

A bivariate index vector for measuring departure from double symmetry in square contingency tables

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Abstract For square contingency tables, a double symmetry model having a matrix structure that combines both symmetry and point symmetry was proposed. Also, an index which represents the degree of departure from double symmetry was proposed. However, this index cannot simultaneously characterize the degree of departure from symmetry and the degree of departure from point symmetry. For measuring the degree of departure from double symmetry, the present paper proposes a bivariate index vector that can simultaneously characterize the degree of departure from symmetry and the degree of departure from point symmetry.

Keywords Confidence region · Double symmetry · Index vector · Visualization

Mathematics Subject Classification 62H17

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1 Introduction

Consider an $r \times r$ square contingency table $\mathbf{P} = (p_{ij})$ with the same row and column nominal classifications. Here p_{ij} denotes the probability that an observation will fall in the i th row and j th column of the table ($i = 1, \dots, r; j = 1, \dots, r$). For the analysis of square contingency tables, we are interested here in whether or not the row classification is symmetric or point-symmetric to the column classification (see Tahata and Tomizawa 2014).

Bowker (1948) proposed the symmetry model defined by

$$p_{ij} = p_{ji} \quad \text{for } i \neq j.$$

This model indicates that the probabilities are symmetric with respect to the main diagonal of the contingency table. Namely, this model indicates that the probability that an observation will fall in row category i and column category j is equal to the probability that the observation falls in row category j and column category i ($i \neq j$). Wall and Lienert (1976) proposed the point symmetry model defined by

$$p_{ij} = p_{r+1-i, r+1-j} \quad \text{for } i, j = 1, \dots, r.$$

This model indicates that the probabilities are point symmetric with respect to the center cell (when r is odd) or center point (when r is even) of the contingency table.

Consider the data in Table 1 taken from Andersen (1997, p. 226). These data show the forecasts for production and prices for the coming three year periods given by experts in July 1956 and the actual production figures for production and prices in May 1959 given from a sample of about 4000 Danish factories. For Table 1, we shall denote, for example, the probability that the forecast and the actual values are ‘‘Higher’’ and ‘‘Lower’’, respectively, by $P(H, L)$, and the probability that those are ‘‘Higher’’ and ‘‘No change’’, respectively, by $P(H, N)$. For these data, we are interested in whether (1) $P(H, N) = P(N, H)$, $P(H, L) = P(L, H)$, and $P(L, N) = P(N, L)$, and (2) $P(H, H) = P(L, L)$, $P(H, N) = P(L, N)$, $P(H, L) = P(L, H)$, and $P(N, H) = P(N, L)$. Note that (1) means the symmetry of cell probabilities with respect to the main diagonal in the table, and (2) means the symmetry of cell probabilities with respect to the center point (the center cell) in the table.

Tomizawa (1985) considered a matrix structure that combines both symmetry and point symmetry. The double symmetry model is defined by

$$p_{ij} = p_{ji} = p_{r+1-i, r+1-j} = p_{r+1-j, r+1-i} \quad \text{for } i, j = 1, \dots, r.$$

This model indicates that the probabilities are symmetric with respect to both the main diagonal and the secondary diagonal of the contingency table.

In case of unknown probabilities p_{ij} these are estimated from an observed $r \times r$ contingency table $\mathbf{N} = (x_{ij})$ obtained from random sampling. Typically, when a given model does not hold, we are interested in checking for an extended model or in analyzing the deviation from the model (from the residuals). On the other hand, we are also interested in measuring the degree of departure from the corresponding

Table 1 The two tables below show the forecasts for production and prices for the coming three year periods given by experts in July 1956 and the actual production figures for production and prices in May 1959 given from a sample of about 4000 Danish factories; from Andersen (1997, p. 226)

Forecast 1956	Actual 1959			Total
	Higher	No change	Lower	
(a) For prices				
Higher	209	169	6	384
No change	190	3073	184	3447
Lower	3	62	81	146
Total	402	3304	271	3977
(b) For production				
Higher	532	394	69	995
No change	447	1727	334	2508
Lower	39	230	231	500
Total	1018	2351	634	4003

model. For evaluating goodness-of-fit of the model, test statistics (e.g., Pearson’s Chi-squared statistic or likelihood ratio statistic) are used. When the model does not hold for several tables, we may be interested in comparing the degrees of departure from the corresponding model. However, test statistics would not be useful for comparing the degrees of departure from the model in several tables because test statistics depend on the dimension r and sample size. Thus, for comparing the degrees of departure from the model in several tables, we are interested in an index that does not depend on the dimension r and sample size.

1.1 Index Φ_S for departure from symmetry

For square contingency tables with nominal categories, Tomizawa (1994) considered the index Φ_S which represents the degree of departure from symmetry. The index Φ_S is independent of the dimension r . Assume that $p_{ij} + p_{ji} > 0$ for all $i \neq j$. The index Φ_S is expressed as follows:

$$\Phi_S = \frac{1}{\log 2} I_S,$$

where

$$I_S = \sum_{i \neq j} \sum a_{ij} \log \left(\frac{a_{ij}}{b_{ij}} \right),$$

with

$$a_{ij} = \frac{p_{ij}}{\delta}, \quad b_{ij} = \frac{p_{ij} + p_{ji}}{2\delta}, \quad \delta = \sum_{i \neq j} p_{ij}.$$

Note that I_S is the Kullback–Leibler information between a_{ij} and b_{ij} for all $i \neq j$. Note that (1) the index Φ_S lies between 0 and 1, (2) $\Phi_S = 0$ if and only if the symmetry model holds, and (3) $\Phi_S = 1$ if and only if the degree of departure from symmetry is maximal, in the sense that for all $i \neq j$ either $p_{ij} = 0$ or $p_{ji} = 0$. We point out that the index Φ_S can be expressed as

$$\Phi_S = \frac{1}{\log 2} \min_{\{e_{ij}\}} \sum_{i \neq j} a_{ij} \log \left(\frac{a_{ij}}{e_{ij}} \right),$$

where minimization is constrained by

$$\sum_{i \neq j} e_{ij} = 1, \quad e_{ij} > 0, \quad e_{ij} = e_{ji}, \quad \text{for } i \neq j.$$

1.2 Index Φ_{PS} for departure from point symmetry

Tomizawa et al. (2007) proposed the index Φ_{PS} which represents the degree of departure from point symmetry. The index Φ_{PS} is independent of the dimension r . Let

$$D = \begin{cases} \{(i, j) | i, j = 1, \dots, r; (i, j) \neq ((r + 1)/2, (r + 1)/2)\} & (r : \text{odd}), \\ \{(i, j) | i, j = 1, \dots, r\} & (r : \text{even}), \end{cases}$$

be the set of cells (i, j) outside of the center of the table. Assume that $p_{ij} + p_{r+1-i, r+1-j} > 0$ for all $(i, j) \in D$. The index Φ_{PS} is expressed as follows:

$$\Phi_{PS} = \frac{1}{\log 2} I_{PS},$$

where

$$I_{PS} = \sum_{(i, j) \in D} \sum A_{ij} \log \left(\frac{A_{ij}}{B_{ij}} \right),$$

with

$$A_{ij} = \frac{p_{ij}}{\Delta}, \quad B_{ij} = \frac{p_{ij} + p_{r+1-i, r+1-j}}{2\Delta}, \quad \Delta = \sum_{(i, j) \in D} p_{ij}.$$

Note that I_{PS} is the Kullback–Leibler information between A_{ij} and B_{ij} for all $(i, j) \in D$. Note that (1) the index Φ_{PS} lies between 0 and 1, (2) $\Phi_{PS} = 0$ if and only if the point symmetry model holds, and (3) $\Phi_{PS} = 1$ if and only if the degree of departure from point symmetry is maximal, in the sense that for all $(i, j) \in D$ either $p_{ij} = 0$ or $p_{r+1-i, r+1-j} = 0$. We point out that the index Φ_{PS} can be expressed as

$$\Phi_{PS} = \frac{1}{\log 2} \min_{\{E_{ij}\}} \sum_{(i,j) \in D} A_{ij} \log \left(\frac{A_{ij}}{E_{ij}} \right),$$

where minimization is constrained by

$$\sum_{(i,j) \in D} E_{ij} = 1, \quad E_{ij} > 0, \quad E_{ij} = E_{r+1-i, r+1-j}, \quad \text{for } (i, j) \in D.$$

1.3 Index Φ_{DS} for departure from double symmetry

Yamamoto et al. (2010) proposed the index Φ_{DS} which represents the degree of departure from double symmetry. Let

$$E_1 = \begin{cases} \{(i, j) | i, j = 1, \dots, r; i = j \text{ or } i + j = r + 1; & (r : \text{odd}), \\ \quad \quad \quad (i, j) \neq ((r + 1)/2, (r + 1)/2)\} \\ \{(i, j) | i, j = 1, \dots, r; i = j \text{ or } i + j = r + 1\} & (r : \text{even}), \end{cases}$$

and, let

$$E_2 = \{(i, j) | i, j = 1, \dots, r; i \neq j \text{ and } i + j \neq r + 1\}.$$

E_1 is the set of cells (i, j) on the diagonal or on the secondary diagonal (without center point if any), while E_2 includes the remaining cells. Assume that $p_{ij} + p_{ji} + p_{r+1-i, r+1-j} + p_{r+1-j, r+1-i} > 0$ for all $(i, j) \in E_1 \cup E_2$. Let for $t = 1, 2$,

$$p_{ij(t)} = \frac{p_{ij}}{\delta_t}, \quad q_{ij(t)} = \frac{p_{ij} + p_{ji} + p_{r+1-i, r+1-j} + p_{r+1-j, r+1-i}}{4\delta_t} \quad \text{for } (i, j) \in E_t,$$

$$\delta_t = \sum_{(i,j) \in E_t} p_{ij}.$$

The index Φ_{DS} is expressed as follows:

$$\Phi_{DS} = \frac{\delta_1 \phi_1 + \delta_2 \phi_2}{\delta_1 + \delta_2},$$

where

$$\phi_t = \frac{1}{\log(2t)} I_t \quad (t = 1, 2),$$

with

$$I_t = \sum_{(i,j) \in E_t} p_{ij(t)} \log \left(\frac{p_{ij(t)}}{q_{ij(t)}} \right).$$

For the analysis of the data, when the double symmetry model does not hold, we may measure the degree of departure from the double symmetry model by using Φ_{DS} . However, the Φ_{DS} cannot simultaneously characterize the degree of departure from symmetry and the degree of departure from point symmetry (although the double symmetry model has a structure that combines both symmetry and point symmetry), see Sect. 4.

2 Index vector and a confidence region

The purpose of the present paper is to propose a bivariate index vector which represents the degree of departure from double symmetry. For measuring the degree of departure from double symmetry, the proposed index vector can simultaneously characterize the degree of departure from symmetry and the degree of departure from point symmetry. Also, the proposed index vector would be useful for visually comparing the degrees of departure from double symmetry using confidence regions.

2.1 Definition of the index vector

Assume that $p_{ij} + p_{ji} > 0$ for all $i \neq j$, and $p_{ij} + p_{r+1-i, r+1-j} > 0$ for all $(i, j) \in D$. In order to represent the degree of departure from double symmetry, we propose in this paper the bivariate index vector

$$\Psi = \begin{pmatrix} \Phi_S \\ \Phi_{PS} \end{pmatrix}; \text{ a } 2 \times 1 \text{ vector.}$$

It has the properties: (1) $\Psi = (0, 0)'$ if and only if both symmetry and point symmetry models hold (namely, the double symmetry model holds), (2) $\Psi = (1, 1)'$ if and only if the degrees of departure from both symmetry and point symmetry are maximal (i.e., the degree of departure from double symmetry is maximal), in the sense that $p_{ij} = p_{r+1-j, r+1-i} = 0$ (then $p_{ji} > 0$ and $p_{r+1-i, r+1-j} > 0$) or $p_{ji} = p_{r+1-i, r+1-j} = 0$ (then $p_{ij} > 0$ and $p_{r+1-j, r+1-i} > 0$) for all $i \neq j$, and either $p_{ii} = 0$ or $p_{r+1-i, r+1-i} = 0$ for $i = 1, \dots, r/2$ (when r is even) or $i = 1, \dots, (r-1)/2$ (when r is odd). Note that the definition of the maximum degree of departure from double symmetry for the proposed index vector Ψ is different from that for the index Φ_{DS} .

2.2 A confidence region for the index vector

Let

$\mathbf{x} = (x_{11}, x_{12}, \dots, x_{1r}, x_{21}, x_{22}, \dots, x_{2r}, \dots, x_{r1}, x_{r2}, \dots, x_{rr})'$; an $r^2 \times 1$ vector,
 $\mathbf{p} = (p_{11}, p_{12}, \dots, p_{1r}, p_{21}, p_{22}, \dots, p_{2r}, \dots, p_{r1}, p_{r2}, \dots, p_{rr})'$; an $r^2 \times 1$ vector.

We assume that \mathbf{x} has a multinomial distribution $\text{Multi}(n; \mathbf{p})$ with sample size n and probability vector \mathbf{p} . Then $\sqrt{n}(\hat{\mathbf{p}} - \mathbf{p})$ has asymptotically a normal distribution with zero mean and covariance matrix $\mathbf{Diag}(\mathbf{p}) - \mathbf{p}\mathbf{p}'$, where $\hat{\mathbf{p}} = \mathbf{x}/n$ and $\mathbf{Diag}(\mathbf{p})$ is diagonal matrix with the elements of \mathbf{p} on the main diagonal (see, e.g., Agresti 2013, p. 590). In order to estimate the indexes, $\hat{\Phi}_S$ and $\hat{\Phi}_{PS}$ are given by Φ_S and Φ_{PS} with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$, respectively. Therefore, the sample version of Ψ , i.e., $\hat{\Psi}$, is given by Ψ with Φ_S and Φ_{PS} replaced by $\hat{\Phi}_S$ and $\hat{\Phi}_{PS}$, respectively. Let $(\partial\Psi/\partial\mathbf{p}')$ denote the $2 \times r^2$ matrix for which the entry in row k and column l is $\partial\Psi_k(\mathbf{p})/\partial p_l$, where Ψ_1 and Ψ_2 denote Φ_S and Φ_{PS} , respectively, and p_l denotes the l th element of \mathbf{p} . For n approaching infinity, the estimated index vector can be approximated by

$$\hat{\Psi} = \Psi + \left(\frac{\partial\Psi}{\partial\mathbf{p}'}\right)(\hat{\mathbf{p}} - \mathbf{p}) + o(\|\hat{\mathbf{p}} - \mathbf{p}\|),$$

where $o(\|\hat{\mathbf{p}} - \mathbf{p}\|)$ tends to $(0, 0)'$. Using the delta method (see Agresti 2013, Sect. 16.1), $\sqrt{n}(\hat{\Psi} - \Psi)$ has asymptotically a bivariate normal distribution with zero mean and covariance matrix

$$\begin{aligned} \Sigma &= \left(\frac{\partial\Psi}{\partial\mathbf{p}'}\right)(\mathbf{Diag}(\mathbf{p}) - \mathbf{p}\mathbf{p}')\left(\frac{\partial\Psi}{\partial\mathbf{p}'}\right)' \\ &= \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}, \end{aligned}$$

with $\sigma_{12} = \sigma_{21}$. The elements σ_{11} , σ_{12} and σ_{22} are expressed as follows:

$$\begin{aligned} \sigma_{11} &= \left(\frac{\partial\Phi_S}{\partial\mathbf{p}'}\right)(\mathbf{Diag}(\mathbf{p}) - \mathbf{p}\mathbf{p}')\left(\frac{\partial\Phi_S}{\partial\mathbf{p}'}\right)' \\ &= \frac{1}{\delta^2} \left[\sum_{i \neq j} \sum p_{ij} (\Omega_{ij})^2 - \delta(\Phi_S)^2 \right], \\ \sigma_{12} &= \left(\frac{\partial\Phi_S}{\partial\mathbf{p}'}\right)(\mathbf{Diag}(\mathbf{p}) - \mathbf{p}\mathbf{p}')\left(\frac{\partial\Phi_{PS}}{\partial\mathbf{p}'}\right)' \\ &= \frac{1}{\delta\Delta} \left[\sum_{i \neq j} \sum p_{ij} (\Omega_{ij} - \Phi_S)(W_{ij} - \Phi_{PS}) \right], \\ \sigma_{22} &= \left(\frac{\partial\Phi_{PS}}{\partial\mathbf{p}'}\right)(\mathbf{Diag}(\mathbf{p}) - \mathbf{p}\mathbf{p}')\left(\frac{\partial\Phi_{PS}}{\partial\mathbf{p}'}\right)' \\ &= \frac{1}{\Delta^2} \left[\sum_{(i,j) \in D} \sum p_{ij} (W_{ij})^2 - \Delta(\Phi_{PS})^2 \right], \end{aligned}$$

where

$$\Omega_{ij} = \frac{1}{\log 2} \log \left(\frac{2p_{ij}}{p_{ij} + p_{ji}} \right), \quad W_{ij} = \frac{1}{\log 2} \log \left(\frac{2p_{ij}}{p_{ij} + p_{r+1-i, r+1-j}} \right).$$

Note that the previous formulas for the asymptotic variances σ_{11} and σ_{22} of Φ_S and Φ_{PS} , respectively, have been derived by Tomizawa (1994) and Tomizawa et al. (2007).

Therefore, an approximate bivariate $100(1 - \alpha)\%$ confidence region for the index vector Ψ is given by

$$n(\widehat{\Psi} - \Psi)' \widehat{\Sigma}^{-1} (\widehat{\Psi} - \Psi) \leq \chi_{(1-\alpha;2)}^2,$$

where $\chi_{(1-\alpha;2)}^2$ is the $1 - \alpha$ quantile of the chi-square distribution with two degrees of freedom and $\widehat{\Sigma}$ is given by Σ with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$.

3 Examples

Consider the data in Table 1 taken from Andersen (1997, p. 226). For these data, we are interested in whether (1) $P(H, N) = P(N, H)$, $P(H, L) = P(L, H)$, and $P(L, N) = P(N, L)$ (symmetry), and (2) $P(H, H) = P(L, L)$, $P(H, N) = P(L, N)$, $P(H, L) = P(L, H)$, and $P(N, H) = P(N, L)$ (point symmetry). Thus, for these data, we are interested in the structure of symmetry and point symmetry (namely, double symmetry).

For Table 1a, b, the estimated index vectors are

$$\widehat{\Psi} = \begin{pmatrix} 0.0770 \\ 0.0887 \end{pmatrix} \quad \text{and} \quad \widehat{\Psi} = \begin{pmatrix} 0.0148 \\ 0.0604 \end{pmatrix},$$

respectively (see Table 2).

We shall compare the degrees of departure from double symmetry in Table 1a, b using the confidence region for Ψ . From Table 1a, b, the estimates of Σ are obtained as

$$\widehat{\Sigma} = \begin{pmatrix} 1.3045 & 0.3148 \\ 0.3148 & 1.0286 \end{pmatrix} \quad \text{and} \quad \widehat{\Sigma} = \begin{pmatrix} 0.1113 & 0.0268 \\ 0.0268 & 0.2902 \end{pmatrix},$$

respectively. Figure 1 shows that the confidence ellipsoids for Ψ do not overlap for the data from Table 1a (prices) and b (production). Therefore, we see that Table 1a, b have a different structure with respect to the degree of departure from double symmetry, in the sense that Table 1a, b have a different structure with respect to the degree of departure from symmetry or point symmetry. From Fig. 1, we can conclude that the degree of departure from symmetry in Table 1a is greater than that in Table 1b. But, we cannot conclude that the degree of departure from point symmetry in Table 1a is greater than that in Table 1b. Thus, it may be difficult to judge whether the degree of departure from double symmetry in Table 1a is greater than that in Table 1b.

Also, we shall compare the degrees of departure from double symmetry in Table 1a, b using the confidence interval of Φ_{DS} . From Table 2, we see that the confidence intervals overlap for Table 1a, b. Thus, we cannot judge whether the degree of departure from double symmetry in Table 1a is greater than that in Table 1b.

Table 2 Estimates of Φ_S , Φ_{PS} and Φ_{DS} , approximate standard errors for $\hat{\Phi}_S$, $\hat{\Phi}_{PS}$ and $\hat{\Phi}_{DS}$, and approximate 95% confidence intervals for Φ_S , Φ_{PS} and Φ_{DS} , for the data of Table 1a, b

	Estimated index	Standard error	Confidence interval
(a) For Table 1a (prices)			
Φ_S	0.0770	0.0181	(0.0415, 0.1125)
Φ_{PS}	0.0887	0.0161	(0.0571, 0.1202)
Φ_{DS}	0.0817	0.0134	(0.0554, 0.1080)
(b) For Table 1b (production)			
Φ_S	0.0148	0.0053	(0.0045, 0.0251)
Φ_{PS}	0.0604	0.0085	(0.0437, 0.0771)
Φ_{DS}	0.0536	0.0075	(0.0389, 0.0684)

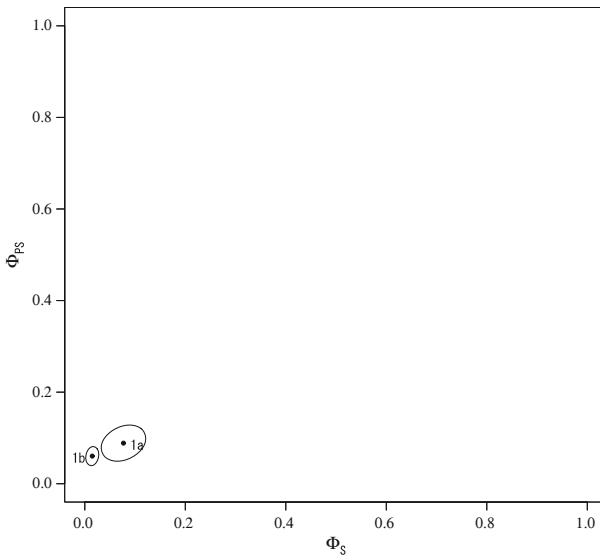


Fig. 1 Approximate 95% confidence regions for Ψ , for the data of Table 1a, b

4 Discussion

We point out that the bivariate index vector is useful for visually comparing the degrees of departure from double symmetry. For example, we consider the artificial data in Table 3a–c. From Table 4a–c, we see that all values of $\hat{\Phi}_{DS}$ for Table 3a–c are equal. However, the value $\hat{\Psi}$ for Table 3a is not equal to those for Table 3b, c.

Also, from Fig. 2, we see the degree of departure from double symmetry, while distinguishing the degree of departure from symmetry and the degree of departure from point symmetry. Thus, we see that (1) for Table 3a, the degree of departure from symmetry is large but the degree of departure from point symmetry is small, (2) for Table 3b, the degree of departure from symmetry is small but the degree of departure

Table 3 Three artificial contingency tables with $n = 4020$ each

(a)	100	10	100	100
	2000	100	100	100
	100	100	100	800
	100	100	10	100
(b)	100	10	100	100
	10	100	100	100
	100	100	100	800
	100	100	2000	100
(c)	100	10	100	100
	2000	100	100	100
	100	100	100	10
	100	100	800	100

Table 4 Estimates of Φ_S , Φ_{PS} and Φ_{DS} , approximate standard errors for $\hat{\Phi}_S$, $\hat{\Phi}_{PS}$ and $\hat{\Phi}_{DS}$, and approximate 95% confidence intervals for Φ_S , Φ_{PS} and Φ_{DS} , for the data of Table 3a–c

	Estimated index	Standard error	Confidence interval
(a) For Table 3a			
Φ_S	0.7324	0.0108	(0.7113, 0.7536)
Φ_{PS}	0.0953	0.0079	(0.0798, 0.1109)
Φ_{DS}	0.3771	0.0068	(0.3637, 0.3905)
(b) For Table 3b			
Φ_S	0.1059	0.0088	(0.0887, 0.1231)
Φ_{PS}	0.6595	0.0103	(0.6393, 0.6798)
Φ_{DS}	0.3771	0.0068	(0.3637, 0.3905)
(c) For Table 3c			
Φ_S	0.7324	0.0108	(0.7113, 0.7536)
Φ_{PS}	0.6595	0.0103	(0.6393, 0.6798)
Φ_{DS}	0.3771	0.0068	(0.3637, 0.3905)

from point symmetry is large, (3) for Table 3c, both the degree of departure from symmetry and the degree of departure from point symmetry are large.

5 Concluding remarks

This paper proposed a bivariate index vector that simultaneously characterizes the degree of departure from symmetry and the degree of departure from point symmetry. Since Φ_S and Φ_{PS} are not independent as shown in Sect. 2.2, we believe that it is important to simultaneously characterizes the degree of departure from symmetry and the degree of departure from point symmetry. In addition, the proposed index

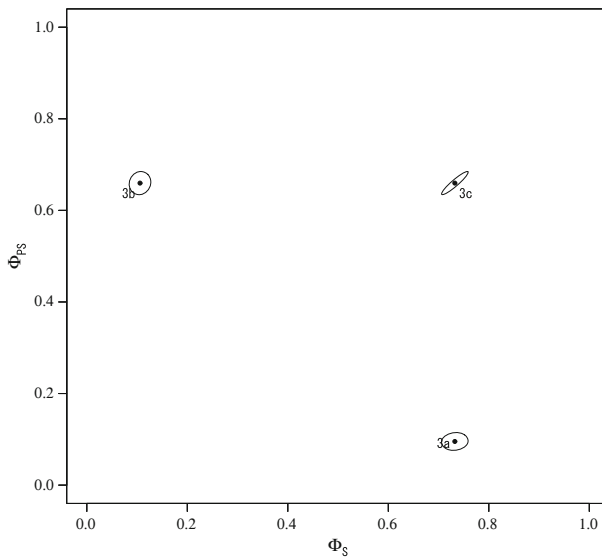


Fig. 2 Approximate 95% confidence regions for Ψ , for the data of Table 3a–c

vector would be useful for visually comparing the degrees of departure from double symmetry using confidence regions. For comparing the degrees of departure from double symmetry in several square contingency tables, we consider that with the proposed index vector it becomes easier to understand the characteristics of the data.

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