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The properties and applications of relative retracts

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Abstract. In this paper, we present relative retracts and we can say that these are multilevel retracts which either retain given properties depending on the level or not. Some properties are constant and are present on every level. These properties are especially important in regard to the theory of coincidence. The class of relative retracts consists of retracts in the sense of Borsuk, multiretracts and many fundamental retracts.

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1. Introduction

The class of retracts in the sense of Borsuk is vast (see [1]). It is proven that the class of multiretracts, although essentially wider (see [11]), differs from the class of retracts only by some spaces that are infinitely and uncountably dimensional so that it could be said that the differences are pathological. In this paper the notion of "relative retract" is introduced. A few levels of retracts were achieved. The first level is obviously the retracts in the sense of Borsuk, the second level is multiretracts and the next levels are relative retracts that lose some properties but retain other important properties. These properties that are present on all levels are particularly important due to their applications (Theorem 3.5).

2. Preliminaries

Throughout this paper, all topological spaces are assumed to be metrizable. A continuous mapping $f : X \to Y$ is called proper if for every compact set $K \subset Y$, the set $f^{-1}(K)$ is nonempty and compact. Let X and Y be two spaces and assume that for every $x \in X$, a nonempty and compact subset

$\varphi(x)$ of Y is given. In such a case we say that $\varphi : X \multimap Y$ is a multivalued mapping. For a multivalued mapping $\varphi : X \multimap Y$ and a subset $A \subset Y$, we let

$$\varphi^{-1}(A) = \{x \in X;\ \varphi(x) \subset A\}.$$

If for every open set $U \subset Y$, the set $\varphi^{-1}(U)$ is open, then φ is called an upper semicontinuous mapping; we shall write that φ is u.s.c. Let H_* be the Čech homology functor with compact carriers and coefficients in the field of rational numbers $\mathbb{Q}$ from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H_*(X) = \{H_k(X)\}$ is a graded vector space, $H_k(X)$ being a k-dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to Y$, $H_*(f)$ is the induced linear map $f_* = \{f_{k*}\}$, where $f_{k*} : H_k(X) \to H_k(Y)$ (see [5]). A space X is acyclic if

(i) X is nonempty,
(ii) $H_k(X) = 0$ for every $k \geq 1$,
(iii) $H_0(X) \approx \mathbb{Q}$.

Proposition 2.1 (See [5]**).** *Assume that, in the category of graded vector spaces, the following diagram commutes:*

$$\begin{array}{ccc} E' & \xrightarrow{\ u\ } & E'' \\ {\scriptstyle u'}\uparrow & \overset{v}{\nwarrow} & \uparrow{\scriptstyle u''} \\ E' & \xrightarrow{\ u\ } & E''. \end{array}$$

Then, if u' or u'' is a Leray endomorphism, so is the other; and, in that case,

$$\Lambda(u') = \Lambda(u'').$$

A proper map $p : X \to Y$ is called Vietoris provided that for every $y \in Y$, the set $p^{-1}(y)$ is acyclic. A proper map $p : X \to Y$ is called cell-like provided that for each $y \in Y$, the set $p^{-1}(y)$ has a trivial shape (in the sense of Borsuk [2]). We know that a compact set of trivial shape is acyclic. Hence if $p : X \to Y$ is a cell-like map, then it is a Vietoris map. The symbol $D(X, Y)$ will denote the set of all diagrams of the form

$$X \xleftarrow{\ p\ } Z \xrightarrow{\ q\ } Y,$$

where $p : Z \to X$ denotes a Vietoris map and $q : Z \to Y$ denotes a continuous map. Each such diagram will be denoted by (p, q). We recall that the composition of two Vietoris mappings is a Vietoris mapping and if $p : X \to Y$ is a Vietoris map, then $p_* : H_*(X) \to H_*(Y)$ is an isomorphism (see [5]).

Definition 2.2 (See [5]**).** Let $(p, q) \in D(X, Y)$ and $(r, s) \in D(Y, T)$. The composition of the diagrams

$$X \xleftarrow{\ p\ } Z_1 \xrightarrow{\ q\ } Y \xleftarrow{\ r\ } Z_2 \xrightarrow{\ s\ } T$$

is called the diagram $(u,v) \in D(X,T)$

$$X \xleftarrow{u} Z_1 \triangle_{qr} Z_2 \xrightarrow{v} T,$$

where

$$Z_1 \triangle_{qr} Z_2 = \{(z_1,z_2) \in Z_1 \times Z_2 :\ q(z_1) = r(z_2)\},$$

$$u = p \circ f_1, \quad v = s \circ f_2,$$

$$Z_1 \xleftarrow{f_1} Z_1 \triangle_{qr} Z_2 \xrightarrow{f_2} Z_2,$$

$$f_1(z_1,z_2) = z_1 \quad \text{(Vietoris map)},$$

$$f_2(z_1,z_2) = z_2 \quad \text{for each } (z_1,z_2) \in Z_1 \triangle_{qr} Z_2.$$

It shall be written as

$$(u,v) = (r,s) \circ (p,q).$$

In the set of all diagrams $D(X,Y)$, the following relation is introduced.

Definition 2.3. Let $(p_1,q_1),(p_2,q_2) \in D(X,Y)$. Then

$$(p_1,q_1) \sim_m (p_2,q_2)$$

if and only if there exist spaces Z, Z_1, Z_2 and Vietoris maps $p_3 : Z \to Z_1$, $p_4 : Z \to Z_2$ such that the following diagram is commutative:

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & Z_1 & \xrightarrow{q_1} & Y \\ \uparrow \scriptstyle{\mathrm{Id}_X} & & \uparrow \scriptstyle{p_3} & & \uparrow \scriptstyle{\mathrm{Id}_Y} \\ X & \xleftarrow{p} & Z & \xrightarrow{q} & Y \\ \downarrow \scriptstyle{\mathrm{Id}_X} & & \downarrow \scriptstyle{p_4} & & \downarrow \scriptstyle{\mathrm{Id}_Y} \\ X & \xleftarrow{p_2} & Z_2 & \xrightarrow{q_2} & Y, \end{array}$$

that is,

$$p = p_1 \circ p_3 = p_2 \circ p_4, \qquad q = q_1 \circ p_3 = q_2 \circ p_4.$$

Proposition 2.4 (See [15]**).** *The relation in the set $D(X,Y)$ introduced in Definition* 2.3 *is an equivalence relation.*

The set of the equivalence classes of the above relation will be denoted by

$$M_m(X,Y) = D(X,Y)_{/\sim_m}.$$

The elements of the space $M_m(X,Y)$ will be called multimorphisms and will be denoted by

$$\varphi_m = [(p,q)]_m,$$

where

$$X \xleftarrow{p} Z \xrightarrow{q} Y.$$

Proposition 2.5 (See [13, 15]**).** *Let $[(p,q)]_m = \varphi_m \in M_m(X,Y)$. Then the following conditions are satisfied:*

(1) $((p_1,q_1),(p_2,q_2)\in\varphi_m)\Rightarrow(q_1(p_1^{-1}(x))=q_2(p_2^{-1}(x))$ *for each* $x\in X)$.
(2) $((p_1,q_1),(p_2,q_2)\in\varphi_m)\Rightarrow(q_{1*}\circ p_{1*}^{-1}=q_{2*}\circ p_{2*}^{-1})$.
(3) *Let* $\psi_m=[(r,s)]_m\in M_m(Y,T)$ *and let*

$$\psi_m\circ\varphi_m=[(r,s)\circ(p,q)]_m\in M_m(X,T)$$

(*see Definition* 2.2). *Then for any* $(p_1,q_1)\in\varphi_m$ *and* $(r_1,s_1)\in\psi_m$ *we have*

$$((r_1,s_1)\circ(p_1,q_1))\in(\psi_m\circ\varphi_m).$$

From Proposition 2.5(1) we get the following definition.

Definition 2.6. For any $\varphi_m\in M_m(X,Y)$, the set $\varphi(x)=q(p^{-1}(x))$, where $\varphi_m=[(p,q)]_m$, is called the image of the point x in a multimorphism φ_m.

Let $\varphi_m\in M_m(X,Y)$. The symbol $\varphi:X\to_m Y$ will denote a multivalued mapping determined by a multimorphism φ_m (see Definition 2.6). We define (see Proposition 2.5(2))

$$\varphi_*=q_*\circ p_*^{-1},\tag{2.1}$$

where $(p,q)\in\varphi_m$; and if $\psi:Y\to_m T$, then $\psi\circ\varphi:X\to_m T$ is a multivalued map determined by $\psi_m\circ\varphi_m$ (see Proposition 2.5(3)) and we have (see [15])

$$(\psi\circ\varphi)_*=\psi_*\circ\varphi_*.\tag{2.2}$$

Let $f:X\to Y$ be a continuous map and let $\mathrm{Id}_X:X\to X$ be an identity map. Then

$$f_m=\big[(\mathrm{Id}_X,f)\big]_m\in M_m(X,Y)$$

and for each $(p,q)\in f_m$ (see [15]),

$$q_*\circ p_*^{-1}=f_*.\tag{2.3}$$

Let $A\subset X$ be a nonempty set and $\varphi:X\to_m Y$. Then the map $\varphi_A:A\multimap Y$ given by the formula

$$\varphi_A(x)=\varphi(x)\quad\text{for each } x\in A\tag{2.4}$$

is determined by a multimorphism $(\varphi_A)_m=[(\widetilde{p},\widetilde{q})]_m$ (see [15]), where

$$A\xleftarrow{\ \widetilde{p}\ }p^{-1}(A)\xrightarrow{\ \widetilde{q}\ }Y$$

and $(\widetilde{p},\widetilde{q})\in D(A,Y)$ is a restriction of some $(p,q)\in\varphi_m$. Hence $\varphi_A:A\to_m Y$ is a multivalued map determined by $(\varphi_A)_m=[(\widetilde{p},\widetilde{q})]_m$.

Definition 2.7. Let $(p_1,q_1),(p_2,q_2)\in D(X,Y)$, where

$$X\xleftarrow{\ p_1\ }Z_1\xrightarrow{\ q_1\ }Y,\qquad X\xleftarrow{\ p_2\ }Z_2\xrightarrow{\ q_2\ }Y.$$

We say that the diagrams (p_1,q_1) and (p_2,q_2) are homotopic, denoted by

$$(p_1,q_1)\sim_{HD}(p_2,q_2),$$

if there exist a space Z and Vietoris maps $p_3:Z\to Z_1$ and $p_4:Z\to Z_2$ such that the following conditions are satisfied:

(1) $p_1\circ p_3=p_2\circ p_4$;

(2) $q_1 \circ p_3 \sim q_2 \circ p_4$, that is, the mappings $q_1 \circ p_3, q_2 \circ p_4 : Z \to Y$ are homotopic.

Definition 2.8. Let $\varphi_m, \psi_m \in M_m(X,Y)$ be multimorphisms. We say that φ_m and ψ_m are homotopic (written $\varphi_m \sim_{HM} \psi_m$) if there exist diagrams $(p_1, q_1) \in \varphi_m$ and $(p_2, q_2) \in \psi_m$ such that

$$(p_1, q_1) \sim_{HD} (p_2, q_2).$$

Proposition 2.9 (See [15]**).** *The homotopy relation introduced in Definition* 2.8 *is an equivalence relation in the set of all multimorphisms $M_m(X,Y)$.*

The following definition is obvious.

Definition 2.10. Let $\varphi, \psi : X \to_m Y$. We say that φ and ψ are homotopic (written $\varphi \sim_{HMF} \psi$) if $\varphi_m \sim_{HM} \psi_m$.

Proposition 2.11 (See [15]**).** *Let $f, g : X \to Y$ be continuous maps. Then*

$$\big[(\mathrm{Id}_X, f)\big]_m = f_m \sim_{HM} g_m = \big[(\mathrm{Id}_X, g)\big]_m$$

if and only if there exist a space Z and a Vietoris mapping $p : Z \to X$ such that

$$f \circ p \sim g \circ p.$$

Definition 2.12. Let X be a metrizable space and $x_0 \in X$. Let $C^{x_0} : X \to X$ be a constant map such that $C^{x_0}(x) = x_0$ for each $x \in X$. We say that a space X is multicontractible to a point x_0 in the context of multimorphisms (written $X \in \mathrm{MCN}_m$) if

$$\big[(\mathrm{Id}_X, \mathrm{Id}_X)\big]_m = \mathrm{Id}_m \sim_{HM} C^{x_0}_m = \big[(\mathrm{Id}_X, C^{x_0})\big]_m.$$

From Proposition 2.11 we get the following fact.

Proposition 2.13 (See [15]**).** *A space $X \in \mathrm{MCN}_m$ if and only if there exist a metrizable space Z and a Vietoris map $p : Z \to X$ such that $p \sim C^{x_0}_1$, where $C^{x_0}_1 : Z \to X$ is a constant map, that is, $C^{x_0}_1(z) = x_0$ for each $z \in Z$.*

Multicontractibility to a point in the context of multimorphisms is essentially more general than regular contractibility to a point (see [12, 15]).

Proposition 2.14 (See [15]**).** *If $X \in \mathrm{MCN}_m$, then X is path connected.*

Definition 2.15. Let X be an ANR and let $X_0 \subset X$ be a closed subset. We say that X_0 is movable in X provided that every neighborhood U of X_0 admits a neighborhood U' of X_0, $U' \subset U$, such that for every neighborhood U'' of X_0, $U'' \subset U$, there exists a homotopy $H \colon U' \times [0,1] \to U$ with $H(x,0) = x$ and $H(x,1) \in U''$, for any $x \in U'$.

Definition 2.16. Let X be a compact metric space. We say that X is movable provided that there exist $Z \in \mathrm{ANR}$ and an embedding $e : X \to Z$ such that $e(X)$ is movable in Z.

Let us notice that the property of being movable is an absolute property; that is, if A is a movable set in some ANR X and $j : A \to X'$ is an embedding into an ANR X', then $j(A)$ is movable in X' (see [2]).

Remark 2.17. We know that movable spaces are of the following types, among others: AR, ANR, AANR (in the sense of Clapp, see [3]), FAR (of trivial shape) and FANR (see [2]).

Definition 2.18. A map $r\colon Y \to X$ from a space Y onto a space X is said to be an mr-map if there is a map $\varphi : X \to_m Y$ such that $r \circ \varphi = \mathrm{Id}_X$.

Definition 2.19. A metric space X is called an absolute multiretract (denoted $X \in$ AMR) provided that there exist a normed space E and an mr-map $r\colon E \to X$ from E onto X.

Definition 2.20. A metric space X is called an absolute neighborhood multiretract (denoted $X \in$ ANMR) provided that there exist an open subset U of some normed space E and an mr-map $r\colon U \to X$ from U onto X.

The classes of metric spaces of types AMR and ANMR are substantially wider than the classes AR and ANR, respectively (see [11]).

Theorem 2.21 (See [5]**).** *Let U be an open subset of a normed space E and let X be a compact subset of U. Then for every $\varepsilon > 0$ there exist a finite polyhedron $K_\varepsilon \subset U$ and a mapping $i_\varepsilon : X \to U$ such that*

(1) $\|x - i_\varepsilon(x)\| < \varepsilon$ *for all* $x \in X$,
(2) $i_\varepsilon(X) \subset K_\varepsilon$,
(3) i_ε *is homotopic to* i, *where* $i : X \hookrightarrow U$ *is an inclusion.*

We recall that a metrizable space X is of finite type if almost all the homologies of X are trivial and for each $k \geq 0$,

$$\dim H_k(X) < \infty.$$

Proposition 2.22. *Let U be an open subset of a normed space E and let X be a compact subset of U. If an inclusion $i : X \hookrightarrow U$ induces a monomorphism $i_* : H_*(X) \to H_*(U)$, then X is of finite type.*

Proof. Let $\varepsilon > 0$ and let $i_\varepsilon : X \to U$ be as in Theorem 2.21. Let $d : X \to K_\varepsilon$ be a map given by the formula $d(x) = i_\varepsilon(x)$ for each $x \in X$, where K_ε is a finite polyhedron (Theorem 2.21(2)). Then we have the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{\ i\ } & U \\ \uparrow{\scriptstyle \mathrm{Id}_X} & & \uparrow{\scriptstyle j} \\ X & \xrightarrow{\ d\ } & K_\varepsilon, \end{array}$$

where $i : X \hookrightarrow U$ and $j : K_\varepsilon \hookrightarrow U$ are inclusions. We observe that from Theorem 2.21(3) we get

$$i_* = i_{\varepsilon *} = (j \circ d)_* = j_* \circ d_*.$$

From the assumption, the map i_* is a monomorphism, so d_* is a monomorphism. Hence X is of finite type and the proof is complete. □

A map $\varphi : X \to_m Y$, determined by $\varphi_m = [(p,q)]_m$, is called compact if $q : Z \to Y$ is a compact map (i.e., $\overline{q(Z)} \subset Y$ is compact). A map $\varphi : X \to_m X$ has a fixed point (written $\mathrm{Fix}(\varphi) \neq \emptyset$) if there exists a point $x \in X$ such that $x \in \varphi(x)$. We recall that a metrizable space X has a fixed point property (i.e., it is a Lefschetz space) if for each compact map $\varphi : X \to_m X$, the following condition is satisfied:

$$(\Lambda(\varphi_*) \neq 0) \Longrightarrow (\mathrm{Fix}(\varphi) \neq \emptyset)$$

provided that the generalized Lefschetz number $\Lambda(\varphi_*)$ of φ_* (see (2.1)) is well defined.

Proposition 2.23 (See [11]**).** *Let $X \in$ ANMR (resp., $X \in$ AMR). Then X has a fixed point property.*

Proposition 2.24 (See [5]**).** *Let $g : X \to Y$ be a proper map and let $\varphi_g : Y \to X$ be a multivalued map given by $\varphi_g(y) = g^{-1}(y)$ for each $y \in Y$. Then φ_g is a u.s.c. map.*

Proposition 2.25 (See [5]**).** *Let X be a compact set in a Hilbert cube Q. For any open neighborhood U of X in Q there exists a compact and locally connected set K (where $K \in$ ANR) such that $X \subset K \subset U$.*

Proposition 2.26 (See [4]**).** *Let X be a compact and locally connected space and let $f : X \to Y$ be a continuous map from X onto Y. Then Y is compact and locally connected.*

3. Relative retracts

In this section, maps determined by multimorphisms $\varphi_m \in M_m(X,Y)$ will be denoted by $\varphi : X \to_m Y$ and will be called multifunctions, while for the single-valued mappings the letters f, g, h, ... are reserved.

Definition 3.1. Let $Z \subset Y$ and let $g : Z \to X$ be a continuous map. A space X is called a g-retract of a space Y (is a retract relative to g) if there exists a continuous map $r : Y \to X$ such that the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\ r\ } & X \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle \mathrm{Id}_X} \\ Z & \xrightarrow{\ g\ } & X \end{array}$$

is commutative with $r \circ i = g$, where $i : Z \hookrightarrow Y$ is an inclusion and Id_X is an identity mapping. The map r will be called a g-retraction.

Let $h : Z \to X$ be a homeomorphism and let $Z \subset Y$. We observe that if X is h-retract of Y, then Z is a retract of Y in the sense of Borsuk. In particular, if $h = \mathrm{Id}_X$, then $X \subset Y$ is a retract of Y in the sense of Borsuk [1]. We observe also that if $g : Z \to X$ is a constant map, then X is a g-retract of Y for each metrizable space Y such that $Z \subset Y$.

Proposition 3.2. *Let $g : Z \to X$ be a proper map and let $Z \subset Y$. If X is a g-retract of Y, then Z is a closed set in Y.*

Proof. Let $(z_n) \subset Z \subset Y$ be a sequence such that $\lim_{n\to\infty} z_n = z_0$. We show that $z_0 \in Z$. It is obvious that

$$\lim_{n\to\infty} g(z_n) = \lim_{n\to\infty} r(z_n) = r(z_0),$$

where $r : Y \to X$ is a g-retraction. From the assumption,

$$g^{-1}\big((g(z_n)) \cup \{r(z_0)\}\big) \subset Z$$

is a nonempty and compact set and

$$(z_n) \subset g^{-1}\big((g(z_n)) \cup \{r(z_0)\}\big).$$

Hence $z_0 \in Z$ and the proof is complete. □

Definition 3.3. Let $g : Z \to X$ and $f_1, f_2 : X \to Y$ be continuous mappings. We say that f_1 and f_2 are g-homotopic (i.e., are homotopic relative to g, written $f_1 \sim_g f_2$, see Proposition 2.11) if there exists a homotopy

$$h : Z \times [0,1] \to Y$$

such that

$$h(\cdot, 0) = f_1 \circ g \quad \text{and} \quad h(\cdot, 1) = f_2 \circ g.$$

We observe that if g is a homeomorphism, then

$$(f_1 \sim_g f_2) \Longleftrightarrow (f_1 \sim f_2);$$

and if $f_1 \sim f_2$, then for each continuous map $g : Z \to X$,

$$f_1 \sim_g f_2.$$

Definition 3.4. Let $g : Z \to X$ be a continuous map. A space X is called g-contractible to a point (i.e., is contractible to a point relative to g) if there exists a point $x_0 \in X$ such that

$$\mathrm{Id}_X \sim_g C^{x_0},$$

where $C^{x_0} : X \to X$ is a constant map given by the formula $C^{x_0}(x) = x_0$ for each $x \in X$.

From Definition 3.3 we get (see Proposition 2.13)

$$(\mathrm{Id}_X \sim_g C^{x_0}) \Longleftrightarrow (g \sim C_1^{x_0}),$$

where $C_1^{x_0} : Z \to X$ is a constant map. Let $\mathbb{R}^{n+1}$ be a Euclidean space, let $\mathbb{K}^{n+1} \subset \mathbb{R}^{n+1}$ be a closed ball with center 0 and radius 1 and let $\mathbb{S}^n \subset \mathbb{K}^{n+1}$ be a sphere. Let $f, g : X \to Y$. We say that f and g have a point of coincidence if there exists a point $x \in X$ such that $f(x) = g(x)$.

In the following three facts we will assume that $n \geq 1$. Firstly, we prove the following theorem (see [6]).

Theorem 3.5. *Let Y be a contractible space and let $g : Z \to \mathbb{S}^n$ be a continuous map, where $Z \subset Y$ is a closed and nonempty set. The following conditions are equivalent:*

(1) $\mathbb{S}^n$ *is not g-contractible to a point;*
(2) *let* $G : Y \to \mathbb{R}^{n+1}$ *be a continuous extension of g. Every continuous map* $F : Y \to \mathbb{R}^{n+1}$ *has at least one of the following properties:*
   (i) F *and* G *have a point of coincidence,*
   (ii) *there is a point* $z \in Z$ *such that* $G(z) = \lambda F(z)$ *for some* $0 < \lambda < 1$;
(3) *let* $F : Y \to \mathbb{R}^{n+1}$ *be a continuous map such that* $F(Z) \subset \mathbb{K}^{n+1}$ *and let* $G : Y \to \mathbb{R}^{n+1}$ *be a continuous extension of g, then F and G have a point of coincidence;*
(4) $\mathbb{S}^n$ *is not a g-retract of Y.*

Proof. (1)⇒(2). Suppose that $G(y) \neq F(y)$ for each $y \in Y$ and $G(z) \neq tF(z)$ for all $0 < t < 1$, $z \in Z$. Then $G(z) \neq tF(z)$ also for $t = 0$ and, by our first hypothesis, for $t = 1$. Let $r : \mathbb{R}^{n+1}\backslash\{0\} \to \mathbb{S}^n$ be a retraction. We define a homotopy $H : Z \times [0,1] \to \mathbb{S}^n$ by the formula

$$H(x,t) = \begin{cases} r\big(G(x) - 2tF(x)\big) & \text{for } 0 \leq t \leq 1/2, \\ r\big(G(h(x,t)) - F(h(x,t))\big) & \text{for } 1/2 \leq t \leq 1, \end{cases}$$

where $h : Y \times [1/2, 1] \to Y$ is a homotopy such that

$$h(x, 1/2) = x \quad \text{and} \quad h(x,1) = y_0$$

for any $x \in Y$. We observe that

$$\begin{aligned} H(x,0) &= r(G(x)) = r(g(x)) = g(x), \\ H(x,1) &= r\big(G(y_0) - F(y_0)\big) \in \mathbb{S}^n. \end{aligned}$$

Hence $\mathbb{S}^n$ is g-contractible to a point, but this contradicts the assumption.

(2)⇒(3). The second possibility in (2) cannot occur since $F(Z) \subset \mathbb{K}^{n+1}$.

(3)⇒(4). Assume that there exists a g-retraction $r : Y \to \mathbb{S}^n$. Then from the definition, r is an extension of g. Let $G = r$ and let $F : Y \to \mathbb{S}^n \subset \mathbb{K}^{n+1}$ be a map given by the formula $F(y) = -G(y)$ for each $y \in Y$. Then G and F have no point of coincidence.

(4)⇒(1). Assume that $\mathbb{S}^n$ is g-contractible to a point. Then there exists a homotopy $h : Z \times [0,1] \to \mathbb{S}^n$, $h(z,0) = g(z)$ and $h(z,1) = d_0 \in \mathbb{S}^n$ for each $z \in Z$. From the theorem on extension of homotopy (see [1]) there exists a continuous extension $H : Y \times [0,1] \to \mathbb{S}^n$ of h such that $H(y,1) = d_0$ for any $y \in Y$. Let $U \subset Y$ be an open set such that $Z \subset U$. We define an Urysohn function $\lambda : Y \to [0,1]$ such that $\lambda(y) = 1$ for $y \in Y\backslash U$ and $\lambda(y) = 0$ for $y \in Z$. Let $r : Y \to \mathbb{S}^n$ be a map given by the formula

$$r(y) = H(y, \lambda(y)) \quad \text{for each } y \in Y.$$

Then r is a g-retraction and the proof is complete. □

Proposition 3.6. *Let Y be an acyclic space (in particular, a contractible space) and let* $g : Z \to \mathbb{S}^n$ *be a continuous map such that* $g_* : H_*(Z) \to H_*(\mathbb{S}^n)$ *is an epimorphism, where* $Z \subset Y$ *is a nonempty set. Then* $\mathbb{S}^n$ *is not a g-retract of Y.*

Proof. Assume to the contrary that $\mathbb{S}^n$ is a g-retract of Y. Then there exists a g-retraction $r : Y \to \mathbb{S}^n$ such that $r \circ i = g$, where $i : Z \hookrightarrow Y$ is an inclusion. We have

$$H_*(Z) \xrightarrow{\ i_*\ } H_*(Y) \xrightarrow{\ r_*\ } H_*(\mathbb{S}^n).$$

Hence for $k \geq 1$, the map $r_{k*} \circ i_{k*} = g_{k*}$ is a zero homomorphism, but this is a contradiction. □

We recall that a continuous map $f : X \to Y$ is universal if for each continuous map $g : X \to Y$, f and g have a point of coincidence (see [7, 8]). Due to Proposition 3.6, Theorem 3.5 is true for a wide class of continuous mappings $g : Z \to \mathbb{S}^n$ inducing epimorphisms on homologies. From Theorem 3.5(3) and Proposition 3.6, the following fact results.

Proposition 3.7. *Let Y be a contractible space and let $G : Y \to \mathbb{K}^{n+1}$ be a continuous map. Assume that there exists a closed and nonempty set $Z \subset Y$ such that $G(Z) \subset \mathbb{S}^n$ and the map $g : Z \to \mathbb{S}^n$, given by the formula $g(x) = G(x)$ for each $x \in Z$, induces an epimorphism $g_* : H_*(Z) \to H_*(\mathbb{S}^n)$. Then G is a universal map.*

We will introduce some definitions. Let $\Re$ denote the set of all metric spaces and let

$$C(\Re) = \{g : Y \to X;\ g \text{ is continuous, } X, Y \in \Re\},$$
$$\Phi(Y, X) = \{g : Y \to X;\ g \text{ is continuous}\},$$
$$\mathbb{H}(Y, X) = \{g : Y \to X;\ g \text{ is a homeomorphism}\}.$$

Let $\Theta \subset C(\Re)$ be a set such that for any $X, Y, Z \in \Re$ the following conditions are satisfied:

$$\mathbb{H}(Y, X) \subset \Theta(Y, X), \tag{3.1}$$
$$(h \in \mathbb{H}(Y, X) \text{ and } g \in \Theta(X, Z)) \Longrightarrow ((g \circ h) \in \Theta(Y, Z)). \tag{3.2}$$

If condition (3.2) replaces the following, stronger, condition:

$$(f \in \Theta(Y, X) \text{ and } g \in \Theta(X, Z)) \Longrightarrow ((g \circ f) \in \Theta(Y, Z)), \tag{3.3}$$

then instead of Θ we write Θ^S. But if the set Θ in addition to conditions (3.1) and (3.2), satisfy the following condition:

$$\text{for each } g \in \Theta(Z, X) \text{ and for each open set } U \subset X \text{ there exists } Z_U \subset Z \tag{3.4}$$

such that $g(Z_U) \subset U$, $g_{Z_U} \in \Theta(Z_U, U)$, where $g_{Z_U}(z) = g(z)$ for each $z \in Z_U$, then instead of Θ we write Θ^L. Let $X, Y \in \Re$. Let us denote

$$\Theta(X) = \{g \in \Phi(Z, X);\ Z \in \Re\},$$
$$\Theta_Y(X) = \{g \in \Phi(Z, X);\ Z \subset Y\}.$$

We can now introduce the following concepts.

Definition 3.8. Let $Z \subset Y$ and let $g : Z \to X$ be a continuous map. A space Z will be called a g-carrier of X in Y (written $Z \in C_Y(X, g)$) if X is a g-retract of Y.

It is easy to observe that X is a g-retract of Y (see Definition 3.1) if and only if there exists $Z \in C_Y(X, g)$. It is obvious that $X \subset Y$ is a retract of Y if and only if $X \in C_Y(X, \mathrm{Id}_X)$.

Definition 3.9. A space X is called a $\Theta_Y(X)$-retract of a space Y (i.e., is a retract relative to the set $\Theta_Y(X)$) if there exist a space $Z \subset Y$ and a map $g : Z \to X$, $g \in \Theta_Y(X)$, such that $Z \in C_Y(X, g)$.

Definition 3.10. We say that X is an absolute relative retract (written $X \in \mathrm{ARR}(\Theta)$) if there exists a space Z such that for each space T and for each closed embedding $h : Z \to T$ there exists $g \in \Theta_T(X)$ such that $h(Z) \in C_T(X, g)$. The space Z will be called an absolute carrier of X relative to the set Θ (written $Z \in \mathrm{AC}(X, \Theta)$).

Let

$$\mathbb{H} = \{g \in C(\Re);\ g \text{ is a homeomorphism}\}.$$

It is clear that

$$X \in \mathrm{ARR}(\mathbb{H}) \Longleftrightarrow X \in \mathrm{AR}$$

and

$$X \in \mathrm{ARR}(\Theta) \Longleftrightarrow \mathrm{AC}(X, \Theta) \neq \emptyset.$$

Definition 3.11. We say that X is an absolute neighborhood relative retract (written $X \in \mathrm{ANRR}(\Theta)$) if there exists a space Z such that for each space T and for each closed embedding $h : Z \to T$ there exist an open set $V \subset T$ and $g \in \Theta_V(X)$ such that $h(Z) \subset V$ and $h(Z) \in C_V(X, g)$. The space Z will be called an absolute neighborhood carrier of X relative to the set Θ (written $Z \in \mathrm{ANC}(X, \Theta)$).

We observe that

$$X \in \mathrm{ANRR}(\mathbb{H}) \Longleftrightarrow X \in \mathrm{ANR}$$

and

$$X \in \mathrm{ANRR}(\Theta) \Longleftrightarrow \mathrm{ANC}(X, \Theta) \neq \emptyset.$$

Definition 3.12. Let $f_1, f_2 : X \to Y$ be continuous maps. We say that f_1 and f_2 are Θ-homotopic (i.e., are homotopic relative to the set $\Theta(X)$, written $f_1 \sim_\Theta f_2$, see Definition 3.3) if there exists a map $g \in \Theta(X)$, $g : Z \to X$ such that

$$f_1 \sim_g f_2.$$

Let $\Theta(X) = \mathbb{H}(X)$. Then

$$(f_1 \sim_\Theta f_2) \Longleftrightarrow (f_1 \sim f_2).$$

Let

$$\mathbb{V} = \{g \in C(\Re);\ g \text{ is a Vietoris map}\}.$$

From Propositions 2.9 and 2.11 it results that a homotopy $\sim_\mathbb{V}$ is an equivalence relation.

Definition 3.13. We say that X is Θ-contractible to a point (i.e., is contractible to a point relative to the set $\Theta(X)$, see Definition 3.4) if there exists $g \in \Theta(X)$, $g : Z \to X$ such that X is g-contractible to a point.

We observe that X is $\mathbb{H}$-contractible to a point if and only if it is contractible to this point. We shall give an example of X such that it is $\mathbb{V}$-contractible and not contractible to a point. Recall that if X is contractible to a point, then X is movable (see Definition 2.15).

Example 3.14. Let X be a nonmovable and compact space such that there exists a Vietoris map $p : Q \to X$, where Q is a Hilbert cube (see [9]). The space X is nonmovable, so X is not contractible to a point. Let $x_0 \in X$ and let $p(z_0) = x_0$. We define a homotopy $h : Q \times [0,1] \to X$ by the formula

$$h(z,t) = p\big((1-t)z + tz_0\big)$$

for each $(z,t) \in Q \times [0,1]$. Hence X is $\mathbb{V}$-contractible to a point.

Definition 3.15 (See [14]). We say that a space X is locally Θ-contractible if for each $x \in X$ and for each open neighborhood $U \subset X$ of x there exist an open neighborhood $V \subset U$ of x and a map $g_V : Z_V \to V$, $g_V \in \Theta(V)$ such that for each open neighborhood $W \subset U$ of x there exists a continuous map $C^W : Z_V \to W$ such that

$$(i_V \circ g_V) \sim \left(i_W \circ C^W\right),$$

where

$$Z_V \xrightarrow{g_V} V \xrightarrow{i_V} U, \qquad Z_V \xrightarrow{C^W} W \xrightarrow{i_W} U$$

and $i_V : V \hookrightarrow U$, $i_W : W \hookrightarrow U$ are inclusions.

It should be noticed that if it is assumed in Definition 3.15 that for some W the map C^W is constant, i.e., $C^W(z) = x \in W \subset U$ for every $z \in Z_V$, then

$$\big((i_V \circ g_V) \sim \left(i_W \circ C^W\right)\big) \Longleftrightarrow \left(i_V \sim_\Theta C^x\right),$$

where $C^x : V \to U$ is a constant map such that $C^x(y) = x$ for each $y \in V$. If in addition we assume that $Z_V = V$ and $g_V = \mathrm{Id}_V$, then we get a local contractibility. We show the following facts.

Proposition 3.16. *Let X be a metrizable space. Then the following conditions are satisfied:*

1. $(X \in \mathrm{ARR}(\Theta)) \Leftrightarrow (X$ *is a* $\Theta_E(X)$*-retract of* $E)$*, where E is some normed space,*
2. $(X \in \mathrm{ANRR}(\Theta)) \Leftrightarrow (X$ *is a* $\Theta_U(X)$*-retract of* $U)$*, where U is some open set in some normed space E.*

Proof. We show condition (2). The proof of condition (1) is analogous. It is obvious that if $X \in \mathrm{ANRR}(\Theta)$, then X in particular is a $\Theta_U(X)$-retract of some open set $U \subset E$ in some normed space E (see Definition 3.11). Assume now that there exist a normed space E and an open set $U \subset E$ such that X is a $\Theta_U(X)$-retract of U. Then we get a space $Z \subset U$, a map $g : Z \to X$,

$g \in \Theta_U(X)$, and a map $r : U \to X$ such that $r \circ i = g$, where $i : Z \hookrightarrow U$ is an inclusion. We have the following diagram:

$$X \xleftarrow{\;g\;} Z \xrightarrow{\;i\;} U \xrightarrow{\;r\;} X.$$

Let $h : Z \to T$ be a closed embedding, where T is some metrizable space, and let $f : h(Z) \to U$ be a map given by the formula

$$f = i \circ h^{-1}.$$

Since $U \in \mathrm{ANR}$, then f has a continuous extension $F : V \to U$, where $V \subset T$ is an open set such that $h(Z) \subset V$. Let $R = r \circ F$. Then we have the diagram

$$X \xleftarrow{\;g\;} Z \xleftarrow{\;h^{-1}\;} h(Z) \xrightarrow{\;j\;} V \xrightarrow{\;R\;} X,$$

where j is an inclusion. We observe that

$$R \circ j = (r \circ F) \circ j = r \circ (F \circ j) = r \circ \big(i \circ h^{-1}\big) = (r \circ i) \circ h^{-1} = g \circ h^{-1}.$$

Hence $h(Z) \in C_V(X, g \circ h^{-1})$ (see Definition 3.11), where $(g \circ h^{-1}) \in \Theta_V(X)$ and the proof is complete. □

Proposition 3.17. *Let $g \in \Theta(X)$ and $g : Z \to X$. Then the following conditions are satisfied:*

(1) $(Z \in \mathrm{AR}) \Rightarrow (X \in \mathrm{ARR}(\Theta))$,
(2) $(Z \in \mathrm{ANR}) \Rightarrow (X \in \mathrm{ANRR}(\Theta))$.

Proof. We show condition (2). The proof of condition (1) is analogous. Let $Z \in \mathrm{ANR}$. Then there exist a normed space E, an open set $U \subset E$, a closed embedding $h : Z \to E$ such that $h(Z) \subset U$ and a continuous map $r' : U \to h(Z)$ such that $r'(y) = y$ for each $y \in h(Z)$. Let $r : U \to X$ be a map given by the formula

$$r = g \circ h^{-1} \circ r'.$$

The map r is a $(g \circ h^{-1})$-retraction, where $(g \circ h^{-1}) \in \Theta_U(X)$. From Proposition 3.16, $X \in \mathrm{ANRR}(\Theta)$ and the proof is complete. □

Proposition 3.18. *Let $X \in \mathrm{ARR}(\Theta)$. Then X is Θ-contractible to a point.*

Proof. Let $X \in \mathrm{ARR}(\Theta)$. From Proposition 3.16 and Definition 3.10 there exist a normed space E, a space $Z \subset E$, a continuous map $r : E \to X$ such that $r \circ i = g$, where $i : Z \hookrightarrow E$ is an inclusion, and $g : Z \to X$, $g \in \Theta_E(X)$. Let $z_0 \in E$. We define a homotopy $h : Z \times [0,1] \to X$ by the formula

$$h(z,t) = r\big((1-t)z + tz_0\big) \quad \text{for each } z \in Z,$$

and the proof is complete. □

Proposition 3.19. *Let $X \in \mathrm{ANRR}(\Theta^S)$. Assume that there exists*

$$Z \in \mathrm{ANC}\left(X, \Theta^S\right)$$

(see Definition 3.11*) such that Z is Θ^S-contractible to a point. Then*

$$X \in \mathrm{ARR}\left(\Theta^S\right).$$

Proof. From Proposition 3.16 we get

$$X \xleftarrow{\ g\ } Z \xrightarrow{\ i\ } U \xrightarrow{\ r'\ } X,$$

where $g : Z \to X$, $g \in \Theta_U^S(X)$, E is a normed space, $U \subset E$ is open in E and $r' \circ i = g$. The space Z is Θ^S-contractible, so there exists a homotopy $h : P \times [0,1] \to Z \subset U$ such that

$$h(x,0) = g'(x) \quad \text{and} \quad h(x,1) = z_0$$

for each $x \in P$ and for some point $z_0 \in Z$, where $g' \in \Theta^S(Z)$. The homotopy h has a continuous extension $H : E' \times [0,1] \to U$ (since $U \in$ ANR, see [1]), where $H(x,1) = z_0$ for all $x \in E'$ and E' is a normed space such that $P \subset E'$. We define a map $r : E' \to X$ by the formula

$$r(x) = r'\big(H(x,0)\big) \quad \text{for each } x \in E'.$$

We observe that if $x \in P$, then

$$r(x) = r'\big(h(x,0)\big) = r'\big(g'(x)\big) = g\big(g'(x)\big).$$

From Proposition 3.16, $X \in \text{ARR}(\Theta^S)$, because $(g \circ g') \in \Theta_{E'}^S(X)$ and the proof is complete. □

Let $f : X \to Y$ be a map and let $A \subset X$ be a nonempty set. We denote by $f_A : A \to Y$ a map given by the formula

$$f_A(x) = f(x) \quad \text{for all } x \in A.$$

Proposition 3.20. *Let $X \in \text{ANRR}(\Theta^L)$ and let $U \subset X$ be an open and nonempty set. Then $U \in \text{ANRR}(\Theta^L)$.*

Proof. From Proposition 3.16 there exist a normed space E and an open set $V \subset E$ such that X is a $(\Theta^L)_V(X)$-retract of V. We have the following diagram:

$$X \xleftarrow{\ g\ } Z \xrightarrow{\ i\ } V \xrightarrow{\ r\ } X,$$

where $g \in (\Theta^L)_V(X)$, $i : Z \hookrightarrow V$ is an inclusion and r is a g-retraction, that is, $r \circ i = g$. Let $U \subset X$ be an open and nonempty set. From (3.4), it results that there exists $Z_U \subset Z$ such that $g(Z_U) \subset U$ and $g_{Z_U} \in \Theta^L(Z_U, U)$. We observe that $g_{Z_U} \in (\Theta^L)_{r^{-1}(U)}(U)$. We have

$$U \xleftarrow{\ g_{Z_U}\ } Z_U \xrightarrow{\ i_{Z_U}\ } r^{-1}(U) \xrightarrow{\ r_{r^{-1}(U)}\ } U,$$

where $r_{r^{-1}(U)} \circ i_{Z_U} = g_{Z_U}$, because for each $x \in X$, $g^{-1}(x) \subset r^{-1}(x)$, and the proof is complete. □

Analyzing the properties contained in Propositions 3.16–3.20 we get a similar property to the properties of Borsuk retracts.

We recall that if $p : X \to Y$ is a Vietoris map, then $p_* : H_*(X) \to H_*(Y)$ is an isomorphism. We say that a proper map $g : X \to Y$ is movable if for each $y \in Y$, the set $g^{-1}(y)$ is movable (see Definition 2.15). We recall that a proper map $f : X \to Y$ is cell-like if for each $y \in Y$, the set $f^{-1}(y)$ has a trivial shape in the sense of Borsuk [2]. We know that if $f : X \to Y$ is a cell-like map, then it is a Vietoris and movable map. A continuous mapping

$f : X \to Y$ will be called multi-right-invertible if there exists a multivalued u.s.c. map $\varphi : Y \multimap X$ (a multi-right-inverse map) such that

$$x \in \varphi\big(f(x)\big) \quad \text{for each } x \in X. \tag{3.5}$$

Remark 3.21. We observe that if $f : X \to Y$ is multi-right-invertible and the space Y is compact, then the space X is compact. A multi-right-inverse map $\varphi : Y \to X$ is u.s.c, so $\varphi(Y) = X$ is a compact space. Let X and Y be compact spaces and let $f : X \to Y$ be a continuous map. Then f is multi-right-invertible. Indeed, let $\varphi : Y \multimap X$ be a map given by the formula $\varphi(y) = X$ for all $y \in Y$, then it is obvious that condition (3.5) is satisfied.

We will need the following two facts.

Proposition 3.22. *Let $f : X \to Y$ be a proper map. Then f is multi-right-invertible.*

Proof. Let $\varphi : Y \multimap X$ be a multivalued map given by the formula

$$\varphi(y) = f^{-1}(y) \quad \text{for each } y \in Y.$$

Then from Proposition 2.24, φ is u.s.c. and condition (3.5) is satisfied. □

Proposition 3.23. *Let $f : X \to Y$ and $g : Y \to Z$ be multi-right-invertible maps. Then $g \circ f : X \to Z$ is a multi-right-invertible map.*

Proof. There exist multi-right-inverse maps $\varphi : Y \multimap X$ and $\psi : Z \multimap Y$ such that $x \in \varphi(f(x))$ for each $x \in X$ and $y \in \psi(g(y))$ for each $y \in Y$. Let $\eta : Z \multimap X$ be a multivalued map given by the formula

$$\eta = \varphi \circ \psi.$$

Then $f(x) \in \psi(g(f(x)))$ for each $x \in X$. Hence for each $x \in X$,

$$x \in \varphi\big(f(x)\big) \subset \varphi\big(\psi(g(f(x)))\big) = \eta\big(g(f(x))\big),$$

and the proof is complete. □

We will denote

$$\begin{aligned}
\mathbb{I} &= \{g \in C(\Re);\ g \text{ is multi-right-invertible and } g_* \text{ is an isomorphism}\},\\
\mathbb{P} &= \{g \in C(\Re);\ g \text{ is proper}\},\\
\mathbb{M} &= \{g \in C(\Re);\ g \text{ is movable}\},\\
\mathbb{V} &= \{g \in C(\Re);\ g \text{ is a Vietoris map}\},\\
\mathbb{H} &= \{g \in C(\Re);\ g \text{ is a homeomorphism}\},\\
\mathbb{CE} &= \{g \in C(\Re);\ g \text{ is a cell-like map}\},\\
\mathbb{IP} &= \mathbb{I} \cap \mathbb{P}, \quad \mathbb{IM} = \mathbb{I} \cap \mathbb{M}, \quad \mathbb{MV} = \mathbb{M} \cap \mathbb{V}.
\end{aligned}$$

We observe that the sets $\mathbb{H}$, $\mathbb{V}$, $\mathbb{P}$, $\mathbb{I}$ and $\mathbb{IP}$ satisfy conditions (3.1) and (3.3) (see Propositions 3.22 and 3.23), whereas the sets $\mathbb{M}$ and $\mathbb{CE}$ satisfy conditions (3.1) and (3.2). Note also that the sets $\mathbb{H}$, $\mathbb{V}$, $\mathbb{P}$, $\mathbb{M}$, $\mathbb{MV}$ and $\mathbb{CE}$ satisfy conditions (3.4) and

$$\mathbb{H} \subset \mathbb{CE} \subset \mathbb{MV} \subset \mathbb{IM} \subset \mathbb{IP} \subset \mathbb{I}, \quad \mathbb{V} \subset \mathbb{IP}, \quad \mathbb{M} \subset \mathbb{P}. \tag{3.6}$$

Now a few interesting properties of retracts relative to the sets (3.6) will be proven. Let Q be a Hilbert cube. In a similar way as Proposition 3.16 one can prove the following fact (see Remark 3.21).

Proposition 3.24. *Let X be a compact space. Then the following conditions are satisfied:*

(1) $(X \in \mathrm{ARR}(\mathbb{I})) \Leftrightarrow (X$ *is an* $\mathbb{I}_Q(X)$*-retract of* $Q)$,
(2) $(X \in \mathrm{ANRR}(\mathbb{I})) \Leftrightarrow (X$ *is an* $\mathbb{I}_U(X)$*-retract of* $U)$*, where U is some open set in Q.*

Proposition 3.25. *Let Y be a compact space. Then*

$$(Y \in \mathrm{ARR}(\mathbb{I})) \Longleftrightarrow (Y \textit{ is an acyclic space}).$$

Proof. Let $Y \in \mathrm{ARR}(\mathbb{I})$. Then Y is an $\mathbb{I}_Q(Y)$-retract of Q (see Proposition 3.24), where Q is a Hilbert cube. Hence there exist a map $g \in \mathbb{I}_Q(Y)$, $g : Z \to Y$, and a g-retraction $r : Q \to Y$ such that $r \circ i = g$, where $i : Z \hookrightarrow Q$ is an inclusion. We have

$$r_* \circ i_* = g_*$$

and from the assumption, g_* is an isomorphism. Hence $i_* : H_*(Z) \to H_*(Q)$ is a monomorphism. The space Q is acyclic, so Z is an acyclic space. The map g_* is an isomorphism, hence Y is acyclic. Assume now that Y is an acyclic space. Let $Z = Y$ and let T be a metrizable space. Assume that $h : Z \to T$ is a closed embedding. Let $g : Z \to Y$ be a constant map, $g(x) = y_0$ for any $x \in Z$, where $y_0 \in Y$ is an arbitrary point. The map g induces an isomorphism $g_* : H_*(Z) \to H_*(Y)$, so $g \in \mathbb{I}(Y)$ (see Remark 3.21). We define a $g \circ h^{-1}$-retraction $r : T \to Y$ (where $(g \circ h^{-1}) \in \mathbb{I}_T(Y)$, h^{-1} is an inverse homeomorphism) by the formula $r(x) = y_0$ for each $x \in T$, and the proof is complete. □

From Proposition 3.18 it results that if $X \in \mathrm{ARR}(\mathbb{P})$, then X is $\mathbb{P}$-contractible.

Proposition 3.26. *Let X be $\mathbb{P}$-contractible. Then X is path connected.*

Proof. Let $x_1 \in X$. By Definition 3.13 we get a map $g \in \mathbb{P}(X)$, $g : Z \to X$, a homotopy $h : Z \times [0, 1] \to X$ such that $h(z, 0) = g(z)$ and $h(z, 1) = x_0$ for each $z \in Z$ and for some point $x_0 \in X$. We can assume that $x_0 \neq x_1$. Define a path $d : [0, 1] \to X$ by the formula

$$d(t) = h(z_1, t),$$

where $z_1 \in Z$ is such that $g(z_1) = x_1$ and the proof is complete. □

Proposition 3.27 (See [14]). *Let $X \in \mathrm{ANRR}(\mathbb{M})$. Then X is locally $\mathbb{M}$-contractible.*

Proof. From the assumption, we have the following diagram:

$$X \xleftarrow{\;g\;} Z \xrightarrow{\;i\;} U \xrightarrow{\;r\;} X,$$

where $U \subset E$ is an open set in a normed space E, $g \in \mathbb{M}_U(X)$, $i : Z \hookrightarrow U$ is an inclusion, r is continuous such that $r \circ i = g$. Let $x_0 \in X$ and let $V \subset X$ be an open set such that $x_0 \in V$. Then we get

$$g^{-1}(x_0) \subset V_1 = r^{-1}(V).$$

From the assumption, the set $g^{-1}(x_0)$ (see Definitions 2.15 and 2.16) is movable, so there exists an open set $V_2 \subset X$ such that

$$g^{-1}(x_0) \subset V_2 \subset V_1$$

and for any open set $V_3 \subset X$ such that

$$g^{-1}(x_0) \subset V_3 \subset V_1$$

there exists a homotopy $H_{V_3} : V_2 \times [0,1] \to V_1$ such that

$$H_{V_3}(x,0) = x, \;\; H_{V_3}(x,1) \in V_3 \quad \text{for each } x \in V_2.$$

From Proposition 2.24 there exists an open set $W \subset X$ such that

$$x_0 \in W \subset V \quad \text{and} \quad g^{-1}(W) \subset V_2.$$

Take any open set $W_1 \subset X$ such that $x_0 \in W_1 \subset V$. Let $W_2 = r^{-1}(W_1)$. It is obvious that $g^{-1}(x_0) \subset W_2$. Let $Z_W = g^{-1}(W)$. We denote by $g_W : Z_W \to W$ the map given by the formula

$$g_W(z) = g(z) \quad \text{for each } z \in Z_W.$$

It is clear that $g_W \in \mathbb{M}(W)$. We define a homotopy $H : Z_W \times [0,1] \to V$ by the formula

$$H(z,t) = r\big(H_{W_2}(z,t)\big) \quad \text{for each } (z,t) \in Z_W \times [0,1].$$

We have

$$H(z,0) = r\big(H_{W_2}(z,0)\big) = r(z) = g(z) = (i_W \circ g_W)(z) \quad \text{for all } z \in Z_W,$$

where $i_W : W \hookrightarrow V$ is an inclusion and

$$H(z,1) = r\big(H_{W_2}(z,1)\big) \in r(W_2) = W_1.$$

Let $C^{W_1} : Z_W \to W_1$ be a map given by the formula

$$C^{W_1}(z) = H(z,1) \quad \text{for each } z \in Z_W.$$

Let $i_{W_1} : W_1 \hookrightarrow V$ be an inclusion. We have the following diagrams:

$$Z_W \xrightarrow{g_W} W \xrightarrow{i_W} V, \qquad Z_W \xrightarrow{C^{W_1}} W_1 \xrightarrow{i_{W_1}} V.$$

Hence the proof is complete. □

Proposition 3.28 (See [14]**).** *If X is locally $\mathbb{P}$-contractible (in particular, locally $\mathbb{M}$-contractible), then X is locally path connected.*

Proof. Let $x_0 \in X$ and let $(t_n) \subset \mathbb{R}$ be an increasing sequence such that $\lim_{n\to\infty} t_n = 1$ and $0 \le t_n < 1$ for each $n \ge 1$. Let $(r_n) \subset \mathbb{R}$ be a decreasing sequence such that $\lim_{n\to\infty} r_n = 0$ and $1 \ge r_n > 0$ for each $n \ge 1$. Let $r_0 = 1$ and $t_0 = 0$. From the assumption, we can construct a family of open balls $\{B_{n-1} = B(y_0, r_{n-1})\}_{n\ge1}$ with center $x_0 \in X$ and radius r_{n-1} such that for $n \ge 1$ there exists a homotopy

$$H_n : Z_n \times [t_{n-1}, t_n] \to B_{n-1}$$

such that

$$H_n(z, t_{n-1}) = g_n(z), \quad H_n(z, t_n) \in B_{n+1} \quad \text{for all } z \in Z_n,$$

where $g_n : Z_n \to B_n$, $g_n \in \mathbb{P}(B_n)$. We take a sequence $(z_n) \subset Z_n$ such that

$$g_n(z_n) = y_n = H_{n-1}(z_{n-1}, t_{n-1})$$

for any $n \ge 2$ and $g_1(z_1) = x_1$, where $x_1 \in B_1$ is an arbitrary point. We define a path $d : [0,1] \to Y$ joining the points x_1 and x_0 by the formula

$$d(t) = \begin{cases} H_n(z_n, t) & \text{if } t \in [t_{n-1}, t_n],\ n \ge 1, \\ x_0 & \text{if } t = 1. \end{cases}$$

It is clear that the path d is well defined. We observe that for each $n \ge 1$,

$$H_n\big(\{z_n\} \times [t_{n-1}, t_n]\big) \subset B_{n-1} \tag{3.7}$$

and

$$H_n(z_n, t_n) = H_{n+1}(z_{n+1}, t_n). \tag{3.8}$$

For points $t \in [0, 1)$ the continuity of d follows from (3.8) and for $t = 1$ the continuity of d follows from (3.7). Hence the path d is continuous, and the proof is complete. □

We recall that a metrizable space X is of finite type if almost all the homologies of X are trivial and for each $k \ge 0$,

$$\dim H_k(X) < \infty.$$

Proposition 3.29. *Let X be a compact space. Assume that $X \in \mathrm{ANRR}(\mathbb{I})$. Then X is of finite type.*

Proof. From Proposition 3.16 there exist a normed space E and an open set $U \subset E$ such that X is an $\mathbb{I}_U(X)$-retract of U. Hence there exist a map $g \in \mathbb{I}_U(X)$, $g : Z \to X$, and a g-retraction $r : U \to X$ such that $r \circ i = g$, where $i : Z \hookrightarrow U$ is an inclusion. From Remark 3.21, Z is a compact space. We have

$$r_* \circ i_* = g_*.$$

From the assumption, g_* is an isomorphism, so i_* is a monomorphism. By Proposition 2.22, Z is of finite type, so X is also of finite type. □

Proposition 3.30. *Let X be a metrizable space. Then the following conditions are satisfied:*

1. $(X \in \mathrm{AMR}) \Leftrightarrow (X \in \mathrm{ARR}(\mathbb{V}))$,
2. $(X \in \mathrm{ANMR}) \Leftrightarrow (X \in \mathrm{ANRR}(\mathbb{V}))$.

Proof. We show condition (2). The proof of condition (1) is analogous. Let $X \in \mathrm{ANMR}$. Then from Definitions 2.18 and 2.20 there exist a normed space E, an open set $U \subset E$, a multifunction $\varphi : X \to_m U$ determined by $\varphi_m = [(p,q)]_m$ (see Definition 2.6) and a continuous map $r : U \to X$ such that

$$r(\varphi(x)) = r(q(p^{-1}(x))) = \{x\} \quad \text{for each } x \in X. \tag{3.9}$$

We have the following diagram:

$$X \xleftarrow{\ p\ } Z \xrightarrow{\ q\ } U \xrightarrow{\ r\ } X.$$

It is clear that $r \circ q = p$ (see (3.9)). Let T be a metrizable space and let $h : Z \to T$ be a closed embedding. The map $f : h(Z) \to U$ given by the formula $f = q \circ h^{-1}$ has a continuous extension $F : U' \to U$ (since $U \in \mathrm{ANR}$), where $U' \subset T$ is an open set such that $h(Z) \subset U'$ and h^{-1} is an inverse homeomorphism. We define a map $R : U' \to X$ by the formula $R = r \circ F$. Then we get the diagram

$$X \xleftarrow{\ p\ } Z \xleftarrow{\ h^{-1}\ } h(Z) \xrightarrow{\ i\ } U' \xrightarrow{\ R\ } X,$$

where i is an inclusion. We have

$$R \circ i = (r \circ F) \circ i = r \circ (F \circ i) = r \circ \left(q \circ h^{-1}\right) = (r \circ q) \circ h^{-1} = p \circ h^{-1}.$$

Hence $X \in \mathrm{ANRR}(\mathbb{V})$, because $h(Z) \in C_{U'}(X, p \circ h^{-1})$ (see Definition 3.11), where $(p \circ h^{-1}) \in \mathbb{V}_{U'}(X)$. Assume now that $X \in \mathrm{ANRR}(\mathbb{V})$. Let $Z \in \mathrm{ANC}(X, \mathbb{V})$ (see Definition 3.11). From the Arens–Eels theorem there exist a normed space E and closed embedding $h : Z \to E$. Then there exist an open set $U' \subset E$ such that $h(Z) \subset U'$, a Vietoris map $p' : h(Z) \to X$, $p' \in \mathbb{V}_{U'}(X)$, and a continuous map $r' : U' \to X$ such that $r' \circ i = p'$, where $i : h(Z) \hookrightarrow U'$ is an inclusion. We define

$$p = p' \circ h, \quad q = i \circ h, \quad r = r'.$$

Then we get $r \circ q = p$. Hence $r(\varphi(x)) = \{x\}$ for each $x \in X$, where $\varphi : X \to_m U'$ is a multifunction determined by $\varphi_m = [(p,q)]_m$, and the proof is complete. □

From Propositions 2.23 and 3.30 we get the following fact.

Proposition 3.31. *Let* $X \in \mathrm{ANRR}(\mathbb{V})$. *Then* X *has a fixed point property.*

Let Q be a Hilbert cube. At the end, let us analyze the following examples.

Example 3.32. Let $p : Q \to X$ be a cell-like map (in particular, a Vietoris map), where X is a nonmovable space (see [9]). From Proposition 3.17, $X \in \mathrm{ARR}(\mathbb{CE}) \subset \mathrm{ARR}(\mathbb{MV})$, but from Remark 2.17, $X \notin \mathrm{ANR}$ and $X \notin \mathrm{FANR}$.

Let $\mathbb{R}^2$ be a plane and let $\mathbb{S}^1 \subset \mathbb{R}^2$ be a circle with center $(0, 0)$ and radius 1.

Example 3.33. Let $f : [1, \infty] \to \mathbb{R}^2$ be a map given by the formula

$$f(t) = \big((1 + e^{-t})\cos(t), (1 + e^{-t})\sin(t)\big) \quad \text{for all } t \in [1, \infty],$$

and let

$$X = f([1, \infty]) \cup \mathbb{S}^1.$$

We know that the set $X \subset \mathbb{R}^2$ is compact, connected and $X \in$ FANR. We show that $X \in$ ANRR($\mathbb{I}$). Let $i : \mathbb{S}^1 \hookrightarrow X$ be an inclusion and let

$$X_n = f([1, \infty]) \cup O_{1/n}(\mathbb{S}^1) \quad \text{for each } n,$$

where

$$O_{1/n}(\mathbb{S}^1) = \{x \in \mathbb{R}^2;\ \text{there exists } y \in \mathbb{S}^1 \text{ such that } \|x - y\| \le 1/n\}.$$

It is obvious that

$$\bigcap_{n=1}^{\infty} X_n = X.$$

We observe that for each n, the inclusion $i_n : \mathbb{S}^1 \hookrightarrow X_n$ induces an isomorphism

$$i_{n*} : H_*(\mathbb{S}^1) \to H_*(X_n).$$

Indeed, for any n, let $r_n : X_n \to \mathbb{S}^1$ be a map given by the formula $r_n(x) = x/\|x\|$ for each $x \in X_n$. Then for each n we have

$$r_n \circ i_n = \mathrm{Id}_{\mathbb{S}^1}, \quad i_n \circ r_n \sim \mathrm{Id}_{X_n}.$$

Let $\{H_*(\mathbb{S}^1), \mathrm{Id}_{H_*(\mathbb{S}^1)}\}$ and $\{H_*(X_n), j^n_{(n+1)*}\}$ be inverse systems, where

$$j^n_{n+1} : X_{n+1} \hookrightarrow X_n$$

is an inclusion for each n. For any n we have a commutative diagram

$$\begin{array}{ccc} H_*(\mathbb{S}^1) & \xrightarrow{i_{n*}} & H_*(X_n) \\ \uparrow{\scriptstyle \mathrm{Id}_{H_*(\mathbb{S}^1)}} & & \uparrow{\scriptstyle j^n_{(n+1)*}} \\ H_*(\mathbb{S}^1) & \xrightarrow{i_{(n+1)*}} & H_*(X_{n+1}). \end{array}$$

Hence and from the continuity of Čech homology we get

$$\left(\lim_{\leftarrow} i_n\right)_* = i_* : H_*(\mathbb{S}^1) \approx \lim_{\leftarrow} H_*(X_n) \approx H_*(X),$$

so X is an i-retract of $\mathbb{S}^1$. From Proposition 3.17, $X \in$ ANRR($\mathbb{I}$). We prove that $X \notin$ ANRR($\mathbb{P}$) (in particular, $X \notin$ ANRR($\mathbb{IP}$)). The space X is not locally connected for points $x \in \mathbb{S}^1$. Assume to the contrary that $X \in$ ANRR($\mathbb{P}$). Then there exist a proper map $g : Z \to X$ (where Z is a compact set and $Z \subset Q$), an open neighborhood U of Z in Q and a g-retraction $r : U \to X$. There exists a compact and locally connected set K such that $Z \subset K \subset U$ (see Proposition 2.25). The map $r_K : K \to X$ given by the formula $r_K(x) = r(x)$ for each $x \in K$ is a surjection (since $r \circ i = g$ and g is a surjection, where $i : Z \hookrightarrow K$ is an inclusion). From Proposition 2.26, X must be locally connected, but this is a contradiction.

Example 3.34. Let

$$S = \left(\bigcup_{n=1}^{\infty} \{1/n\} \times [0,1] \right) \cup (\{0\} \times [0,1]) \cup ([0,1] \times \{0\}),$$

$$T = bd\big([-1,0] \times [0,1]\big),$$

and let $X = T \cup S$, where T denotes a boundary square. We know that $X \subset \mathbb{R}^2$ is an AANR in the sense of Noguchi (see [10]) and it is not locally connected. It is clear that the set S is compact and contractible. Similarly as in Example 3.33, it can be shown that $X \in \mathrm{ANRR}(\mathbb{I})$ and $X \notin \mathrm{ANRR}(\mathbb{P})$.

Based on Propositions 3.25–3.30, a few levels of relative retracts can be differentiated. The first level is $\mathbb{H}$-retracts, i.e., the retracts in the sense of Borsuk. The second level is $\mathbb{MV}$-retracts, i.e., multiretracts (Proposition 3.30) that retain the finite type, acyclicity, the fixed point property, path connectedness and locally path connectedness, but cannot be movable spaces (Example 3.32). On the next, third, level $\mathbb{IM}$-retracts can be distinguished. They retain the finite type, acyclicity, path connectedness and locally path connectedness. The fourth level is $\mathbb{IP}$-retracts that retain the finite type, acyclicity and path connectedness. On the final level, there are $\mathbb{I}$-retracts that retain acyclicity. Obviously, the number of levels of relative retracts is conventional but the level of $\mathbb{I}$-retracts should be final, because a sphere is not a relative retract of a ball (Proposition 3.6) and a compact space $X \in \mathrm{ARR}(\mathbb{I})$ if and only if X is an acyclic space (Proposition 3.25). Theorem 3.5 deserves a scrutiny on its own as it is important in regard to the application of coincidence theory (Proposition 3.7). Finally, it shall be noticed that $\mathbb{I}$-retracts encompass the class of retracts in the sense of Borsuk, multiretracts (Proposition 3.30), the space of FAR type (see Proposition 3.25) and all the spaces of FANR (AANR) type that have the homologies of space of ANR type (isomorphisms determined by continuous mappings, see Proposition 3.17 and Examples 3.33 and 3.34).

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