Cube-like complexes, Steinhaus' chains and the Poincaré–Miranda theorem

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Abstract. We discuss combinatorial results allowing to prove a new, more general version of the Poincaré–Miranda fixed point theorem. The main tool is the Steinhaus' chain property.

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1. Introduction

In 1883–1884 Henri Poincaré announced the following result without proof (see [10, 11]).

Theorem 1.1 (Poincaré theorem). *Let $f_1; \dots; f_n$ be n continuous functions of n variables $x_1; \dots; x_n$; the variable x_i is subjected to vary between the limits a_i and $-a_i$. Let us suppose that for $x_i = a_i$ the function f_i is constantly positive, and that for $x_i = -a_i$ the function f_i is constantly negative; I say there will exist a system of values of x which all f 's vanish.*

In 1940 Miranda [8] rediscovered the Poincaré theorem and showed that it is equivalent to the Brouwer fixed point theorem [1, 2]. More information about the history and connections between Brouwer's and Poincaré's theorems, proof and consequences the reader can find in [5].

The chain property problem was formulated by Steinhaus in [13] as follows.

Let some segments of the chessboard be mined. Assume that the king cannot go across the chessboard from the left edge to the right one without meeting a mined square. Then the rook can go from upper edge to the lower one moving exclusively on mined segments.

In 1940's Nash [9] announced a similar observation for the game of Hex. In the game of Hex the first player always has a winning strategy. Nash's theorem says that the first player, having assigned black or white color, can

always form a connected path with tiles in his color, which shall join opposite faces of the hexagonal chessboard, regardless of how the second player moves (for more details see [8]). Gale [4] demonstrated that an extension of Nash's observation to arbitrary dimension is, in fact, equivalent to the Brouwer fixed point theorem. Steinhaus' and Nash's observations are almost the same. In fact Steinhaus' theorem is more general. In Hex, players move only in one manner—across the common face of two hexagons. The rook and the king do not move in the same way. According to Shashkin and Surówka [12, 14] several proofs of the Steinhaus theorem seem to be incomplete or use induction on the size of the chessboard. In [6] the Steinhaus theorem was generalized assuming that the chessboard is divided into arbitrary polygons and there was showed an algorithm allowing to find rook's or king's route between chosen opposite edges of the chessboard.

In [15] Tkacz and Turzański proved the following theorem.

Theorem 1.2 (*n*-dimensional version of Steinhaus' theorem). *For an arbitrary decomposition $T(k)$ of n -dimensional cube I^n into k^n cubes and for arbitrary coloring function $F: T(k) \rightarrow \{1, \dots, n\}$ for some $i \in \{1, \dots, n\}$ there exists an i th colored chain of cubes $P_1, \dots, P_r$ ($P_j \cap P_{j+1} \neq \emptyset$ for $j = 1, 2, \dots, r-1$) such that the i th opposite faces are connected.*

Moreover, they proved the equivalence of this theorem, the Poincaré–Miranda theorem and the Brouwer fixed point theorem [16]. The main tool in the Tkacz–Turzański papers was Steinhaus' chain property.

For $n = 3$, in [15] the following strong version of Theorem 1.2 has been proved (the i th colored chain is called *i*-connected if $\dim[P_j \cap P_{j+1}] = n - i$ for $j = 1, \dots, r-1$).

Theorem 1.3 (On the existence of the *i*-connected chain). *For an arbitrary decomposition of the n -dimensional cube I^n into k^n cubes and an arbitrary coloring function $F: T(k) \rightarrow \{1, \dots, n\}$ for some natural number $i \in \{1, \dots, n\}$ there exists an i th colored *i*-connected chain $P_1, \dots, P_r$ such that $P_1 \cap I_i^+ \neq \emptyset$ and $P_r \cap I_i^- \neq \emptyset$.*

We can formulate the following problem.

Problem 1. Is the strong version of the Steinhaus theorem true for an arbitrary natural number greater than 3?

The aim of our presentation is to formulate and prove the Steinhaus chain property in a discrete form and use it for obtaining a wider and more general version of the Poincaré–Miranda fixed point theorem. This gives an extension of some classical results due to Bolzano, Poincaré, Bohl and Brouwer. For this purpose we define a new class of complexes—*n*-cube-like complexes. They are the generalization of *n*-cubes. It is interesting that each continuous map satisfying Poincaré's boundary conditions on the support of *n*-cube-like polyhedron achieves zero, but there exists an *n*-cube-like polyhedron without fixed point property (see Figure 1).

2. Notation and elementary properties

Our terminology follows [3]. Let A be a finite set. Denote by $P(A)$ the family of all subsets of A and by $P_{n+1}(A)$ the family of all subsets of A of cardinality $n + 1$. The elements of $P_{n+1}(A)$ are called n -simplexes defined on the set A . Let $S \in P_{n+1}(A)$. Then $T \in P_{k+1}(S)$ is called a k -face of the n -simplex S .

Definition 2.1. The family $\mathcal{K} \subset P(A)$ is said to be an *abstract complex* if it satisfies the following condition:

$$\text{if } V \in \mathcal{K} \text{ then } P(V) \subset \mathcal{K}.$$

The *support* of the abstract complex $\mathcal{K}$ is defined by the formula

$$|\mathcal{K}| := \bigcup \{V : V \in \mathcal{K}\}.$$

The elements of $|\mathcal{K}|$ are called *vertices*.

Let us now define the complex generated by the given family.

Definition 2.2. Let $\mathcal{S} \subset P(A)$ be a nonempty family. Then $\mathcal{K}(\mathcal{S})$ is said to be a *complex generated by the family $\mathcal{S}$* if

$$\mathcal{K}(\mathcal{S}) := \bigcup_{S \in \mathcal{S}} P(S).$$

Definition 2.3. If $\mathcal{S} \subset P_{n+1}(A)$, then the *boundary of the complex $\mathcal{K}(\mathcal{S})$* is the subcomplex $\partial\mathcal{K}(\mathcal{S})$ generated by the family

$$\mathcal{B} = \{T \in P_n(A) : \exists! S \in \mathcal{S} \text{ such that } T \subset S\}.$$

3. Cube-like complex

Let us consider an n -dimensional cube I^n in $\mathbb{R}^n$. Observe that the boundary ∂I^n is the union of n pairs of opposite faces ($\partial I^n = \bigcup_{i=1}^n I_i^- \cup I_i^+$) and each face is $(n - 1)$ -dimensional cube. Moreover, for all $i \in \{1, \dots, n\}$ and all $\epsilon \in \{-, +\}$ the opposite faces of I_i^ϵ have the following form:

$$I_i^\epsilon \cap I_j^-, \quad I_i^\epsilon \cap I_j^+ \quad \text{for } j \neq i.$$

Let us fix a finite, nonempty set A .

Definition 3.1. Let $\mathcal{K}^0 = \{a\}$, where a is an arbitrary element of A . The complex $\mathcal{K}^n$ generated by the family $\mathcal{S} \subset P_{n+1}(A)$ is said to be *n -cube-like* if

- (A) for every $(n - 1)$ -face $T \in \mathcal{K}^n \setminus \partial\mathcal{K}^n$ there exist exactly two n -simplexes $S, S' \in \mathcal{K}^n$ such that $S \cap S' = T$;
- (B) there exists a sequence of n pairs of subcomplexes $\mathcal{F}_i^-, \mathcal{F}_i^+$ called *i th opposite faces* such that
 - (B₁) $\partial\mathcal{K}^n = \bigcup_{i=1}^n \mathcal{F}_i^- \cup \mathcal{F}_i^+$,
 - (B₂) $\mathcal{F}_i^- \cap \mathcal{F}_i^+ = \emptyset$ for $i \in \{1, \dots, n\}$,
 - (B₃) for all $i \in \{1, \dots, n\}$ and all $\epsilon \in \{+, -\}$, $\mathcal{F}_i^\epsilon$ is an $(n - 1)$ -cube-like complex, such that its opposite faces have form $\mathcal{F}_i^\epsilon \cap \mathcal{F}_j^-, \mathcal{F}_i^\epsilon \cap \mathcal{F}_j^+$ for $j \neq i$.

Let $(\bar{K}, \bar{\mathcal{K}})$ be a polyhedron, where $\bar{\mathcal{K}}$ is a simplicial complex and $\bar{K}$ is the support of $\bar{\mathcal{K}}$. Each polyhedron determines an abstract complex $\mathcal{K}$ called its vertex-scheme: $\mathcal{K}$ consists of subsets of vertices that span the simplexes of $\bar{\mathcal{K}}$. $|\mathcal{K}|$ is the set of vertices of $(\bar{K}, \bar{\mathcal{K}})$; see [3].

Definition 3.2. The polyhedron $(\bar{K}, \bar{\mathcal{K}})$ in $\mathbb{R}^m$ is said to be n -cube-like if its vertex-scheme abstract complex $\mathcal{K}$ is n -cube-like. The opposite faces of $\bar{\mathcal{K}}$ correspond to the faces of $\mathcal{K}$ and they are denoted by $\mathcal{F}_i^-$ and $\mathcal{F}_i^+$ (the supports are denoted by F_i^- and F_i^+) for $i \in \{1, \dots, n\}$.

Example 3.3. Figure 1 represents an example of two cube-like complexes.

FIGURE 1. Example of two cube-like complexes.

Example 3.4. The Möbius strip can be represented as 2-cube-like complex. The boundary of the Möbius strip is a circle, so we can divide it into faces in the same way as the boundary of the square.

Example 3.5. Solid torus can be represented as a 3-cube-like complex (see Figure 2).

FIGURE 2. Solid torus and its boundary.

The following proposition shows how large is the class of cube-like complexes.

Proposition 3.6. *For each pair of natural number n, k there exists an n -cube-like polyhedron $(\bar{K}, \bar{\mathcal{K}})$ such that $H_{n-1}(\bar{K}) = \mathbb{Z}^k$ and $H_i(\bar{K}) = 0$ for $i \neq n-1$.*

Proof. Let us fix two natural numbers n and k . Consider the n -dimensional cube I^n . Its opposite faces are denoted by I_i^-, I_i^+ for $i \in \{1, \dots, n\}$. Fix a triangulation $\bar{\mathcal{L}}$ such that there exists a family of different n -simplexes $\{\sigma_1, \dots, \sigma_k\}$ with properties

$$\sigma_i \cap \sigma_j = \emptyset \quad \text{for } i \neq j$$

and

$$\sigma_i \subset I^n \setminus \partial I^n \quad \text{for } i \in \{1, \dots, k\}.$$

Let

$$\bar{K} = I^n \setminus \bigcup_{i \in \{1, 2, \dots, k\}} \text{Int } \sigma_i \quad \text{and} \quad \bar{\mathcal{K}} = \bar{\mathcal{L}} \setminus \bigcup_{i \in \{1, 2, \dots, k\}} \sigma_i.$$

Since obviously $\hat{K}$ satisfies the homological assertions, it is enough to prove that $(\bar{K}, \bar{\mathcal{K}})$ is an n -cube-like polyhedron. Let $\mathcal{K}$ be a vertex-scheme abstract complex of $\bar{\mathcal{K}}$. The vertex-scheme realizations of triangulated faces I_i^-, I_i^+ are denoted by $\mathcal{F}'_i, \mathcal{F}'_i^+$ for $i \in \{1, \dots, n\}$. Now define the faces of $\mathcal{K}$ as follows:

$$\mathcal{F}_1^- = \mathcal{F}'_1^-, \quad \mathcal{F}_1^+ = \mathcal{F}'_1^+ \cup \bigcup_{i \in \{1, 2, \dots, k\}} E_i,$$

where E_i is the vertex-scheme of the complex generated by σ_i and

$$\mathcal{F}_i^- = \mathcal{F}'_i^- \quad \text{for } i \in \{2, \dots, n\}.$$

It is easy to observe that the family

$$\{\mathcal{F}_i^\epsilon : 0 \leq i \leq n, \epsilon \in \{-, +\}\}$$

satisfies conditions (B₁)–(B₃) from Definition 3.1. □

Observation 3.7. Let $\mathcal{K}^n$ be an n -cube-like complex. Then for each sequence $1 \leq i_1 < i_2 < \dots < i_k \leq n$, the intersection $\mathcal{F}_{i_1}^{\epsilon_1} \cap \dots \cap \mathcal{F}_{i_k}^{\epsilon_k}$ is $(n-k)$ -cube-like complex, where $\epsilon_i \in \{-, +\}$ for $i \in \{1, \dots, k\}$.

Proof. The proof is by induction on k . Of course $\mathcal{F}_{i_1}^{\epsilon_1}$ is $(n-1)$ -cube-like complex. Let us observe that

$$\mathcal{F}_{i_1}^{\epsilon_1} \cap \dots \cap \mathcal{F}_{i_k}^{\epsilon_k} = \left(\mathcal{F}_{i_1}^{\epsilon_1} \cap \dots \cap \mathcal{F}_{i_{k-1}}^{\epsilon_{k-1}} \right) \cap \left(\mathcal{F}_{i_1}^{\epsilon_1} \cap \dots \cap \mathcal{F}_{i_{k-2}}^{\epsilon_{k-2}} \cap \mathcal{F}_{i_k}^{\epsilon_k} \right).$$

So we have the intersection of two $(n-k+1)$ -cube-like complexes. Moreover, they are the faces of $(n-k+2)$ -cube-like complex:

$$\mathcal{F}_{i_1}^{\epsilon_1} \cap \dots \cap \mathcal{F}_{i_{k-2}}^{\epsilon_{k-2}}.$$

Condition (B₃) from Definition 3.1 completes the proof. □

Observation 3.8. Let $\mathcal{K}^n$ be an n -cube-like complex. Then for each sequence $1 \leq i_1 < i_2 < \dots < i_k \leq n$ and for each one of its proper subsequences $1 \leq i_{k_1} < i_{k_2} < \dots < i_{k_l} \leq n$, we have

$$\mathcal{F}_{i_1}^{\epsilon_1} \cap \dots \cap \mathcal{F}_{i_k}^{\epsilon_k} \subset \partial \left(\mathcal{F}_{i_{k_1}}^{\epsilon_{k_1}} \cap \dots \cap \mathcal{F}_{i_{k_l}}^{\epsilon_{k_l}} \right).$$

Proof. Let us observe that there exists $j \in \{i_1, \dots, i_k\}$ such that

$$j \notin \{i_{k_1}, \dots, i_{k_l}\}.$$

We have

$$\begin{aligned} \mathcal{F}_{i_1}^{\epsilon_1} \cap \dots \cap \mathcal{F}_{i_k}^{\epsilon_k} &\subset \mathcal{F}_{i_{k_1}}^{\epsilon_{k_1}} \cap \dots \cap \mathcal{F}_{i_{k_l}}^{\epsilon_{k_l}} \cap \mathcal{F}_j^{\epsilon_j} \\ &= \left(\mathcal{F}_{i_{k_1}}^{\epsilon_{k_1}} \cap \dots \cap \mathcal{F}_{i_{k_l}}^{\epsilon_{k_l}} \right) \cap \left(\mathcal{F}_{i_{k_1}}^{\epsilon_{k_1}} \cap \dots \cap \mathcal{F}_{i_{k_l-1}}^{\epsilon_{k_l-1}} \cap \mathcal{F}_j^{\epsilon_j} \right) \\ &\subset \partial \left(\mathcal{F}_{i_{k_1}}^{\epsilon_{k_1}} \cap \dots \cap \mathcal{F}_{i_{k_l}}^{\epsilon_{k_l}} \right). \end{aligned} \quad \square$$

4. Steinhaus' chain property

In this section the generalization of Steinhaus' chain property is considered. Let $\mathcal{K}^n$ be an n -cube-like complex.

Definition 4.1. The map $\phi : |\mathcal{K}^n| \rightarrow \{1, \dots, n\}$ is said to be a *coloring function* of $|\mathcal{K}^n|$.

Definition 4.2. The sequence $\{s_1, \dots, s_m\} \subset |\mathcal{K}^n|$ is a *chain* if for each $i \in \{1, \dots, m-1\}$ we have $\{s_i, s_{i+1}\} \in \mathcal{K}^n$.

Definition 4.3. The chain $\{s_1, \dots, s_m\} \subset |\mathcal{K}^n|$ is said to be *i th colored* if for each $j \in \{1, \dots, m\}$, $\phi(s_j) = i$.

Definition 4.4. The set $B \subset |\mathcal{K}^n|$ is said to be *proper* if $\phi(B) = \{1, 2, \dots, n\}$.

Theorem 4.5 (Steinhaus' chain property). *Let $\mathcal{K}^n$ be an n -cube-like complex. Then for every map $\phi : |\mathcal{K}^n| \rightarrow \{1, \dots, n\}$ there exist $i \in \{1, \dots, n\}$ and an i th colored chain $\{s_1, \dots, s_m\} \subset |\mathcal{K}^n|$ such that $s_1 \in \mathcal{F}_i^-$ and $s_m \in \mathcal{F}_i^+$.*

Since the case $n = 1$ is trivial, we can assume that $n \geq 2$. Before we prove Theorem 4.5 let us first introduce some definitions and lemmas.

Lemma 4.6. *If $\mathcal{K}^n$ is an n -cube-like complex, then for every $(n-2)$ -face F in $\partial\mathcal{K}^n$ there exist exactly two $(n-1)$ -faces in $\partial\mathcal{K}^n$ such that F is their common face.*

Proof. Let us take the $(n-2)$ -face $F \in \partial\mathcal{K}^n$. From condition (B₁) there exists i such that $F \in \mathcal{F}_i^\epsilon$.

Let us assume that $F \in \partial\mathcal{F}_i^\epsilon$. Then there exists exactly one $(n-1)$ -face $T_1 \in \mathcal{F}_i^\epsilon$ such that $F \subset T_1$. Moreover, from condition (B₃) there exists a face $\mathcal{F}_j^{\epsilon'}$ such that

$$F \in \mathcal{F}_i^\epsilon \cap \mathcal{F}_j^{\epsilon'}.$$

We obtain the $(n-1)$ -face $T_2 \in \mathcal{F}_j^{\epsilon'}$ such that $F \subset T_2$. Now, in this case it is enough to prove that there are only two $(n-1)$ -faces $T_1, T_2 \subset \partial\mathcal{K}^n$ such that F is their common face. Since it is obvious for $n=2$, let us assume that $n > 2$ and that there exist $(n-1)$ -faces $T_1, T_2, T_3 \subset \partial\mathcal{K}^n$ such that

$$F \subset T_1 \cap T_2 \cap T_3.$$

Since $F \in \partial\mathcal{F}_i^\epsilon$, there exist

$$\mathcal{F}_{i_1}^{\epsilon_1} \neq \mathcal{F}_{i_2}^{\epsilon_2} \neq \mathcal{F}_{i_3}^{\epsilon_3}$$

such that $T_j \in \mathcal{F}_{i_j}^{\epsilon_j}$ for $j=1, 2, 3$. Let us observe that

$$\mathcal{F}_{i_1}^{\epsilon_1} \cap \mathcal{F}_{i_2}^{\epsilon_2} \cap \mathcal{F}_{i_3}^{\epsilon_3}$$

is $(n-3)$ -cube-like (complex generated by $(n-3)$ -simplexes). Since F is an $(n-2)$ -face and $F \in \mathcal{F}_{i_1}^{\epsilon_1} \cap \mathcal{F}_{i_2}^{\epsilon_2} \cap \mathcal{F}_{i_3}^{\epsilon_3}$, we get a contradiction.

Let us assume that $F \notin \partial\mathcal{F}_i^\epsilon$. Since $\mathcal{F}_i^\epsilon$ is an $(n-1)$ -cube-like, conditions (A) and (B₃) finish the proof. $\square$

Definition 4.7. Let $S = \{v_0, v_1, \dots, v_n\}$ be an n -simplex. An abstract complex contained in $P(\{v_0, v_1, \dots, v_n\} \times \{0, 1\})$ generated by the family

$$\{(v_0, 0), (v_1, 0), \dots, (v_i, 0), (v_i, 1), (v_{i+1}, 1), \dots, (v_n, 1) : i \in \{0, 1, \dots, n\}\}$$

is called an *S-doubled complex*. Let us denote it by $\text{dc}(S)$.

Let us remark that an *S-doubled complex* depends on the orientation $\{v_0, v_1, \dots, v_n\}$ of the simplex S .

Let $\mathcal{K}$ be an abstract complex. Put

$$|\mathcal{K}| = \{w_0, w_1, \dots, w_m\}.$$

Each k -simplex $T \in \mathcal{K}$ has a unique representation

$$T = \{w_{i_0}, w_{i_1}, \dots, w_{i_k}\},$$

where $0 \leq i_0 < i_1 < \dots < i_k \leq m$. From now, by writing

$$T = \{v_0, v_1, \dots, v_k\}$$

we mean the orientation of T agrees with the enumeration of $\mathcal{K}$ ($v_0 = w_{i_0}$, $v_1 = w_{i_1}, \dots, v_k = w_{i_k}$).

Definition 4.8. The abstract complex

$$\mathcal{CK} = \mathcal{K} \times \{0\} \cup \bigcup \{\text{dc}(S) : S \in \partial\mathcal{K}\},$$

where the orientation of each simplex S agrees with the orientation of $|\mathcal{K}| = \{w_0, w_1, \dots, w_m\}$, is called an *extension of the complex* $\mathcal{K}$.

We identify the elements of $\mathcal{K}$ with the points of $\mathcal{K} \times \{0\}$. Figure 3 illustrates the concept of an extension of 2-cube-like complex.

FIGURE 3. Cube-like complex $\mathcal{K}^2$ and its extension $\mathcal{CK}^2$.

Lemma 4.9. *If $\mathcal{K}^n$ is an n -cube-like complex, then an extension $\mathcal{CK}^n$ is also an n -cube-like complex.*

Proof. Let us observe that $\partial(\mathcal{CK}^n)$ is a copy of $\partial\mathcal{K}^n$. Therefore condition (B) from Definition 3.1 is satisfied.

To end the proof it remains to show that if T is $(n-1)$ -face of

$$\mathcal{CK}^n \setminus \partial(\mathcal{CK}^n),$$

then there exist exactly two n -simplexes $P, P' \in \mathcal{CK}^n$ such that $P \cap P' = T$. Let T be an $(n-1)$ -face of $\mathcal{CK}^n \setminus \partial(\mathcal{CK}^n)$. Since $\mathcal{K}^n$ is an n -cube like complex, if

$$T \in \mathcal{K}^n \setminus \partial\mathcal{K}^n,$$

then T meets this condition.

Let us assume now that

$$T \in \bigcup \{\text{dc}(S) : S \in \partial\mathcal{K}^n\}.$$

Therefore there exists $S \in \partial\mathcal{K}^n$ such that $T \in \text{dc}(S)$. Without loss of generality, we can assume that T is an $(n-1)$ -face of the n simplex

$$\{(v_1, 0), \dots, (v_i, 0), (v_i, 1), (v_{i+1}, 1), \dots, (v_n, 1)\}.$$

Let us assume now that

$$T = \{(v_1, 0), \dots, (v_i, 0), \widehat{(v_i, 1)}, (v_{i+1}, 1), \dots, (v_n, 1)\}.$$

If $i = n$, then $T \in \partial\mathcal{K}^n$ and (from the definition of boundary) there exists exactly one n -simplex $P \in \mathcal{K}^n$ such that T is its $(n-1)$ -face. Obviously T is also an $(n-1)$ -face of the simplex

$$P' = \{(v_1, 0), \dots, (v_i, 0), (v_{i+1}, 0), \dots, (v_n, 0), (v_n, 1)\} \in (\mathcal{CK}^n) \setminus \mathcal{K}^n.$$

If $i \neq n$, then T is a common face of the simplexes

$$P = \{(v_1, 0), \dots, (v_i, 0), (v_i, 1), (v_{i+1}, 1), \dots, (v_n, 1)\}$$

and

$$P' = \{(v_1, 0), \dots, (v_i, 0), (v_{i+1}, 0), (v_{i+1}, 1), \dots, (v_n, 1)\}.$$

It is easy to observe that T does not belong to other n -simplexes in $\mathcal{CK}^n$. Similar arguments apply to the case

$$T = \{(v_1, 0), \dots, (\widehat{v_i, 0}), (v_i, 1), (v_{i+1}, 1), \dots, (v_n, 1)\}.$$

Let us assume now that there exists

$$j \in \{1, 2, \dots, i-1, i+1, \dots, n\}$$

such that

$$T = \{(v_1, 0), \dots, (v_{j-1}, 0), (\widehat{v_j, 0}), (v_{j+1}, 0), \dots, (v_i, 0), \\ (v_i, 1), (v_{i+1}, 1), \dots, (v_n, 1)\}$$

From Lemma 4.6, for the $(n-2)$ -face

$$\{(v_1, 0), \dots, (v_{j-1}, 0), (v_{j+1}, 0), \dots, (v_i, 0), (v_{i+1}, 0), \dots, (v_n, 0)\},$$

there exist exactly two $(n-1)$ -faces

$$\{(v_1, 0), \dots, (v_{j-1}, 0), (v_j, 0), (v_{j+1}, 0), \dots, (v_i, 0), (v_{i+1}, 0), \dots, (v_n, 0)\}$$

and

$$\{(v_1, 0), \dots, (v_{j-1}, 0), (v_{j+1}, 0), \dots, (v_k, 0), \dots, (v_i, 0), (v_{i+1}, 0), \dots, (v_n, 0)\}$$

such that

$$\{(v_1, 0), \dots, (v_{j-1}, 0), (v_{j+1}, 0), \dots, (v_i, 0), (v_{i+1}, 0), \dots, (v_n, 0)\}$$

is their common face. Therefore, T is a common face of exactly two n -simplexes:

$$P = \{(v_1, 0), \dots, (v_{j-1}, 0), (v_j, 0), (v_{j+1}, 0), \dots, \\ (v_i, 0), (v_i, 1), (v_{i+1}, 1), \dots, (v_n, 1)\}$$

and

$$P' = \{(v_1, 0), \dots, (v_{j-1}, 0), (v_{j+1}, 0), \dots, (v_k, 0), \dots, \\ (v_i, 0), (v_i, 1), (v_{i+1}, 1), \dots, (v_n, 1)\}.$$

□

Let us define in $\mathcal{CK}^n$ the sets

$$\begin{aligned} \mathcal{F}_1 &= \mathcal{F}_1^- \times \{1\} \cup \mathcal{F}_2^+ \times \{1\}, \\ \mathcal{F}_2 &= (\mathcal{F}_2^- \times \{1\} \cup \mathcal{F}_3^+ \times \{1\}) \setminus \mathcal{F}_1, \\ \mathcal{F}_3 &= (\mathcal{F}_3^- \times \{1\} \cup \mathcal{F}_4^+ \times \{1\}) \setminus (\mathcal{F}_1 \cup \mathcal{F}_2), \\ &\vdots \\ \mathcal{F}_n &= (\mathcal{F}_n^- \times \{1\} \cup \mathcal{F}_1^+ \times \{1\}) \setminus \bigcup_{i=1}^{n-1} \mathcal{F}_i, \end{aligned}$$

and define the coloring map

$$\phi^\partial: |\partial(\mathcal{CK}^n)| \rightarrow \{1, 2, \dots, n\}$$

by the formula

$$\forall i \in \{1, \dots, n\}, \phi^\partial(x) = i \quad \text{for } x \in |\mathcal{F}_i|$$

Illustration. Figure 4 illustrates a 2-cube-like complex whose boundary is colored by ϕ^∂ . One can observe that there are exactly two proper faces.

FIGURE 4. The coloring map of boundary and two proper faces.

Lemma 4.10. *For the coloring map defined above there exist exactly two proper $(n-1)$ -faces in $\partial(\mathcal{CK}^n)$.*

Proof. Let us define two decreasing finite sequences of sets:

$$\begin{aligned} S_1^- &= \{T \in (\mathcal{F}_n^- \times \{1\}) : T \in P_n(A)\}, \\ S_2^- &= \{T \in S_1^- : T \cap (\mathcal{F}_{n-1}^- \times \{1\}) \in P_{n-1}(A)\}, \\ S_3^- &= \{T \in S_2^- : T \cap (\mathcal{F}_{n-2}^- \times \{1\}) \in P_{n-2}(A)\}, \\ &\vdots \\ S_n^- &= \{T \in S_{n-1}^- : T \cap (\mathcal{F}_1^- \times \{1\}) \in P_1(A)\} \end{aligned}$$

and

$$\begin{aligned} S_1^+ &= \{T \in (\mathcal{F}_1^+ \times \{1\}) : T \in P_n(A)\}, \\ S_2^+ &= \{T \in S_1^+ : T \cap (\mathcal{F}_n^+ \times \{1\}) \in P_{n-1}(A)\}, \\ S_3^+ &= \{T \in S_2^+ : T \cap (\mathcal{F}_{n-1}^+ \times \{1\}) \in P_{n-2}(A)\}, \\ &\vdots \\ S_n^+ &= \{T \in S_{n-1}^+ : T \cap (\mathcal{F}_2^+ \times \{1\}) \in P_1(A)\}. \end{aligned}$$

By Observation 3.7 the sets S_i^- and S_i^+ are nonempty for $i = 1, 2, \dots, n$. Moreover, if T is a proper $(n-1)$ -face in $\partial(\mathcal{CK}^n)$, then (from definitions of

the map ϕ^∂ and the sets S_i^-, S_i^+) we conclude that T is an element of S_n^- or S_n^+ .

Let us observe that S_n^- has exactly one element. Assume now that there exist two different $(n-1)$ -faces $T, T' \in S_n^-$. By Observation 3.8, the sets

$$T \cap (\mathcal{F}_{n-1}^- \times \{1\}) \quad \text{and} \quad T' \cap (\mathcal{F}_{n-1}^- \times \{1\})$$

are different $(n-2)$ -faces. Analogously, the sets

$$T \cap ((\mathcal{F}_{n-1}^- \cap \mathcal{F}_{n-2}^-) \times \{1\}) \quad \text{and} \quad T' \cap ((\mathcal{F}_{n-1}^- \cap \mathcal{F}_{n-2}^-) \times \{1\})$$

are different $(n-3)$ -faces. Proceeding in this way we obtain that the sets

$$T \cap ((\mathcal{F}_{n-1}^- \cap \mathcal{F}_{n-2}^- \cap \cdots \cap \mathcal{F}_1^-) \times \{1\})$$

and

$$T' \cap ((\mathcal{F}_{n-1}^- \cap \mathcal{F}_{n-2}^- \cap \cdots \cap \mathcal{F}_1^-) \times \{1\})$$

are two different 0-faces. Since $T \in \mathcal{F}_n^- \times \{1\}$ and $T' \in \mathcal{F}_n^- \times \{1\}$, we obtain that there are two different points in

$$(\mathcal{F}_n^- \cap \mathcal{F}_{n-1}^- \cap \mathcal{F}_{n-2}^- \cap \cdots \cap \mathcal{F}_1^-) \times \{1\},$$

which contradicts the fact that

$$(\mathcal{F}_n^- \cap \mathcal{F}_{n-1}^- \cap \mathcal{F}_{n-2}^- \cap \cdots \cap \mathcal{F}_1^-) \times \{1\}$$

is 0-cube-like, so a point. Analogously, we can show that there exists exactly one element of the set S_n^+ that is also a proper $(n-1)$ -face. $\square$

Proof of Theorem 4.5. Let us consider the extension $\mathcal{CK}^n$ of the n -cube-like complex $\mathcal{K}^n$ and define a coloring map: $\phi': |\mathcal{CK}^n| \rightarrow \{1, \dots, n\}$ as follows:

$$\phi'(x) := \begin{cases} \phi(x) & \text{for } x \in |\mathcal{K}^n|, \\ \phi^\partial & \text{for } x \in |\partial\mathcal{CK}^n|. \end{cases}$$

Observe that each proper n -simplex in $\mathcal{CK}^n \setminus \partial\mathcal{CK}^n$ has exactly two proper faces. Moreover, each face is the face of exactly one or two n -simplexes from $\mathcal{CK}^n$ depending whether or not it lies on $\partial\mathcal{CK}^n$.

Algorithm for finding the chain

Let us take a unique proper simplex S_1 such that $S_1 \cap (\mathcal{F}_n^- \times \{1\})$ is a proper $(n-1)$ -face. This face is called “used.” Take an “unused” proper $(n-1)$ -face of S_1 . If it lies on $\partial\mathcal{CK}^n$, then end. Otherwise, it is the face of exactly one simplex S_2 different from S_1 . Now this face is called “used.” The procedure can be continued for S_2 .

After a finite number of iterations we obtain the sequence $S_1, \dots, S_l$ such that $\phi'(S_l \cap (\mathcal{F}_1^+ \times \{1\})) = \{1, \dots, n\}$ because on the boundary of $\mathcal{CK}^n$ lies only two proper faces. Let us observe that for all $i \in \{1, \dots, n\}$,

$$S_1 \cap \mathcal{F}_i^- \neq \emptyset \neq S_l \cap \mathcal{F}_i^+,$$

hence we can take the smallest index $l^1 \in \{1, 2, \dots, m\}$ such that

$$S_{l^1} \cap (\mathcal{F}_i^+ \times \{1\}) \neq \emptyset$$

for some $i \in \{1, \dots, n\}$. Then let us find the biggest index $l^2 \in \{1, 2, \dots, l^1\}$ such that

$$S_{l^2} \cap (\mathcal{F}_i^- \times \{1\}) \neq \emptyset.$$

Then from the sequence $S_{l^2+1}, \dots, S_{l^1}$ choose points $s_1, s_2, \dots, s_m$ successively such that $\phi'(s_j) = i$ and $s_j \in \mathcal{K}^n$ for $j = 1, 2, \dots, m$. Moreover, we can choose them such that $s_j \neq s_{j+1}$ for $j = 1, 2, \dots, m-1$.

The sequence $s_1, s_2, \dots, s_m$ is the i th colored chain connecting $\mathcal{F}_i^-$ and $\mathcal{F}_i^+$. $\square$

5. Corollaries

Let k be an arbitrary natural number. The set $C = \{1, 2, \dots, k\}^n$ is said to be the n -dimensional combinatorial cube. Its i th opposite faces are defined as follows:

$$C_i^- = \{z \in C : z(i) = 1\}, \quad C_i^+ = \{z \in C : z(i) = k\}.$$

Let $e_i = (0, \dots, 0, 1, 0, \dots, 0)$, $e_i(i) = 1$, be the i th basic vector. An ordered set $S = [z_0, \dots, z_n] \subset C$ is said to be an n -simplex if there exists a permutation α of the set $\{1, \dots, n\}$ such that

$$z_1 = z_0 + e_{\alpha(1)}, \dots, z_n = z_{n-1} + e_{\alpha(n)}.$$

Let $\mathcal{S} := \{S \subset C : S \text{ is } n\text{-simplex}\}$. Observe that the complex $\mathcal{K}(\mathcal{S})$ is n -cube-like ($\mathcal{K}^n$). The faces of $\mathcal{K}^n$ are generated by the faces of the combinatorial cube C . The vertices of $\mathcal{K}^n$ can be interpreted as the centers of cubes from the decomposition of I^n onto k^n cubes. Using Theorem 4.5 (Steinhaus' chain property) we obtain the following corollary.

Corollary 5.1 (n -dimensional version of Steinhaus' theorem). *For an arbitrary decomposition of the n -dimensional cube I^n onto k^n cubes and for an arbitrary coloring function $F: T(k) \rightarrow \{1, \dots, n\}$ for some natural number $i \in \{1, \dots, n\}$ there exists an i th colored chain $P_1, \dots, P_r$ such that*

$$P_1 \cap I_i^+ \neq \emptyset \quad \text{and} \quad P_r \cap I_i^- \neq \emptyset.$$

Let us consider a Hex board and a related cube-like complex (see Figure 5). The points of the combinatorial cube are interpreted as centers of hexagons on Hex board.

Then we get the following corollary.

Corollary 5.2 (Hex theorem). *If every tile of the Hex board is marked by either a white or a black, then there is a white path connecting regions W and W' or a black path connecting B and B'.*

FIGURE 5. Hex board and related cube-like complex.

6. Poincaré–Miranda and Brouwer theorems

Lemma 6.1. *Let $(\bar{K}, \bar{\mathcal{K}})$ be an n -cube-like polyhedron in $\mathbb{R}^m$ and let $U_1, \dots, U_n$ be an open cover of $\bar{K}$. Then there exist $i \in \{1, \dots, n\}$ and a continuum $W \subset U_i$ such that $W \cap \bar{F}_i^- \neq \emptyset \neq W \cap \bar{F}_i^+$.*

Proof. Let $\lambda > 0$ be the Lebesgue number for covering $U_1, \dots, U_n$. Let us consider the simplicial subdivision $\bar{\mathcal{K}}'$ of $\bar{\mathcal{K}}$ such that for each $s \in |\bar{\mathcal{K}}'|$,

$$\text{diam} \left(\bigcup \{ \sigma \in \bar{\mathcal{K}}'; s \in \sigma \} \right) < \lambda.$$

Let us define a coloring function $\phi : |\bar{\mathcal{K}}'| \rightarrow \{1, \dots, n\}$ as follows:

$$\phi(s) = \text{maximal natural number } i \text{ such that } \bigcup \{ \sigma \in \bar{\mathcal{K}}' : s \in \sigma \} \subset U_i.$$

Using Theorem 4.5 we obtain the i th colored chain $(s_1, \dots, s_m)$ such that $s_1 \in \mathcal{F}_i^-$ and $s_m \in \mathcal{F}_i^+$. It is easy to see that the set

$$W = \bigcup_{j=1}^{m-1} \sigma_j,$$

where $\sigma_j \in \bar{\mathcal{K}}'$ is a 1-face spanned by $\{s_j, s_{j+1}\}$ for $j \in \{1, \dots, m-1\}$, is closed and connected. Since each set σ_j is contained in U_i , we have $W \subset U_i$. We leave to the reader to verify that $W \cap \bar{F}_i^- \neq \emptyset \neq W \cap \bar{F}_i^+$. $\square$

Theorem 6.2 (Generalization of the Poincaré–Miranda theorem). *Let $(\bar{K}, \bar{\mathcal{K}})$ be an n -cube-like polyhedron, and let $f : \bar{K} \rightarrow \mathbb{R}^n$, $f = (f_1, \dots, f_n)$, be a continuous map such that*

$$f_i \left(\bar{F}_i^- \right) \subset (-\infty, 0] \quad \text{and} \quad f_i \left(\bar{F}_i^+ \right) \subset [0, \infty) \quad \text{for } i = 1, \dots, n.$$

Then there exists $c \in \bar{K}$ such that $f(c) = (0, \dots, 0)$.

Proof. Conversely, suppose that $f(x) \neq 0$ for each $x \in \bar{K}$. Let us define sets

$$U_i = \{x \in \bar{K} : f_i(x) \neq 0\} \quad \text{for } i = 1, \dots, n.$$

We have that $\bar{K} = U_1 \cup \dots \cup U_n$ and U_i is opened for $i = 1, \dots, n$. By Lemma 6.1, there exist $i \in \{1, \dots, n\}$ and a continuum $W \subset U_i$ such that

$$W \cap \bar{F}_i^- \neq \emptyset \neq W \cap \bar{F}_i^+.$$

Hence there exist $x, y \in W$ such that

$$f_i(x) < 0 \quad \text{and} \quad f_i(y) > 0.$$

Since f is a continuous map, $f_i(W)$ is connected in $\mathbb{R}$, i.e., $f_i(W)$ is an interval containing $[f_i(x), f_i(y)]$. Therefore there exists $c \in W$ such that $f_i(c) = 0$. Since $c \in W \subset U_i$, we obtain a contradiction. $\square$

Remark 6.3. As we have announced in the introduction, the Poincaré and Brouwer theorems are equivalent on the cube I^n . It is interesting that each continuous map satisfying Poincaré's boundary conditions on the support of an n -cube-like polyhedron achieves zero but there exists an n -cube-like polyhedron without fixed point property (see Figure 1).

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