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DISCUSSION

Comments on: Remarkable polyhedra related to set functions, games and capacities

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A set function assigns a real number to any subset of a universal set. Set functions arise in many fields of mathematics and applied mathematics, but this survey focusses on cooperative games with transferable utility and on capacities where, formally speaking, the latter are a subclass of the former. Within these applications, the paper focusses on (i) the structure of polyhedra generated by capacities and several special classes of capacities; (ii) the structure of polyhedra generated by supermodular games; (iii) the structure of the core and variations on the core of capacities and games.

A game for player set N is a function $v : 2^N \to \mathbb{R}$ satisfying $v(\emptyset) = 0$. A capacity is a monotonic game (that is, $S \subseteq T$ implies $v(S) \leq v(T)$) such that $v(N) = 1$. A common interpretation of a capacity is that now the set N is the set of states (of the world) and the numbers $v(S)$ are probabilities, so v is a possibly non-additive probability measure.

A first result in the paper (Theorem 1, attributed to Stanley 1986) states that the set of capacities is polytope with the set of simple games (capacities with $v(S) \in \{0, 1\}$) as extreme points. The number of simple games increases exponentially with the cardinality of N and, according to the paper, is only known for $n \leq 9$, where $n = |N|$.

A p-additive capacity ($p \in \{1, \ldots, n\}$) is a capacity such that its Möbius transform attaches value 0 to all sets S with $|S| > p$. Such a capacity is completely determined by its values on sets with cardinality at most p. For instance, for $p = 1$ one obtains additive capacities. The set of 2-additive capacities is again a polytope, with n^2 extreme points, but nothing seems to be known for $p > 2$ (Miranda et al. 2006).

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A k-ary capacity ($k \in \mathbb{N}$) assigns to each vector in $\{0, 1, \ldots, k\}^N$ a real number. The case $k = 1$ returns the usual capacities, since we can identify a vector with zeros and ones with a coalition. This concept corresponds to the concept of a multichoice game in game theory. The extreme points of the polytope of k-ary capacities are the k-ary simple games (Theorem 4, which thus extends Theorem 1). This result still holds for 2-additive k-ary capacities: the extreme points of the polytope of 2-additive k-ary capacities are the 0–1-valued 2-additive k-ary capacities (Theorem 5).

The remainder of the survey focusses on games. The extremal rays of the cone of zero-normalized supermodular (convex) games are described (Section 6). Sections 7 and 8 focus on the core: the core of a game v is the set $\{x \in \mathbb{R}^N \mid x(S) := \sum_{i \in S} x_i \geq v(S) \, \forall S \in 2^N, \; x(N) = v(N)\}$. The Bondareva–Shapley theorem is reviewed and the relation with the Weber set. The paper (Section 7) mentions several results on the core of restricted games, which means that the core inequalities only have to hold for a subset of 2^N. In such a case, the core is not necessarily bounded. Special attention is paid to supermodular games with restricted cooperation in Theorems 13 and 14 (Grabisch and Sudhölter 2012).

The core of a game v can be seen as a collection of (1-)additive games which pointwise (per coalition) dominate v and are efficient. Section 8 presents an extension to the so-called k-additive core, consisting of k-additive games that pointwise dominate v (Vassil'ev 1978; Miranda and Grabisch 2010). For $k \geq 2$ the k-additive core is always nonempty. In fact, the k-additive core turns out to be a very large set (Theorem 17), a result which is derived by establishing a connection with the selectope (Derks et al. 2000). The survey concludes with an examination of the vertices of the k-additive core.

This is an excellent survey. In relatively few pages, the author manages to discuss many topics in much more detail and depth than one would expect in such a short paper. Most of the concepts considered here have well-understood and clear interpretations in game theory, although this is perhaps less true of the k-additive core, as the author also notes. As far as capacities are concerned, the interpretation in decision theory is that of a generalized, not per se additive probability measure (used, e.g., in the Choquet integral), and it would be interesting to reflect on the question whether the concepts studied here can be interpreted in a similar vein. For instance, how to interpret a k-ary capacity if the elements of N are states of the world and the values of the capacity are probabilities? Is the Möbius transform of a capacity just a technical device, or does it also have an interesting interpretation in decision theory, similarly as it has an interpretation in terms of dividends in the game model? It is well known and also mentioned in the survey that the core of a capacity and in particular a supermodular capacity has interesting applications in decision theory, e.g., Schmeidler (1986, 1989).

It may also be added that set functions and more precisely (pseudo-) Boolean functions have applications in social choice theory. In particular, they can be used to prove quantitative versions of the famous Arrow and Gibbard–Satterthwaite impossibility theorems. For instance, in the latter case such a result tells us how close we get to dictatorship, depending on the amount of manipulability we are willing to accept. See the studies by Friedgut et al. (2011) and Kalai (2002).

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