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An Ontological Basis for the Diffusion Theory

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Abstract Fick's diffusion equation represents physical reality that has been interpreted by Einstein and Smoluchowski. In this way, the question of interpretation of diffusion is answered in the affirmative. It gives rise to a new question critical for the understanding of our world: how broad is the spectrum of physical reality that diffusion could in principle give a complete account. The answer in this work is based on the elegant mathematical foundations formulated three decades before Fick by French mathematician Augustin Cauchy ($\sim$ 1822). It will be shown that the diffusion equation is a consequence of his model of the ideal elastic continuum. Namely, a product of the classical energy and momentum balance equations and their solutions. This demonstrates that the complete ontological construal of the diffusion theory exists. Explicitly, the interpretation of both, the diffusion equation and the flux constative formulae exist. The two terms in the flux equations, the driving forces defined by the potential gradients and the kinetic coefficients in front of the driving forces, are derived in this paper. Some fundamental consequences of all derived equations and relations for physics, chemistry and the prospects are presented. The ontological interpretation of the diffusion equation presented here provides evidence of the common roots of the chemistry and physics.

Keywords Cauchy-elastic solid · diffusion equation · Klein–Gordon equation · relativistic quaternion mechanics

1 Introduction

This paper is dedicated to the memory of John Morral. I met John several times; he was not only a brilliant scientist, but the way in which he interacted with others made him very special. I remember a seminar talk in which he presented the construction of multicomponent phase diagrams based on limited experimental data. After his talk he suggested that we should try to implement multicomponent Darken method in the problem of multiphase interdiffusion, i.e., the *multi-multi problem*. Later, in 2004 I invited John to Cracow (DIMAT 2004) and arranged the dinner with Richard Ghez (author of *Primer of Diffusion Problems*). We were having fun to argue about correlations between diffusion and gravity. John was in favor of Ghez's speculations and suggested a more systematic investigation of the broad spectrum of physical reality that could, in principle, fully represent diffusion.

John's first suggestion, multiphase interdiffusion, turned out to be extremely stimulating and solvable. We confirmed the existence of the "type zero boundaries", "zigzag" diffusion paths in interdiffusion microstructures[1] and established a model of diffusive interaction between two-phase alloys in two-dimensional geometry.[2] The *multi-multi* experience raised confidence in math. The diffusion in the Planck-Kleinert crystal (P-KC), which was the second problem discussed with John, was published in

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2007.[3] It took the next 15 years to formulate and proof the self-consistent formulation of the energy transport phenomena in an ideal elastic continuum. In 2021 the Diffusion Fundamentals conference was planned in Kraków, we were hoping to discuss with John our solution of the 2004 diffusion puzzle. Regrettably he was already not able to accept our invitation. This paper presents the results encouraged by Prof. John Morral in 2004.

The diffusion equation has been interpreted by Einstein and Smoluchowski. This gives rise to a new question critical for the understanding of our world: how broad is the spectrum of physical reality that could, in principle, fully represent diffusion. The answer is based on the elegant mathematical foundations formulated three decades before Fick ($\sim$ 1822) by French mathematician Augustin Cauchy. It will be shown that the diffusion equation is a consequence of his model of the ideal elastic continuum. Namely, a product of the classical energy and momentum balance equations and their solutions. We use the concepts of the quaternion quantum mechanics, QQM.[4] The QQM is ontic in the sense that it answers the central questions of interpretation: it is based on being (the Cauchy elastic continuum) and relates the basic categories of being and their relationships.

1.1 Quaternion Quantum Mechanics Today

The first suggestion of quaternion quantum mechanics came from Birkhoff and von Neumann.[5] Already in 1936 they mentioned that quaternion quantum mechanics has greater logical consistency than classical (complex) quantum mechanics. Finkelstein et al.[6] have shown that a quaternion calculus exists. Summing up, QQM has many new features which make it a much richer theory. It is caused generally by the noncommutativity of quaternion-valued wave functions. The quaternion Klein–Gordon[7] and Schrödinger equations[4] have much more physical information than their complex equivalents and make QQM a much richer theory.

1.2 Crystalline Aether

The Cauchy model of elastic solids was already published[8] when Maxwell speculated on the crystal hypothesis. He constructed the equations of electromagnetism by modelling a lattice of elastic cells, etc.[9] In "*A Dynamical Theory of the Electromagnetic Field*"[10] Maxwell explicitly remarked on the aether:

> *On our theory, it* (energy)*… may be described according to a very probable hypothesis, as the motion and the strain of one and the same medium* (elastic aether).[11]

Less known, if not entirely forgotten, is the remark on gravity where Maxwell wrote:

> *…assumption, therefore, that gravitation arises from the action of the surrounding medium leads to the conclusion that every part of this medium possesses, when undisturbed, an enormous intrinsic energy. As I am unable to understand in what way a medium can possess such properties, I cannot go any further in this direction in searching for the cause of gravitation.*[10]

Maxwell's idea of solid aether showing "enormous intrinsic energy" was unimaginable in the nineteenth century.

The original arguments to implement the classical mechanics equations in the field of wave mechanics in crystalline, granular aether were given by Kleinert.[12,13] The Kleinert concept linked with the Cauchy model of the elastic continuum has later been analyzed with the arbitrary assumption of the complex potential field.[3] Recent results confirm that quantum gravity effects, when applied to a non-relativistic particle in a one-dimensional box, imply the quantization of length,[14] i.e., confirm the fundamental discreteness of space itself. Namely imply that all measurable lengths are quantized in units of a fundamental Planck length. The Cauchy theory was rigorously combined with the Helmholtz decomposition and Planck-Kleinert crystal hypothesis. The quaternion representation the Cauchy displacement field produced the Klein–Gordon wave equation[7] and the Schrödinger and Poisson equations.[4,15]

In this work, we combine the model of the Cauchy elastic continuum with the Planck–Kleinert crystal hypothesis and derive the diffusion equation. The earlier results allow presenting the problem from a different point of view and show the fundamentals of the diffusion, with the specification of what the diffusion theory is basically about. In what follows we consider aether as the Planck-Kleinert crystal. The building blocks of the crystal are Planck particles m_P, that obey the laws of mass, momentum and energy conservation.[4,12] The properties of such a continuum are approximated by the Cauchy model of an ideal elastic solid continuum in the quaternion representation. Using the Hamilton algebra of quaternions[16] allows combining the Planck-Kleinert crystal hypothesis, the Cauchy theory of an ideal elastic solid and the conclusions from the quaternion wave equation.[3,4,7] The outline of this article is:

Section 2 reports the basic definitions and formulae of the quaternion numbers and functions used in the paper. In 2.1 the notation that is precise and convenient in a case of an ideal elastic Cauchy continuum is shown (can by omitted by experienced readers).

Section 3 presents the mechanical reactions in the real FCC crystal assessed by means of Cauchy continuum theory in vectorial form. In 3.1. the quaternion form of Cauchy model is derived.

Section 4 combines the quaternion model of elastic continuum with P-KC hypothesis.

Section 5 presents the particles in the ideal elastic continuum of the P-KC.

Section 6 derives the diffusion equation.

Section 7 shows the time dependent diffusion equation and quaternionic fluxes.

2 The Quaternions

Quaternions were created by Hamilton as the analog of the complex numbers; his unquestionable motivation was the mechanics of solids and liquids. The reformulation of the basic principles of mechanics of solids in terms of quaternion algebra allows better understanding of the classical mechanics and diffusion. In Hamilton's own words[17]:

> *Time is said to have only one dimension, and space to have three dimensions. The mathematical quaternion partakes of both these elements; in technical language it may be said to be "time plus space"… and in this sense it has, or at least involves a reference to, four dimensions.*

The review of basic definitions and formulae of the quaternion numbers and functions is limited to those used in the paper. The algebra of quaternions Q retains all laws of algebra yet has unique properties.[18] The essentials here are:

(1) **The multiplication of quaternions is noncommutative,** *as a result it allows to differentiate between left and right twists in elastic continuum.*

(2) **The quaternion displacement potential** (the displacement four-potential) combines both a compression scalar potential (pressure) and a torsion vector potential (twist) into a single quaternion (four-vector). Thus, it implicates a reference to the four-dimensional space-time continuum, i.e., Hamilton's "time plus space".

(3) **The quaternion displacement potential** is Lorentz invariant.

In the original Hamilton notation, a quaternion is regarded as the sum of real (scalar) and imaginary (vector) parts: $q = q_0\mathbf{1} + \hat{q} = [q_0, \hat{q}] \in \mathrm{Q}$. An arbitrary quaternion $q \in \mathrm{Q}$ can also be written in terms of its basic components

$$q = (q_0, q_1, q_2, q_3) \in \mathrm{Q} \qquad \text{(Eq 1)}$$

where the four q_0, q_i coefficients are real.[1] The following algebraical notation is useful: $e_0 = 1$, $e_1 = i$, $e_2 = j$, $e_3 = k$. Thus an arbitrary quaternion q, can be written in terms of its basis components as already shown in Eq 1,

$$\begin{aligned} q &= (q_0, q_1, q_2, q_3) = q_0\mathbf{1} + q_1 i + q_2 j + q_3 k \\ &= q_0 + q_1 i + q_2 j + q_3 k \\ &\in \mathrm{Q} \end{aligned} \qquad \text{(Eq 2)}$$

Because it is reserved for the scalar quantity (real), for the first component the notation **l** is used. The unit vector **1**, behaves like the ordinary unit and can be ignored as a factor. The remaining unit vectors i, j, k are usually called imaginary units.

The multiplication is the fundamental operation that is defined by the multiplication of the unit vectors:

- The real quaternion **1** is the identity element.
- The real quaternions commute with all other quaternions, that is $a \cdot q = q \cdot a$ for every quaternion q and every real quaternion a.
- The quaternion multiplication is not commutative: $p \cdot q \neq q \cdot p$, but is associative. Thus the quaternions form an associative algebra: $(p \cdot q) \cdot r = p \cdot (q \cdot r)$ over the real numbers.

The non-commutativity of multiplication is the key property that makes quaternions different from a real and complex numbers. The unit vectors obey the following relations:

$$\begin{aligned} i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j, \\ i\mathbf{1} = \mathbf{1}i = i, \quad j\mathbf{1} = \mathbf{1}j = j, \quad k\mathbf{1} = \mathbf{1}k = k. \end{aligned} \qquad \text{(Eq 3)}$$

A conjugate quaternion is defined as

$$q^* = q_0 - q_1 i - q_2 j - q_3 k \qquad \text{(Eq 4)}$$

where the asterisk means the following: one goes over to the "conjugate" of the quaternion, one gives the imaginary units the opposite sign. For the conjugate mean one gives the vector components (the space part): $\hat{q} = q_1 i + q_2 j + q_3 k$, the opposite sign:

$$q^* = q_0 - \hat{q} = q_0 - q_1 i - q_2 j - q_3 k \qquad \text{(Eq 5)}$$

It is easy to see that the quantity $q \cdot q^*$ is simply a scalar number.

[1] Rigorously, in the mathematical way, the quaternion algebra $q \in \mathrm{Q} := R \otimes P$ can be defined as the four-dimensional Euclidean vector space with:

- the orthonormal basis $\{e_0, e_1, e_2, e_3\}$ and.
- the three-dimensional vector subspace $P = span\{e_1, e_2, e_3\}$.

From Eqs 3–5 it results that $q \cdot q^* = q^* \cdot q = \sum_{i=0}^{3} q_i^2$. Thus Q is a normed algebra with the Euclidian norm

$$\|q\| = \sqrt{q^* \cdot q} \quad \text{(Eq 6)}$$

The multiplication is given by a convenient formula

$$p \cdot q = (p_0 q_0 - \hat{p} \circ \hat{q})e_0 + \hat{p} \times \hat{q} + p_0 \hat{q} + q_0 \hat{p} \quad \text{(Eq 7)}$$

where $p = \sum_{i=0}^{3} p_i e_i$, $q = \sum_{i=0}^{3} q_i e_i \in \mathrm{R}^4$; $\hat{p} = \sum_{i=1}^{3} p_i e_i$, $\hat{q} = \sum_{i=1}^{3} q_i e_i \in P$ and $\circ$, $\times$ mean the scalar and vector products in P, respectively:

$\hat{p} \circ \hat{q} = \sum_{i=1}^{3} p_i q_i$,

$$\hat{p} \times \hat{q} = \det \begin{bmatrix} e_1 & e_2 & e_3 \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{bmatrix}$$

The cross product of p and q relative to the orientation determined by the ordered basis i, j, k is

$$p \times q = \frac{1}{2}(p \cdot q - q \cdot p) \quad \text{(Eq 8)}$$

The vector space R^4 with the multiplication (7) is a noncommutative algebra with unity usually denoted by Q and it is named the quaternion algebra. The quaternion can be represented as the exponential function with its trigonometrical representation:

$$e^q = e^{q_0}(\cos|\hat{q}| + \hat{q}/|\hat{q}| \sin|\hat{q}|) \quad \text{(Eq 9)}$$

Like functions of a complex variable, functions of a quaternion variable suggest useful physical models. For example, the original electric and magnetic fields described by Maxwell can be expressed as functions of a quaternion variable.[18] The so-called Q-valued functions may be written as

$$q(x) = q_0(x)1 + q_1(x)i + q_2(x)j + q_3(x)k, \quad \text{where } x = (x_1, x_2, x_3) \in \Omega \quad \text{(Eq 10)}$$

where the functions $q_0(x), q_l(x)$, $l = 1, 2, 3$ are real-valued and Ω is volume.

Similarly, the functions $q(t, x)$ depending on time t may be considered. Properties such as continuity, differentiability, integrability and so on, which are ascribed to q must be possessed by all the components $q_0(t, x), q_l(t, x)$, $l = 1, 2, 3$.

2.1 Quaternions in Cauchy Elastic Continuum

We introduce now the notation that is precise and convenient in a case of an ideal elastic continuum where only the compression, $q_0 = \sigma_0$, and twist vector (torsion), $\hat{q} = \hat{\phi}$, emerge: $\sigma = \sigma_0 + \hat{\phi}$ explicitly: $q_1 = \phi_1$, $q_2 = \phi_2$ and $q_3 = \phi_3$.

In the following sections we use the Cauchy-Riemann operator D acting on the quaternion valued functions σ:

$$\mathrm{D}\sigma = -\mathrm{div}\hat{\phi} + \mathrm{grad}\sigma_0 + \mathrm{rot}\hat{\phi} \quad \text{(Eq 11)}$$

where $\mathrm{grad}\sigma_0 = \frac{\partial \sigma_0}{\partial x_1} i + \frac{\partial \sigma_0}{\partial x_2} j + \frac{\partial \sigma_0}{\partial x_3} k$, $\mathrm{div}\hat{\phi} = \frac{\partial \phi_1}{\partial x_1} + \frac{\partial \phi_2}{\partial x_2} + \frac{\partial \phi_3}{\partial x_3}$ and

$$\mathrm{rot}\hat{\phi} = \det \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ \phi_1 & \phi_2 & \phi_3 \end{bmatrix}$$

.

Under the constraint $\mathrm{div}\hat{\phi} = 0$, fundamental in the ideal elastic continuum, the Cauchy-Riemann operator D equals

$$\mathrm{D}\sigma = \mathrm{grad}\sigma_0 + \mathrm{rot}\hat{\phi} \quad \text{(Eq 12)}$$

Note that $\mathrm{DD}\sigma = \Delta\sigma$ and hence in the Cauchy elastic continum D corresponds physically to the gradient in $\mathbb{R}^3$.

The exponent function has its trigonometrical representation

$$e^{\sigma} = e^{\sigma_0}\left(\cos\left|\hat{\phi}\right| + \frac{\hat{\phi}}{\left|\hat{\phi}\right|}\sin\left|\hat{\phi}\right|\right), \quad \text{(Eq 13)}$$

where σ is a Q-valued function.

We also will introduce a displacement four-potential, the relativistic function from which the displacement field will be defined. It combines both a compression scalar potential and a torsion vector potential into a single quaternion (four-vector) defined as

$$\sigma = \left(\sigma_0, \hat{\phi}\right) \quad \text{(Eq 14)}$$

The first component of the displacement four-potential is the compression scalar potential, and the other three components make up the twist vector potential. Note that while both the scalar and vector potential depend upon the frame, the displacement four-potential is Lorentz invariant.

3 Cauchy Classical Theory of Elasticity

In the following the mechanical reactions in the real FCC crystal are assessed by means of Cauchy continuum theory, i.e., we approximate the grainy continuum of the crystal by field variables. The Cauchy model of the ideal elastic continuum[8] constitutes a consistent base used here since:

- The macroscopic phenomena are expressed in terms of field variables.[19]

- From its beginning was effectively applied in studying the elementary waves.[20]
- The proof of uniqueness of solutions[21] and completeness proof are done.[22]

We follow Planck-Kleinert crystal hypothesis[3] and consider FCC structure, where the Poisson number $\nu = 0.25$, l_P denotes the dimension of the FCC elementary cell that consists of four interacting Planck particles showing the mass m_P. The continuum is treated as a closed system occupying the constant volume $\Omega \subset \mathrm{R}^3$. The displacement vector $\mathbf{u}$ has the standard definition

$$\mathbf{u} := \mathbf{x}(\mathbf{X}, t) - \mathbf{X} \qquad \text{(Eq 15)}$$

where $\mathbf{X}$ denotes the position vectors of material points at $t = 0$ and $\mathbf{x}$ spatial position at other time t of the point that moved and was at $\mathbf{X}$ at $t = 0$. The Cauchy theory describes the case when any infinitesimal line element $\mathrm{d}\mathbf{X}$ of the reference configuration undergoes extremely small rotations and fractional change in length, in deforming the corresponding line element $\mathrm{d}\mathbf{x}$, i.e., when $\left|\partial u_i/\partial X_j\right| << 1$. In the Planck-Kleinert crystal continuum, P-KC, we assume:

1. The continuum density, ρ_P, is enormously high and we consider the small deformation limit only, $l_P = const.$, thus the density changes are negligible and $\rho_P = 4m_P/l_P^3 = const.$
2. The small deformation limit implies the invariant wave velocities, e.g., the transverse wave velocity: $c = \sqrt{0.4Y/\rho_P} = const.$, where Y is the Young modulus,[23] Eq 16.
3. We consider here the long evolution times, $t >> t_P$, where t_P is the Planck time;
4. The quasi-stationary wave exists that exhibits the velocity of its mass center, υ.[4]

In such a continuum, the equation of motion relates to local acceleration due to the displacement $\mathbf{u}$, with the field variables, the compression $(\mathrm{div}\,\mathbf{u})$ and twist $(\mathrm{rot}\,\mathbf{u})$

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = 3c^2 \mathrm{grad}\,\mathrm{div}\,\mathbf{u} - c^2 \mathrm{rotrot}\,\mathbf{u} \qquad \text{(Eq 16)}$$

where we only for the sake of simplicity do not consider the external fields.

From Eq 16, the energy per mass unit in the deformation field follows[23,24]

$$e = \frac{\rho_E}{\rho_P} = \frac{1}{2}\mathbf{u}_t \circ \mathbf{u}_t + \frac{3}{2}c^2(\mathrm{div}\mathbf{u})^2 + \frac{1}{2}c^2 \mathrm{rot}\mathbf{u} \circ \mathrm{rot}\mathbf{u} \quad \text{where} \quad \frac{\partial \mathbf{u}}{\partial t} = \mathbf{u}_t \qquad \text{(Eq 17)}$$

Equation 16 and relation 17 obey the Euler–Lagrange relation, $\frac{\partial e}{\partial \mathbf{u}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial e}{\partial \mathbf{u}_t}\right) = 0$, and are sufficient to describe every deformation in the ideal elastic continuum.

3.1 Quaternion Form of Cauchy Model

The Helmholtz theorem allows using quaternion algebra and Cauchy-Riemann operator. Strong formulation of the Helmholtz's decomposition theorem introduces four-potential $\mathbf{A}$ and allows expressing displacement by the Cauchy-Riemann operator of the quaternion potential $\mathbf{A}$, using Eq 12:

$$\mathbf{u} = \mathrm{D}\mathbf{A} = \mathrm{grad}A_0 + \mathrm{rot}\mathbf{A}_\phi = \mathrm{grad}\sigma_0 + \mathrm{rot}\hat{\phi} \qquad \text{(Eq 18)}$$

where $\mathrm{div}\mathbf{A}_\phi = \mathrm{div}\hat{\phi} = 0$ and $\mathbf{A} = \sigma$ denotes four-potential (quaternion potential):

$$\mathbf{A} = A_0 + \mathbf{A}_\phi = A_0\mathbf{1} + iA_1 + jA_2 + kA_3. \qquad \text{(Eq 19)}$$

We use here the weak formulation of the Helmholtz's decomposition theorem. Every deformation can be expressed by the curl-free component, $\mathbf{u}_0$, and a divergence-free component, $\mathbf{u}_\phi$, and if $\mathbf{u}$ belongs to the C^3 class of functions then $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_\phi$, where $\mathrm{rot}\mathbf{u}_0 = 0$ and $\mathrm{div}\mathbf{u}_\phi = 0$.[25] As a result, upon acting on Eq 16 by the divergence and rotation operators, we get two equations: the vector transverse, $\mathbf{a}_{tt} = k\,\Delta\mathbf{a}$ and the longitudinal scalar equation $a_{tt} = k\,\Delta a$:

$$\begin{aligned}
&\mathrm{div}\left(\frac{\partial^2 \mathbf{u}}{\partial t^2} = 3c^2\mathrm{graddiv}\mathbf{u} - c^2\mathrm{rotrot}\mathbf{u}\right) \quad \Rightarrow \quad \frac{\partial^2}{\partial t^2}\mathrm{div}\mathbf{u}_0 = 3c^2\,\Delta\mathrm{div}\mathbf{u}_0,\\
&\mathrm{rot}\left(\frac{\partial^2 \mathbf{u}}{\partial t^2} = 3c^2\mathrm{graddiv}\mathbf{u} - c^2\mathrm{rotrot}\mathbf{u}\right) \quad \overset{\Delta\mathbf{u}=\nabla(\nabla\cdot\mathbf{u})-\nabla\times(\nabla\times\mathbf{u})}{\Rightarrow}\\
&\frac{\partial^2}{\partial t^2}\mathrm{rot}\mathbf{u}_\phi = c^2\,\Delta\mathrm{rot}\mathbf{u}_\phi.
\end{aligned} \qquad \text{(Eq 20)}$$

Thus, the Cauchy equation of motion combined with the Helmholtz decomposition theorem leads to four second-order scalar differential equations, "quattro cluster" (20), that implies the transverse and longitudinal waves in the Cauchy elastic solid. Note that these equations remain coupled by the relation of the energy density (17), however, the more complex wave phenomena are not yet obvious in Eq 20.

The Cauchy deformation field in the quaternion representation shows the deepness of physical reality: the correlation with Hamilton time–space continuum and the complexity of wave phenomena. The Hamilton algebra Q allows coupling the the curl-free and a divergence-free components that are separated in (20). Upon denoting $\sigma_0 = \mathrm{div}\mathbf{u}_0$ and $\hat{\phi} = \mathrm{rot}\mathbf{u}_\phi$ we get

$$\begin{aligned} \frac{\partial^2 \sigma_0}{\partial t^2} &= 3c^2\,\Delta\sigma_0, \\ \frac{\partial^2 \hat{\phi}}{\partial t^2} &= c^2\,\Delta\hat{\phi}, \end{aligned} \qquad \text{(Eq 21)}$$

and the energy density formula (17) takes the form

$$e = 1/2\mathbf{u}_t \circ \mathbf{u}_t + 3/2c^2\sigma_0^2 + 1/2c^2\hat{\phi}\circ\hat{\phi} \qquad \text{(Eq 22)}$$

The decomposition $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_\phi$ in (20) and change of variables results in four equations in 21 and allows using the Hamilton quaternions. Namely, it implies the existence of the deformation field $\sigma = \sigma_0 + \hat{\phi}$ that represents the twist and compression fields as a superposition of real (scalar compression σ_0) and imaginary (twist vector $\hat{\phi}$) field parts at each point

$$\sigma = \sigma_0 + \hat{\phi} \in \mathrm{Q} \quad \text{and} \quad \sigma^* = \sigma_0 - \hat{\phi} \quad \in \mathrm{Q} \qquad \text{(Eq 23)}$$

where the Helmholtz decomposition implies the following constraint

$$\mathrm{div}\hat{\phi} = \mathrm{divrot}\mathbf{u}_\phi = 0. \qquad \text{(Eq 24)}$$

The Helmholtz decomposition theorem,[24] states that any sufficiently continuous vector field can be represented as the sum of the gradient of a scalar potential plus the curl of a vector potential. The gradient term in the decomposition has a zero rotation and is referred to as the *irrotational* part, while the rot term has no divergence and is called *solenoidal*. This representation specifies three displacement components in terms of four potential components, and furthermore the divergence of $\hat{\phi}$ is an arbitrary constant. It is common to choose $\hat{\phi}$ with zero divergence: $\mathrm{div}\hat{\phi} = 0$.

Adding equations in 21 and from 23, we get the quaternion form of the motion equation

$$\frac{1}{c^2}\frac{\partial^2\sigma}{\partial t^2} - \Delta\sigma - 2\Delta\sigma_0 = 0 \qquad \text{(Eq 25)}$$

where $\sigma = \sigma_0 + \hat{\phi}$ and $\hat{\phi}$ must obey the constraint from (24).

The above equation requires boundary conditions for a solution. Since $\mathbf{u}_t \circ \mathbf{u}_t = \hat{u}_t \circ \hat{u}_t = -\hat{u}_t \cdot \hat{u}_t = \hat{u}_t \cdot \hat{u}_t^*$ where $\hat{u}_t = \hat{u}_{t1}i + \hat{u}_{t2}j + \hat{u}_{t3}k$ and $\mathbf{u}_t = (u_{t1}, u_{t2}, u_{t3})$, the overall energy of the deformation field, the formula (22) becomes in quaternion form

$$e = \tfrac{1}{2}\hat{u}_t \cdot \hat{u}_t^* + \tfrac{1}{2}c^2\sigma\cdot\sigma^* + c^2\sigma_0^2 \qquad \text{(Eq 26)}$$

The energy is conserved, hence (26) leads to the non-local boundary condition for Eq (25).[7]

The Cauchy theory of elastic continuum combined with the Helmholtz decomposition theorem and quaternion algebra results in second-order differential Eq (25) and constraint (24). It infers the transverse, longitudinal and multiple complex forms of waves and shows Lorentz invariance. Equation (25) and the relation (26) satisfy the Euler–Lagrange differential equation, i.e., satisfy the fundamental equation of the calculus of variations. Paraphrasing Hamilton:

> *Pressure is said to have only one, and twist to have three dimensions. The mathematical quaternion partakes of both these elements; in technical language it may be said to be "pressure plus twist": and in this sense it has, or at least involves a reference to, four dimensions and is analogue of time-space continuum, i.e., the Hamilton's time-space.*

4 Quaternion Quantum Mechanics

In simple words, we regard aether as an analog to the Cauchy elastic solid. The properties of ideal elastic continuum are presented in Table 1.

The fundamental new results, explicitly the ontology of Quaternion Quantum Mechanics and the family of waves in elastic continua were already published.[4] In Sect. 3, upon adding Eqs (21), we obtained quaternion form of the motion equation. The relativistic waves in the Cauchy continuum arise from Eq (25).[7] Let us consider the stable wave showing the constant energy due to the motion and the strain of the medium: $E_n = \text{const}$. Upon splitting Eq (25) into the system of the wave and Poisson type equations, the non-linear form of the wave equation follows[4,7]:

$$\begin{cases} \dfrac{1}{c^2}\dfrac{\partial^2\sigma}{\partial t^2} - \Delta\sigma + k^2\sigma\cdot\sigma^* = 0, \\ -2\Delta\sigma_0 - k^2\sigma\cdot\sigma^* = 0, \end{cases} \qquad \text{(Eq 27)}$$

where $k = 1/\lambda$ and $\lambda = f(E)$ denotes the wavelength.

That is, the quaternion motion equation produces the non-linear stationary waves, Eq (27). The system (27) is a hyperbolic-elliptic quaternion representation of a the wave and has solutions of the form

$$\sigma(t,x) = \sigma_0 + \hat{\phi} = \sigma_0 + \phi_1 i + \phi_2 j + \phi_3 k \in \mathrm{Q} \qquad \text{(Eq 28)}$$

The second equation in (27) is the Poisson type equation, and it describes/defines the compression potential as a function of energy density. The first equation in (27) is the wave equation, namely the quaternion form of the Klein–Gordon type equation[26]

$$\begin{cases} \left(\dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2} - \Delta\right)\sigma + k_n^2\sigma^*\cdot\sigma = 0, \\ 2\Delta\sigma_0 = -k_n^2\sigma^*\cdot\sigma. \end{cases} \qquad \text{(Eq 29)}$$

Table 1 The physical constants of the Planck–Kleinert crystal (fcc ideal isotropic crystal): Nat'l Institute of Standards and Technology (2018); http://physics.nist.gov

Label used in this work	Planck constants	Symbol for unit	Value	SI unit	References
Lattice parameter	Planck length	l_P	$1.616229(38) \times 10^{-35}$	m	NIST
Poisson ratio		ν	0.25	…	3
Mass of the Planck particle	Planck mass	m_P	$2.176470(51) \times 10^{-8}$	kg	NIST
Planck–Kleinert crystal density		ρ_P	2.062072×10^{97}	kg m^{-3}	NIST
Duration of the internal process	Planck time	t_P	$5.39116(13) \times 10^{-44}$	s	NIST
Young modulus	Intrinsic energy density	Y	$4.6332447 \times 10^{114}$	kg m^{-1} s^{-2}	$c = \sqrt{0.4Y/\rho_P}$
Transverse wave velocity	Light velocity in vacuum	c	2.99792458×10^{8}	m s^{-1}	NIST

In a case of the stable particle showing the constant overall energy (equivalent mass) we have: $m = E\,c^{-2} = const$, where E denotes overall wave energy and $c = l_P/t_P$. It was shown that system (29) can be written as[7]:

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2} - c^2\Delta\right)\sigma + \frac{8\pi m}{m_P t_P^2}\sigma^* \cdot \sigma = 0, \\ 2c^2\Delta\sigma_0 = -\frac{8\pi m}{m_P t_P^2}\sigma \cdot \sigma^*. \end{cases} \quad \text{(Eq 30)}$$

The first equation above is the Klein–Gordon equation that in the equivalent conventional form equals

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\sigma + \frac{8\pi m}{h t_P}\sigma^* \cdot \sigma = 0 \quad \text{(Eq 31)}$$

where $h = m_P\,c^2\,t_P = 1.0545727 \times 10^{-34}[\mathrm{kg} \cdot \mathrm{m}^2\mathrm{s}^{-1}]$.

The energy computed using formula (26) as well as Eq (31) is per definition always positive due to the constraint (24). The second equation in (30) describes the irrotational, e.g., compression, potential in the displacement field. When expressed as a function of the local mass density[7]: $\rho = \rho_E/c^2$, where $\rho_E = m\,c^2\sigma \cdot \sigma^*/l_P^3$ we get

$$c^2\Delta\sigma_0 = -4\pi\rho\frac{l_P^3}{m_P t_P^2} = -4\pi\,\rho\,G \quad \text{(Eq 32)}$$

Using the data in Table 1, the gravitational constant equals:

$$G = l_P^3/\left(t_P^2 m_P\right) = 6.674082 \times 10^{-11}$$

$\left[\mathrm{m}^3 \cdot \mathrm{kg}^{-1}\mathrm{s}^{-2}\right]$. The more detailed analysis of the Poisson equation can be found in previous papers.[4,7]

Ulrych confirmed the existence of spin quaternion waves in the Klein-Gordon theory.[26] Recently Gantner demonstrated the equivalence of complex and quaternionic quantum mechanics.[27] These results validate the postulate of existence of particles (stable waves) in the P-KC continuum, Eq (30).

5 The Particles in the Cauchy Continuum of the P-KC

In this section we derive the diffusion equation. The wave will be treated as a particle in an arbitrary positive volume Ω that has a mass center and translation velocity υ.[4] The overall wave energy E is affected be external fields, $E = E^0 + Q$, where E^0 and Q are the ground and excess (e.g., kinetic) energies, and the overall energy density is defined as usual $E = \int_\Omega \rho_E \mathrm{d}x$. The equivalent mass m is correlated to the wave overall energy[28];

The local movements of the continuum itself, i.e., the local kinetic energy density, k, is caused by the lattice local velocity, $\hat{u}_t$. From relation (26), upon substituting $\sigma^3 = \sqrt{3}\sigma_0$, the overall wave energy is expressed by

$$\begin{aligned} E = E^0 + Q = \int_\Omega \rho_E \mathrm{d}x &= \int_\Omega \left(\frac{1}{2}\rho_P \hat{u}_t \cdot \hat{u}_t^* + \rho_P \frac{1}{2}c^2\sigma^3 \cdot \sigma^{3*}\right)\mathrm{d}x \\ &= c^2 \int_\Omega \left(\frac{1}{2}\frac{\rho_P}{c^2}\hat{u}_t \cdot \hat{u}_t^* + \frac{1}{2}\rho_P \sigma^3 \cdot \sigma^{3*}\right)\mathrm{d}x \\ &= c^2 \int_\Omega \rho_P k \mathrm{d}x + c^2 \int_\Omega \rho_P s \mathrm{d}x \\ &= K + S, \end{aligned} \quad \text{(Eq 33)}$$

where $\sigma^3 = \sigma_0^3 + \hat{\phi} = \sqrt{3}\sigma_0 + \hat{\phi}$.

Upon dividing relation (33) by c^2, the overall mass of the particle, m, and the particle mass density, ρ, follow,

$$m = \int_\Omega \rho \mathrm{d}x = \int_\Omega \rho_P \left(\frac{1}{2c^2}\hat{u}_t \cdot \hat{u}_t^* + \frac{1}{2}\sigma^3\sigma^{3*}\right)\mathrm{d}x \quad \text{(Eq 34)}$$

where $\rho = \rho_E/c^2$ denotes the density of an equivalent mass of the wave, i.e., the local particle density in Ω.

The quasi-stationary wave has to satisfy the constraint (24) and at every position, the energy density is a sum of the motion k, and strain s, energy terms:

$$\rho_E = c^2\rho_P(k + s) = 0.4Y(k + s) \quad \text{(Eq 35)}$$

We conclude that the "the overall particle mass" follows from the relations (33) and (34): $m = E/c^2$. The *mass* is ontic in the sense that the particle mass is a result of the overall energy of the wave in elastic continuum of the Planck particles, see also.[28] Note that in general when $Q > 0$, m differs from the mass at the ground state. By using the extremum principle, namely the action concept, one can quantify the elementary properties of such waves. At every position within the particle (in Ω):

1. the existence assumption of the quasi-stationary waves implies an equal duration of the s- and k-periodic cycles in the whole volume occupied by the wave, $\Delta t = const$. Consequently this implies, that the s- and k-actions are equal everywhere

$$\int_t^{t+\Delta t} s(\tau, x)\,\mathrm{d}\tau = \int_t^{t+\Delta t} k(\tau, x)\,\mathrm{d}\tau = \gamma(x)\,\Delta t \qquad \text{(Eq 36)}$$

2. the sum of the overall strain, S, and the kinetic energy, K, in relation (33) equals the overall wave energy $E = E^0 + Q$, and, e.g., is time-invariant in the case of a free particle;
3. spans of the strain and the motion energy terms are equal,

$$\begin{aligned}[0, \max\{k(t,x)\}] &= [0, \max\{s(t,x)\}] \\ &= \left[0, \rho_E(t,x)/\left(c^2\rho_P\right)\right] \\ &= [0, \rho_E(t,x)/(0.4\,Y)]\end{aligned}$$

The relation (36) is valid for the whole Ω so

$$\int_t^{t+\Delta t}\int_\Omega s(\tau,x)\mathrm{d}x\,\mathrm{d}\tau = \int_t^{t+\Delta t}\int_\Omega k(\tau,x)\mathrm{d}x\,\mathrm{d}\tau = \gamma_\Omega\,\Delta t \qquad \text{(Eq 37)}$$

and also for an arbitrary number of cycles: $t = n\,\Delta t$. Thus, from assumptions 2 and 3 above and the relation (33), it follows that both actions in Ω can be approximated by the discrete formula

$$\gamma_\Omega\, n\,\Delta t = \int_0^{n\Delta t}\int_\Omega s\,\mathrm{d}x\,\mathrm{d}\tau = \int_0^{n\Delta t}\int_\Omega k\,\mathrm{d}x\,\mathrm{d}\tau \qquad \text{(Eq 38)}$$

Taking into account that we consider time evolution in a case when $t >> \Delta t$, the continuous expressions for both actions is allowed

$$\gamma_\Omega\, t = \int_0^t\int_\Omega s\,\mathrm{d}x\,\mathrm{d}\tau = \int_0^t\int_\Omega k\,\mathrm{d}x\,\mathrm{d}\tau \qquad \text{(Eq 39)}$$

and time derivative of the relations in (39) equals:

$$\int_\Omega s\,\mathrm{d}x = \int_\Omega k\,\mathrm{d}x \text{ for } t >> \Delta t \qquad \text{(Eq 40)}$$

Terms s and k, in (40) oscillate and depend on time and position. We normalize the displacement term s in (33) with respect to the overall particle mass, relation (34). From the formulae, (33), (34) and (40), the normalized mass density due to the motion of mass k, and strain energy density s, within the particle (in Ω) equals

$$\int_\Omega \frac{\rho_P}{m}\sigma^3\cdot\sigma^{3*}\mathrm{d}x = \int_\Omega \psi\cdot\psi^*\,\mathrm{d}x = 1,\ \text{where } \psi = \sqrt{\frac{\rho_P}{m}}\sigma^3 \qquad \text{(Eq 41)}$$

Obviously, both terms, $\psi = \sqrt{\frac{\rho_P}{m}}\sigma^3$ and $\psi\cdot\psi^*$, vary in time. Consequently from (33), (40) and (41)

$$0 = \int_\Omega \left(\rho_P c^2\sigma^3\cdot\sigma^{3*} - E\,\psi\cdot\psi^*\right)\mathrm{d}x \qquad \text{(Eq 42)}$$

and

$$0 = \int_\Omega \left(\rho_P\,\hat{u}_t\cdot\hat{u}_t^* - E\,\psi\cdot\psi^*\right)\mathrm{d}x \qquad \text{(Eq 43)}$$

The relations (33), (34) and (42) imply the relation between the overall energy of the wave, the overall wave mass: $E = mc^2$. The wave periodicity implies that by solving the relation (43), one should expect only discrete values if excess translation energy $Q > 0$. Both, the excess Q, and the ground E^0, energies are entangled in (42) and (43).

6 Particle in the Time Invariant Potential Field

Let us consider now the evolution of the wave as in the relation (33) in the time invariant potential field, e.g., the wave in the field generated by other particles when velocity of the wave mass center is low. The overall energy is now a sum of the ground and excess energy Q,

$$\begin{aligned}E &= E^0 + Q \\ &= \int_\Omega \left(\frac{1}{2}\rho_P\,\hat{u}_t\cdot\hat{u}_t^* + \frac{1}{2}\rho_P c^2\sigma^3\cdot\sigma^{3*} + V(x)\,\psi\cdot\psi^*\right)\mathrm{d}x\end{aligned} \qquad \text{(Eq 44)}$$

where $\sqrt{\sigma^3\cdot\sigma^{3*}} = \left|\sigma^3\right|$.

We consider the low excess energies only, $\upsilon << c$. Consequently the impact of Q on the overall particle mass is marginal, $m \cong m_0$, and the displacement σ^3 in (44) can be normalized using the formula (41). Thus relation (44) becomes

$$\begin{aligned}E = E^0 + Q &= \int_\Omega \left(\frac{1}{2}\rho_P\hat{u}_t\cdot\hat{u}_t^* + \frac{1}{2}m\,c^2\psi\cdot\psi^* + V(x)\,\psi\cdot\psi^*\right)\mathrm{d}x \\ &= \frac{1}{2}m\,c^2 + \int_\Omega \left(\frac{1}{2}\rho_P\,\hat{u}_t\cdot\hat{u}_t^* + V(x)\,\psi\cdot\psi^*\right)\mathrm{d}x.\end{aligned} \qquad \text{(Eq 45)}$$

Both, the E^0 and m are constant, thus it is enough to minimize the relation

$$Q = \int_\Omega \left(\frac{1}{2} \rho_P \, \hat{u}_t \cdot \hat{u}_t^* + V(x) \, \psi \cdot \psi^* \right) \mathrm{d}x \qquad \text{(Eq 46)}$$

Above relation contains the unknown velocity due to the potential $V(x)$, i.e., contains two unknowns: $\hat{u}_t$ and ψ. The Cauchy–Riemann operator of the deformation, $D\sigma^3$, can be understood, by means of the relation (12), as an analogy of the gradient in R^3. In the classical dynamics, the potential gradient results in acceleration. For the quaternion representation of the deformation field it is reasonable to guess that the local mass momentum in the lattice, $\hat{p} = m_p \hat{u}_t$, is related to the Cauchy–Riemann operator of the quaternion displacement, $D\sigma^3$. Namely, the local lattice velocity $\hat{u}_t$ is proportional to the force that is the normalized Cauchy–Riemann derivative of local displacement $l_P D\sigma^3$, and the transverse wave velocity c. Accordingly

$$\hat{p} = -m_P c l_P \, D\sigma^3 = -h \, D\sigma^3 \qquad \text{(Eq 47)}$$

where we introduced the Planck constant $h = m_P c^2 t_P$ and t_P is the time that transverse wave travels at the lattice distance: $l_P = c t_p$.

The overall momentum balance requires

$$\hat{u}_t = \frac{\hat{p}}{m} = -\frac{h}{m} D\sigma^3 \qquad \text{(Eq 48)}$$

By introducing (48), the relation (46) becomes

$$Q = \int_\Omega \left[\rho_P \, \frac{h^2}{2m^2} \left(D\sigma^3\right) \cdot \left(D\sigma^3\right)^* + V(x) \, \psi \cdot \psi^* \right] \mathrm{d}x \qquad \text{(Eq 49)}$$

Finally, the normalization using (41) results in the functional

$$Q[\psi] = \int_\Omega \left[\frac{h^2}{2m} (D\psi) \cdot (D\psi)^* + V(x) \, \psi \cdot \psi^* \right] \mathrm{d}x \qquad \text{(Eq 50)}$$

There are numerous methods solving above problem, e.g., the path integrals, the Hamilton–Jacobi equation, etc. The functional $Q[\psi]$, that is the integral above, was minimized with respect to a quaternion function such that ψ satisfies the normalization introduced in the relation (41).

In simple words, we look for a differential equation that has to be satisfied by the ψ function to extremize (here minimize) the energies allowed by (50). Subsequently we will show that the extremum problem leads to the quaternion analog of the time–independent diffusion equation.

Given the functional (50) and the constraint in (24), the conditional extreme was found using the Lagrange coefficient method and the Du Bois-Reymond variational lemma.[29] The whole procedure is presented in.[4] It was found that there exists a multiplayer $\lambda \neq 0$ such that ψ minimalizes the functional

$$\hat{Q}[\psi] = \int_\Omega \left[\frac{h^2}{2m} (D\psi) \cdot (D\psi)^* + V(x) \, \psi \cdot \psi^* + \lambda \left(\frac{1}{|\Omega|} - \psi \cdot \psi^* \right) \right] \mathrm{d}x. \qquad \text{(Eq 51)}$$

In such a case ψ satisfies the time invariant diffusion equation

$$-\frac{h^2}{2m} \Delta\psi + [V(x) - \lambda] \, \psi = 0 \qquad \text{(Eq 52)}$$

A constant factor λ on the right–hand side can be considered as an extra energy of the particle as a result of the field $V = V(x)$. For $E = \lambda$, Eq (52) is clearly the time independent Schrödinger equation satisfied by the particle in the ground state of the energy E,

$$-\frac{h}{2m} \Delta\psi + \frac{1}{h} [V(x) - E] \, \psi = 0 \qquad \text{(Eq 53)}$$

It has to be satisfied together with the condition

$$\mathrm{div}\hat{\psi} = 0 \text{ where } \psi = \psi_0 + \hat{\psi} \qquad \text{(Eq 54)}$$

Upon using the NIST data[26] of Planck's natural units m_P, l_P, t_p and the light velocity c, the computed constant equals the Planck constant $h = m_P c^2 t_P = 6.626069311 \times 10^{-34}$.

7 Time Dependent Diffusion Equation

By analogy to the complex time-dependent Schrödinger equation $i \frac{\partial \Psi}{\partial t} = -\frac{h}{2m} \Delta\Psi + \frac{1}{h} V(x) \, \Psi$, one may propose a quaternion form

$$\frac{i + j + k}{3} \frac{\partial \Psi}{\partial t} = -\frac{h}{2m} \Delta\Psi + \frac{1}{h} V(x) \, \Psi. \qquad \text{(Eq 55)}$$

It was demonstrated that in the diagonal case the quaternion (55) and complex time dependent Schrödinger equations are equivalent in the same sense. Moreover, it was shown that by suitable natural substitution, the time-dependent Schrödinger equation, Eq (55), implies the quaternion stationary Schrödinger Equation (53).[4]

By multiplying by $-(i + j + k)$, the Eq (55) can be expressed also as

$$\left(\frac{i + j + k}{3} \frac{\partial \Psi}{\partial t} = -\frac{h}{2m} \Delta\Psi + \frac{1}{h} V(x) \, \Psi \right) \times (-i - j - k)$$
$$\Rightarrow \quad \frac{\partial \Psi}{\partial t} = \Theta_P \, \Delta\Psi - \frac{i + j + k}{h} V(x) \, \Psi \qquad \text{(Eq 56)}$$

where $\Theta_P = (i + j + k) \frac{h}{2m} \left[\frac{m^2}{s} \right]$ denotes the imaginary diffusion coefficient.

When the external potential $V(x)$ is negligible, then we generated a quaternion form of the diffusion equation

$$\frac{\partial \Psi}{\partial t} = \Theta_P \, \Delta \Psi \qquad \text{(Eq 57)}$$

Due to identity $\Delta\Psi = -DD\Psi$ it can be written in the flux form

$$\frac{\partial \Psi}{\partial t} = -\Theta_P \, DD\Psi = D(-\Theta_P \, D\Psi), \qquad \text{(Eq 58)}$$
$$\frac{\partial \Psi}{\partial t} = D\,[J(\Psi)] \text{ where } J(\Psi) = -\Theta_P \, D\Psi.$$

We demonstrated that the energy conservation in the elastic Cauchy continuum implies a quaternion form of the diffusion equation, Eq (57), and allows defining diffusive fluxes, Eq (58). Both can be regarded as the fundamental diffusion equations.

8 Summary

In this article we evaded using matrices, restricted our attention to the quaternion algebra and assumed rigorous identity: *the particle = the overall energy of stable real wave*.

The aim of our work has been to show the ontology of the diffusion equation. The Cauchy model of an ideal elastic solid with the Helmholtz decomposition theorem and the Hamilton's quaternion algebra generates the transverse, longitudinal and multiple forms of waves. The quaternionic analog of vector formulation of the Cauchy model elucidates the coupling between the irrotational and solenoidal displacements in the deformation field (compression and torsion) and allows for a physical interpretation of the wave mechanics. The wave, i.e., the collective movement of the ideal elastic continuum, is considered equivalent to the particle.

By further combining the quaternion representation of the Cauchy model with the Planck–Kleinert crystal postulate we presented the approach that allowed the self–consistent classical interpretation of the basic transport phenomena. We demonstrated that the energy conservation in the elastic Cauchy continuum implies a quaternion form of the diffusion equation and that it can be regarded as the fundamental formulation. Our derivation provides new evidence that there is a rigorously defined mathematical connection between classical and quantum mechanics.

What is more, the results demonstrate that quaternions are much more comfortable than vectors, have huge advantages in the calculation of twist (and rotations) and can be regarded as the most concise representation of physical reality. Not only at the Planck scale, not only helpful and convenient, but quaternions also allow us to understand the processes in continua.

To conclude this discussion, one may cite opinions by Nobel Prize winners that may incite unconvinced readers to reconsider their positions. We follow Sir Arthur Eddington's remark.

> *I cannot believe that anything so ugly as multiplication of matrices is an essential part of the scheme of nature.*[30]

Schrödinger did not like the "probability" interpretation of the wave function and always considered the wave to be a real wave:

> *Let me say at the outset, that in this discourse, I am opposing not a few special statements of quantum physics held today (1950), I am opposing as it were the whole of it, I am opposing its basic views that have been shaped 25 years ago, when Max Born put forward his probability interpretation, which was accepted by almost everybody.*[31]

There are widely known remarks; by Richard Feynman in 1964[32]:

> *It is safe to say that no one understands quantum mechanics*

and Murray Gell-Mann in his lecture at the 1976 Nobel Conference[33]:

> *Niels Bohr brainwashed the whole generation of theorists into thinking that the job (of finding an interpretation of quantum mechanics) was done 50 years ago.*

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