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DISCUSSION

Comments on: Recent progress on the combinatorial diameter of polytopes and simplicial complexes

Friedrich Eisenbrand

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Abstract This paper features two main contributions. On the one hand, it gives an impressive survey on the progress on the *diameter problem*, including the breakthrough of the author with his disproof of the *Hirsch conjecture* among many other recent results. On the other hand, it features new results (exponential lower bounds) on the diameter of simplicial complexes and re-interprets the recent activities of the polymath 3 project that has been coordinated by Gil Kalai.

The diameter problem

One of the most prominent mysteries in optimization and discrete geometry is the question whether the diameter of a polyhedron can be bounded by a polynomial in the dimension and the number of its defining inequalities. What precisely is this question?

A *polyhedron* is a set of the form $P = \{x \in \mathbb{R}^d : Ax \leq b\}$, where $A \in \mathbb{R}^{n \times d}$ is a matrix and $b \in \mathbb{R}^n$ is an n-dimensional vector. A *vertex* of P is a point $x^* \in P$ such that there exist d linearly independent rows of A whose corresponding inequalities of $Ax \leq b$ are satisfied by x^* with equality.

Two different vertices x^* and y^* are *neighbors* if they are the endpoints of an *edge* of the polyhedron, i.e., there exist $n - 1$ linearly independent rows of A whose corresponding inequalities of $Ax \leq b$ are satisfied with equality both by x^* and y^*. In this way, one obtains the undirected *polyhedral graph* with edges being pairs of neighboring vertices of P. The *diameter* of P is the smallest natural number that bounds the length of a shortest path between any pair of vertices in this graph. The question is now as follows:

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F. Eisenbrand (✉)
EPFL, Lausanne, Switzerland
e-mail: friedrich.eisenbrand@epfl.ch

What is the largest diameter of a polyhedron $P \subseteq \mathbb{R}^d$ with n facets?

The belief that the answer to this question implies a polynomial upper bound in n and d is the *polynomial Hirsch conjecture*. Clearly, the motivation for this question lies in another important problem in computational complexity, which is whether there exists a variant of the *simplex algorithm* that runs in *strongly polynomial time*. This means in time polynomial in the dimension and the number of inequalities only, where arithmetic operations only account for unit time. The classical Hirsch conjecture stated that this diameter is bounded by $n - d$. It was disproved for polyhedra by Klee and Walkup (1967) and in a celebrated paper of Santos (2012) for polytopes.

I cannot review all the content of this exhaustive survey. Instead I want to highlight two topics that are close to my heart.

Abstractions and simplicial complexes

It is not hard to see that the maximum diameter is attained at a *simple polyhedron*. A polyhedron with vertices is simple if each vertex is defined by *exactly one* d-element subset of the set of n inequalities.

Now, a *pure simplicial complex* of dimension $d - 1$ is a family of d-subsets of a set of *vertices* (typically $[n] = \{1, \ldots, n\}$). Interpreting vertices of a polyhedron $P \subseteq \mathbb{R}^d$ with n facets via their defining inequalities, one obtains a *simplicial complex* of dimension $d - 1$ on n vertices, where the vertices of the complex are the labels $\{1, \ldots, n\}$ of the facets of the polyhedron.

Two vertices of a polyhedron are adjacent if their corresponding defining sets differ by exactly one element. This notion of adjacency carries over to a simplicial complex. Two d-sets of the complex are adjacent if their intersection has $d - 1$ elements.

In the paper Eisenbrand et al. (2010), we considered the diameter of pure simplicial complexes of dimension $d - 1$ that satisfy a certain connectivity property. If one fixes a set $u \subseteq [n]$, then the subcomplex consisting of all d-sets containing u is connected. It turns out that all so far best-known asymptotic upper bounds for the diameter of polyhedra such as the $n^{\log d + 1}$-bound of Kalai and Kleitman (1992) and the linear bounds in fixed dimension of Barnette (1974) and Larman (1970) also hold in this case. Such a simplicial complex is called *normal* or *locally strongly connected*. In fact, these bounds do also hold if the adjacency between d-sets is not specified, as it is in the case of simplicial complexes; see Eisenbrand et al. (2010).

The author of the paper under discussion shows that the diameter of pure simplicial complexes can be exponential in n and d. This proves that the normality is essential to derive the aforementioned upper bounds. This is a very nice result.

The author then discusses a conjecture of Hähnle which, in the setting of normal simplicial complexes states that their diameter is bounded by

$$d(n - 1). \tag{1}$$

This conjecture implies the polynomial Hirsch conjecture and seems to capture much of the essence of the difficulty of the latter.

Diameter of polyhedra defined by structured matrices

In light of the importance and apparent difficulty of settling the polynomial Hirsch conjecture, many researchers have shown that it can be answered in an affirmative way in some special cases. Naddef (1989) proved that the Hirsch conjecture holds true for $0/1$-polytopes. Orlin (1997) provided a quadratic upper bound for flow-polytopes. Brightwell et al. (2006) showed that the diameter of the transportation polytope is linear in n and d, and a similar result holds for the dual of a transportation polytope (Balinski 1984) and the axial 3-way transportation polytope (De Loera et al. 2009).

The results on flow polytopes and classical transportation polytopes concern polyhedra defined by *totally unimodular matrices*, i.e., integer matrices whose subdeterminants are 0, ± 1. For such polyhedra, Dyer and Frieze (1994) have shown that the diameter is bounded by a polynomial in d and n. Their bound is $O(n^{16}d^3(\log dn)^3)$. Their result is also algorithmic: They show that there exists a randomized simplex-algorithm that solves linear programs defined by totally unimodular matrices in polynomial time.

Francisco Santos describes a recent result of ours (Bonifas et al. 2012). The diameter of a polyhedron $P = \{x \in \mathbb{R}^d : Ax \le b\}$, with $A \in \mathbb{Z}^{n \times d}$ is bounded by $O(\Delta^2 d^4 \log d\Delta)$. Here, Δ denotes the largest absolute value of a *sub-determinant* of A. If $\Delta = 1$, then one is in the regime of totally unimodular matrices, and thus one obtains a significant improvement of the result of Dyer and Frieze mentioned above.

The proof proceeds via a *volume expansion* argument that is often used in randomized algorithms and Markov chains. However, an important question still remains.

> Is there a polynomial-time simplex algorithm for linear programs defined by integer constraint-matrices with a bounded subdeterminant?

There might be such an algorithm and the excellent survey of Francisco Santos will find interesting readers that will work on this and many of the other highly visible and important problems that are still open around the polynomial Hirsch conjecture.

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