epsilon(\mathbb{D}^2)$, hence Conditions (ii) and (iii) of Lemma 5.3 imply Properties (vi) and (vii). □

Remark 5.5. The chained Moser isotopies in Proposition 5.1 can be extended with two more parameters $r \geq 0$ and $\rho \geq 0$. Consider the decoposition

$$\widehat{F} = \widehat{F}_d \circ \cdots \circ \widehat{F}_1 \circ \underbrace{\Psi_r^{\ell_r} \circ \cdots \circ \Psi_r^{\ell_r}}_{r} \circ \underbrace{\Upsilon_\rho^{\ell_\rho} \circ \cdots \circ \Upsilon_\rho^{\ell_\rho}}_{\rho}, \tag{34}$$

where $\Psi_r^{\ell_r}|_{\mathbb{D}^2} = \mathrm{id}$ and $\ell_r \geq 3$, and $\Upsilon_\rho^{\ell_\rho}|_{\mathbb{D}^2} = \mathrm{id}$ and $\ell_\rho \geq 3$. We can again define an Moser isotopy as in (33) with d replaced by $d + r\ell_r + \rho\ell_\rho$. The isotopy is again called a chained Moser isotopy and denoted by ϕ_t, and the extended period will again be denoted by d. The strands $\phi_t(z_\pm)$ link with the cylinder $[0, 1] \times \mathbb{D}^2$ and with each other with linking number 2ρ.

5.3. The discrete action functional

Let $F \in \mathrm{Symp}(\mathbb{D}^2)$ be the given symplectomorphism of the 2-disc and let $\{\phi_t\}_{t \in \mathbb{R}}$ and $\{t_j\}_{j \in \{0,\ldots,d\}}$ be the associated continuous isotopy and sequence of discretization times as given in Proposition 5.2 for the extension $\widehat{F}$. The isotopy is extended periodically, that is, $\phi_{t+s} = \phi_t \circ \phi_1^s$ and $t_{j+sd} = s + t_j$ for all $s \in \mathbb{Z}$. The decomposition of $\widehat{F}$ given by Proposition 5.2 yields a periodic sequence of positive twist symplectomorphisms $\{\widehat{F}_j\}$, with $\widehat{F}_j = \phi_{t_{j+1}} \circ \phi_{t_j}^{-1} \in SV(\mathbb{R}^2)$ and $\widehat{F}_{j+d} = \widehat{F}_j$.

Definition 5.6. A sequence $\{(x_j, y_j)\}_{j \in \mathbb{Z}}$ is a *full orbit* for the system $\{\widehat{F}_j\}$ if

$$(x_{j+1}, y_{j+1}) = \widehat{F}_j(x_j, y_j), \quad j \in \mathbb{Z}.$$

If $(x_{j+n}, y_{j+d}) = (x_j, y_j)$ for all j, then $\{(x_j, y_j)\}_{j \in \mathbb{Z}}$ is called an *d-periodic sequence* for the system $\{\widehat{F}_j\}$.

For every twist symplectomorphism $\widehat{F}_j \in SV(\mathbb{R}^2)$, we assign a generating function $\boldsymbol{h}_j = \boldsymbol{h}_j(x_j, x_{j+1})$ on the x-coordinates, which implies that $y_j = -\partial_1 \boldsymbol{h}_j$ and $y_{j+1} = \partial_2 \boldsymbol{h}_j$. From the twist property it follows that

$$\partial_1 \partial_2 \boldsymbol{h}_j < 0, \quad \forall j \in \mathbb{Z}. \tag{35}$$

Note that the sequence $\{\boldsymbol{h}_j\}$ is d-periodic.

Define the *action functional* $W_d : \mathbb{R}^{\mathbb{Z}/d\mathbb{Z}} \to \mathbb{R}$ by

$$W_d(\{x_j\}) := \sum_{j=0}^{d-1} \boldsymbol{h}_j(x_j, x_{j+1}). \tag{36}$$

A sequence $\{x_j\}$ is a critical point of W_d if and only if

$$\mathcal{R}_j(x_{j-1}, x_j, x_{j+1}) := -\partial_2 \boldsymbol{h}_{j-1}(x_{j-1}, x_j) - \partial_1 \boldsymbol{h}_j(x_j, x_{j+1}) = 0, \tag{37}$$

for all $j \in \mathbb{Z}$. The y-coordinates satisfy $y_j = \partial_2 \boldsymbol{h}_{j-1}(x_{j-1}, x_j)$.

Periodicity and exactness of $\mathcal{R}_j$ is immediate. The monotonicity follows directly from inequality (35). A periodic point z, i.e., $F^d(z) = z$, is equivalent to the periodic sequence $\{(x_j, y_j)\}_{j \in \mathbb{Z}}$, with $z = (x_0, y_0) \in \mathbb{D}^2$. Since $z = (x_0, y_0) \in \mathbb{D}^2$, the invariance of $\mathbb{D}^2$ under F implies that $(x_j, y_j) \in \mathbb{D}^2$ for all j. The above considerations yield the following variational principle.

Proposition 5.7. *A d-periodic sequence $\{(x_j, y_j)\}_{j \in \mathbb{Z}}$ is an d-periodic orbit for the system $\{\widehat{F}_j\}$ if and only if the sequence of x-coordinates $\{x_j\}$ is a critical point of W_d.*

The idea of periodic sequences can be generalized to periodic configurations. Let $\{B_j\}_{j \in \mathbb{Z}}$, $B_j = \{(x_j^\mu, y_j^\mu) \mid \mu = 1, \ldots, m\} \in \mathcal{C}_m(\mathbb{R}^2)$ and $B_{j+d} = B_j$ for all j. Such a sequence $\{B_j\}$ is a d-periodic sequence for $\{\widehat{F}_j\}$ if $\widehat{F}_j(B_j) = B_{j+1}$ for all $j \in \mathbb{Z}$.

For a d-periodic sequence $\{B_j\}$, the x-projection yields a discretized braid $\boldsymbol{b} = \{\boldsymbol{b}^\mu\} = \{x_j^\mu\}$, cf. Definition 3.1. The above action functional can be extended to the space of discretized braids $\mathscr{D}_m^d$:

$$W_d(\boldsymbol{b}) := \sum_{\mu=1}^{m} W_d(\boldsymbol{b}^\mu), \tag{38}$$

where $W_d(\boldsymbol{b}^\mu)$ is given by (36). This yields the following extension of the variational principle.

Proposition 5.8. *A d-periodic sequence $\{B_j\}_{j \in \mathbb{Z}}$, $B_j \in \mathcal{C}_m(\mathbb{R}^2)$, is a d-periodic sequence of configurations for the system $\{\widehat{F}_j\}$ if and only if the sequence of x-coordinates $\boldsymbol{b} = \{x_j^\mu\}$ is a critical point of W_d on $\mathscr{D}_m^d$.*

A discretized braid $\boldsymbol{b}$ that is stationary for W_d if it satisfies the parabolic recurrence relations in (37) for all μ and the periodicity condition in Definition 3.1(b). In Sect. 6, we show that d-periodic sequences of configurations $\{B_j\}$ for the system $\{\widehat{F}_j\}$ yields geometric braids.

6. Braiding of periodic points

For symplectomorphisms $F \in \mathrm{Symp}(\mathbb{D}^2)$, with a finite invariant set $B \subset \mathrm{int}\,\mathbb{D}^2$, the mapping class can be identified via a chained Moser isotopy.

Proposition 6.1. *Let $B \subset \mathrm{int}\,\mathbb{D}^2$, with $\#B = m$, be a finite invariant set for $F \in \mathrm{Symp}(\mathbb{D}^2)$ and let ϕ_t be a chained Moser isotopy given in Proposition 5.2. Then, $\boldsymbol{\beta}(t) = \phi_t(B)$ represents a geometric braid based at $B \in \mathcal{C}_m\mathbb{D}^2$ with only positive crossings and $\beta = \iota_B\big([\boldsymbol{\beta}]_B\big)$ is a positive word in the braid monoid $\mathscr{B}_m^+$. The x-projection $\boldsymbol{b}(t) = \pi_x\boldsymbol{\beta}(t)$ on the (t, x)-plane is a (continuous) piecewise linear braid diagram.*

Proposition 2.3 implies that the associated positive braid word $\beta \in \mathcal{B}_m$, derived from the braid diagram $\pi_x\phi_t(B)$ determines the mapping class of F relative to B. If the based path $\boldsymbol{\beta}(t) = \phi_t(B)$ is regarded as a *free loop* $\mathbb{S}^1 \to \mathcal{C}_m\mathbb{D}^2$, i.e., discarding the base point, then $\boldsymbol{\beta}$ is referred to as a *closed* geometric braid in $\mathbb{D}^2$.

Definition 6.2. Let $\boldsymbol{\beta}$ be a geometric braid in $\mathbb{D}^2$. A component of $\boldsymbol{\beta}' \subset \boldsymbol{\beta}$ is called *cylindrical in* $\boldsymbol{\beta}$ if $\boldsymbol{\beta}'$ can be deformed onto $\partial\mathbb{D}^2$ as a closed geometric braid. Otherwise, $\boldsymbol{\beta}'$ is called *acylindrical*. A union of components $\boldsymbol{\beta}'$ is called cylindrical/acylindrical in $\boldsymbol{\beta}$ if all members are.

Remark 6.3. A positive conjugacy class $[\![\gamma, \mathrm{A}]\!]$ is associated with braid classes in $[\boldsymbol{a} \operatorname{rel} \boldsymbol{b}]_{\pm}$ in $\mathscr{D}_{n,m}^{d+q}$, cf. Sect. 3.3. If for a representative $\boldsymbol{a} \operatorname{rel} \boldsymbol{b}$ it holds that $\mathrm{B}(\boldsymbol{a})$ is cylindrical/acylindrical in $\boldsymbol{\gamma} = \mathrm{B}(\boldsymbol{a}) \operatorname{rel} \mathrm{B}(\boldsymbol{b})$, then $[\![\gamma, \mathrm{A}]\!]$ is said to cylindrical/acylindrical, cf. Definition 3.9.

Let $z, z' \in \mathbb{R}^2$ be distinct points with the property that $\widehat{F}^n(z) = z$ and $\widehat{F}^n(z') = z'$, for some $n \geq 1$, and where $\widehat{F} = \phi_1$ and ϕ_t a chained Moser isotopy constructed in Proposition 5.2. Define the continuous functions $z(t) = \phi_t(z)$ and $z'(t) = \phi_t(z')$ and let $x(t)$ and $x'(t)$ the x-projection of $z(t)$ and $z'(t)$, respectively. By Proposition 5.2, $x(t)$ and $x'(t)$ are (continuous) piecewise linear functions that are uniquely determined by the sequence $\{t_j\}_{j=0}^{nd}$, $t_j = j/d$.

Lemma 6.4. (cf. [1]) *The two x-projections $x(t)$ and $x'(t)$ form a (piecewise linear) braid diagram, i.e., no tangencies. The intersection number $\iota\big(x(t), x'(t)\big)$, given as the total number of intersections of the graphs of $x(t)$ and $x'(t)$ on the interval $t \in [0, n]$, is well-defined and even.*

Proof. Let $x_j = x(t_j)$ and $x_j' = x'(t_j)$, $j = 0, \ldots, nd$ and by the theory in Sect. 5.3 the sequences satisfy the parabolic recurrence relations $\mathcal{R}_j(x_{j-1}, x_j, x_{j+1}) = 0$ and $\mathcal{R}_j(x_{j-1}', x_j', x_{j+1}') = 0$. Suppose the sequences $\{x_j\}$ and $\{x_j'\}$ have a tangency at $x_j = x_j'$ (but are not identically equal). Then, either $x_{j-1}' < x_{j-1}$ and $x_{j+1}' \leq x_{j+1}$, or $x_{j-1}' \leq x_{j-1}$ and $x_{j+1}' < x_{j+1}$, and similar with the role of $\{x_j\}$ and $\{x_j'\}$, are reversed. Since functions $\mathcal{R}_j$ are strictly increasing in the first and third variables and since $x_j = x_j'$, both evaluations of $\mathcal{R}_j$ cannot be zero simultaneously which contradicts the existence of tangencies. All intersections of $x(t)$ and $x'(t)$ are therefore transverse in the sense of Definition 3.1(c), and thus $\iota\big(x(t), x'(t)\big)$ is well-defined and even. $\square$

The curves $z(t)$ and $z'(t)$ may be regarded as three-dimensional (continuous) curves $z, z' \colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$, $t \mapsto (t, z(t))$ and $t \mapsto (t, z'(t))$. Due to the special properties of the chained Moser isotopy ϕ_t, we have:

Lemma 6.5. *The graphs $t \mapsto (t, z(t))$ and $t \mapsto (t, z'(t))$ form a positive 2-strand braid. Intersections in the x, x'-braid diagram correspond to positive crossings in the z, z'-braid. The linking number is given by* $\mathrm{Link}\big(z(t), z'(t)\big) = \frac{1}{2}\iota\big(x(t), x'(t)\big)$.

Proof. By Lemma 6.4 the projection graphs $t \mapsto x(t)$ and $t \mapsto x'(t)$ form a braid diagram. To show that the graphs $t \mapsto (t, z(t))$ and $t \mapsto (t, z'(t))$ form a positive 2-strand braid, we examine the intersections in the x, x'-braid diagram.

(i) Consider an intersection on the interval $[t_j, t_{j+1}] \subset [0, n]$ for which $x_j < x'_j$ and $x_{j+1} > x'_{j+1}$. Let $\tau \in [t_j, t_{j+1}]$ be the intersection point and $x(\tau) = x'(\tau) = x_*$. After rescaling and shifting to the interval $[0, 1]$ we have $x(s(t)) = x_j + (x_{j+1} - x_j)s(t)$, $s(t) = d(t - t_j) \in [0, 1]$ and the same for $x'(s(t))$. Recall that ϕ_t is given by (33), and therefore by (23),

$$y(s(\tau)) = \partial_p L^j \left(s(\tau), x_*, x_{j+1} - x_j \right),$$
$$y'(s(\tau)) = \partial_p L^j \left(s(\tau), x_*, x'_{j+1} - x'_j \right),$$

where L^j are the Lagrangians for the Moser isotopies ψ_{t, H^j} in Proposition 5.1. Since $\partial^2_{pp} L^j \geq \delta > 0$ and $x_{j+1} - x_j > x'_{j+1} - x'_j$, we conclude that $y(s(\tau)) > y'(s(\tau))$. By reversing the role of x and x', i.e., $x_j > x'_j$ and $x_{j+1} < x'_{j+1}$, we obtain $y(s(\tau)) < y'(s(\tau))$, which shows that an intersection in the x, x'-diagram corresponds to a positive crossing in the z, z'-braid.

(ii) Consider an intersection at x_j, for which $x_{j-1} < x'_{j-1}$, $x_j = x'_j = x_*$ and $x_{j+1} > x'_{j+1}$. As in the previous case

$$y(s(\tau)) = \partial_p L^{j-1} \left(1, x_*, x_* - x_{j-1} \right) = \partial_p L^j \left(0, x_*, x_{j+1} - x_* \right);$$
$$y'(s(\tau)) = \partial_p L^{j-1} \left(1, x_*, x_* - x'_{j-1} \right) = \partial_p L^j \left(0, x_*, x'_{j+1} - x_* \right),$$

and since $x_* - x_{j-1} > x_* - x'_{j-1}$ (and $x_{j+1} - x_* > x'_{j+1} - x_*$) we conclude that $y(s(\tau)) > y'(s(\tau))$. Reversing the role of x and x' yields $y(s(\tau)) < y'(s(\tau))$. In this case, we also conclude that an intersection in the x, x'-diagram corresponds to a positive crossing in the z, z'-braid, which concludes the proof. □

Proof of Proposition 6.1. Since ϕ_t is an isotopy and $F(B) = \phi_1(B) = B$, the path $t \mapsto \phi_t(B)$ in $\mathcal{C}_m \mathbb{D}^2$ represents a geometric braid $\boldsymbol{\beta}(t)$. By Lemma 6.5 all crossings in $\boldsymbol{\beta}(t)$ are positive. Indeed, if we consider the m-fold cover $\widehat{\boldsymbol{\beta}}(t)$ of $\boldsymbol{\beta}$, i.e., $t \mapsto \phi_t(B)$, $t \in [0, m]$, then all pairs of strands satisfy the hypotheses of Lemma 6.5. Consequently, the presentation of $\beta \in \mathcal{B}_m$ of $\boldsymbol{\beta}$ is unique and consists of only positive letters. □

Our decomposition automatically selects a braid word in $\mathcal{B}_m^+$ which is positive and which may be represented as a positive piecewise linear braid diagram. This allows us to use the theory of parabolic recurrence relations for finding additional periodic points for F. In the following lemma $\widehat{F}$ is an extension of F to $\mathbb{R}^2$ given by Proposition 5.2.

Lemma 6.6. *Let $A, B \subset \mathbb{R}^2$ be finite, disjoint sets and $B \subset \text{int } \mathbb{D}^2$. Let $F \in \text{Symp}(\mathbb{D}^2)$ with $F(B) = B$ and let ϕ_t be a chained Moser isotopy given by Proposition 5.2 with $\phi_1 = \widehat{F}$. Suppose A is an invariant set for $\widehat{F}$ and $\boldsymbol{\alpha}(t) = \phi_t(A)$ is acylindrical in $\boldsymbol{\alpha}$ rel $\boldsymbol{\beta}$. Then, A is an invariant set for F with $A \subset \text{int } \mathbb{D}^2$.*

Proof. The set $A = \boldsymbol{\alpha}(0)$ is an invariant set for $\widehat{F}$. Assume without loss of generality that $\boldsymbol{\alpha}(t)$ is a single component braid. Let $\boldsymbol{\alpha}^\dagger(t)$ be the n-fold cover of $\boldsymbol{\alpha}$, with $t \in [0, n]$. If $\boldsymbol{\alpha}^\dagger(t_j) \in \mathbb{R}^2 \setminus \text{int } \mathbb{D}^2$, $t_j = j/d$, for some j, then $\boldsymbol{\alpha}^\dagger(t_j) \in \mathbb{R}^2 \setminus \text{int } \mathbb{D}^2$ for all j, since the set $\phi_t(\mathbb{D}^2)$ separates the points inside

and outside $\partial \mathbb{D}^2$ under the isotopy ϕ_t. Therefore, $\boldsymbol{\alpha}^\dagger(t) \in \mathbb{R}^2 \backslash \operatorname{int} \phi_t(\mathbb{D}^2)$ for all $t \in [0, n]$, and thus $\boldsymbol{\alpha}(t) \in \mathbb{R}^2 \backslash \operatorname{int} \phi_t(\mathbb{D}^2)$ for all $t \in [0, 1]$. By assumption $\boldsymbol{\beta}(t) \in \phi_t(\mathbb{D}^2)$ for all $t \in [0, 1]$, and therefore that $\boldsymbol{\alpha}$ can be contracted onto $\partial \phi_t(\mathbb{D}^2)$ which contradicts the assumption that $\boldsymbol{\alpha} \operatorname{rel} \boldsymbol{\beta}$ is acylindrical. Consequently, $\boldsymbol{\alpha}(t) \in \phi_t(\mathbb{D}^2)$ for all $t \in [0, 1]$. The latter implies $A \subset \operatorname{int} \mathbb{D}^2$, which completes the proof. $\square$

7. The main theorem

Given $F \in \operatorname{Symp}(\mathbb{D}^2)$ and a finite invariant set $B \subset \operatorname{int} \mathbb{D}^2$, then

$$\jmath_B([F]) = \beta \bmod \square \in \mathscr{B}_m / Z(\mathscr{B}_m). \tag{39}$$

The braid word β can be chosen positive by adding full twists and can be used to force additional invariant sets as quantified by the following result.

Theorem 7.1. *Let $B \subset \operatorname{int} \mathbb{D}^2$ be a finite invariant set for a symplectomorphism $F : \mathbb{D}^2 \to \mathbb{D}^2$ and let $\beta \in \mathscr{B}_m^+$ be a positive braid word representing the mapping class of F in $\operatorname{Mod}(\mathbb{D}^2 \operatorname{rel} B)$. Consider a coloring $\textsc{a} \subset \{1, \ldots, n+m\}$ of length n and a 2-colored braid $(\gamma, \textsc{a}) \in \pi_\textsc{a}^{-1}(\beta) \subset \mathscr{B}_{n,m,\textsc{a}}^+$ and assume that $[\![\gamma, \textsc{a}]\!]$ is proper and acylindrical,[3] and $\operatorname{H}[\![\gamma, \textsc{a}]\!] \neq 0$. Then,*

(i) *there exists a finite invariant set $A \subset \operatorname{int} \mathbb{D}^2$ of F such that the mapping class of F satisfies $\jmath_{A \cup B}([F]) = \gamma \bmod \square$;*

(ii) *the number of distinct invariants sets A in (i) is bounded from below by $|P_t[\![\gamma, \textsc{a}]\!]|$.*

Proof. Recall from Sect. 3.3 that $[\![\gamma, \textsc{a}]\!]$ determines a 2-colored discretized braid class $[\boldsymbol{a} \operatorname{rel} \boldsymbol{b}]_{\pm}$ via the mapping $\gamma \mapsto \mathcal{E}_0(\gamma) \in \mathscr{D}_{n+m}^{d_0}$, with d_0 the number of generators in γ. The representative $\boldsymbol{a} \operatorname{rel} \boldsymbol{b} \in \mathscr{D}_{n,m}^{d_0}$ is obtained from the coloring $\textsc{a}$.

Consider a chained Moser isotopy ϕ_t with $r = \rho = 0$, then

$$\imath_B([\phi_t(B)]) \bmod \square = \beta,$$

where $\ell \geq 3$ and k in Proposition 5.2 are fixed. Let $\boldsymbol{b}_B := \{\pi_x \phi_{t_j}(B)\}$, $j = 0, \ldots, d$ and choose κ large enough such that $[\boldsymbol{b}_B]$ defines a free discrete braid class. The following three cases can be distinguished:

$$\boldsymbol{b} \doteq \boldsymbol{b}_B, \quad \boldsymbol{b} \doteq \mathbb{T}^{2\lambda} \boldsymbol{b}_B, \quad \text{and} \quad \mathbb{T}^{2\lambda} \boldsymbol{b} \doteq \boldsymbol{b}_B,$$

for some $\lambda \geq 1$. The choices of ℓ, k and κ determine d.

Case I The integers $q \geq 0$ and κ are chosen large enough such that $\mathbb{E}^q \boldsymbol{b}^* \sim \boldsymbol{b}_B^*$ and $[\mathbb{E}^q(\boldsymbol{a} \operatorname{rel} \boldsymbol{b}^*)]$ is free, where $\boldsymbol{b}_B^* = \boldsymbol{b}_B \cup \boldsymbol{b}_B^- \cup \boldsymbol{b}_B^+$, with $\boldsymbol{b}_B^\pm = \{\pm 2\}$. Note that $\boldsymbol{b}_B^* = \{\pi_x \phi_{t_j}(B^*)\}$, with $B^* = B \cup \{-2, +2\}$. Choose $\boldsymbol{a}_B \operatorname{rel} \boldsymbol{b}_B^* \sim \mathbb{E}^q(\boldsymbol{a} \operatorname{rel} \boldsymbol{b}^*)$, cf. Fig. 5, then by the invariance of the discrete braid invariant we have that

$$\operatorname{H}(\boldsymbol{a}_B \operatorname{rel} \boldsymbol{b}_B^*) = \operatorname{H}(\mathbb{E}^q(\boldsymbol{a} \operatorname{rel} \boldsymbol{b}^*)) = \operatorname{H}(\boldsymbol{a} \operatorname{rel} \boldsymbol{b}^*) \neq 0,$$

which proves that $P_t(\boldsymbol{a}_B \operatorname{rel} \boldsymbol{b}_B^*) = P_t(\boldsymbol{a} \operatorname{rel} \boldsymbol{b}^*) = P_t[\![\gamma, \textsc{a}]\!] \neq 0$.

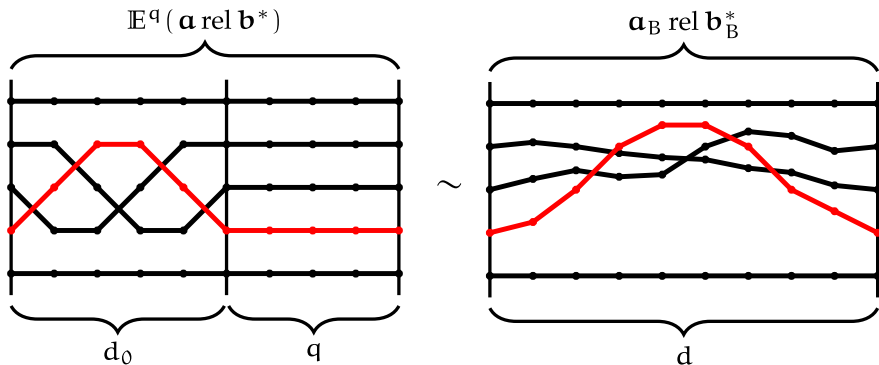[3]See Definition 3.9 and Remark 6.3.

FIGURE 5. Two representatives for Case I. Numbers of discretization points are linked by $d = d_0 + q$
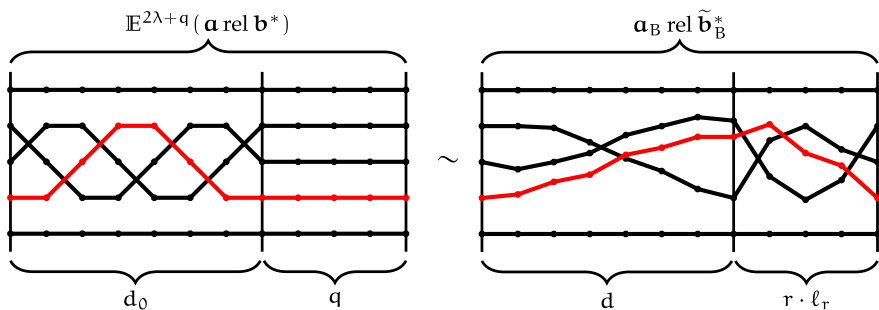

FIGURE 6. Two representatives for Case II. Numbers of discretization points are linked by $d + r\ell_r = d_0 + q$

From the Morse Theory in [1, Lemma 35], we derive that the number of stationary discrete braid diagrams $\boldsymbol{a}_B \operatorname{rel} \boldsymbol{b}_B$ for the action W_d in (36) induced by the Moser isotopy ϕ_t is bounded below by $|P_t[\![\gamma, \mathrm{A}]\!]|$. The associated piecewise linear braid diagrams $\mathrm{B}(\boldsymbol{a}_B \operatorname{rel} \boldsymbol{b}_B)$ lift to 2-colored braids $\boldsymbol{\alpha}_B(t) \operatorname{rel} \boldsymbol{\beta}_B(t)$ by constructing the y-component via (24). The stationary braids $\boldsymbol{\alpha}_B(t) \operatorname{rel} \boldsymbol{\beta}_B(t)$ are in fact braids in $\mathbb{D}^2$ by Proposition 6.6 and $A = \boldsymbol{\alpha}_B(0) \subset \operatorname{int} \mathbb{D}^2$ is an invariant set for F. Moreover, $\boldsymbol{\alpha}_B(t) \operatorname{rel} \boldsymbol{\beta}_B(t)$ determines the mapping class of F, which completes the proof in Case I.

Case II By Remark 5.5 we use a(n) (extended) chained Moser isotopy, denoted by $\widetilde{\phi}_t$, with $r = \lambda$, $\ell_r \geq 3$ and $\rho = 0$ and we denote the associated discrete braid by $\widetilde{\boldsymbol{b}}_B := \{\pi_x \widetilde{\phi}_{t_j}(B)\}$. By construction $\widetilde{\boldsymbol{b}}_B \doteq \mathbb{T}^{2\lambda} \boldsymbol{b}_B \doteq \boldsymbol{b}$. The integers $q \geq 0$ and κ are chosen large enough such that $\mathbb{E}^{q+2\lambda} \boldsymbol{b}^* \sim \widetilde{\boldsymbol{b}}_B^*$ and $[\mathbb{E}^{2\lambda+q}(\boldsymbol{a} \operatorname{rel} \boldsymbol{b}^*)]$ is free, where $\widetilde{\boldsymbol{b}}_B^* = \widetilde{\boldsymbol{b}}_B \cup \boldsymbol{b}_B^- \cup \boldsymbol{b}_B^+ = \{\pi_x \widetilde{\phi}_{t_j}(B^*)\}$. As in Case I we choose $\boldsymbol{a}_B \operatorname{rel} \widetilde{\boldsymbol{b}}_B^* \sim \mathbb{E}^{2\lambda+q}(\boldsymbol{a} \operatorname{rel} \boldsymbol{b}^*)$, cf. Fig. 6, such that

$$\mathrm{H}\big(\boldsymbol{a}_B \operatorname{rel} \widetilde{\boldsymbol{b}}_B^*\big) = \mathrm{H}\big(\mathbb{E}^{2\lambda+q}(\boldsymbol{a} \operatorname{rel} \boldsymbol{b}^*)\big) = \mathrm{H}(\boldsymbol{a} \operatorname{rel} \boldsymbol{b}^*) \neq 0.$$

This proves the theorem in Case II.

$\mathbb{E}^q \boldsymbol{a} \operatorname{rel}(\mathbb{E}^q \boldsymbol{b}^*)^{\#}$ $\boldsymbol{a}_B \operatorname{rel} \widetilde{\boldsymbol{b}}_B^{*\#}$

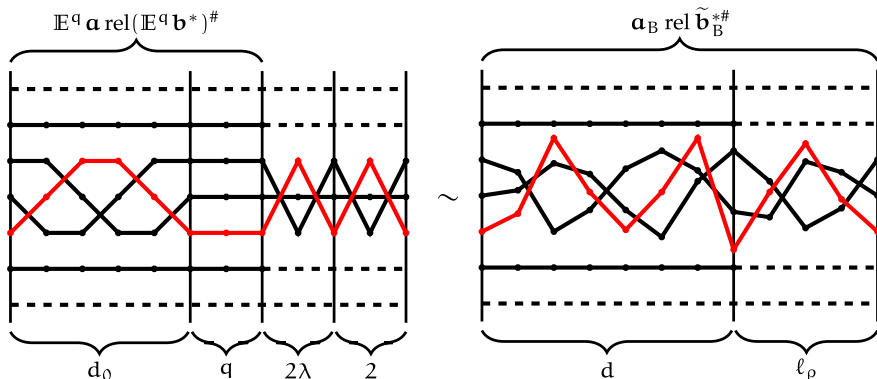d_0 q 2λ 2 d ℓ_ρ

FIGURE 7. Two representatives for Case III. Numbers of discretization points are linked by $d + \ell_\rho = d_0 + q + 2\lambda + 2$. The *dashed lines* indicate the oscillating strands due to the action of $\mathbb{T}$ and the augmentation of the skeletal by $^{\#}$

Case III By Remark 5.5 we use a(n) (extended) chained Moser isotopy, denoted by $\widetilde{\phi}_t$, with $r = 0$ and $\rho = 1$ and $\ell_\rho = 2\lambda + 2 \geq 4$. We denote the associated discrete braid by $\widetilde{\boldsymbol{b}}_B := \{\pi_x \widetilde{\phi}_{t_j}(B)\}$. By construction $\widetilde{\boldsymbol{b}}_B \doteq \mathbb{T}^2 \boldsymbol{b}_B \doteq \mathbb{T}^{2\lambda+2} \boldsymbol{b}$. The integers $q \geq 0$ and κ are chosen large enough such that $\mathbb{T}^{2\lambda+2}(\mathbb{E}^q \boldsymbol{b}^*) \sim \widetilde{\boldsymbol{b}}_B^*$ and $[\mathbb{T}^{2\lambda+2}\mathbb{E}^q(\boldsymbol{a} \operatorname{rel} \boldsymbol{b}^*)]$ is free, and where $\boldsymbol{b}_B^* = \{\pi_x \widetilde{\phi}_{t_j}(B^*)\}$. Define $\widetilde{\boldsymbol{b}}_B^{*\#} = \{\pi_x \widetilde{\phi}_{t_j}(B^{*\#})\}$, with $B^{*\#} = B \cup \{-4, -2, 2, 4\}$, which is equivalent to augmentating $\widetilde{\boldsymbol{b}}_B^*$ with the strands $\boldsymbol{b}_B^s = \{(-1)^{j+1}4\}_j$ and $\boldsymbol{b}_B^n = \{(-1)^j 4\}_j$. Choose $\boldsymbol{a}_B \operatorname{rel} \widetilde{\boldsymbol{b}}_B^* \sim \mathbb{T}^{2\lambda+2}\mathbb{E}^q(\boldsymbol{a} \operatorname{rel} \boldsymbol{b}^*)$. From (14) and the fact that $\mathbb{E}^q(\boldsymbol{a} \operatorname{rel} \boldsymbol{b}) = \mathbb{E}^q \boldsymbol{a} \operatorname{rel} \mathbb{E}^q \boldsymbol{b}$ and $\mathbb{T}^{2\lambda+2}(\mathbb{E}^q \boldsymbol{a} \operatorname{rel}(\mathbb{E}^q \boldsymbol{b}^*)^{\#}) \sim \boldsymbol{a}_B \operatorname{rel} \widetilde{\boldsymbol{b}}_B^{*\#}$, cf. Fig. 7, we derive that

$$0 \neq \mathrm{H}(\boldsymbol{a} \ \operatorname{rel} \boldsymbol{b}^*) \wedge \mathbb{S}^{2n(\lambda+1)}$$
$$= \mathrm{H}(\mathbb{E}^q(\boldsymbol{a} \operatorname{rel} \boldsymbol{b}^*)) \wedge \mathbb{S}^{2n(\lambda+1)} = \mathrm{H}(\mathbb{E}^q \boldsymbol{a} \operatorname{rel}(\mathbb{E}^q \boldsymbol{b}^*)^{\#}) \wedge \mathbb{S}^{2n(\lambda+1)}$$
$$= \mathrm{H}(\mathbb{T}^{2\lambda+2}(\mathbb{E}^q \boldsymbol{a} \operatorname{rel}(\mathbb{E}^q \boldsymbol{b}^*)^{\#})) = \mathrm{H}(\boldsymbol{a}_B \operatorname{rel} \widetilde{\boldsymbol{b}}_B^{*\#}),$$

which implies that $\mathrm{H}(\boldsymbol{a}_B \operatorname{rel} \widetilde{\boldsymbol{b}}_B^{*\#}) \neq 0$. As in the previous case the non-triviality of the braid invariant proves the theorem in Case III since the braid class $[\boldsymbol{a}_B \operatorname{rel} \widetilde{\boldsymbol{b}}_B^{*\#}]$ is free, bounded and proper and stationary braids of W_d in $[\boldsymbol{a}_B \operatorname{rel} \widetilde{\boldsymbol{b}}_B^{*\#}]$ yield invariant sets $A \subset \operatorname{int} \mathbb{D}^2$ by Proposition 6.6. $\qquad \square$

Example 7.2. Consider $F \in \operatorname{Symp}(\mathbb{D}^2)$ and assume that F has a four point invariant sets B and mapping class $[F]$ relative to B is given by $\beta = \sigma_3 \sigma_1 \sigma_2^2 \sigma_1 \sigma_3$, cf. Fig. 8. Consider the 2-colored braid class $[\![\gamma, \mathrm{A}]\!] \in \mathscr{CB}_{1,4}^+$, with

$$\gamma = \sigma_4 \sigma_1 \sigma_2 \sigma_3 \sigma_2^2 \sigma_3 \sigma_2 \sigma_1 \sigma_4, \quad \mathrm{A} = \{3\}.$$

Then, $(\gamma, \mathrm{A}) \mapsto \beta$ and the positive conjugacy class is proper and acylindrical. In [1, Sect. 4.5] the braid invariant is given: $\mathrm{H}[\![\gamma, \mathrm{A}]\!] = \mathbb{S}^1$, which by
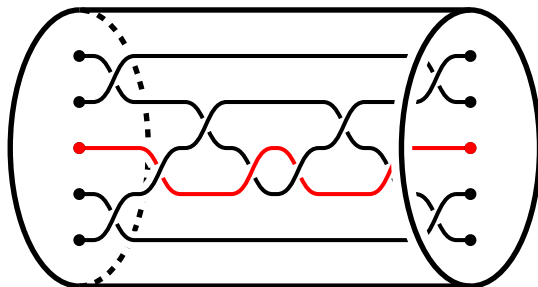
FIGURE 8. The canonical representation of the 2-colored braid class $[\![\gamma, \mathrm{A}]\!] \in \mathscr{CB}_{1,4}^+$ with $\gamma = \sigma_4\sigma_1\sigma_2\sigma_3\sigma_2^2\sigma_3\sigma_2\sigma_1\sigma_4$ and $\mathrm{A} = \{3\}$

Theorem 7.1 yields another fixed point. The index calculations in [1] suggest that we can find more periodic points for F. Instead of using multi-strand braids with $n > 1$, we use F^k instead. Consider three different 2-colored braid words: $\gamma_0 = \gamma$ as above, $\gamma_{-1} = \sigma_4\sigma_1\sigma_2\sigma_3^2\sigma_2\sigma_1\sigma_4$, and $\gamma_1 = \sigma_4\sigma_1\sigma_3\sigma_2^2\sigma_3\sigma_1\sigma_4$. For all three cases the skeletal word is given by β, the coloring is given by $\mathrm{A} = \{3\}$. Consider a symbolic sequence $\{a_i\}_{i=1}^k$, $a_i \in \{-1, 0, 1\}$, then the positive conjugacy class (γ, A), with $\mathrm{A} = \{3\}$, and

$$\gamma = \gamma_{a_0} \cdot \gamma_{a_1} \cdots \gamma_{a_{k-1}} \cdot \gamma_{a_k},$$

is proper and acylindrical, except for $a_i = -1$, or $a_i = 1$ for all i. It follows that $(\gamma, \mathrm{A}) \mapsto \beta^k$. In [1, Sect. 4.5] the braid invariant is given by $\mathrm{H}[\![\gamma, \mathrm{A}]\!] = \mathbb{S}^r$, where r is the number of zeroes in $\{a_i\}$. This procedure produces many k-periodic point for F that are forced by the invariant set B. As matter of fact one obtains a lower bound on the topological entropy of F.

Example 7.3. Let $F \in \mathrm{Symp}(\mathbb{D}^2)$ possess an invariant set B consisting of three points, and let the mapping class of $[F]$ relative to B be represented by a positive braid word $\beta = i_m^{-1}[\boldsymbol{\beta}]$, with $\boldsymbol{\beta} = \{\boldsymbol{\beta}^1(t), \boldsymbol{\beta}^2(t), \boldsymbol{\beta}^3(t)\}$ being a geometric representative of

$$\beta = \sigma_1\sigma_2^2\sigma_1^2\sigma_2^2\sigma_1\sigma_2\sigma_1\sigma_2^2\sigma_1,$$

cf. Fig. 9. For the intersection numbers it holds that $\iota(\boldsymbol{\beta}^1(t), \boldsymbol{\beta}^2(t)) = \iota(\boldsymbol{\beta}^1(t), \boldsymbol{\beta}^3(t)) = 6$ and $\iota(\boldsymbol{\beta}^2(t), \boldsymbol{\beta}^3(t)) = 1 < 6$. From considerations in [1, Sect. 9.2] it follows that through the black strands $\boldsymbol{\beta}$ one can plot a single red strand $\boldsymbol{\alpha}$, such that the 2-colored braid class $[\![\gamma, \mathrm{A}]\!]$ represented by their union

$$\gamma = \sigma_2\sigma_1\sigma_2\sigma_3\sigma_2\sigma_3\sigma_2\sigma_3\sigma_1\sigma_2\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2\sigma_3\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2\sigma_3\sigma_2\sigma_1\sigma_2,$$

with $\mathrm{A} = \{2\}$, cf. Fig. 9 is proper, acylindrical and of nontrivial index. The intersection numbers for $\boldsymbol{\alpha}$ are $\iota(\boldsymbol{\alpha}(t), \boldsymbol{\beta}^2(t)) = \iota(\boldsymbol{\alpha}(t), \boldsymbol{\beta}^3(t)) = 4$. If the intersection numbers are chosen more generally, i.e., $\iota(\boldsymbol{\beta}^1(t), \boldsymbol{\beta}^2(t)) = \iota(\boldsymbol{\beta}^3(t), \boldsymbol{\beta}^3(t)) = 2p$ and $\iota(\boldsymbol{\beta}^2(t), \boldsymbol{\beta}^3(t)) = r < 2p$, and $\iota(\boldsymbol{\alpha}(t), \boldsymbol{\beta}^2(t)) =$
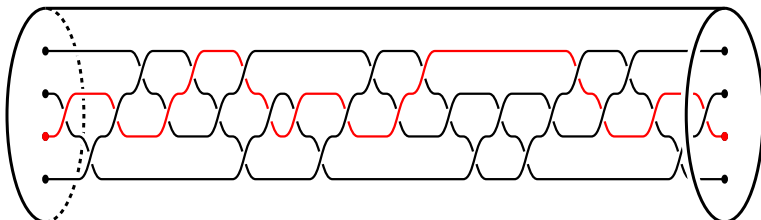
FIGURE 9. A geometric representative of the braid for Example 7.3 with $p = 3$, $q = 2$, and $r = 1$

$\iota\big(\boldsymbol{\alpha}(t), \boldsymbol{\beta}^3(t)\big) = 2q$, where $r > 0$ and $p \geq 2$. If $r < 2q < 2p$, then the singular homology of $[\![\gamma, \mathrm{A}]\!]$ is given by

$$H_k\big(\mathrm{H}[\![\gamma, \mathrm{A}]\!]\big) = \begin{cases} \mathbb{R} : k = 2q, \quad 2q + 1, \\ 0 : \text{elsewise.} \end{cases}$$

By Theorem 7.1 we conclude that there are at least two additional distinct fixed points A_1, A_2 with $\jmath_{A_i \cup B}([F]) = \gamma \bmod \square$, $i = 1, 2$. In addition, via concatenating braid diagrams one can produce an infinite number of periodic solutions of different periods, cf. [1, Lemma 47]. The above forcing result is specific for area-preserving mapping of the 2-disc, or $\mathbb{R}^2$ and *not* true for arbitrary diffeomorphisms of the 2-disc.

For example, consider the time-1 mapping $F\colon \mathbb{D}^2 \to \mathbb{D}^2$ given by the differential equation $\dot{r} = r(r - a_1)(r - a_2)(r - 1)$ and $\dot{\theta} = g(r) > 0$, with $g(a_1) = \pi$ and $g(a_2) = 6\pi$, and $0 < a_1 < a_2 < 1$. The set $B = \{-a_1, a_1, a_2\}$ is invariant and the mapping class of F relative to B is given by the skeleton $\boldsymbol{\beta}$. The mapping F has no invariant set matching the braid γ. This implies that the invariants in [4] are void for this example and the braid invariant introduced in this paper add additional information to existing invariants, in particular in the area-preserving case.

Remark 7.4. One can also consider 2-colored words (γ, A), $\gamma \in \mathscr{B}_{n+m}$. Braid words can be expressed in normal form. The fundamental element, or Garside element in $\mathscr{B}_m^+$ is denoted by $\triangle := (\sigma_1 \dots \sigma_{m-1})(\sigma_1 \dots \sigma_{m-2}) \dots (\sigma_1 \sigma_2) \sigma_1$, and is also referred to as a *half twist*. The element $\triangle$ is a factor of a positive word β, if $\beta = \beta' \triangle \beta''$, β', β'' positive (possibly trivial) words. If not, β is said to be *prime* to $\triangle$. If we use the lexicographical order on the presentations of a positive word β, then the smallest positive word β', positively equal to β, is called the *base* of β. For positive words β, prime to $\triangle$, the base is denoted by $\bar{\beta}$. Following [6, 13], every word $\beta \in \mathscr{B}_m$ is uniquely presented by a word $\triangle^\varrho \bar{\delta}$, with $\varrho \in \mathbb{Z}$ and $\bar{\delta} \in \mathscr{B}_m^+$, which is called *left Garside normal form*. Via the relation $\triangle \beta = r(\beta) \triangle$ we obtain the *right Garside normal form* $\beta = r(\bar{\delta}) \triangle^\varrho$.

For the 2-colored braid words the Garside normal form is not the appropriate normal form, since odd powers of $\triangle$ do not represent trivial permutations. In the case that the Garside power of γ is even, then $\triangle^2 = \square$ defines the identity permutation, and therefore the associated base $(\bar{\delta}, \mathrm{A})$,

with $\bar{\delta} \in \mathscr{B}_{n+m}^+$, is a positive 2-colored braid word. In this case, the left and right normal form are the same since $\square$ is at the center of the braid group $\mathscr{B}_{n+m}$. When the Garside power is odd, we argue as follows. Let $\varrho = 2\lambda + 1$, then

$$\gamma = \triangle^{2\lambda+1}\bar{\delta} = \square^\lambda \triangle \bar{\delta} = \square^\lambda r(\bar{\delta})\triangle = r(\bar{\delta})\triangle\square^\lambda = \bar{\epsilon}\square^\lambda = \square^\lambda\bar{\epsilon}, \qquad (40)$$

where $\bar{\epsilon} = \triangle\bar{\delta} = r(\bar{\delta})\triangle$ is the base and $\bar{\epsilon}$ is prime to $\square$. The power λ is related to the Garside power via $\lambda = \lambda(\gamma) = \lfloor \varrho/2 \rfloor$ and is referred to as the *symmetric Garside power* of γ. For 2-colored braid words (γ, A) this yields the following symmetric normal form: $\gamma = \square^\lambda\bar{\epsilon} = \bar{\epsilon}\square^\lambda$, with $\lambda \in \mathbb{Z}$ and $(\bar{\epsilon}, \mathrm{A})$ a positive 2-colored braid word. Since mapping classes of F are given modulo full twists the latter normal form suggests that we capture all forcing via positive braid words. If conjugacy is incorporated, the power λ can be optimized with respect to different conjugate braid words.

Acknowledgements

Appendix A: Mapping classes and braids

We give overview of various known and less known facts about mapping class groups of the 2-disc and the 2-disc with a finite number of marked points.

A.1 Mapping class groups of the 2-disc

By the 'Alexander trick', cf. [14, pp. 48], $\mathrm{Homeo}_0^+(\mathbb{D}^2)$ is contractible, and thus $\mathrm{Mod}_0(\mathbb{D}^2)$ is trivial. The case of $\mathrm{Homeo}^+(\mathbb{D}^2)$ is more involved. Consider the Serre fibration $\mathrm{Homeo}^+(\mathbb{D}^2) \to \mathrm{Homeo}^+(\partial\mathbb{D}^2)$ with fiber $\mathrm{Homeo}_0^+(\mathbb{D}^2)$, cf. [15, pp. 536] and the associated homotopy long exact sequence

$$\begin{aligned}
\cdots \longrightarrow \pi_1\big(\mathrm{Homeo}_0^+(\mathbb{D}^2)\big) &\longrightarrow \pi_1\big(\mathrm{Homeo}^+(\mathbb{D}^2)\big) \\
\longrightarrow \pi_1\big(\mathrm{Homeo}^+(\partial\mathbb{D}^2)\big) &\longrightarrow \pi_0\big(\mathrm{Homeo}_0^+(\mathbb{D}^2)\big) \\
\longrightarrow \pi_0\big(\mathrm{Homeo}^+(\mathbb{D}^2)\big) &\longrightarrow \pi_0\big(\mathrm{Homeo}^+(\partial\mathbb{D}^2)\big).
\end{aligned}$$

Since $\pi_k\big(\mathrm{Homeo}_0^+(\mathbb{D}^2)\big) \cong 1$, for all $k \geq 0$, and $\pi_1\big(\mathrm{Homeo}^+(\partial\mathbb{D}^2)\big) \cong \mathbb{Z}$ and $\pi_0\big(\mathrm{Homeo}^+(\partial\mathbb{D}^2)\big) \cong 1$, we have the short exact sequences

$$1 \longrightarrow \pi_1\big(\mathrm{Homeo}^+(\mathbb{D}^2)\big) \longrightarrow \mathbb{Z} \longrightarrow 1,$$

which yields that $\pi_1\big(\mathrm{Homeo}^+(\mathbb{D}^2)\big) \cong \mathbb{Z}$, and

$$1 \longrightarrow \pi_0\big(\mathrm{Homeo}^+(\mathbb{D}^2)\big) \longrightarrow 1,$$

which shows that $\mathrm{Mod}(\mathbb{D}^2) \cong 1$ and completes the proof of Proposition 2.1.

Remark A.1. Mapping classes can also be defined for orientation preserving diffeomorphisms of the 2-disc, denoted by $\mathrm{Diff}^+(\mathbb{D}^2)$ and $\mathrm{Diff}_0^+(\mathbb{D}^2)$. Due to a result by Smale [16], $\mathrm{Diff}_0^+(\mathbb{D}^2)$ is contractible, and therefore the Serre fibration $\mathrm{Diff}^+(\mathbb{D}^2) \to \mathrm{Diff}^+(\partial\mathbb{D}^2)$ with fiber $\mathrm{Diff}_0^+(\mathbb{D}^2)$, yields that $\pi_0\big(\mathrm{Diff}^+(\mathbb{D}^2)\big) \cong 1$ and $\pi_0\big(\mathrm{Diff}_0^+(\mathbb{D}^2)\big) \cong 1$, and thus the above mapping class groups can also be defined via diffeomorphisms, cf. [17–19].

For a treatment of mapping class groups via symplectic and Hamiltonian diffeomorphisms, see Appendix B.

A.2 Braids and mapping classes

To determine the mapping class group in Proposition 2.3, we consider the fiber bundle

$$\mathrm{Homeo}_0^+(\mathbb{D}^2 \,\mathrm{rel}\, B) \xrightarrow{i} \mathrm{Homeo}_0^+(\mathbb{D}^2) \xrightarrow{\epsilon} \mathcal{C}_m\mathbb{D}^2,$$

cf. [14, pp. 245]. The associated homotopy long exact sequence with base point B, yields

$$\cdots \longrightarrow \pi_1\big(\mathrm{Homeo}_0^+(\mathbb{D}^2)\big) \xrightarrow{\epsilon_*} \pi_1\big(\mathcal{C}_m\mathbb{D}^2\big)$$
$$\xrightarrow{d_*} \pi_0\big(\mathrm{Homeo}_0^+(\mathbb{D}^2 \,\mathrm{rel}\, B)\big) \xrightarrow{i_*} \pi_0\big(\mathrm{Homeo}_0^+(\mathbb{D}^2)\big) \longrightarrow 1.$$

Since $\pi_k\big(\mathrm{Homeo}_0^+(\mathbb{D}^2)\big) \cong 1$, for all $k \geq 0$, we obtain the short exact sequence

$$1 \longrightarrow \pi_1\big(\mathcal{C}_m\mathbb{D}^2\big) \xrightarrow{d_*} \pi_0\big(\mathrm{Homeo}_0^+(\mathbb{D}^2 \,\mathrm{rel}\, B)\big) \longrightarrow 1,$$

which proves that $\pi_1\big(\mathcal{C}_m\mathbb{D}^2\big) \xrightarrow{d_*} \pi_0\big(\mathrm{Homeo}_0^+(\mathbb{D}^2 \,\mathrm{rel}\, B)\big)$ is an isomorphism and $\mathrm{Mod}_0(\mathbb{D}^2 \,\mathrm{rel}\, B) \cong \mathscr{B}_m$ via $\iota_B \circ d_*^{-1}$. Since $\mathrm{Diff}_0^+(\mathbb{D}^2)$ is contractible, the same arguments yield $\pi_0\big(\mathrm{Diff}_0^+(\mathbb{D}^2 \,\mathrm{rel}\, B)\big) \cong \mathscr{B}_m$, which implies that diffeomorphisms define the same mapping class group. For a detailed study of relations between braids and mapping class groups the reader is also referred to a comprehensive treatise by Birman [6,7]. See also [5,18].

For homeomorphisms of the 2-disc preserving $\partial\mathbb{D}^2$ setwise, we consider the fiber bundle

$$\mathrm{Homeo}^+(\mathbb{D}^2 \,\mathrm{rel}\, B) \xrightarrow{i} \mathrm{Homeo}^+(\mathbb{D}^2) \xrightarrow{\epsilon} \mathcal{C}_m\mathbb{D}^2,$$

which implies the following homotopy long exact sequence

$$\cdots \longrightarrow \pi_1\big(\mathrm{Homeo}^+(\mathbb{D}^2)\big) \xrightarrow{\epsilon_*} \pi_1\big(\mathcal{C}_m\mathbb{D}^2\big)$$
$$\xrightarrow{d_*} \pi_0\big(\mathrm{Homeo}^+(\mathbb{D}^2 \,\mathrm{rel}\, B)\big) \xrightarrow{i_*} \pi_0\big(\mathrm{Homeo}^+(\mathbb{D}^2)\big) \longrightarrow 1,$$

with base point B. From exactness we deduce that $\mathcal{B}_m\mathbb{D}^2/\ker d_* \cong \mathrm{im}\, d_* = \ker i_*$. For the kernel we have $\ker d_* = Z(\mathcal{B}_m\mathbb{D}^2)$, cf. [6, Theorem 4.3]. Since, $\pi_0\big(\mathrm{Homeo}^+(\mathbb{D}^2)\big) \cong 1$ we have that $\ker i_* = \mathrm{Mod}(\mathbb{D}^2 \,\mathrm{rel}\, B)$ and

$$\longrightarrow \pi_1\big(\mathrm{Homeo}^+(\mathbb{D}^2)\big) \xrightarrow{\epsilon_*} \mathcal{B}_m\mathbb{D}^2 \xrightarrow{d_*} \mathrm{Mod}(\mathbb{D}^2 \,\mathrm{rel}\, B) \xrightarrow{i_*} 1,$$

which implies that $\mathrm{Mod}(\mathbb{D}^2 \operatorname{rel} B) \cong \mathscr{B}_m/Z(\mathscr{B}_m)$.

The same can be derived using diffeomorphism instead of homeomorphisms, i.e., $\pi_0\big(\mathrm{Diff}^+(\mathbb{D}^2 \operatorname{rel} B)\big) = \mathrm{Mod}(\mathbb{D}^2 \operatorname{rel} B) \cong \mathscr{B}_m/Z(\mathscr{B}_m)$, cf. [18].

Appendix B: Symplectic mapping classes

Two symplectomorphisms $F, G \in \mathrm{Symp}(\mathbb{D}^2)$ are symplectically isotopic if there exists an isotopy ϕ_t, with $\phi_t \in \mathrm{Symp}(\mathbb{D}^2)$ for all $t \in [0,1]$,[4] such that $\phi_0 = F$ and $\phi_1 = G$. The equivalence classes in $\mathrm{Symp}(\mathbb{D}^2)/\sim$ are called *symplectic mapping classes*.

Proposition B.1. $\pi_0\big(\mathrm{Symp}(\mathbb{D}^2)\big) = \mathrm{Mod}(\mathbb{D}^2) \cong 1$.

Proof. Since $\mathrm{Mod}(\mathbb{D}^2) \cong 1$, two symplectomorphisms $F, G \in \mathrm{Symp}(\mathbb{D}^2)$ are isotopic in $\mathrm{Diff}^+(\mathbb{D}^2)$. Therefore, $F^{-1}G$ is isotopic to the identity via an isotopy ξ_t. The isotopy ξ_t does not necessarily preserve ω. Define $\omega_t := \xi_t^* \omega$, which is a based loop in $\Omega^2(\mathbb{D}^2)$, since $\omega_t = \omega$ at $t = 0$ and $t = 1$. Observe that $\int_{\mathbb{D}^2} \omega_t = \int_{\mathbb{D}^2} \omega = \pi$ for all $t \in [0,1]$. Indeed, since ξ_t is smooth 1-parameter family of diffeomorphisms it holds that $\int_{\mathbb{D}^2} \omega_t = \pi \deg(\xi_t) = \pi$, since $\deg(\xi_t) = 1$ for all $t \in [0,1]$.

Write $\omega_t = a_t(x,y)\mathrm{d}x \wedge \mathrm{d}y$, with $a_t(x,y) > 0$ on $[0,1] \times \mathbb{D}^2$ and $a_0 = a_1 = 1$. To construct a symplectic isotopy we invoke Moser's stability argument, cf. [20,21], Sect. 3.2. Consider potential functions $\Phi_t \colon \mathbb{D}^2 \times [0,1] \to \mathbb{R}$ and define the vector fields $X_t = \frac{1}{a_t(x,y)} \nabla \Phi_t$, with $\langle X_t(x), \mathbf{n} \rangle = 0$ for $x \in \partial \mathbb{D}^2$, $\mathbf{n}$ the outward pointing normal. The boundary condition guarantees that $\mathbb{D}^2$ is invariant for the associated flow χ_t. Furthermore, define 1-forms $\theta_t = -\iota_{X_t} \omega_t$. To apply Moser stability, we seek potentials Φ_t such that $\frac{\mathrm{d}\omega_t}{\mathrm{d}t} = \mathrm{d}\theta_t$, which is equivalent to the Neumann problem

$$-\Delta \Phi_t = \frac{\mathrm{d}a_t(x,y)}{\mathrm{d}t}, \quad \partial_{\mathbf{n}} \Phi_t\big|_{\partial \mathbb{D}^2} = 0. \tag{41}$$

Since the forms ω_t are cohomologous, Stokes' theorem implies that

$$0 = \int_{\mathbb{D}^2} \frac{\mathrm{d}\omega_t}{\mathrm{d}t} = \int_{\mathbb{D}^2} -\Delta \Phi_t \omega = \oint_{\partial \mathbb{D}^2} \theta_t = \oint_{\partial \mathbb{D}^2} \partial_{\mathbf{n}} \Phi_t$$

which shows that the Neumann problem is well-posed and has a unique solution (up to an additive constant), which depends smoothly on t.

By construction $\chi_t^* \omega_t = \omega$ and the desired symplectic isotopy is given by $\xi_t \circ \chi_t$. The symplectic isotopy $\phi_t := F \circ \xi_t \circ \chi_t$ is an isotopy between F and G, which proves that F and G are symplectically isotopic, and thus $\pi_0\big(\mathrm{Symp}(\mathbb{D}^2)\big) = \mathrm{Mod}(\mathbb{D}^2)$. $\square$

As for homeomorphisms we can also consider relative symplectic mapping classes. Two symplectomorphisms $F, G \in \mathrm{Symp}(\mathbb{D}^2)$ are of the same relative symplectic mapping class if there exists an isotopy ϕ_t, with $\phi_t \in \mathrm{Symp}(\mathbb{D}^2)$ and $\phi_t(B) = B$ for all $t \in [0,1]$, such that $\phi_0 = F$ and $\phi_1 = G$. The subgroup of such symplectomorphisms is denoted by $\mathrm{Symp}(\mathbb{D}^2 \operatorname{rel} B)$.

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[4]The condition $\phi_t \in \mathrm{Symp}(\mathbb{D}^2)$ is equivalent to $\phi_t^* \omega = \omega$.

Lemma B.2. *Let $F, G \in \mathrm{Symp}(\mathbb{D}^2)$ be isotopic in $\mathrm{Diff}^+(\mathbb{D}^2 \, \mathrm{rel} \, B)$, then they are isotopic in $\mathrm{Symp}(\mathbb{D}^2 \, \mathrm{rel} \, B)$.*

Proof. Let $F, G \in \mathrm{Symp}(\mathbb{D}^2)$ be isotopic in $\mathrm{Diff}^+(\mathbb{D}^2 \, \mathrm{rel} \, B)$, then $F^{-1}G$ is isotopic to the identity via a smooth isotopy ξ_t with the additional condition that $\xi_t(B) = B$ for all $t \in [0, 1]$. To find a symplectic isotopy we repeat the proof of Proposition B.1 with a few modifications.

Let $X_t = \frac{1}{a_t(x,y)} \nabla \Phi_t + \frac{1}{a_t(x,y)} J \nabla \lambda_t(x)$, where $\lambda_t \colon [0, 1] \times \mathbb{D}^2 \to \mathbb{R}$ is a smooth function compactly supported in $\mathrm{int}\,\mathbb{D}^2$, where $\mathbf{t} = J\mathbf{n}$ is a unit tangent such that $\{\mathbf{n}, \mathbf{t}\}$ is positively oriented. In order for the associated flow χ_t to restrict to a flow on $\mathbb{D}^2$ we need that $\frac{\partial \lambda_t}{\partial \mathbf{t}} + \frac{\partial \Phi_t}{\partial \mathbf{n}} = 0$ at $\partial \mathbb{D}^2$. Since λ_t is compactly supported in $\mathrm{int}\,\mathbb{D}^2$ we use the Neumann boundary condition for Φ_t. For the 1-forms we obtain

$$\theta_t = -\iota_{X_t}\omega_t = -\partial_x \Phi_t \mathrm{d}y + \partial_y \Phi_t \mathrm{d}x - \mathrm{d}\lambda_t.$$

As in the proof of Proposition B.1 the potential Φ_t is determined by the Neumann problem in (41). Since λ_t can be chosen arbitrarily we define $\lambda_t(x) = \langle J\nabla\Phi_t(z_0), x \rangle$ on neighborhoods of $z_0 \in B$ and a smooth extension outside, compactly supported in $\mathrm{int}\,\mathbb{D}^2$. This guarantees that the isotopies χ_t preserve B point wise, which completes the proof. $\qquad\square$

Let ϕ_t be a path in $\mathrm{Symp}(\mathbb{D}^2)$, then ϕ_t satisfies the initial value problem $\frac{\mathrm{d}}{\mathrm{d}t}\phi_t = X_t \circ \phi_t$, where $X_t = \frac{\mathrm{d}}{\mathrm{d}t}\phi_t \circ \phi_t^{-1}$ is a time-dependent vector field. Since $\phi_t^*\omega = \omega$ it holds that $\mathrm{d}\iota_{X_t}\omega = 0$ for all $t \in [0, 1]$, cf. [21, Proposition 3.2]. A symplectic isotopy is Hamiltonian is there exists a smooth function $H \colon [0, 1] \times \mathbb{D}^2 \to \mathbb{R}$ such that $\iota_{X_t}\omega = -\mathrm{d}H(t, \cdot)$. In this case, $F = \phi_1$ is a Hamiltonian symplectomorphism.

By construction $\langle X_t, \mathbf{n} \rangle = 0$ and

$$-\mathrm{d}H(t, \cdot)(\mathbf{t}) = \iota_{X_t}\omega(\mathbf{t}) = \omega(X_t, \mathbf{t}) = \omega(X_t, J\mathbf{n}) = \langle X_t, \mathbf{n} \rangle = 0,$$

which proves that $H(t, \cdot)|_{\partial \mathbb{D}^2} = \mathrm{const}$. Since the 2-disc $\mathbb{D}^2$ is contractible such a Hamiltonian exists, which shows that symplectomorphisms of the 2-disc are Hamiltonian.

Let $\mathrm{Symp}(\mathbb{D}^2 \, \mathrm{rel} \, B)$ be the symplectomorphisms which leave a set $B \subset \mathrm{int}\,\mathbb{D}^2$ invariant. The subgroup of Hamiltonian symplectomorphisms is denoted by $\mathrm{Ham}(\mathbb{D}^2 \, \mathrm{rel} \, B)$. The above procedure yields the following result.

Proposition B.3. $\mathrm{Symp}(\mathbb{D}^2 \, \mathrm{rel} \, B) = \mathrm{Ham}(\mathbb{D}^2 \, \mathrm{rel} \, B)$.

Proof. From the previous it follows that there exists a Hamiltonian such that $\iota_{X_t}\omega = -\mathrm{d}H(t, \cdot)$. Since the points in B are rest points for X_t it follows that $X_t(z_0) = 0$ for all t and for all $z_0 \in B$. Consequently, the points in B are critical points of H. $\qquad\square$

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