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Selections of set-valued functions satisfying the general linear inclusion

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Abstract. Applying the classical Banach fixed point theorem we prove that a set-valued function satisfying a general linear functional inclusion admits a unique selection fulfilling the corresponding functional equation. We also adopt the method of the proof for investigating the Rassias stability of general linear equation.

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1. Introduction

Let $(Y, \|\cdot\|)$ be a real normed space. We denote by $n(Y)$ the family of all nonempty subsets of Y and by $\mathrm{cl}(Y)$, $\mathrm{ccl}(Y)$, $\mathrm{c}(Y)$ and $\mathrm{cp}(Y)$ we denote collections of all closed, convex closed, compact and complete members of $n(Y)$, respectively.

The number

$$\mathrm{diam}\, A := \sup\{\|a - b\| : a, b \in A\}$$

is said to be the *diameter* of $A \in n(Y)$. We say that a set-valued function $F \colon K \to n(Y)$ (an "s.v. function" for abbreviation) is *with bounded diameter* if the function $K \ni x \mapsto \mathrm{diam}\, F(x) \in \mathbb{R}$ is bounded.

In [5], the authors proved that if S is a commutative semigroup with zero and $(Y, \|\cdot\|)$ is a real Banach space, then $F \colon S \to \mathrm{ccl}(Y)$ is a *subadditive* s.v. function; i.e.,

$$F(x + y) \subset F(x) + F(y), \quad x, y \in S,$$

with bounded diameter admits a unique additive selection. Popa in [11] proved that if $\emptyset \neq K$ is a *convex cone* in a real vector space X (i.e., $t_1 K + t_2 K \subset K$ for all $t_1, t_2 \geq 0$) and $F \colon K \to \mathrm{ccl}(Y)$ (where $(Y, \|\cdot\|)$ is a real Banach space) is an s.v. function with bounded diameter fulfilling the inclusion

$$F(\alpha x + \beta y) \subset \alpha F(x) + \beta F(y), \quad x, y \in K,$$

for $\alpha, \beta > 0$, $\alpha + \beta \neq 1$, then there exists exactly one additive selection of F.

Nikodem and Popa in [9] and Piszczek in [10] proved the following theorem.

Theorem 1.1. *Let K be a convex cone in a real vector space X, $(Y, \|\cdot\|)$ a real Banach space and $\alpha, \beta, p, q > 0$. Consider an s.v. function $F\colon K \to \mathrm{ccl}(Y)$ with bounded diameter fulfilling the inclusion*

$$\alpha F(x) + \beta F(y) \subset F(px + qy), \quad x, y \in K.$$

If $\alpha + \beta < 1$, then there exists a unique selection $f\colon K \to Y$ of F satisfying the equation

$$\alpha f(x) + \beta f(y) = f(px + qy), \quad x, y \in K.$$

If $\alpha + \beta > 1$, then F is single valued.

The case of $p + q = 1$ was investigated by Popa in [13], Inoan and Popa in [7] and recently, by Smajdor and Szczawińska in [14].

In this paper, we determine the conditions for which an s.v. function $F\colon K \to n(Y)$ satisfying the inclusion

$$\alpha F(x) + \beta F(y) \subset F(px + qy), \quad x, y \in K, \tag{1.1}$$

where $|\alpha| < p$ and $\operatorname{diam} F(x) \le M\|x\|$, $x \in K$ (for some $M > 0$) admits a selection satisfying the corresponding functional equation. It is easy to check that if F satisfies the opposite inclusion, then it is single valued. Theorem 2.1 and also the method of its proof are used for the investigation of the Rassias stability of general linear functional equation.

2. The main theorem

Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be real normed spaces and let K be a nonempty subset of X. Consider an s.v. function $F\colon K \to n(Y)$. A function $f\colon K \to Y$ is a *selection* of the s.v. function F if and only if $f(x) \in F(x)$, $x \in K$. Let

$$\mathrm{Sel}(F) := \{f\colon K \to Y : f(x) \in F(x),\ x \in K\}.$$

It is easy to check that if there exists a constant $M > 0$ such that $\operatorname{diam} F(x) \le M\|x\|$ for all $x \in K$, then the function

$$d(f, g) := \sup\left\{\frac{\|f(x) - g(x)\|}{\|x\|},\ x \in K \setminus \{0\}\right\}, \quad f, g \in \mathrm{Sel}(F),$$

is a metric in $\mathrm{Sel}(F)$. Moreover, if $F(x)$ is complete for every $x \in K$, the metric space $(\mathrm{Sel}(F), d)$ is complete. Obviously, the convergence in the space $(\mathrm{Sel}(F), d)$ implies the pointwise convergence on the set K.

Theorem 2.1. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be real normed spaces and let $K \subset X$ be such that $0 \in K$. Assume that $p, q > 0$ and $\alpha, \beta \in \mathbb{R}$ are fixed and one of the following conditions holds:*

(i) *$|\alpha| < p$ and $K \subset pK$,*
(ii) *$|\beta| < q$ and $K \subset qK$.*

Consider an s.v. function $F\colon K \to \mathrm{cp}(Y)$ such that $0 \in F(0)$ and

$$\mathrm{diam}\, F(x) \leq M\|x\|, \quad x \in K,$$

for some positive constant M. If

$$\alpha F(x) + \beta F(y) \subset F(px + qy), \quad x, y \in K,\, px + qy \in K, \qquad (2.1)$$

then there exists a unique selection $f\colon K \to Y$ of F such that

$$\alpha f(x) + \beta f(y) = f(px + qy), \quad x, y \in K,\, px + qy \in K.$$

Proof. Assume that $|\alpha| < p$ and $K \subset pK$. Since $\mathrm{diam}\, F(0) = 0$ and $0 \in F(0)$, $F(0) = \{0\}$. Putting $y = 0$ in (2.1), we obtain

$$\alpha F\left(\frac{x}{p}\right) \subset F(x), \quad x \in K. \qquad (2.2)$$

Let

$$\mathcal{T}(g)(x) := \alpha g\left(\frac{x}{p}\right), \quad x \in K,\, g \in \mathrm{Sel}(G).$$

By (2.2), $\mathcal{T}(g) \in \mathrm{Sel}(F)$, $g \in \mathrm{Sel}(F)$. Moreover, for every $g_1, g_2 \in \mathrm{Sel}(F)$,

$$d\big(\mathcal{T}(g_1), \mathcal{T}(g_2)\big) = |\alpha| \sup\left\{ \frac{\|g_1\left(\frac{x}{p}\right) - g_2\left(\frac{x}{p}\right)\|}{\|x\|}, \, x \in K \setminus \{0\} \right\}$$

$$= \frac{|\alpha|}{p} \sup\left\{ \frac{\|g_1\left(\frac{x}{p}\right) - g_2\left(\frac{x}{p}\right)\|}{\|\frac{x}{p}\|}, \, x \in K \setminus \{0\} \right\}$$

$$\leq \frac{|\alpha|}{p}\, d(g_1, g_2).$$

Since $|\alpha| < p$, the map $\mathcal{T}\colon \mathrm{Sel}(F) \to \mathrm{Sel}(F)$ is contractive in the complete metric space $(\mathrm{Sel}(F), d)$, so by the Banach theorem, it has a unique fixed point f and $\lim_{n\to\infty} T^n(g) = f$ for each $g \in \mathrm{Sel}(F)$. Hence $f\colon K \to Y$ is the unique selection of the s.v. function F such that

$$f(x) = \alpha f\left(\frac{x}{p}\right), \quad x \in K.$$

Fix $g \in \mathrm{Sel}(F)$ and $x, y \in K$ such that $px + qy \in K$. Then

$$\frac{x}{p}, \frac{y}{p}, \frac{px+qy}{p} \in K.$$

By (2.1),

$$\alpha g\left(\frac{x}{p}\right) + \beta g\left(\frac{y}{p}\right),\, g\left(\frac{px+qy}{p}\right) \in F\left(\frac{px+qy}{p}\right).$$

Hence

$$\left\| \alpha g\left(\frac{x}{p}\right) + \beta g\left(\frac{y}{p}\right) - g\left(\frac{px+qy}{p}\right) \right\| \leq \mathrm{diam}\, F\left(\frac{px+qy}{p}\right)$$

$$\leq M \left\| \frac{px+qy}{p} \right\|.$$

Thus

$$\left\| \alpha \mathcal{T}(g)(x) + \beta \mathcal{T}(g)(y) - \mathcal{T}(g)(px + qy) \right\| \leq \frac{|\alpha|}{p} M \|px + qy\|$$

for every $x, y \in K$ such that $px + qy \in K$. Proceeding by induction, we get

$$\left\| \alpha \mathcal{T}^n(g)(x) + \beta \mathcal{T}^n(g)(y) - \mathcal{T}^n(g)(px + qy) \right\| \leq \left(\frac{|\alpha|}{p} \right)^n M \|px + qy\|$$

for every $n \in \mathbb{N}$ and all $x, y \in K$ with $px + qy \in K$. Letting $n \to \infty$, we obtain

$$\alpha f(x) + \beta f(y) = f(px + qy), \quad x, y \in K, \ px + qy \in K. \qquad \square$$

Let X and Y be real vector spaces and K a convex cone in X. It is easy to check that if $f \colon K \to Y$ satisfies the equation

$$\alpha f(x) + \beta f(y) = f(px + qy), \quad x, y \in K,$$

and $f(0) = 0$, then f is additive; i.e.,

$$f(x) + f(y) = f(x + y), \quad x, y \in K.$$

Corollary 2.2. *Let $p, q > 0$ and $|\alpha| < p$ (or $|\beta| < q$). Assume that $(X, \| \cdot \|)$ and $(Y, \| \cdot \|)$ are normed spaces, K a convex cone in X and $F \colon K \to \mathrm{cp}(Y)$ an s.v. function such that $0 \in F(0)$ and*

$$\operatorname{diam} F(x) \leq M \|x\|, \quad x \in K,$$

for some constant $M > 0$. If

$$\alpha F(x) + \beta F(y) \subset F(px + qy), \quad x, y \in K,$$

then the unique selection $f \colon K \to Y$ of F fulfilling the equation

$$\alpha f(x) + \beta f(y) = f(px + qy), \quad x, y \in K,$$

is additive.

Remark 2.3. If $|\alpha| = p$ and $|\beta| = q$, the above corollary is not true.

Proof. Let $K = [0, +\infty)$ and let $F \colon K \to \mathrm{c}(\mathbb{R})$ be an s.v. function defined by

$$F(x) = [0, x], \quad x \in K,$$

and $\operatorname{diam} F(x) = |x|$, $x \in K$. Then

$$F(x + y) = F(x) + F(y), \quad x, y \in K,$$

and for every $a \in [0, 1]$, the function $f(x) = ax$, $x \in K$, is an additive selection of F. $\qquad \square$

3. Stability results

In the first part of this section we present an application of Theorem 2.1 to the investigation of the Rassias stability of the general linear functional equation

$$\alpha f(x) + \beta f(y) = f(px + qy). \tag{3.1}$$

For definition and more results in the Rassias stability theory see, for instance, [6, 8]. The general linear equation was considered by several authors (for example, [1, 2, 12]). A set-valued approach can be found in [3].

Gajda showed in [4] (see also [8, Theorem 2.6]) that for $\varepsilon > 0$ there is a function $f \colon \mathbb{R} \to \mathbb{R}$ satisfying the inequality

$$\big|f(x) + f(y) - f(x+y)\big| \leq \varepsilon(|x| + |y|), \quad x, y \in \mathbb{R},$$

while there is no constant $M \geq 0$ and no additive function $f_0 \colon \mathbb{R} \to \mathbb{R}$ such that

$$\big|f(x) - f_0(x)\big| \leq M\,|x|, \quad x \in \mathbb{R}.$$

We apply the method used in the proof of Theorem 2.1 to obtain the stability result for equation (3.1). We will denote by $\mathbb{R}_+$ and $\mathbb{Q}_+$ the set of all nonnegative real and rational numbers, respectively.

Theorem 3.1. *Let K be a convex cone in a real normed space $(X, \| \cdot \|)$, $(Y, \| \cdot \|)$ a real Banach space and $\varepsilon \colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ an $\mathbb{R}_+$-homogenous function. Assume that $p, q > 0$, $\alpha, \beta \in \mathbb{R}$ are fixed and $|\alpha| < p$. If a function $f \colon K \to Y$ satisfies*

$$\|\alpha f(x) + \beta f(y) - f(px + qy)\| \leq \varepsilon(\|x\|, \|y\|), \quad x, y \in K,$$

then there exists a unique function $f_0 \colon K \to Y$ fulfilling the functional equation

$$\alpha f_0(x) + \beta f_0(y) = f_0(px + qy), \quad x, y \in K,$$

and such that

$$\|f(x) - f_0(x) - f(0)\| \leq M\|x\|, \quad x \in K,$$

for some $M > 0$. Moreover,

$$\|f(x) - f_0(x) - f(0)\| \leq \frac{\varepsilon(1,0)}{p - \alpha}\,\|x\|, \quad x \in K,$$

and f_0 is an additive function. In particular, if $\varepsilon(1,0) = 0$, then

$$f(x) = f_0(x) - f(0), \quad x \in K.$$

Proof. Let $g \colon K \to Y$ be given by $g(x) = f(x) - f(0)$, $x \in K$. It is easy to check that $g(0) = 0$ and

$$\|\alpha g(x) + \beta g(y) - g(px + qy)\| \leq \varepsilon(\|x\|, \|y\|), \quad x, y \in K. \tag{3.2}$$

Hence, for all $x, y \in K$,

$$\alpha g(x) + \beta g(y) \in g(px + qy) + \varepsilon(\|x\|, \|y\|)B, \tag{3.3}$$

where B denotes the unit closed ball in the space Y. Setting $y = 0$ and replacing x by x/p in (3.3), we obtain

$$\alpha g\left(\frac{x}{p}\right) \in g(x) + \varepsilon\left(\frac{\|x\|}{p}, 0\right) B = g(x) + \frac{\varepsilon(1,0)}{p}\|x\|B, \quad x \in K.$$

Consider an s.v. function $G\colon K \to \mathrm{cl}(Y)$ given by

$$G(x) = g(x) + \frac{\varepsilon(1,0)}{p-\alpha}\|x\|B, \quad x \in K.$$

Then

$$\mathrm{diam}\, G(x) = \frac{2\varepsilon(1,0)}{p-\alpha}\|x\|, \quad x \in K,$$

and

$$\alpha G\left(\frac{x}{p}\right) = \alpha g\left(\frac{x}{p}\right) + \alpha\frac{\varepsilon(1,0)}{p-\alpha}\left\|\frac{x}{p}\right\|B$$
$$\subset g(x) + \frac{\varepsilon(1,0)}{p}\left(1 + \frac{\alpha}{p-\alpha}\right)\|x\|B = G(x)$$

for all $x \in K$. The idea of the proof is the same as before so we only give a sketch. The function $T\colon \mathrm{Sel}(G) \to \mathrm{Sel}(G)$, given by

$$T(h)(x) := \alpha h\left(\frac{x}{p}\right), \quad x \in K,\, h \in \mathrm{Sel}(G),$$

is contraction with the constant $|\alpha|/p$. By the Banach theorem, there exists a unique function $f_0 \in \mathrm{Sel}(G)$ such that

$$f_0(px) = \alpha f_0(x), \quad x \in K,$$

and $\lim_{n\to\infty} T^n(g) = f_0$. By the definition of G,

$$\|f_0(x) - g(x)\| \leq \frac{\varepsilon(1,0)}{p-\alpha}\|x\|, \quad x \in K.$$

Since g satisfies (3.2) and ε is positively homogeneous,

$$\big\|\alpha T(g)(x) + \beta T(g)(y) - T(g)(px+qy)\big\| \leq \frac{|\alpha|}{p}\varepsilon(\|x\|, \|y\|)$$

for all $x, y \in K$. Proceeding by induction, we get

$$\big\|\alpha T^n(g)(x) + \beta T^n(g)(y) - T^n(g)(px+qy)\big\| \leq \left(\frac{|\alpha|}{p}\right)^n \varepsilon(\|x\|, \|y\|)$$

for every $x, y \in K$ and $n \geq 1$. It follows that

$$\alpha f_0(x) + \beta f_0(y) = f_0(px+qy), \quad x, y \in K.$$

Since $f_0(0) = 0$, f_0 is additive.

For the end assume that $f_1\colon K \to Y$ is a solution of equation (3.1) such that

$$\|f_1(x) - g(x)\| \leq M\|x\|, \quad x \in K,$$

for some positive M. Then

$$f_1(0) = 0, \quad \alpha f_1\left(\frac{x}{p}\right) = f_1(x), \quad x \in K$$

and

$$\|f_1(x) - f_0(x)\| \le \left(M + \frac{\varepsilon(1,0)}{p - \alpha}\right)\|x\|, \quad x \in K.$$

It is easy to check that

$$\|f_1(x) - f_0(x)\| \le \left(\frac{|\alpha|}{p}\right)^n \left(M + \frac{\varepsilon(1,0)}{p - \alpha}\right)\|x\|, \quad x \in K,\ n \in \mathbb{N}.$$

Hence $f_1 = f_0$, which ends the proof. $\qquad\qquad\square$

In the second part of this section, we apply Theorem 2.1 to obtain a stability result for equation (3.1).

Theorem 3.2. *Let K be a convex cone in a real normed space $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ a real Banach space and $p, q > 0$, $\alpha, \beta \in \mathbb{R}$ fixed constants such that $|\alpha| < p$. Consider $\eta \ge 0$ and a function $f\colon K \to Y$ satisfying the inequality*

$$\|\alpha f(x) + \beta f(y) - f(px + qy)\| \le \eta\,(\|x\| + \|y\|), \quad x, y \in K.$$

If there exists a constant $\lambda > 0$ such that

$$(1 + \lambda\alpha)\|x\| + (1 + \lambda\beta)\|y\| \le \lambda\|px + qy\|, \quad x, y \in K, \qquad (3.4)$$

then there exists a unique function $f_0\colon K \to Y$ fulfilling the functional equation

$$\alpha f_0(x) + \beta f_0(y) = f_0(px + qy), \quad x, y \in K,$$

and such that

$$\|f(x) - f_0(x) - f(0)\| \le M\|x\|, \quad x \in K,$$

for some $M > 0$. Moreover,

$$\|f(x) - f_0(x) - f(0)\| \le \lambda\eta\|x\|, \quad x \in K,$$

and f_0 is an additive function.

Proof. Let $g\colon K \to Y$ be given by $g(x) = f(x) - f(0)$, $x \in K$. As in the proof of Theorem 3.1, $g(0) = 0$ and

$$\|\alpha g(x) + \beta g(y) - g(px + qy)\| \le \eta\,(\|x\| + \|y\|), \quad x, y \in K.$$

In particular, for all $x, y \in K$,

$$\alpha g(x) + \beta g(y) \in g(px + qy) + \eta(\|x\| + \|y\|)B,$$

where B is the unit closed ball in Y. Define an s.v. function $G\colon K \to \mathrm{cl}(Y)$ as follows:

$$G(x) = g(x) + \lambda\eta\|x\|B, \quad x \in K.$$

Then $0 \in G(0)$, $\mathrm{diam}\,G(x) = 2\eta\lambda\|x\|$, $x \in K$, and

$$\alpha G(x) + \beta G(y) \subset G(px + qy), \quad x, y \in K.$$

Indeed, let $x, y \in K$ be fixed. By condition (3.4),

$$
\begin{aligned}
\alpha G(x) + \beta G(y) &= \alpha g(x) + \beta g(y) + \eta \left(\lambda \alpha \|x\| + \lambda \beta \|y\| \right) B \\
&\subset g(px + qy) + \eta (\|x\| + \|y\|) B + \eta \left(\lambda \alpha \|x\| + \lambda \beta \|y\| \right) B \\
&\subset g(px + qy) + \eta \lambda \|px + qy\| B \\
&= G(px + qy).
\end{aligned}
$$

By Theorem 2.1, there exists exactly one selection $f_0 \colon K \to Y$ of the s.v. function G satisfying equation (3.1). Hence

$$
\|f(x) - f_0(x) - f(0)\| \leq \lambda \eta \|x\|, \quad x \in K.
$$

Since $f_0(0) = 0$, f_0 must be additive. The proof that f_0 is the only solution of equation (3.1) such that

$$
\|f(x) - f_0(x) - f(0)\| \leq M \|x\|, \quad x \in K,
$$

for some $M > 0$ runs as before. $\qquad \square$

Example. Let $|\alpha| < p$ and $|\beta| < q$. Every convex cone

$$
K \subset \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \colon x_1 \geq 0 \wedge \cdots \wedge x_n \geq 0 \vee x_1 \leq 0 \wedge \cdots \wedge x_n \leq 0 \right\}
$$

satisfies condition (3.4) with

$$
\lambda \geq \max \left\{ \frac{1}{p - \alpha}, \frac{1}{q - \beta} \right\}
$$

and with the norm in $\mathbb{R}^n$ given by

$$
\|(x_1, \ldots, x_n)\| = |x_1| + \cdots + |x_n|, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n.
$$

4. The selection theorem

In this section we present a certain consequence of Theorem 2.1 for the case of a compact-valued s.v. function F satisfying the inclusion (1.1) and such that the function $x \mapsto \operatorname{diam} F(x)$ maps bounded sets onto bounded ones.

Lemma 4.1. *Let $p, q > 0$ and $\alpha, \beta \in \mathbb{R}$ be fixed. Assume that X and Y are real vector spaces, K a convex cone in X and $\mathcal{K}$ a collection of nonempty subsets of Y closed under intersections of chains. Let $F \colon K \to \mathcal{K}$ be an s.v. function fulfilling the inclusion (1.1) and $0 \in F(0)$. There exists a minimal s.v. function $F_0 \colon K \to \mathcal{K}$ such that*

  (i) *$F_0(x) \subset F(x)$, $x \in K$,*
  (ii) *$0 \in F_0(0)$,*
  (iii) *$\alpha F_0(x) + \beta F_0(y) \subset F_0(px + qy)$, $x, y \in K$.*

Proof. Define the set $\mathcal{F}$ of all s.v. functions $H \colon K \to \mathcal{K}$ such that

$$
H(x) \subset F(x), \quad x \in K,
$$

$0 \in H(0)$ and

$$
\alpha H(x) + \beta H(y) \subset H(px + qy), \quad x, y \in K.
$$

The set $\mathcal{F}$ is nonempty, because $F \in \mathcal{F}$, and partially ordered by the relation

$$H_1 \preceq H_2 \Longleftrightarrow H_1(x) \subset H_2(x), \quad x \in K, \, H_1, H_2 \in \mathcal{F}.$$

It is enough to prove that there is a minimal element in $\mathcal{F}$. Let $\mathcal{C}$ be a chain in $\mathcal{F}$. Put

$$H_0(x) = \bigcap \{H(x) : H \in \mathcal{C}\}, \quad x \in K.$$

Since $\mathcal{K}$ is closed under intersections, $H_0(x) \in \mathcal{K}$, $x \in K$. Obviously, $0 \in H_0(0)$ and $H_0(x) \subset H(x)$, $x \in K$. Fix now $x, y \in K$. For every $H \in \mathcal{C}$,

$$\alpha H_0(x) + \beta H_0(y) \subset \alpha H(x) + \beta H(y) \subset H(px + qy),$$

hence

$$\alpha H_0(x) + \beta H_0(y) \subset H_0(px + qy).$$

Therefore, the s.v. function $H_0 \colon K \to \mathcal{K}$ belongs to $\mathcal{F}$ and $H_0 \preceq H$, $H \in \mathcal{C}$. By the Kuratowski–Zorn lemma , there exists a minimal element F_0 in the set $\mathcal{F}$. $\qquad\square$

Lemma 4.2. *Let $p, q > 0$ and $\alpha, \beta \in \mathbb{R}$ be fixed. Assume that X and Y are real vector spaces, K a convex cone in X and $\mathcal{K}$ a family of nonempty subsets of Y such that $\alpha\mathcal{K} \subset \mathcal{K}$ and $\lambda\mathcal{K} \subset \mathcal{K}$, $\lambda \in \mathbb{Q}_+$. Consider an s.v. function $F \colon K \to \mathcal{K}$. If $F_0 \colon K \to \mathcal{K}$ is a minimal s.v. function such that*

  (i) $F_0(x) \subset F(x)$, $x \in K$,
  (ii) $0 \in F_0(0)$,
 (iii) $\alpha F_0(x) + \beta F_0(y) \subset F_0(px + qy)$, $x, y \in K$,

then F_0 is superadditive and $\mathbb{Q}_+$-homogeneous; i.e.,

$$F_0(x) + F_0(y) \subset F_0(x + y), \quad x, y \in K,$$

and $F_0(\lambda x) = \lambda F_0(x)$, $x \in K$, $\lambda \in \mathbb{Q}_+$.

Proof. Putting $y = 0$ in (iii) and by (ii),

$$\alpha F_0 \left(\frac{x}{p} \right) \subset F_0(x), \quad x \in K.$$

Define

$$H(x) = \alpha F_0 \left(\frac{x}{p} \right), \quad x \in K.$$

Then $H(x) \in \mathcal{K}$, $H(x) \subset F_0(x) \subset F(x)$, $x \in K$ and $0 \in \alpha F_0(0) = H(0)$. Observe that the s.v. function $H \colon K \to \mathcal{K}$ satisfies the inclusion (iii). Indeed, let $x, y \in K$ be fixed. Then

$$\alpha H(x) + \beta H(y) = \alpha \left(\alpha F_0 \left(\frac{x}{p} \right) + \beta F_0 \left(\frac{y}{p} \right) \right)$$

$$\subset \alpha F_0 \left(\frac{px + qy}{p} \right) = H(px + qy).$$

Thus, by the minimality of F_0, $F_0(x) = H(x)$, $x \in K$; i.e.,

$$F_0(x) = \alpha F_0 \left(\frac{x}{p} \right), \quad x \in K.$$

Similarly we get

$$F_0(y) = \beta F_0\left(\frac{y}{q}\right), \quad y \in K.$$

Hence, by (iii),

$$F_0(x) + F_0(y) = \alpha F_0\left(\frac{x}{p}\right) + \beta F_0\left(\frac{y}{q}\right) \subset F_0(x+y)$$

for every $x, y \in K$; i.e., F_0 is superadditive. In particular,

$$nF_0\left(\frac{x}{n}\right) \subset F_0(x), \quad x \in K, n \in \mathbb{N}.$$

Fix now $n \in \mathbb{N}$ and consider the s.v. function

$$K \ni x \mapsto nF_0\left(\frac{x}{n}\right) \in \mathcal{K}.$$

We have

$$nF_0\left(\frac{x}{n}\right) \subset F_0(x) \subset F(x), \quad x \in K,$$

$0 \in nF_0(0)$ and

$$\alpha nF_0\left(\frac{x}{n}\right) + \beta nF_0\left(\frac{y}{n}\right) \subset nF_0\left(\frac{px+qy}{n}\right), \quad x, y \in K.$$

Therefore,

$$nF_0\left(\frac{x}{n}\right) = F_0(x), \quad x \in K,$$

by the minimality of F_0. Consequently,

$$F_0(\lambda x) = \lambda F_0(x), \quad x \in K, \lambda \in \mathbb{Q}_+. \qquad \square$$

Remark 4.3. Let $(X, \|\cdot\|)$ be a real normed space and $K \subset X$ a cone. Assume that $f \colon K \to \mathbb{R}$ is $\mathbb{Q}_+$-homogeneous. The function f maps bounded subsets of K onto bounded sets if and only if there exists a positive constant M such that

$$|f(x)| \leq M\|x\|, \quad x \in K.$$

Proof. Assume that $f \colon K \to \mathbb{R}$ is $\mathbb{Q}_+$-homogeneous (in particular, $f(0) = 0$) and maps bounded subsets of K onto bounded sets. Let

$$M = \sup\{|f(x)| : x \in K \wedge \|x\| \leq 1\}.$$

Fix $x \in K \setminus \{0\}$. There exists a decreasing sequence $(\lambda_n)_n$ such that

$$\lim_{n\to\infty} \lambda_n = 1 \quad \text{and} \quad \lambda_n\|x\| \in \mathbb{Q}_+, \quad n \in \mathbb{N}.$$

Since

$$\left\|\frac{x}{\lambda_n\|x\|}\right\| = \frac{1}{\lambda_n} < 1, \quad n \in \mathbb{N},$$

we have

$$\left|f\left(\frac{x}{\lambda_n\|x\|}\right)\right| \leq M, \quad n \in \mathbb{N}.$$

Consequently $|f(x)| \leq M\|x\|$.

The opposite implication is obvious. $\qquad \square$

Theorem 4.4. *Let $p, q > 0$, $\alpha, \beta \in \mathbb{R}$ and $|\alpha| < p$. Assume that $(X, \| \cdot \|)$ and $(Y, \| \cdot \|)$ are real normed spaces, K is a convex cone in X and $F \colon K \to \mathrm{c}(Y)$ an s.v. function such that $0 \in F(0)$ and*

$$\alpha F(x) + \beta F(y) \subset F(px + qy), \quad x, y \in K.$$

If $K \ni x \mapsto \operatorname{diam} F(x) \in \mathbb{R}$ maps bounded sets onto bounded sets, then there exists a unique selection $f \colon K \to Y$ of F fulfilling the equation

$$\alpha f(x) + \beta f(y) = f(px + qy), \quad x, y \in K.$$

The selection f is additive.

Proof. The family $\mathrm{c}(Y)$ is closed under intersections of chains and

$$\lambda \, \mathrm{c}(Y) \subset \mathrm{c}(Y), \quad \lambda \in \mathbb{R}.$$

By Lemma 4.1, there exists a minimal s.v. function $F_0 \colon K \to \mathrm{c}(Y)$ such that $F_0(x) \subset F(x)$, $x \in K$, $0 \in F_0(0)$ and

$$\alpha F_0(x) + \beta F_0(y) \subset F_0(px + qy), \quad x, y \in K.$$

Lemma 4.2 shows now that F_0 is superadditive and $\mathbb{Q}_+$-homogeneous. Since $\operatorname{diam} F_0(x) \leq \operatorname{diam} F(x)$, $x \in K$, the function $K \ni x \mapsto \operatorname{diam} F_0(x) \in \mathbb{R}$ maps bounded sets onto bounded sets and it is $\mathbb{Q}_+$-homogeneous. Hence, by Remark 4.3, there exists a positive constant M such that

$$\operatorname{diam} F_0(x) \leq M \|x\|, \quad x \in K.$$

Consequently, by Theorem 2.1, the s.v. function F_0 admits a unique selection $f_0 \colon K \to Y$ such that

$$\alpha f_0(x) + \beta f_0(y) = f_0(px + qy), \quad x, y \in K.$$

Since $f_0(0) = 0$ and f_0 satisfies the above equation, f_0 must be additive. Assume that $f_1, f_2 \colon K \to Y$ are selections of F such that

$$\alpha f_i(x) + \beta f_i(y) = f_i(px + qy), \quad x, y \in K, \ i = 1, 2.$$

They are additive and

$$f_i(x) = \alpha f_i \left(\frac{x}{p} \right), \quad x \in K, \ i = 1, 2.$$

Hence, for every $n \in \mathbb{N}$,

$$f_i(x) = \alpha^n f_i \left(\frac{x}{p^n} \right), \quad x \in K, \ i = 1, 2.$$

Fix $x \in K$ arbitrary. Since $K \ni z \mapsto \operatorname{diam} F(z) \in \mathbb{R}$ maps bounded sets onto bounded sets, there exists a constant M such that

$$\operatorname{diam} F(y) \leq M, \quad \|y\| \leq 2\|x\|.$$

For every $n \in \mathbb{N}$ there exists a number $q_n \in (1, 2)$ such that $q_n p^n \in \mathbb{Q}_+$. Obviously, $\|q_n x\| \leq 2\|x\|$, $n \in \mathbb{N}$. Hence, for every $n \in \mathbb{N}$,

$$\|f_1(x) - f_2(x)\| = |\alpha|^n \left\| f_1\left(\frac{x}{p^n}\right) - f_2\left(\frac{x}{p^n}\right) \right\|$$

$$= |\alpha|^n \left\| f_1\left(\frac{q_n x}{q_n p^n}\right) - f_2\left(\frac{q_n x}{q_n p^n}\right) \right\|$$

$$= \frac{1}{q_n} \left(\frac{|\alpha|}{p}\right)^n \|f_1(q_n x) - f_2(q_n x)\|$$

$$\leq \left(\frac{|\alpha|}{p}\right)^n \operatorname{diam} F(q_n x) \leq \left(\frac{|\alpha|}{p}\right)^n M.$$

Letting $n \to +\infty$ ends the proof. □

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